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An Operator Theoretic Approach to the Prime Number Theorem

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Dedicated to a free Ukraine

We establish an operator theoretic version of the Wiener–Ikehara Tauberian theorem and use it to obtain a short proof of the Prime number theorem that should be accessible to anyone with a basic knowledge of operator theory and Fourier analysis.

Key words: prime number theorem, Tauberian theorems, integral operators

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1. Introduction

We begin by stating a version of Ikehara's Tauberian theorem, due to Korevaar [4]. To this end, we remark that by a *pseudo-function*, we mean the distributional Fourier transform of a function $v \in L^\infty(\mathbb{R})$ such that $\lim_{|x|\to\infty} v(x) = 0$.

Theorem 1.1 (Ikehara–Wiener–Korevaar)**.** *Let $S(u)$ be a non-decreasing function with support in $[0,\infty)$, and suppose that the Laplace transform*

$$\mathscr{L}S(s) = \int_0^\infty S(u)\mathrm{e}^{-su}\,\mathrm{d}u$$

exists for $\operatorname{Re} s > 1$, and, for some constant A, let

$$g(s) = \mathscr{L}S(s) - \frac{A}{s-1}.$$

If $g(s)$ coincides with a pseudo-function on every bounded interval on the abscissa $\operatorname{Re} s = 1$ then

$$\lim_{u\to\infty} \frac{S(u)}{\mathrm{e}^u} = A.$$

Conversely, if this limit holds, then g extends to a pseudo-function on $\operatorname{Re} s = 1$.

We point out that Ikehara, a student of Wiener, originally established his Tauberian theorem in order to find a simple analytic proof for the Prime number theorem.

Our aim is to state and prove an operator theoretic generalisation of the Ikehara–Korevaar theorem. Since the machinery of operator theory allows us to avoid the delicate manipulations of limits required to prove the classical Ikehara–Korevaar theorem, this provides a more direct route to the Prime number theorem for anyone with a basic familiarity of operator theory and Fourier analysis (we mention that J.-P. Kahane has an ingenious functional analytic proof of the Prime number theorem [3]).

To motivate our approach, we recall some ideas from [5]. Specifically, we define, for intervals $I \subset \mathbb{R}$ symmetric with respect to the origin, the following operator on $L^2(I)$ (which we consider as a subspace of $L^2(\mathbb{R})$):

$$W_I : f \longmapsto \lim_{\epsilon\to 0^+} \frac{1}{\pi}\int_I f(\tau)\,\mathrm{Re}\,\frac{\zeta\big(1+\epsilon+\mathrm{i}(t-\tau)\big)}{1+\epsilon+\mathrm{i}(t-\tau)}\,\mathrm{d}\tau,$$

where $\zeta(s)=\sum_{n\in\mathbb{N}} n^{-s}$ is the Riemann zeta function. Now, on the one hand, it is well-known that

$$\frac{\zeta(s)}{s} = \frac{1}{s-1} + \psi(s), \tag{1.1}$$

where ψ is analytic on $\mathbb{C}\backslash\{0\}$. Plugging (1.1) into the formula for W_I, and noting that the term $1/(s-1)$ leads to the appearance of the Poisson kernel, we obtain

$$W_I f = f + \Psi_I f, \tag{1.2}$$

where Ψ_I is a compact operator on $L^2(I)$. On the other hand, $\zeta(s)/s$ is the Laplace transform of $\pi_{\mathbb{N}}(\mathrm{e}^u)$, where we let $\pi_{\mathbb{N}}$ denote the counting function of the integers. It therefore follows by Plancherel's theorem that

$$W_I f(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\pi_{\mathbb{N}}(\mathrm{e}^{|u|})}{\mathrm{e}^{|u|}}\hat{f}(u)\mathrm{e}^{\mathrm{i}ut}\,\mathrm{d}u. \tag{1.3}$$

By the Fourier inversion formula, in combination with (1.2) with (1.3), we obtain

$$\Psi_I f(t) = W_I f(t) - f(t) = \frac{1}{2\pi}\int_{\mathbb{R}}\left(\frac{\pi_{\mathbb{N}}(\mathrm{e}^{|u|})}{\mathrm{e}^{|u|}} - 1\right)\hat{f}(u)\mathrm{e}^{\mathrm{i}ut}\,\mathrm{d}u. \tag{1.4}$$

Inspired by this computation, we now formulate our main result, where Id denotes the identity operator on $L^2(I)$.

