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Creating and controlling band gaps in periodic media with small resonators

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Dedicated to Professor V.A. Marchenko
on the occasion of his 100th birthday

We investigate spectral properties of the Neumann Laplacian $\mathscr{A}_\varepsilon$ on a periodic unbounded domain Ω_ε depending on a small parameter $\varepsilon > 0$. The domain Ω_ε is obtained by removing from $\mathbb{R}^n$ $m \in \mathbb{N}$ families of ε-periodically distributed small resonators. We prove that the spectrum of $\mathscr{A}_\varepsilon$ has at least m gaps. The first m gaps converge as $\varepsilon \to 0$ to some intervals whose location and lengths can be controlled by a suitable choice of the resonators; other gaps (if any) go to infinity. An application to the theory of photonic crystals is discussed.

Key words: periodic media, resonators, Neumann Laplacian, spectral gaps

Mathematical Subject Classification 2020: 35B27, 35P05, 47A75

1. Introduction

The problem addressed in this paper belongs to spectral analysis of periodic differential operators. It is well-known (see, e.g., [9,29,30]) that the spectrum of such operators has the form of a locally finite union of compact intervals (*bands*). In general the bands may touch each other and even (in the multidimensional case) overlap. The open interval (α, β) is called a *gap* if it has an empty intersection with the spectrum, but its endpoints belong to it.

The presence of gaps in the spectrum is not guaranteed. For example, the spectrum of the Laplacian on $\mathbb{R}^n$ has no gaps: $\sigma(-\Delta_{\mathbb{R}^n}) = [0, \infty)$. Therefore the natural and interesting problem arises here: to construct examples of periodic operators with non-void spectral gaps. This problem has been actively studied since mid of the 90th and currently a lot of examples for various classes of periodic operators are available in the literature. We refer to some pioneer articles [12–14,16,38], further references can be found in the overviews [17,30].

The problem of the spectral gaps opening received a strong motivation coming from the advances in investigation of novel materials of various sorts, in particular, the so-called *photonic crystals* — periodic dielectric nanostructure whose characteristic feature is that they strongly affect the propagation of light waves

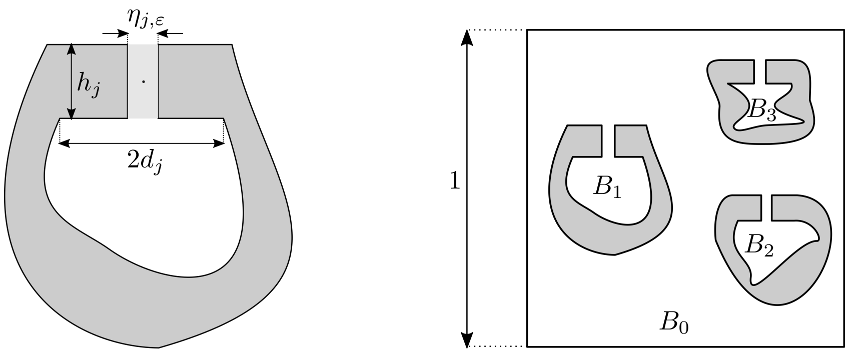

Fig. 1.1: *Left:* The sets $R_{j,\varepsilon}$ (dark gray color) and $T_{j,\varepsilon}$ (light gray color). The black dot in the center of $T_{j,\varepsilon}$ corresponds to the point z_j. *Right:* The set Y_ε (period cell before re-scaling); here $m = 3$.

at certain optical frequencies, which is caused by gaps in the spectrum of the Maxwell operator or related scalar operators. In practice, one deals with artificially fabricated photonic crystals, which are composed of relatively simple materials (dielectrics or metals) in a tricky way and have a spatially periodic structure, typically at the length scale of hundred nanometers. For more details on mathematics of photonic crystals we refer to [8].

Fig. 1.2: The domain Ω_ε, here $m = 3$. The dotted lattice separates different period cells.

The next important question in spectral theory of periodic operators concerns the possibility of engineering a prescribed gap structure (i.e., opening of spectral gaps with prescribed locations and lengths) — by choosing a properly devised material texture. In the present paper, we investigate this problem for the Neumann Laplacian $\mathscr{A}_\varepsilon$ on unbounded periodic domain $\Omega_\varepsilon \subset \mathbb{R}^n$ ($n \geq 2$), which is obtained by removing from $\mathbb{R}^n$ m families of ε-periodically distributed small resonators (see Figure 1.2):

$$\Omega_\varepsilon = \mathbb{R}^n \setminus \overline{\bigcup_{i\in\mathbb{Z}^n} \bigcup_{j=1}^{m} \varepsilon(R_{j,\varepsilon} + i)}, \quad m \in \mathbb{N}. \tag{1.1}$$

Here $\varepsilon > 0$ is a small parameter, the sets $R_{j,\varepsilon}$ (*resonators*), $j \in \{1, \dots, m\}$ have the form

$$R_{j,\varepsilon} = (F_j \setminus \overline{B_j}) \setminus T_{j,\varepsilon}$$

where $\overline{B_j} \subset F_j \subset (0,1)^n$ and $T_{j,\varepsilon}$ are thin passages connecting the opposite sides of the domain $F_j \setminus \overline{B_j}$ (see Figure 1.1, left picture). The passages diameters $\eta_{j,\varepsilon}$ are chosen as follows,

$$\eta_{j,\varepsilon} = \mathcal{O}(\varepsilon^{\frac{2}{n-1}}). \tag{1.2}$$

We demonstrate that the spectrum $\sigma(\mathscr{A}_\varepsilon)$ of $\mathscr{A}_\varepsilon$ has the following properties:

- there is $\Lambda > 0$ such that $\sigma(\mathscr{A}_\varepsilon)$ has m gaps withing the interval $[0, \Lambda\varepsilon^{-2}]$ for sufficiently small ε,
- the endpoints of these m gaps converges to the endpoints of some pairwise disjoint intervals (α_j, β_j). The numbers α_j and β_j depend on the lengths of the passages, the (re-scaled) areas of their cross-sections and the volumes of the domains F_j and B_j,
- one can choose the domains F_j, B_j and the constants standing at the expressions for $\eta_{j,\varepsilon}$ (see (2.6)) in such a way that (α_j, β_j) do coincide with predefined intervals.

Similar problem was treated by the first author in [25] for "zero-thickness" resonators, i.e. when Ω_ε is again of the form (1.1), but the sets $R_{j,\varepsilon}$ have the form $R_{j,\varepsilon} = \partial B_j \setminus T_{j,\varepsilon}$, where B_j are bounded domains and $T_{j,\varepsilon}$ are small subsets of ∂B_j. In this case the critical scaling for the diameters $\eta_{j,\varepsilon}$ of $T_{j,\varepsilon}$ is

$$\eta_{j,\varepsilon} = \mathcal{O}(\varepsilon^{\frac{2}{n-2}}) \text{ as } n > 2 \quad \text{and} \quad |\ln(\varepsilon\eta_{j,\varepsilon})|^{-1} = \mathcal{O}(\varepsilon^2) \qquad \text{as } n = 2.$$

The problem of opening of spectral gaps having prescribed locations and lengths was also considered in [22] for Laplace–Beltrami operators on periodic Riemannian manifolds, in [24] for periodic elliptic operators on $\mathbb{R}^n$, in [2, 26] for periodic quantum graphs, and in [10] for periodic Schrödinger operators with singular potentials. The proofs in [22, 24] rely on methods of homogenization theory, while in [2, 10, 25, 26] the approach is close to the one of the present paper (asymptotic analysis of the quasi-periodic, Neumann and Dirichlet eigenvalue problems on a smallest period cell).

Peculiar effects caused by inserting small resonators are long known. For example, a suitably scaled resonator may abruptly change the spectrum (even if the resonator diameter is very small) — the first example goes back to the Courant–Hilbert monograph [6], further investigations were carried out in [1, 36]. Another remarkable application of small resonators is the possibility to construct materials with frequency-dependent effective properties, with large and/or negative permittivities [32], materials with memory [33], etc. (see the overview [37] for more details).

In the next section we formulate the problem more precisely and present the main results. We also demonstrate how to apply these results for photonic crystals design.

2. Problem setting and main results

Let $\varepsilon > 0$ (the small parameter), $n \in \mathbb{N} \setminus \{1\}$ (the space dimension), $m \in \mathbb{N}$ (the number of resonators per a period cell). We set

$$\mathbb{M} \coloneqq \{1, \dots, m\}, \quad \mathbb{M}_0 \coloneqq \{0, \dots, m\}. \tag{2.1}$$

In the following, by $x' = (x^1, \dots, x^{n-1})$ and $x = (x', x^n)$ we denote the Cartesian coordinates in $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$, respectively.