Theorem 1.2. *Let $S(u)$ be a non-decreasing function on $[0,\infty)$ such that the Laplace transform $\mathscr{L}S(s)$ exists for $\mathrm{Re}\,s>1$ and, for intervals $I\subset\mathbb{R}$ symmetric with respect to the origin, consider*

$$W_{S,I} : f\in L^2(I) \longmapsto \lim_{\epsilon\to 0^+}\frac{1}{\pi}\int_I f(\tau)\,\mathrm{Re}\,\mathscr{L}S\Big(1+\epsilon+\mathrm{i}(t-\tau)\Big)\,\mathrm{d}\tau.$$

Then the following holds:

(i) *For all I, $W_{S,I}$ is a well-defined bounded operator on $L^2(I)$ if and only if*

$$\sup_{u\geq 0}\frac{S(u)}{\mathrm{e}^u} < \infty.$$

(ii) *For all I sufficiently large, $W_{S,I} = A\mathrm{Id} + \Psi_{S,I}$, for some constant A and compact operator $\Psi_{S,I}$, if and only if*

$$\lim_{u\to\infty} \frac{S(u)}{\mathrm{e}^u} = A.$$

Before discussing the proof of the above result, we explain how the Prime number theorem follows from Theorem 1.2. This part of the argument should be clear to anyone familiar with Ikehara's theorem.

Corollary 1.3 (The Prime number theorem)**.** *Let $\pi_{\mathbb{P}}$ be the counting function for the prime numbers. Then*

$$\lim_{x\to\infty} \pi_{\mathbb{P}}(x)\frac{\ln x}{x} = 1.$$

Proof. Let $\zeta_{\mathbb{P}}(s) = \sum_{p \text{ prime}} p^{-s}$. Then,

$$\frac{\zeta_{\mathbb{P}}(s)}{s} = \mathscr{L}\Big\{\pi_{\mathbb{P}}(\mathrm{e}^u)\Big\}(s). \tag{1.5}$$

By taking the logarithm of the Euler product formula

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}},$$

and using the first order Taylor approximation of $\log(1-x)$, we find that

$$\log\zeta(s) = \zeta_{\mathbb{P}}(s) + \sum_{p \text{ prime}} \mathcal{O}\left(p^{-2s}\right).$$

In particular, since $\zeta(s)$ has no zeroes on $\mathrm{Re}\, s = 1$, as was independently proved by de la Vallée Poussin and Hadamard as a key step of their proofs for the Prime number theorem (see [1, 2]), it follows from (1.1) that

$$\frac{\zeta_{\mathbb{P}}(s)}{s} = \frac{1}{s}\log\frac{1}{s-1} + \psi_{\mathbb{P}}(s), \tag{1.6}$$

where $\psi_{\mathbb{P}}(s)$ is analytic in a neighbourhood of $\mathrm{Re}\, s \geq 1$. We now combine formulas (1.5) and (1.6), and differentiate, to obtain the relation

$$\mathscr{L}\Big\{u\pi_{\mathbb{P}}(\mathrm{e}^u)\Big\}(s) = \frac{1}{s-1} + \varphi_{\mathbb{P}}(s),$$

where $\varphi_{\mathbb{P}}$ is analytic in a neighbourhood of $\{\mathrm{Re}\, s \geq 1\}\backslash\{s = 1\}$ and locally integrable on $\mathrm{Re}\, s = 1$ (in the sense that $\phi_{\mathbb{P}}(\epsilon + it)$ converges in $L^1(I)$ as $\epsilon \to 0^+$). Finally, by the same reasoning used to deduce (1.2) from (1.1), this implies that for $S(u) = u\pi_{\mathbb{P}}(\mathrm{e}^u)$, we have

$$W_{S,I}f = \mathrm{Id} + \Psi_{S,I}f,$$

where $\Psi_{S,I}$ is readily seen to be compact (e.g., by Lemma 2 in [5]). And so, by Theorem 1.2, the Prime number theorem follows. □

2. Proof of Theorem 1.2

Proof of part **(i)**. First, suppose that $W_{S,I}$ is a bounded operator on $L^2(I)$, and let $\{e_n\}_{n\in\mathbb{Z}}$ denote the standard orthonormal exponential basis for $L^2(I)$. A straight-forward application of the monotone convergence theorem yields

$$\begin{aligned}\langle W_{S,I}e_n, e_n\rangle &= \lim_{\epsilon\to 0^+}\frac{1}{2\pi}\int_{\mathbb{R}}\frac{S(|u|)}{\mathrm{e}^{|u|}}\,|\hat{e}_n(u)|^2\,\mathrm{e}^{-\epsilon|u|}\,\mathrm{d}u\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\frac{S(|u|)}{\mathrm{e}^{|u|}}\,|I|\left(\frac{\sin(u\,|I|\,/2)}{u\,|I|\,/2-\pi n}\right)^2\mathrm{d}u.\end{aligned}$$

In particular, the latter integral converges. To obtain a contradiction, suppose that there exists a sequence u_k of positive numbers tending to infinity, such that $S(u_k)/\mathrm{e}^{u_k} \geq k$. Since S is non-decreasing, we get, for any $\Delta u > 0$, that

$$\frac{S(u_k+\Delta u)}{\mathrm{e}^{u_k+\Delta u}} \geq \frac{S(u_k)}{\mathrm{e}^{u_k}}\frac{1}{\mathrm{e}^{\Delta u}} \geq \frac{k}{\mathrm{e}^{\Delta u}}.$$

But this immediately implies that the sequence $\langle W_{S,I}e_n, e_n\rangle$ is unbounded, which is absurd.

To obtain the converse, suppose that $S(u)/\mathrm{e}^u$ is bounded. In particular, $S(|u|)/\mathrm{e}^{|u|}$ is a bounded Fourier multiplier on $L^2(\mathbb{R})$. As we view $L^2(I)$ as a subspace of $L^2(\mathbb{R})$, this implies the boundedness of $W_{S,I}$. Indeed,

$$\|W_{S,I}f\|_{L^2(I)} \leq \|W_{S,I}f\|_{L^2(\mathbb{R})} = \left\|\frac{S(|u|)}{\mathrm{e}^{|u|}}\hat{f}\right\|_{L^2(\mathbb{R})} \leq \sup_{u\geq 0}\frac{S(u)}{\mathrm{e}^u}\|f\|_{L^2(I)},$$

where we used the monotone convergence theorem in the second step. □

Proof of part **(ii)**. We first note that if $\Psi_{S,I}$ is compact for some interval I, then

$$|\langle \Psi_{S,I}e_n, e_n\rangle| \underset{n\to\infty}{\longrightarrow} 0. \tag{2.1}$$

As $\Psi_{S,I}$ compact implies $W_{S,I}$ bounded, it follows by the lemma that $S(u)/\mathrm{e}^u$ is bounded. So, by Lebesgue's theorem on dominated convergence,

$$\langle \Psi_{S,I}e_n, e_n\rangle = \frac{1}{2\pi}\int_{\mathbb{R}}\left(\frac{S(|u|)}{\mathrm{e}^{|u|}}-A\right)|I|\left(\frac{\sin(u\,|I|\,/2)}{u\,|I|\,/2-\pi n}\right)^2\mathrm{d}u. \tag{2.2}$$

Set $h(u) := \frac{S(|u|)}{\mathrm{e}^{|u|}} - A$. To arrive at a contradiction, suppose that $h(u)$ does not decay to 0 as $|u| \to \infty$. There are now two cases. In the first, we suppose that there exists an $\epsilon > 0$ and an unbounded sequence u_k of real numbers so that

$$h(u_k) = \frac{S(|u_k|)}{\mathrm{e}^{|u_k|}} - A \geq \epsilon.$$

Without loss of generality, we may assume that all $u_k > 0$. Since S is non-decreasing, this implies that

$$\frac{S(u_k+\Delta u)}{\mathrm{e}^{u_k+\Delta u}} \geq \frac{S(u_k)}{\mathrm{e}^{u_k}}\frac{1}{\mathrm{e}^{\Delta u}},$$

from which it follows that there exists a fixed $\Delta u > 0$ so that for all $k \in \mathbb{N}$ and $u \in [u_k, u_k + \Delta u]$ we have

$$h(u) \geq \frac{\epsilon}{2}.$$

Since $S(u)/\mathrm{e}^u$ is bounded, it is now straight-forward to apply this estimate to the integral expression in (2.2) to see that for all I large enough, there exists a constant $c = c(I) > 0$ so that for infinitely many n, we have

$$|\langle \Psi_{S,I} e_n, e_n \rangle| \geq c.$$

This contradicts (2.1). In the remaining case, we suppose that there exist an unbounded sequence of real numbers u_k so that $h(u) \leq -\epsilon$. This case is settled as above with the adjustment that we consider intervals of the type $[u_k - \Delta u, u_k]$.

To obtain the reverse implication (which is not needed to prove the Prime number theorem), we suppose that $S(u)/\mathrm{e}^u \longrightarrow A$ as $|u| \to \infty$. In the same way that we arrived at (1.4), with h as above, we obtain

$$\Psi_{S,I} = \frac{1}{2\pi} \int_{\mathbb{R}} h(u) \hat{f}(u) \mathrm{e}^{uit} \, \mathrm{d}u. \tag{2.3}$$

As $h(u)$ decays to 0, it readily follows that $\Psi_{S,I}$ is compact (e.g., by the proof of Lemma 2 in [5]). □

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Теоретико-операторний підхід до теореми про розподіл простих чисел

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Ми встановлюємо теоретико-операторну версію теореми Вінера–Ікегара–Таубера та використовуємо її для одержання короткого доведення теореми про розподіл простих чисел, яке має бути доступним будь-кому, хто володіє базовими знаннями з теорії операторів і аналізу Фур'є.

Ключові слова: теорема про розподіл простих чисел, тауберови теореми, інтегральні оператори