Let $(F_j)_{j\in\mathbb{M}}$, $(B_j)_{j\in\mathbb{M}}$ be Lipschitz domains in $\mathbb{R}^n$ satisfying

$$\overline{B_j} \subset F_j, \quad \overline{F_j} \subset Y \coloneqq (0,1)^n, \quad j \in \mathbb{M}, \qquad \overline{F_i} \cap \overline{F_j} = \varnothing, \ i \neq j.$$

Furthermore, we assume that for each $j \in \mathbb{M}$ there exist $z_j = (z_j', z_j^n) \in \mathbb{R}^n$ and positive numbers h_j and d_j such that

$$\left\{x = (x', x^n) \in \mathbb{R}^n : \ |x^n - z_j^n| < h_j/2, \ \|x' - z_j'\|_{\mathbb{R}^{n-1}} \le d_j\right\} \subset F_j \setminus \overline{B_j}, \tag{2.2}$$
$$\left\{x = (x', x^n) \in \mathbb{R}^n : \ x^n - z_j^n = h_j/2, \ \|x' - z_j'\|_{\mathbb{R}^{n-1}} \le d_j\right\} \subset \partial F_j, \tag{2.3}$$
$$\left\{x = (x', x^n) \in \mathbb{R}^n : \ x^n - z_j^n = -h_j/2, \ \|x' - z_j'\|_{\mathbb{R}^{n-1}} \le d_j\right\} \subset \partial B_j \tag{2.4}$$

(here $\|\cdot\|_{\mathbb{R}^{n-1}}$ stands for the Euclidean distance in $\mathbb{R}^{n-1}$). We define the passage $T_{j,\varepsilon}$ connecting the opposite sides of $F_j \setminus \overline{B_j}$ via

$$T_{j,\varepsilon} \coloneqq \left\{x = (x', x^n) : \ |x^n - z_j^n| \le h_j/2, \ x' - z_j' \in \eta_{j,\varepsilon} D_j\right\}, \tag{2.5}$$

where

$$\eta_{j,\varepsilon} = \eta_j \varepsilon^{2/(n-1)}, \ \eta_j > 0, \tag{2.6}$$

$(D_j)_{j\in\mathbb{M}}$ are connected Lipschitz domains in $\mathbb{R}^{n-1}$ satisfying $0 \in D_j$ and ε is sufficiently small in order to have $\eta_{j,\varepsilon}\overline{D_j} \subset \{x' \in \mathbb{R}^{n-1} : \ \|x'\|_{\mathbb{R}^{n-1}} < d_j\}$. Finally, we define the domain Ω_ε (see Figure 1.2):

$$\Omega_\varepsilon = \mathbb{R}^n \setminus \overline{\bigcup_{i\in\mathbb{Z}^n} \bigcup_{j\in\mathbb{M}} \varepsilon(R_{j,\varepsilon} + i)}.$$

where the sets $R_{j,\varepsilon}$, $j \in \mathbb{M}$, which will play role of the resonators before scaling (see Figure 1.1, left picture), are defined via

$$R_{j,\varepsilon} = (F_j \setminus \overline{B_j}) \setminus \overline{T_{j,\varepsilon}}.$$

The set Ω_ε is $\mathbb{Z}^n$-periodic with a period cell $\varepsilon Y_\varepsilon$, where (see Figure 1.1, right picture)

$$Y_\varepsilon \coloneqq Y \setminus \overline{\bigcup_{j\in\mathbb{M}} R_{j,\varepsilon}}.$$

Now, we define the (minus) Neumann Laplacian $\mathscr{A}_\varepsilon$ on Ω_ε. In the space $\mathsf{L}^2(\Omega_\varepsilon)$ we introduce the sesquilinear form $\mathfrak{a}_\varepsilon$ via

$$\mathfrak{a}_\varepsilon[u, v] = \int_{\Omega_\varepsilon} \nabla u \cdot \overline{\nabla v} \, \mathrm{d}x, \quad \mathrm{dom}(\mathfrak{a}_\varepsilon) = \mathsf{H}^1(\Omega_\varepsilon)$$

This form is densely defined, closed, and positive, hence by the first representation theorem [21, Chapter 6, Theorem 2.1] there is a the unique self-adjoint and positive operator $\mathscr{A}_\varepsilon$ satisfying $\mathrm{dom}(\mathscr{A}_\varepsilon) \subset \mathrm{dom}(\mathfrak{a}_\varepsilon)$ and

$$(\mathscr{A}_\varepsilon u, v)_{\mathsf{L}^2(\Omega_\varepsilon)} = \mathfrak{a}_\varepsilon[u, v], \quad \forall u \in \mathrm{dom}(\mathscr{A}_\varepsilon),\ \forall v \in \mathrm{dom}(\mathfrak{a}_\varepsilon).$$

Our goal is to describe the behaviour of the spectrum $\sigma(\mathscr{A}_\varepsilon)$ of $\mathscr{A}_\varepsilon$ as $\varepsilon \to 0$. To state the results we have to introduce some notations.

For $j \in \mathbb{M}$ we denote

$$\alpha_j := \frac{\eta_j^{n-1}|D_j|}{h_j|B_j|}, \tag{2.7}$$

where the notation $|\cdot|$ stands for the volume of either a domain in $\mathbb{R}^n$ (as B_j) or a domain in $\mathbb{R}^{n-1}$ (as D_j). We assume that the numbers α_j are pairwise distinct; without loss of generality we may assume that

$$\alpha_j < \alpha_{j+1}, \quad j \in \{1, \dots, m-1\}. \tag{2.8}$$

Remark 2.1. The quantity (2.7) is connected with the so-called *Helmholtz resonance* — the phenomenon of air resonance in a cavity, which one can experience, for example, blowing across the top of an empty bottle. The Helmholtz resonator consists of a rigid container with a small neck — so, in substance, it has the same shape as the resonators we treat in this work.

Apparently, the first mathematically rigorous derivation of the resonator frequency formula was performed in [36]. In this paper the author analized the spectrum of the Laplace operator on a domain obtained by removing a small set from a bounded domain $\Omega \subset \mathbb{R}^n$ with the removed set being congruent, up to minor differences, to the re-scaled resonator $\varepsilon R_{1,\varepsilon}$ (hereinafter, we assume for simplicity that $m = 1$). It was shown in [36] that the spectrum of the Laplacian subject to the Neumann boundary conditions on the removed set boundary is asymptotically close (as $\varepsilon \to 0$) to the union of the spectrum of the Laplacian on Ω and an additional point ω^2, where ω (Helmholtz resonance frequency) is calculated as follows,

$$\omega = \lim_{\varepsilon \to 0} \sqrt{\frac{A_\varepsilon}{L_\varepsilon V_\varepsilon}}. \tag{2.9}$$

Here $V_\varepsilon = \varepsilon^n |B_1|$ is the resonator volume, $L_\varepsilon = \varepsilon h_1$ is the channel length, $A_\varepsilon = \varepsilon^{n-1}|D_{1,\varepsilon}| = \varepsilon^{n-1}\eta_{1,\varepsilon}^{n-1}|D_1|$ is the channel cross section area. Evidently, under the scaling (2.6), the right-hand-side of (2.9) coincides with α_1 given in (2.7).

Further, we consider the following function:

$$F(\lambda) := 1 + \sum_{j \in \mathbb{M}} \frac{\alpha_j |B_j|}{|B_0|(\alpha_j - \lambda)}, \tag{2.10}$$

where the set B_0 is defined by

$$B_0 := Y \setminus \overline{\cup_{j\in\mathbb{M}} F_j}.$$

It is easy to see that $F(\lambda)$ has exactly m zeros, they are real and interlace with α_j provided (2.8) holds. We denote them β_j, $j \in \mathbb{M}$ assuming that they are renumbered in the ascending order; then one has

$$\alpha_j < \beta_j < \alpha_{j+1}, \quad j \in \{1,\dots,m-1\}, \quad \alpha_m < \beta_m < \infty. \tag{2.11}$$

Remark 2.2. For $m = 1$, we have $\beta_1 = \alpha_1(|B_0| + |B_1|)|B_0|^{-1}$. For $m = 2$ and $j = 1, 2$, one has

$$\beta_j = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_1 k_1 + \alpha_2 k_2) + \frac{1}{2}(-1)^j \left((\alpha_1 + \alpha_2 + \alpha_1 k_1 + \alpha_2 k_2)^2 - 4\alpha_1\alpha_2(1 + k_1 + k_2)\right)^{1/2},$$

where $k_j := |B_j||B_0|^{-1}$. Further, using Cardano–Tartaglia and Ferrari's formulae, one can write down the exact expressions for β_j as $m = 3, 4$. It seems impossible to present exact formulae for β_j as $n \geq 5$. However, one is able to solve a kind of an inverse problem — knowing the zeros of $F(\lambda)$, to reconstruct the coefficients entering the expression for the function $F(\lambda)$ – see Theorem 2.4 and its proof for more precise statement.

We are now in position to formulate the main results of this work.

Theorem 2.3. *There exists $\Lambda > 0$ depending on the set B_0 only such that the the spectrum of $\mathscr{A}_\varepsilon$ has the following form within the interval $[0, \Lambda\varepsilon^{-2}]$ for sufficiently small ε:*

$$\sigma(\mathscr{A}_\varepsilon) \cap [0, \Lambda\varepsilon^{-2}] = [0, \Lambda\varepsilon^{-2}] \setminus \left(\bigcup_{j\in\mathbb{M}} (\alpha_{j,\varepsilon}, \beta_{j,\varepsilon})\right) \tag{2.12}$$

The closures of the intervals $(\alpha_{j,\varepsilon}, \beta_{j,\varepsilon}) \subset (0, \Lambda\varepsilon^{-2})$ are pairwise disjoint and their endpoints satisfy

$$\lim_{\varepsilon\to 0} \alpha_{j,\varepsilon} = \alpha_j, \quad \lim_{\varepsilon\to 0} \beta_{j,\varepsilon} = \beta_j. \tag{2.13}$$

Our second result states that one can choose the resonators in such a way that the limiting intervals (α_j, β_j) coincide with prescribed intervals.

Theorem 2.4. *Let $(\widetilde{\alpha}_j)_{j\in\mathbb{M}}$ and $(\widetilde{\beta}_j)_{j\in\mathbb{M}}$ be positive numbers satisfying*

$$\widetilde{\alpha}_j < \widetilde{\beta}_j < \widetilde{\alpha}_{j+1}, \quad j \in \{1,\dots,m-1\}, \quad \widetilde{\alpha}_m < \widetilde{\beta}_m < \infty. \tag{2.14}$$

Then one can choose the domains F_j, B_j and the numbers η_j in such a way that

$$\alpha_j = \widetilde{\alpha}_j, \quad \beta_j = \widetilde{\beta}_j, \quad j \in \mathbb{M}.$$

Remark 2.5. The design of domains with prescribed spectral properties is an interesting problem of spectral theory – see, e.g., [5, 15, 18] and the overview [3]. In this connection, one may ask the following natural question: is it possible to achieve *the precise coincidence* of spectral gaps of our operator $\mathscr{A}_\varepsilon$ with prescribed intervals for some fixed small $\varepsilon > 0$. Similar problem was addressed and partly solved in by the first author in [26]. In this paper the periodic Hamiltonian on a given periodic metric graph was constructed such that its spectrum has at least m gaps ($m \in \mathbb{N}$ is given), and their asymptotic behavior can be completely controlled through a suitable choice of coupling constants standing in the vertex conditions. Moreover, it was shown that for fixed (small enough) ε one can ensure the precise coincidence of *the left endpoints* of the first m spectral gaps with predefined numbers; the main ingredient in the proof of the latter result is the multi-dimensional version of the intermediate value theorem established in [15]. Unfortunately, for the operators we treat in the current work this theorem from [15] cannot be utilized; the reason is the lack of the monotonicity with respect to the parameter η_j of the left endpoints $\alpha_{j,\varepsilon}$ of the spectral gaps (for fixed ε), while, this monotonicity is one of the prerequisites to apply the above mentioned theorem from [15]. (To be precise, this monotonocity may take place, but is not at all obvious — when one deals with the *Neumann* Laplacian, the monotonicity of its eigenvalues with respect to domain variations is unclear, and usually it can be justified only for some particular cases.) Note, however, that for "zero-thickness" resonators considered in [25] such monotonicity takes place, and consequently, by using the same proof methods as in [26], one is able to achieve the precise coincidence of the left endpoints of the first m spectral gaps with predefined numbers.

Before to proceed to the proof of the above results we briefly demonstrate how to apply them for constructing periodic $2D$ photonic crystals; for more details see [27].

We introduce the following sets in $\mathbb{R}^3$ (see Figure 2.1):

$$\widetilde{\Omega}_\varepsilon = \left\{(x^1, x^2, z) \in \mathbb{R}^3 : \ x = (x^1, x^2) \in \Omega_\varepsilon, \ z \in \mathbb{R}\right\}, \quad \widetilde{R}_\varepsilon = \mathbb{R}^3 \setminus \widetilde{\Omega}_\varepsilon,$$

where $\Omega_\varepsilon \subset \mathbb{R}^2$ is a periodic domain being defined above. We assume that $\widetilde{\Omega}_\varepsilon$ is occupied by a dielectric medium with the electric permittivity and the magnetic permeability being equal to 1, while the set $\widetilde{R}_\varepsilon$ is made of a perfectly conducting material.

It is well-known that the propagation of electromagnetic waves in the dielectric $\widetilde{\Omega}_\varepsilon$ is governed by the Maxwell operator $\mathscr{M}_\varepsilon$ acting on $U = (E, H)$ (E and H are the electric and magnetic fields, respectively) as follows,

$$\mathscr{M}_\varepsilon U = (i\,\nabla \times H, \ -i\,\nabla \times E)\,,$$

subject to the conditions

$$\nabla \cdot E = \nabla \cdot H = 0 \text{ in } \widetilde{\Omega}_\varepsilon, \quad E_\tau = 0, \ H_\nu = 0 \text{ on } \partial\widetilde{R}_\varepsilon.$$

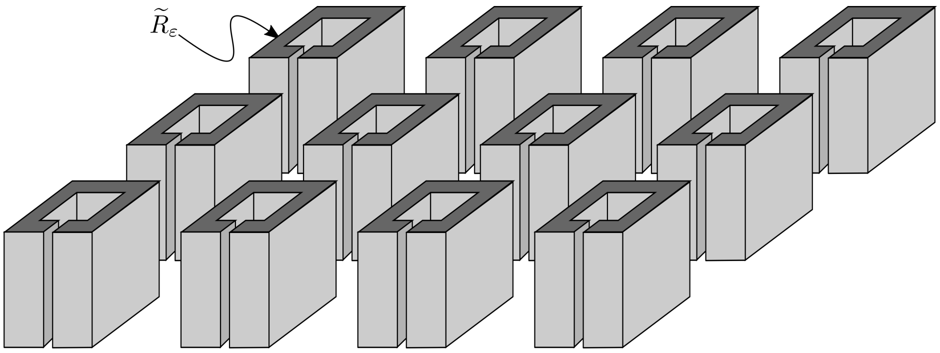

Fig. 2.1: $2D$ photonic crystal. The union of vertical columns $\widetilde{R}_\varepsilon$ is made from a perfectly conducting material, while the rest space is occupied by a dielectric

Here E_τ and H_ν are the tangential and normal components of E and H, respectively.

In the following we focus on the case when E, H depends on x_1, x_2 only, i.e. the waves propagated along the plane $\{z = 0\}$. It is known that if the medium is periodic in two directions and homogeneous with respect to the third one ($2D$ *medium*), then the analysis of the Maxwell operator reduces to the analysis of scalar elliptic operators on the two-dimensional cross-section Ω_ε. Namely, we denote

$$\begin{aligned} J &= \big\{(E,H):\ \nabla\cdot E = \nabla\cdot H = 0 \text{ in } \widetilde{\Omega}_\varepsilon,\ E_\tau = H_\mu = 0 \text{ on } \partial\widetilde{R}_\varepsilon\big\},\\ J_E &= \{(E,H)\in J:\ E_1 = E_2 = H_3 = 0\},\\ J_H &= \{(E,H)\in J:\ H_1 = H_2 = E_3 = 0\}. \end{aligned}$$

The elements of J_E and J_H are called TE (Transverse Electric)- and TM(Transverse Magnetic)-polarized waves, respectively. J_E and J_H are invariant subspaces of $\mathscr{M}_\varepsilon$, they are L^2-orthogonal, and each $U \in J$ can be represented in unique way as $U = U_E + U_H$ with $U_E \in J_E$, $U_H \in J_H$. Consequently, one has

$$\sigma(\mathscr{M}_\varepsilon) = \sigma(\mathscr{M}_\varepsilon|_{J_E}) \cup \sigma(\mathscr{M}_\varepsilon|_{J_H}). \tag{2.15}$$

We denote by $\mathcal{A}_\varepsilon^D$ and $\mathcal{A}_\varepsilon$, respectively, the Dirichlet and the Neumann Laplacians on Ω_ε. One can be show (see, e.g, [19]) that

$$\omega \in \sigma(\mathscr{M}_\varepsilon|_{J_E}) \ \Leftrightarrow\ \omega^2 \in \sigma(\mathcal{A}_\varepsilon^D) \quad \text{and} \quad \omega \in \sigma(\mathscr{M}_\varepsilon|_{J_H}) \ \Leftrightarrow\ \omega^2 \in \sigma(\mathcal{A}_\varepsilon). \tag{2.16}$$

The spectrum of $\mathscr{A}_\varepsilon$ is already described, see Theorems 2.3–2.4. As for the spectrum of $\mathscr{A}_\varepsilon^D$, one can easily derive (cf. [27, Lemma 3.1]) the following Poincare-type inequality:

$$\mathfrak{a}_\varepsilon^D[u,u] \ge C\varepsilon^{-2}\|u\|^2_{\mathsf{L}^2(\Omega_\varepsilon)}, \quad \forall u \in \mathrm{dom}(\mathfrak{a}_\varepsilon^D)$$

(the constant $C > 0$ is independent of ε, $\mathfrak{a}_\varepsilon^D$ is the form associated with $\mathscr{A}_\varepsilon^D$). Consequently,

$$\inf \sigma(\mathcal{A}_\varepsilon^D) \to \infty, \quad \varepsilon \to 0. \tag{2.17}$$

Then by virtue of Theorem 2.3, (2.15)–(2.17) we conclude that for an arbitrary (large enough) $L > 0$ the Maxwell operator $\mathscr{M}_\varepsilon$ has $2m$ gaps in $[-L, L]$ when ε is sufficiently small. These gaps converge to the intervals $\pm(\sqrt{\alpha_j}, \sqrt{\beta_j})$, whose location and lengths can be controlled via a suitable choice of the resonators (see Theorem 2.4).

The rest of the paper is devoted to the proof of the main results. In Section 3 we prove Theorem 2.3: in Subsection 3.1 we sketch some elements of Floquet–Bloch theory establishing a relationship between the spectrum of $\mathscr{A}_\varepsilon$ and the spectra of certain operators on Y_ε; in Subsection 3.2 we detect $\Lambda > 0$ such that $\sigma(\mathscr{A}_\varepsilon)$ has at most m gaps within $[0, \Lambda\varepsilon^{-2}]$; in Subsection 3.3 we recall the abstract result from [20] serving to describe the convergence of eigenvalues of operators in varying Hilbert spaces; using this abstract result we complete the proof in Subsections 3.4–3.6. In Section 4 we prove Theorem 2.4.

3. Proof of Theorem 2.3

In the following, if $\mathscr{A}$ is a self-adjoint operator with purely discrete spectrum bounded from below and accumulating at ∞, we denote by $\lambda_j(\mathscr{A})$ its kth eigenvalue, where, as usual, the eigenvalues are arranged in the ascending order and repeated according to their multiplicities.

By $C, C_1, \ldots$ we denote generic constants being independent of ε and of functions appearing in the estimates and equalities where these constants occur.

3.1. Preliminaries. The operator $\mathscr{A}_\varepsilon$ is $\mathbb{Z}^n$-periodic with the period cell $\varepsilon Y_\varepsilon$. It is convenient to work further with a period cell Y_ε, whose internal boundary ∂Y is ε-independent. Thereby, we set

$$\Xi_\varepsilon := \varepsilon^{-1}\Omega_\varepsilon = \mathbb{R}^n \setminus \overline{\bigcup_{i\in\mathbb{Z}^n}\bigcup_{j\in\mathbb{M}}(R_{j,\varepsilon} + i)},$$

and introduce the operator $\mathbf{A}_\varepsilon$ in $\mathsf{L}^2(\Xi_\varepsilon)$ via

$$\mathbf{A}_\varepsilon = -\varepsilon^{-2}\Delta_{\Xi_\varepsilon},$$

where Δ_{Ξ_ε} is the Neumann Laplacian on Ξ_ε. The operator $\mathbf{A}_\varepsilon$ is periodic with respect to the period cell Y_ε, and it is easy to see that

$$\sigma(\mathbf{A}_\varepsilon) = \sigma(\mathscr{A}_\varepsilon). \tag{3.1}$$

The Floquet–Bloch theory (see, e.g., [9, 29, 30]) establishes a relationship between the spectrum of $\mathbf{A}_\varepsilon$ and the spectra of certain operators on Y_ε. Namely, let

$$\theta = (\theta_1, \ldots, \theta_n) \in [0, 2\pi)^n.$$

We introduce the space $\mathsf{H}^{1,\theta}(Y_\varepsilon)$, which consists of functions from $\mathsf{H}^1(Y_\varepsilon)$ satisfying the following conditions at the opposite faces of ∂Y:

$$\forall k \in \{1, \ldots, n\}: \ u(x + e_k) = \exp(i\theta_k)u(x) \text{ for } x = (x_1, x_2, \ldots, \underset{k\text{-th place}}{\underset{\uparrow}{0}}, \ldots, x_n), \tag{3.2}$$

where $e_k = (0, 0, \ldots, 1, \ldots, 0)$. In the space $\mathsf{L}^2(Y_\varepsilon)$ we introduce the sesquilinear form $\mathbf{a}_\varepsilon^\theta$,

$$\mathbf{a}_\varepsilon^\theta[u,v] = \varepsilon^{-2}\int_{Y_\varepsilon} \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x, \quad \mathrm{dom}(\mathbf{a}_\varepsilon^\theta) = \mathsf{H}^{1,\theta}(Y_\varepsilon).$$

Let $\mathbf{A}_\varepsilon^\theta$ be the associated with this form self-adjoint operator. This operator acts as

$$-\varepsilon^{-2}\Delta.$$

The functions $u \in \mathrm{dom}(\mathbf{A}_\varepsilon^\theta)$ belong to $\mathsf{H}^2_{\mathrm{loc}}(Y_\varepsilon)$ and, besides (3.2), satisfy

$$\forall k \in \{1,\ldots,n\}: \ \frac{\partial u}{\partial x_k}(x + e_k) = \exp(i\theta_k)\frac{\partial u}{\partial x_k}(x) \quad \text{for } x = (x_1, x_2, \ldots, \underset{\substack{\uparrow \\ k\text{-th place}}}{0}, \ldots, x_n). \tag{3.3}$$

The spectrum of $\mathbf{A}_\varepsilon^\theta$ is purely discrete. The Floquet–Bloch theory (see, e.g., the overview [30] and the articles [11, 31, 34] treating the case of periodically perforated domains) yields

$$\sigma(\mathbf{A}_\varepsilon) = \bigcup_{k\in\mathbb{N}} L_{k,\varepsilon}, \ \text{ where } L_{k,\varepsilon} \coloneqq \cup_{\theta\in[0,2\pi)^n}\left\{\lambda_j(\mathbf{A}_\varepsilon^\theta)\right\}, \tag{3.4}$$

and moreover, for any fixed $k \in \mathbb{N}$ the set $L_{k,\varepsilon}$ is a compact interval (*the* k*th spectral band*).

Along with the operators $\mathbf{A}_\varepsilon^\theta$ we also introduce the operators $\mathbf{A}_\varepsilon^N$ and $\mathbf{A}_\varepsilon^D$, which differ from $\mathbf{A}_\varepsilon^\theta$ only by the boundary conditions at ∂Y: instead of the θ-conditions we impose the Neumann and the Dirichlet ones, respectively. More precisely, let $\mathbf{A}_\varepsilon^N$ and $\mathbf{A}_\varepsilon^D$ be the operators in $\mathsf{L}^2(Y_\varepsilon)$ being associated with the sesquilinear forms $\mathbf{a}_\varepsilon^N$ and $\mathbf{a}_\varepsilon^D$ with the domains

$$\mathbf{a}_\varepsilon^N[u,v] = \mathbf{a}_\varepsilon^D[u,v] = \varepsilon^{-2}\int_{Y_\varepsilon} \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x,$$
$$\mathrm{dom}\left(\mathbf{a}_\varepsilon^N\right) = \mathsf{H}^1(Y_\varepsilon) \quad \text{and} \quad \mathrm{dom}\left(\mathbf{a}_\varepsilon^D\right) = \left\{u \in \mathsf{H}^1(Y_\varepsilon):\ u\restriction_{\partial Y} = 0\right\}$$

The spectra of these operators are purely discrete. One has

$$\forall\theta \in [0,2\pi)^n: \quad \mathrm{dom}\left(\mathbf{a}_\varepsilon^N\right) \supset \mathrm{dom}\left(\mathbf{a}_\varepsilon^\theta\right) \supset \mathrm{dom}\left(\mathbf{a}_\varepsilon^D\right),$$

whence, by the min-max principle [7, Section 4.5], we get

$$\forall k \in \mathbb{N},\ \forall\theta \in [0,2\pi)^n: \quad \lambda_k\left(\mathbf{A}_\varepsilon^N\right) \le \lambda_k\left(\mathbf{A}_\varepsilon^\theta\right) \le \lambda_k\left(\mathbf{A}_\varepsilon^D\right). \tag{3.5}$$

We will see further that the band edges are asymptotically (as $\varepsilon \to 0$) reached by the eigenvalues of the Neumann and Dirichlet operators introduced above.

3.2. Determination of Λ. In this subsection we detect $\Lambda > 0$ such that the spectrum of $\mathscr{A}_\varepsilon$ (or, equivalently, the spectrum of $\mathbf{A}_\varepsilon$, cf. (3.1)) has at most m gaps in the interval $[0, \Lambda\varepsilon^{-2}]$.

Let $\theta \in [0, 2\pi)^n$.

For $j \in \mathbb{M}$ we denote

$$Q_{j,\varepsilon} := Y_\varepsilon \setminus \overline{B_0 \cup (\cup_{i\in\mathbb{M}:\, i\neq j} T_{i,\varepsilon} \cup B_i)}.$$

In the space $\mathsf{L}^2(Y_\varepsilon)$ we introduce the form $\mathbf{a}_\varepsilon^{\theta,\mathrm{dec}}$ by

$$\mathbf{a}_\varepsilon^{\theta,\mathrm{dec}}[u,v] = \varepsilon^{-2}\int_{B_0} \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x + \varepsilon^{-2}\sum_{j\in\mathbb{M}}\int_{Q_{j,\varepsilon}} \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x$$

on the domain

$$\mathrm{dom}(\mathbf{a}_\varepsilon^{\theta,\mathrm{dec}}) = \big\{u \in \mathsf{L}^2(Y_\varepsilon) : \\ u \in \mathsf{H}^1(B_0),\ u \text{ satisfies (3.2)},\ u \in \mathsf{H}^1(Q_{j,\varepsilon}),\ \forall j \in \mathbb{M}\big\}.$$

Let $\mathbf{A}_\varepsilon^{\theta,\mathrm{dec}}$ be the self-adjoint operator being associated with this form. Evidently, with respect to the decomposition $\mathsf{L}^2(Y_\varepsilon) = \mathsf{L}^2(B_0) \oplus \big(\oplus_{j\in\mathbb{M}} \mathsf{L}^2(Q_{j,\varepsilon})\big)$, one has

$$\mathbf{A}_\varepsilon^{\theta,\mathrm{dec}} = \left(-\varepsilon^{-2}\Delta_{B_0}^\theta\right) \oplus \left(\oplus_{k=1}^m \left(-\varepsilon^{-2}\Delta_{Q_{j,\varepsilon}}\right)\right),$$

where $\Delta_{B_0}^\theta$ is the Laplace operator on B_0 subject to the Neumann conditions on $\cup_{j\in\mathbb{M}}\partial F_j$ and conditions (3.2), (3.3) on ∂Y, $\Delta_{Q_{j,\varepsilon}}$ is the Neumann Laplacian on $Q_{j,\varepsilon}$. In fact, $\mathbf{A}_\varepsilon^{\theta,\mathrm{dec}}$ differs from $\mathbf{A}_\varepsilon^\theta$ by introducing the Neumann boundary conditions from both sides of $\overline{T_{j,\varepsilon}} \cap \overline{B_0}$, $j \in \mathbb{M}$.

It is easy to see that

$$\mathrm{dom}(\mathbf{a}_\varepsilon^{\theta,\mathrm{dec}}) \supset \mathrm{dom}(\mathbf{a}_\varepsilon^\theta) \quad\text{and}\quad \mathbf{a}_\varepsilon^{\theta,\mathrm{dec}}[u,u] = \mathbf{a}_\varepsilon^\theta[u,u], \quad \forall u \in \mathrm{dom}(\mathbf{a}_\varepsilon^\theta),$$

whence, by the min-max principle, we get

$$\forall k \in \mathbb{M}: \quad \lambda_k(\mathbf{A}_\varepsilon^{\theta,\mathrm{dec}}) \le \lambda_k(\mathbf{A}_\varepsilon^\theta).$$

The first m eigenvalues of $\mathbf{A}_\varepsilon^{\theta,\mathrm{dec}}$ are equal to zero, while the $(m+1)$th eigenvalue equals $\varepsilon^{-2}\Lambda^\theta$, where Λ^θ is the smallest eigenvalue of the operator $-\Delta^\theta(B_0)$; note that $\Lambda^\theta > 0$ if $\theta \neq (0, 0, \dots, 0)$. Hence we obtain the estimate

$$\forall\theta \in [0, 2\pi)^n:\ \varepsilon^{-2}\Lambda^\theta \le \lambda_{m+1}(\mathbf{A}_\varepsilon^\theta) \le \sup L_{m+1,\varepsilon}$$

(recall that $L_{m+1,\varepsilon}$ is the $(m+1)$th band of $\sigma(\mathbf{A}_\varepsilon)$, see (3.4)), whence

$$\varepsilon^{-2}\Lambda \le \sup L_{m+1,\varepsilon},\ \text{ where } \Lambda := \max_{\theta\in[0,2\pi)^n} \Lambda^\theta. \tag{3.6}$$

Note that Λ depends only on the set B_0.

From (3.6) and (3.4) we immediately conclude the following result.

Lemma 3.1. *The spectrum $\sigma(\mathscr{A}_\varepsilon)$ has at most m gaps within the interval $[0, \Lambda\varepsilon^{-2}]$.*

3.3. Abstract scheme. To describe the behaviour of the eigenvalues of the operators $\mathbf{A}_\varepsilon^N$, $\mathbf{A}_\varepsilon^D$ and $\mathbf{A}_\varepsilon^\theta$ as $\varepsilon \to 0$, we utilize the abstract result from [20] (see also [35] for more detailed proofs) concerning the convergence of eigenvalues of compact self-adjoint operators in varying Hilbert spaces.

Let $\mathscr{H}_\varepsilon$ and $\mathscr{H}$ be separable Hilbert spaces, and $\mathscr{R}_\varepsilon$ and $\mathscr{R}$ be linear compact self-adjoint non-negative operators in $\mathscr{H}_\varepsilon$ and $\mathscr{H}$, respectively. We denote by $\{\mu_{k,\varepsilon}\}_{k\in\mathbb{N}}$ and $\{\mu_k\}_{k\in\mathbb{N}}$ the sequences of eigenvalues of the operators $\mathscr{R}_\varepsilon$ and $\mathscr{R}$, respectively, being renumbered in the descending order and with account of their multiplicity.

Theorem 3.2 ([20, Lemma 1])**.** *Let the following conditions* (A_1)–(A_4) *hold:*

(A_1) *There exists linear bounded operator* $\mathscr{J}_\varepsilon\colon \mathscr{H} \to \mathscr{H}_\varepsilon$ *such that*

$$\forall f \in \mathscr{H}: \quad \|\mathscr{J}_\varepsilon f\|_{\mathscr{H}_\varepsilon} \to \|f\|_{\mathscr{H}} \quad \text{as } \varepsilon \to 0.$$

(A_2) *The operators* $\mathscr{R}_\varepsilon$ *are bounded uniformly in* ε*.*

(A_3) *For any* $f \in \mathscr{H}$ *one has*

$$\|\mathscr{R}_\varepsilon \mathscr{J}_\varepsilon f - \mathscr{J}_\varepsilon \mathscr{R} f\|_{\mathscr{H}_\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0.$$

(A_4) *For any family* $\{f_\varepsilon \in \mathscr{H}_\varepsilon\}_\varepsilon$ *with* $\sup_\varepsilon \|f_\varepsilon\|_{\mathscr{H}_\varepsilon} < \infty$ *there exists a subsequence* $(f_{\varepsilon_m})_{m\in\mathbb{N}}$ *with* $\varepsilon_m \to 0$ *as* $m \to \infty$ *and* $w \in \mathscr{H}$ *such that*

$$\|\mathscr{R}_{\varepsilon_m} f_{\varepsilon_m} - \mathscr{J}_{\varepsilon_m} w\|_{\mathscr{H}_{\varepsilon_m}} \to 0 \quad \text{as } m \to \infty.$$

Then for any $k \in \mathbb{N}$ *we have*

$$\mu_{k,\varepsilon} \to \mu_k \quad \text{as } \varepsilon \to 0.$$

Remark 3.3. The above result was established under the assumption $\dim \mathscr{H} = \dim \mathscr{H}_\varepsilon = \infty$. Tracing its proof in [20, 35] one can easily see that for $\dim \mathscr{H} < \infty$ and $\mathscr{R}$ being a self-adjoint operator in $\mathscr{H}$ with

$$\sigma(\mathscr{R}) = \{\mu_1 \ge \mu_2 \ge \cdots \ge \mu_{\dim(\mathscr{H})} > 0\},$$

the result reads as follows:

$$\text{conditions (A}_1\text{)–(A}_4\text{) imply } \lim_{\varepsilon\to 0} \mu_{k,\varepsilon} = \mu_k \text{ for } k \in \{1, \dots, \dim \mathscr{H}\}.$$

3.4. Asymptotic behavior of Neumann and periodic eigenvalue problems. One has:

$$\lambda_1(\mathbf{A}_\varepsilon^N) = 0. \tag{3.7}$$

For the next eigenvalues one has the following convergence result. Recall that the numbers β_j are the zeros of the function $F(\lambda)$ (2.10) being arranged according to (2.11).

Lemma 3.4. *For any $k \in \{2,\dots,m+1\}$ one has*

$$\lambda_k(\mathbf{A}_\varepsilon^N) \to \beta_{k-1}, \quad \varepsilon \to 0. \tag{3.8}$$

Proof. Let $\mathscr{H}^N$ be the space $\mathbb{C}^{m+1}$ equipped with the weighted scalar product,

$$(\mathbf{u},\mathbf{v})_{\mathscr{H}^N} = \sum_{j\in\mathbb{M}_0} u_j\overline{v_j}|B_j| \tag{3.9}$$

(recall that the notations $\mathbb{M}$ and $\mathbb{M}_0$ are defined in (2.1)). Hereinafter the elements of $\mathscr{H}^N$ are denoted by bold letters, and their entries are enumerated starting from zero:

$$\mathbf{u} \in \mathscr{H}^N \ \Rightarrow\ \mathbf{u} = (u_0,\dots,u_m) \text{ with } u_j \in \mathbb{C}.$$

In this space we introduce the sesquilinear form $\mathbf{a}^N$ via

$$\mathbf{a}^N[\mathbf{u},\mathbf{v}] = \sum_{j\in\mathbb{M}} \alpha_j|B_j|(u_j-u_0)\overline{(v_j-v_0)}, \quad \operatorname{dom}(\mathbf{a}^N) = \mathscr{H}^N.$$

Let $\mathbf{A}^N$ be the operator in $\mathscr{H}^N$ associated with this form. It is represented by the $(m+1)\times(m+1)$ symmetric (with respect to the scalar product (3.9)) matrix

$$\mathbf{A}^N = \begin{pmatrix} \sum\limits_{k=1}^m \alpha_j|B_j||B_0|^{-1} & -\alpha_1|B_1||B_0|^{-1} & -\alpha_2|B_2||B_0|^{-1} & \dots & -\alpha_m|B_m||B_0|^{-1} \\ -\alpha_1 & \alpha_1 & 0 & \dots & 0 \\ -\alpha_2 & 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_m & 0 & 0 & \dots & \alpha_m \end{pmatrix}. \tag{3.10}$$

Remark 3.5. The matrix $\mathbf{A}^N$ has an interesting interpretation. Let Γ be a combinatorial star graph with $m+1$ vertices. We denote by $\mathscr{V} = (v_k,\ k\in\mathbb{M}_0)$ the set of these vertices; here v_0 is the internal vertex and v_k, $k\in\mathbb{M}$ are the boundary vertices. We equip the graph with the measure $m:\mathscr{V}\to\mathbb{C}$, which is defined by $m(v_k) = |B_k|$, $k\in\mathbb{M}_0$. Then (see, e.g., [28, Chapter 2]) the matrix $\mathbf{A}^N$ corresponds to the weighted discrete Laplacian on (Γ, m), where the weight function (i.e., the function assigning a number to a pair of connected vertices) is defined by $b_{v_0,v_k} = \alpha_k|B_k|$, $k\in\mathbb{M}$.

We denote the eigenvalues of $\mathbf{A}^N$ by $\lambda_1(\mathbf{A}^N) \le \lambda_2(\mathbf{A}^N) \le \dots \le \lambda_{m+1}(\mathbf{A}^N)$. The eigenvalue problem $\mathbf{A}^N\mathbf{u} = \lambda\mathbf{u}$ is equivalent to the following system of $m+1$ linear equations with unknowns $u_0,\dots,u_m$:

$$\left(\sum_{i\in\mathbb{M}} \alpha_i|B_i||B_0|^{-1}\right)u_0 - \sum_{j\in\mathbb{M}} \alpha_j|B_j||B_0|^{-1}u_j = \lambda u_0, \tag{3.11}$$

$$-\alpha_k u_0 + \alpha_k u_k = \lambda u_k, \quad k \in \mathbb{M}. \tag{3.12}$$

If $\lambda = \alpha_k$ for some $k \in \mathbb{M}$, then $u_0 = 0$ due to (3.12); consequently, by virtue of (3.11), $u_k = 0$ for each $k \in \mathbb{M}$, which contradics to $\mathbf{u} \neq 0$. Thus $\lambda \notin \{\alpha_k, k \in \mathbb{M}\}$. Taking this fact into account, we conclude from (3.12):

$$u_k = \frac{\alpha_k}{\alpha_k - \lambda} u_0, \quad k \in \mathbb{M}. \tag{3.13}$$

Inserting (3.13) into (3.11), we arrive at the equality

$$\lambda F(\lambda) u_0 = 0,$$

where the function $F(\lambda)$ is defined by (2.10). Note that $u_0 \neq 0$ (otherwise, due to (3.13), $\mathbf{u}$ would be zero vector). Thus λ is an eigenvalue of $\mathbf{A}^N$ iff either $\lambda = 0$ or $F(\lambda) = 0$. whence (cf. (2.11))

$$\lambda_1(\mathbf{A}^N) = 0, \quad \lambda_k(\mathbf{A}^N) = \beta_{k-1}, \quad k = 2, \dots, m+1. \tag{3.14}$$

Our goal is to show that for $k = 1, \dots, m+1$ one has

$$\lambda_k(\mathbf{A}_\varepsilon^N) \to \lambda_k(\mathbf{A}^N) \quad \text{as } \varepsilon \to 0. \tag{3.15}$$

Then the desired convergence result (3.8) follows immediately from (3.14)–(3.15).

To prove (3.15) we utilize abstract Theorem 3.2. We denote $\mathscr{H}_\varepsilon := \mathsf{L}^2(Y_\varepsilon)$ and introduce the operators $\mathscr{R}_\varepsilon^N$ and $\mathscr{R}^N$ acting in $\mathscr{H}_\varepsilon$ and $\mathscr{H}^N$, respectively:

$$\mathscr{R}_\varepsilon^N := (\mathbf{A}_\varepsilon^N + \mathrm{I})^{-1}, \quad \mathscr{R}^N := (\mathbf{A}^N + \mathrm{I})^{-1}$$

(hereinafter I stands for an identity operator). The operators $\mathscr{R}_\varepsilon^N$, $\mathscr{R}^N$ are compact, non-negative, moreover, one has

$$\|\mathscr{R}_\varepsilon^N\| \le 1. \tag{3.16}$$

We denote by $\{\mu_{k,\varepsilon}^N\}_{k\in\mathbb{N}}$ the set of the eigenvalues of $\mathscr{R}_\varepsilon^N$ being renumbered in the descending order and with account of their multiplicity. By virtue of spectral mapping theorem one has

$$\mu_{k,\varepsilon}^N = (\lambda_k(\mathbf{A}_\varepsilon^N) + \mathrm{I})^{-1}, \quad k \in \mathbb{N}. \tag{3.17}$$

Similarly, we have

$$\mu_k^N = (\lambda_k(\mathbf{A}^N) + \mathrm{I})^{-1}, \quad k = 1, \dots, m+1, \tag{3.18}$$

where $1 = \mu_1^N \ge \mu_2^N \ge \dots \ge \mu_{m+1}^N > 0$ are the eigenvalues of the operator $\mathscr{R}^N$. Finally, we introduce the linear operator $\mathscr{J}_\varepsilon^N : \mathscr{H}^N \to \mathscr{H}_\varepsilon$ acting on $\mathbf{f} = (f_0, \dots, f_m)$ as follows,

$$(\mathscr{J}_\varepsilon^N \mathbf{f})(x) = \begin{cases} f_j, & x \in B_j,\ j \in \mathbb{M}_0 \\ 0, & x \in T_{j,\varepsilon},\ j \in \mathbb{M} \end{cases}.$$

It is easy to see that for each $\mathbf{f} \in \mathscr{H}^N$ one has

$$\|\mathscr{J}_\varepsilon^N \mathbf{f}\|_{\mathscr{H}_\varepsilon} = \|\mathbf{f}\|_{\mathscr{H}^N}. \tag{3.19}$$

Below we demonstrate that the operators $\mathscr{R}_\varepsilon^N$, $\mathscr{R}^N$ and $\mathscr{J}_\varepsilon^N$ satisfy the conditions (A$_3$) and (A$_4$) of Theorem 3.2 (the other two conditions (A$_1$) and (A$_2$) are fulfilled due to (3.19) and (3.16)); then, by virtue of Theorem 3.2 and Remark 3.3 we get

$$\mu_{k,\varepsilon}^N \to \mu_k^N \quad \text{as } \varepsilon \to 0$$

for $k = 1, \ldots, m+1$, whence, owing to (3.17)–(3.18), the desired convergence (3.15) follows.

Let us check the fulfillment of (A$_3$). Let $\mathbf{f} \in \mathscr{H}^N$. We denote $f_\varepsilon := \mathscr{J}_\varepsilon^N \mathbf{f}$ and

$$u_\varepsilon := \mathscr{R}_\varepsilon^N f_\varepsilon. \tag{3.20}$$

Then $\mathbf{A}_\varepsilon^N u_\varepsilon + u_\varepsilon = f_\varepsilon$, whence, $u_\varepsilon \in \mathsf{H}^1(Y_\varepsilon)$ and

$$\mathbf{a}_\varepsilon^N[u_\varepsilon, v_\varepsilon] + (u_\varepsilon, v_\varepsilon)_{\mathscr{H}_\varepsilon} = (f_\varepsilon, v_\varepsilon)_{\mathscr{H}_\varepsilon}, \ \forall v_\varepsilon \in \mathsf{H}^1(Y_\varepsilon). \tag{3.21}$$

The equality (3.21) implies easily the estimate

$$\varepsilon^{-2}\|\nabla u_\varepsilon\|^2_{\mathsf{L}^2(Y_\varepsilon)} + \|u_\varepsilon\|^2_{\mathsf{L}^2(Y_\varepsilon)} \le \|f_\varepsilon\|^2_{\mathsf{L}^2(Y_\varepsilon)} = \|\mathbf{f}\|^2_{\mathscr{H}^N}. \tag{3.22}$$

In particular, it follows from (3.22) that the norms $\|u_\varepsilon\|_{\mathsf{H}^1(Y_\varepsilon)}$ are uniformly bounded with respect to $\varepsilon \in (0,1]$. Hence, by virtue of Banach–Alaoglu and Rellich–Kondrachov theorems, there exists a sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ with $\varepsilon_m \searrow 0$ as $m \to \infty$ and $u_j \in \mathsf{H}^1(B_j)$, $j \in \mathbb{M}_0$ such that

$$\nabla u_{\varepsilon_m} \to \nabla u_j \quad \text{weakly in } \mathsf{L}^2(B_j), \tag{3.23}$$

$$u_{\varepsilon_m} \to u_j \quad \text{strongly in } \mathsf{L}^2(B_j), \tag{3.24}$$

as $m \to \infty$. Furthermore, using (3.22), (3.23), we get

$$\|\nabla u_j\|^2_{\mathsf{L}^2(B_j)} \le \liminf_{m\to\infty} \|\nabla u_{\varepsilon_m}\|_{\mathsf{L}^2(B_j)} \le \lim_{m\to\infty} (\varepsilon_m)^2 \|\mathbf{f}\|_{\mathscr{H}^N} = 0,$$

whence u_j are constant functions, and we can regard $\mathbf{u} = (u_0, \ldots, u_m)$ as the element of $\mathscr{H}^N$. Note that (3.24) implies

$$\lim_{m\to\infty} \langle u_{\varepsilon_m} \rangle_{B_j} = \langle u_j \rangle_{B_j} = u_j, \quad j \in \mathbb{M}_0. \tag{3.25}$$

where by $\langle u \rangle_B$ we denote the mean value of the function $u(x)$ over the domain B, i.e.,

$$\langle u \rangle_B = \frac{1}{|B|} \int_B u(x)\,\mathrm{d}x.$$

The same notation will be used further (see (3.29)) for the mean value of a function defined on a subset S of an $(n-1)$-dimensional hyperplane, i.e,

$$\langle u \rangle_S = \frac{1}{|S|} \int_S u\,\mathrm{d}s, \quad |S| = \int_S \mathrm{d}s,$$

where $\mathrm{d}s$ stands for the density of the surface measure on S.

Let $\mathbf{v} = (v_0, \dots, v_m) \in \mathscr{H}^N$. We define the function $v_\varepsilon \in \mathsf{H}^1(Y_\varepsilon)$ via

$$v_\varepsilon(x) := \begin{cases} v_j, & x \in B_j,\ j \in \mathbb{M}_0 \\ \frac{v_0 - v_j}{h_j}(x^n - z_j^n) + \frac{v_0 + v_j}{2}, & x \in T_{j,\varepsilon},\ j \in \mathbb{M} \end{cases}$$

(recall that $z_j = (z_j', z_j^n) \in \mathbb{R}^n$ is a point around which we built the passage $T_{j,\varepsilon}$, see (2.5)). Inserting this v_ε into (3.21), we arrive at the equality

$$\varepsilon^{-2} \sum_{j\in\mathbb{M}} \left(\int_{T_{j,\varepsilon}} \frac{\partial u_\varepsilon}{\partial x^n}\, \mathrm{d}x \right) \overline{\frac{v_0 - v_j}{h_j}} + \sum_{j\in\mathbb{M}_0} \langle u_\varepsilon \rangle_{B_j} \overline{v_j} |B_j| \\ + \sum_{j\in\mathbb{M}} (u_\varepsilon, v_\varepsilon)_{\mathsf{L}^2(T_{j,\varepsilon})} = (\mathbf{f}, \mathbf{v})_{\mathscr{H}^N}. \qquad (3.26)$$

Using (3.25) we get

$$\lim_{m\to\infty} \sum_{j\in\mathbb{M}_0} \langle u_{\varepsilon_m} \rangle_{B_j} \overline{v_j} |B_j| = \sum_{j\in\mathbb{M}_0} u_j \overline{v_j} |B_j| = (\mathbf{u}, \mathbf{v})_{\mathscr{H}^N}, \qquad (3.27)$$

Further, one has

$$|(u_\varepsilon, v_\varepsilon)_{\mathsf{L}^2(T_{j,\varepsilon})}| \le \|u_\varepsilon\|_{\mathsf{L}^2(T_{j,\varepsilon})} \|v_\varepsilon\|_{\mathsf{L}^2(T_{j,\varepsilon})} \le \|u_\varepsilon\|_{\mathsf{L}^2(Y_\varepsilon)} \max\{|v_j|;\ |v_0|\} |T_{j,\varepsilon}|^{1/2},$$

whence, taking into account that $\|u_\varepsilon\|_{\mathsf{L}^2(Y_\varepsilon)} \le \|f\|_{\mathscr{H}^N}$ and $|T_{j,\varepsilon}| \to 0$, we conclude

$$\lim_{\varepsilon\to 0} \sum_{j\in\mathbb{M}} (u_\varepsilon, v_\varepsilon)_{\mathsf{L}^2(T_{j,\varepsilon})} = 0. \qquad (3.28)$$

Now, let us inspect the first term in the left-hand-side of (3.26). We denote by $S_{j,\varepsilon}^\pm$ the top and bottom faces of the passage $T_{j,\varepsilon}$, i.e.,

$$S_{j,\varepsilon}^\pm := \left\{ x = (x', x^n) \in \mathbb{R}^n :\ x^n - z_j^n = \pm h_j/2,\ x' - z_j' \in \eta_{j,\varepsilon} D_j \right\}.$$

Then, integrating by parts, we obtain:

$$\varepsilon^{-2} \sum_{j\in\mathbb{M}} \left(\int_{T_{j,\varepsilon}} \frac{\partial u_\varepsilon}{\partial x^n}\, \mathrm{d}x \right) \overline{\frac{v_0 - v_j}{h_j}} = \varepsilon^{-2} \sum_{j\in\mathbb{M}} \left(\int_{S_{j,\varepsilon}^+} u\, \mathrm{d}s - \int_{S_{j,\varepsilon}^-} u\, \mathrm{d}s \right) \overline{\frac{v_0 - v_j}{h_j}} \\ = \sum_{j\in\mathbb{M}} \alpha_j |B_j| \left(\langle u_\varepsilon \rangle_{S_{j,\varepsilon}^+} - \langle u_\varepsilon \rangle_{S_{j,\varepsilon}^-} \right) (\overline{v_0 - v_j}) \qquad (3.29)$$

(on the last step we use $|S_{j,\varepsilon}^\pm| = (\eta_{j,\varepsilon})^{n-1} |D_j| = (\eta_j)^{n-1} \varepsilon^2 |D_j| = \alpha_j |B_j| h_j \varepsilon^2$, see (2.6) and (2.7)). One has the following estimates for $j \in \mathbb{M}$:

$$|\langle u_\varepsilon \rangle_{S_{j,\varepsilon}^+} - \langle u_\varepsilon \rangle_{B_0}|^2 \le C \|\nabla u_\varepsilon\|^2_{\mathsf{L}^2(B_0)} \begin{cases} (\eta_{k,\varepsilon})^{2-n}, & n \ge 3 \\ |\ln \eta_{k,\varepsilon}|, & n = 2 \end{cases}, \qquad (3.30)$$

$$|\langle u_\varepsilon\rangle_{S^-_{j,\varepsilon}} - \langle u_\varepsilon\rangle_{B_j}|^2 \le C\|\nabla u_\varepsilon\|^2_{\mathsf{L}^2(B_j)} \begin{cases}(\eta_{k,\varepsilon})^{2-n}, & n\ge 3\\ |\ln\eta_{k,\varepsilon}|, & n=2\end{cases}. \tag{3.31}$$

The proof is similar to the proof of [23, Lemma 2.1]. From (2.6), (3.22), (3.30), (3.31) we get

$$|\langle u_\varepsilon\rangle_{S^+_{j,\varepsilon}} - \langle u_\varepsilon\rangle_{B_0}|^2 + |\langle u_\varepsilon\rangle_{S^-_{j,\varepsilon}} - \langle u_\varepsilon\rangle_{B_j}|^2 = \left.\begin{cases}\mathcal{O}(\varepsilon^{2/(n-1)}), & n\ge 3\\ \mathcal{O}(\varepsilon^2|\ln\varepsilon|), & n=2\end{cases}\right\} \to 0 \quad \text{as } \varepsilon\to 0. \tag{3.32}$$

Using (3.25) and (3.32), we finally conclude from (3.29):

$$\lim_{m\to\infty}(\varepsilon_m)^{-2}\sum_{j\in\mathbb{M}}\left(\int_{T_{j,\varepsilon_m}}\frac{\partial u_{\varepsilon_m}}{\partial x^n}\,\mathrm{d}x\right)\frac{\overline{v_0-v_j}}{h_j} = \sum_{j\in\mathbb{M}}\alpha_j|B_j|(u_0-u_j)(\overline{v_0-v_j}) = \mathbf{a}^N[\mathbf{u},\mathbf{v}]. \tag{3.33}$$

Combining (3.26), (3.27), (3.28), (3.33) we arrive at the equality

$$\mathbf{a}^N[\mathbf{u},\mathbf{v}] + (\mathbf{u},\mathbf{v})_{\mathscr{H}^N} = (\mathbf{f},\mathbf{v})_{\mathscr{H}^N}, \quad \forall v\in\mathscr{H}^N,$$

whence we get

$$\mathbf{u} = \mathscr{R}^N\mathbf{f}. \tag{3.34}$$

The limiting vector $\mathbf{u}$ is independent of the sequence u_{ε_m} satisfying (3.23)–(3.24) ($\mathbf{u}$ is defined in a unique way by (3.34)), whence we conclude that the whole family u_ε converges to $\mathbf{u}$:

$$u_\varepsilon \to u_j \text{ strongly in } \mathsf{L}^2(B_j) \text{ as } \varepsilon\to 0, \quad j\in\mathbb{M}_0. \tag{3.35}$$

Finally, using (3.20), (3.34) and the definition of the operator $\mathscr{J}^N_\varepsilon$, we get

$$\|\mathscr{R}^N_\varepsilon\mathscr{J}^N_\varepsilon\mathbf{f} - \mathscr{J}^N_\varepsilon\mathscr{R}^N\mathbf{f}\|^2_{\mathscr{H}_\varepsilon} = \sum_{j\in\mathbb{M}_0}\|u_\varepsilon - u_j\|^2_{\mathsf{L}^2(B_j)} + \sum_{j\in\mathbb{M}}\|u_\varepsilon\|^2_{\mathsf{L}^2(T_{j,\varepsilon})}. \tag{3.36}$$

Due to (3.35) the first term in the right-hand-side of (3.36) tends to zero as $\varepsilon\to 0$. Furthermore, one has the following estimate:

$$\forall u\in\mathsf{H}^1(T_{j,\varepsilon}\cup B_0):$$
$$\|u\|^2_{\mathsf{L}^2(T_{j,\varepsilon})} \le C\left(\eta^{n-1}_{j,\varepsilon}\|u\|^2_{\mathsf{L}^2(B_0)} + \eta_{j,\varepsilon}\kappa_{j,\varepsilon}\|\nabla u\|^2_{\mathsf{L}^2(B_0)} + \|\nabla u\|^2_{\mathsf{L}^2(T_{j,\varepsilon})}\right), \tag{3.37}$$

where $\kappa_{j,\varepsilon} := 1$ as $n\ge 3$ and $\kappa_{j,\varepsilon} := |\ln\eta_{j,\varepsilon}|$ as $n=2$. The proof of (3.37) is similar to the proof of inequality (5.16) in [4]. It follows from (2.6), (3.22), (3.37) that

$$\|u_\varepsilon\|^2_{\mathsf{L}^2(T_{j,\varepsilon})} \le C\left(\eta^{n-1}_{j,\varepsilon}\|u_\varepsilon\|^2_{\mathsf{L}^2(Y_\varepsilon)} + \|\nabla u_\varepsilon\|^2_{\mathsf{L}^2(Y_\varepsilon)}\right) \le C_1\varepsilon^2\|\mathbf{f}\|^2_{\mathscr{H}^N}. \tag{3.38}$$

Thus the second term in the right-hand-side of (3.36) goes to zero too; consequently, condition (A$_3$) is fulfilled.

Finally, we check the fulfillment of (A$_4$). Let $f_\varepsilon \in \mathscr{H}_\varepsilon$ with $\|f_\varepsilon\|_{\mathscr{H}_\varepsilon} \le C$. We set

$$u_\varepsilon := \mathscr{R}_\varepsilon^N f_\varepsilon. \tag{3.39}$$

The function u_ε belongs to $\mathsf{H}^1(Y_\varepsilon)$, it satisfies (3.21) and the estimate (3.22) holds true. From this estimate we conclude that there exist a sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ with $\varepsilon_m \searrow 0$ as $m \to \infty$ and $w_j \in \mathsf{H}^1(B_j)$, $j \in \mathbb{M}_0$ such that

$$u_{\varepsilon_m} \to w_j \text{ strongly in } \mathsf{L}^2(B_j) \tag{3.40}$$

as $m \to \infty$; furthermore, the functions w_j are constants (so, we can regard $\mathbf{w} = (w_0, \dots, w_m)$ as the element of $\mathscr{H}^N$). Also, similarly to (3.38), we get

$$\|u_\varepsilon\|^2_{\mathsf{L}^2(T_{j,\varepsilon})} \le C\varepsilon^2 \|f_\varepsilon\|^2_{\mathscr{H}_\varepsilon} \to 0 \text{ as } \varepsilon \to 0. \tag{3.41}$$

It follows from (3.39)–(3.41) and the definition of the operator $\mathscr{J}_\varepsilon^N$ that

$$\left\|\mathscr{R}^N_{\varepsilon_m} f_{\varepsilon_m} - \mathscr{J}^N_{\varepsilon_m}\mathbf{w}\right\|_{\mathscr{H}_{\varepsilon_m}} \to 0 \text{ as } m \to \infty.$$

Hence condition (A$_4$) is also fulfilled. This completes the proof Lemma 3.4. □

Similar result holds for the eigenvalues of the operator $\mathbf{A}_\varepsilon^\theta$ with $\theta = (0, 0, \dots, 0)$, which corresponds to the periodic conditions on ∂Y. One has:

$$\lambda_1\big(\mathbf{A}_\varepsilon^\theta\big) = 0 \quad \text{if } \theta = (0, 0, \dots, 0), \tag{3.42}$$

while for the next eigenvalues one has the following lemma.

Lemma 3.6. *Let $\theta = (0, 0, \dots, 0)$. Then for any $k \in \{2, \dots, m+1\}$ one has*

$$\lambda_k\big(\mathbf{A}_\varepsilon^\theta\big) \to \beta_{k-1}, \quad \varepsilon \to 0.$$

The proof of Lemma 3.6 repeats verbatim the proof of Lemma 3.4. Note that the boundary conditions (3.2) imply no restrictions on the limiting constant u_0 (see (3.23)–(3.24)), since any constant function satisfies (3.2) if $\theta = (0, 0, \dots, 0)$.

3.5. Asymptotic behavior of Dirichlet and antiperiodic eigenvalue problems.

Recall that the numbers α_j are given in (2.7) and satisfy (2.8).

Lemma 3.7. *For any $k \in \{1, \dots, m\}$ one has*

$$\lambda_k(\mathbf{A}_\varepsilon^D) \to \alpha_k, \quad \varepsilon \to 0. \tag{3.43}$$

Proof. The proof resembles the one of Lemma 3.4, thus we underline only the principal differences. Let $\mathscr{H}^D$ be the space $\mathbb{C}^m$ equipped with the scalar product

$$(\mathbf{u},\mathbf{v})_{\mathscr{H}^D} = \sum_{j\in\mathbb{M}} u_j\overline{v_j}|B_j| \tag{3.44}$$

(its elements are denoted by bold letters, their entries are enumerated from 1 to m). In the space $\mathscr{H}^D$ we introduce the sesquilinear form $\mathbf{a}^D$ via

$$\mathbf{a}^D[\mathbf{u},\mathbf{v}] = \sum_{j\in\mathbb{M}} \alpha_j|B_j|u_j\overline{v_j},\quad \mathrm{dom}\left(\mathbf{a}^D\right) = \mathscr{H}^D.$$

The associated (with respect to the scalar product (3.44)) operator $\mathbf{A}^D$ is represented by the $m\times m$ matrix

$$\mathbf{A}^D = \mathrm{diag}(\alpha_1,\dots,\alpha_m).$$

Its eigenvalues are given by $\lambda_1(\mathbf{A}^D)\le\lambda_2(\mathbf{A}^D)\le\dots\le\lambda_m(\mathbf{A}^D)$, and we have

$$\lambda_k\left(\mathbf{A}^D\right) = \alpha_k,\quad k=1,\dots,m. \tag{3.45}$$

Below we demonstrate that for $k=1,\dots,m$ one has

$$\lambda_k\left(\mathbf{A}^D_\varepsilon\right)\to\lambda_k(\mathbf{A}^D),\quad \varepsilon\to 0. \tag{3.46}$$

Then the desired convergence result (3.43) follows immediately from (3.45)–(3.46).

For the proof of (3.46) we again use Theorem 3.2. As before, we denote $\mathscr{H}_\varepsilon := \mathsf{L}^2(Y_\varepsilon)$, and introduce compact non-negative operators $\mathscr{R}^D_\varepsilon := \left(\mathbf{A}^D_\varepsilon + \mathrm{I}\right)^{-1}$ and $\mathscr{R}^D := \left(\mathbf{A}^D + \mathrm{I}\right)^{-1}$ acting in $\mathscr{H}_\varepsilon$ and $\mathscr{H}^D$, respectively. One has

$$\left\|\mathscr{R}^D_\varepsilon\right\| \le 1. \tag{3.47}$$

We denote by $\{\mu^D_{k,\varepsilon}\}_{k\in\mathbb{N}}$ the set of the eigenvalues of $\mathscr{R}^D_\varepsilon$ being renumbered in the descending order and with account of their multiplicity; similarly, $\mu^D_1\ge\mu^D_2\ge\dots\ge\mu^D_{m+1}$ stand for the eigenvalues of the operator $\mathscr{R}^D$. One has

$$\mu^D_{k,\varepsilon} = \left(\lambda_k\left(\mathbf{A}^D_\varepsilon\right)+\mathrm{I}\right)^{-1},\ k\in\mathbb{N},\quad \mu^D_k = \left(\lambda_k\left(\mathbf{A}^D\right)+\mathrm{I}\right)^{-1},\ k=1,\dots,m, \tag{3.48}$$

Finally, we introduce the operator $\mathscr{J}^D_\varepsilon : \mathscr{H}^D\to\mathscr{H}_\varepsilon$ acting on $\mathbf{f}=(f_0,\dots,f_m)$ as follows,

$$\left(\mathscr{J}^D_\varepsilon\mathbf{f}\right)(x) = \begin{cases} f_j, & x\in B_j,\ j\in\mathbb{M},\\ 0, & x\in B_0\cup\left(\cup_{j\in\mathbb{M}}T_{j,\varepsilon}\right).\end{cases}$$

Obviously, for each $\mathbf{f}\in\mathscr{H}^D$ we have

$$\left\|\mathscr{J}^D_\varepsilon\mathbf{f}\right\|_{\mathscr{H}_\varepsilon} = \|\mathbf{f}\|_{\mathscr{H}^D}. \tag{3.49}$$

The conditions (A_1) and (A_2) of Theorem 3.2 are fulfilled, see (3.47) and (3.49). Let us check the fulfillment of the condition (A_3). Let $\mathbf{f} \in \mathscr{H}^D$, $f_\varepsilon := \mathscr{J}_\varepsilon^D \mathbf{f}$ and

$$u_\varepsilon := \mathscr{R}_\varepsilon^D f_\varepsilon. \tag{3.50}$$

Then $u_\varepsilon \in \mathsf{H}^1(Y_\varepsilon)$, $u_\varepsilon = 0$ on ∂Y,

$$\mathbf{a}_\varepsilon^D[u_\varepsilon, v_\varepsilon] + (u_\varepsilon, v_\varepsilon)_{\mathscr{H}_\varepsilon} = (f_\varepsilon, v_\varepsilon)_{\mathscr{H}_\varepsilon}, \quad \forall v_\varepsilon \in \mathrm{dom}(\mathbf{a}_\varepsilon^D). \tag{3.51}$$

and the estimate

$$\varepsilon^{-2}\|\nabla u_\varepsilon\|^2_{\mathsf{L}^2(Y_\varepsilon)} + \|u_\varepsilon\|^2_{\mathsf{L}^2(Y_\varepsilon)} \le \|\mathbf{f}\|_{\mathscr{H}^D} \tag{3.52}$$

holds true. It follows from (3.52) that the norms $\|u_\varepsilon\|_{\mathsf{H}^1(Y_\varepsilon)}$ are uniformly bounded with respect to $\varepsilon \in (0,1]$, whence there exists a sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ with $\varepsilon_m \searrow 0$ as $m \to \infty$ and the constant functions u_j, $j \in \mathbb{M}_0$ such that (3.23)–(3.24) hold. Moreover, $u_{\varepsilon_m} \to u_0$ strongly in $\mathsf{L}^2(\partial Y)$, hence $u_0 = 0$ a.e. on ∂Y, and, since u_0 is a constant function, we conclude

$$u_0 \equiv 0. \tag{3.53}$$

In the following we regard $\mathbf{u} = (u_1, \dots, u_m)$ as the element of $\mathscr{H}^D$. We fix an arbitrary $\mathbf{v} = (v_1, \dots, v_m) \in \mathscr{H}^D$, and define the function $v_\varepsilon \in \mathrm{dom}(\mathbf{a}_\varepsilon^D)$ by

$$v_\varepsilon(x) := \begin{cases} v_j, & x \in B_j,\ j \in \mathbb{M}, \\ 0, & x \in B_0, \\ -\frac{v_j}{h_j}(x^n - z_j^n) + \frac{v_j}{2}, & x \in T_{j,\varepsilon},\ j \in \mathbb{M}. \end{cases}$$

Inserting v_ε into (3.51), we obtain the equality

$$-\varepsilon^{-2}\sum_{j\in\mathbb{M}}\left(\int_{T_{j,\varepsilon}} \frac{\partial u_\varepsilon}{\partial x^n}\,\mathrm{d}x\right)\frac{\overline{v_j}}{h_j} + \sum_{j\in\mathbb{M}}\langle u_\varepsilon\rangle_{B_j}\overline{v_j}|B_j| \\ + \sum_{j\in\mathbb{M}}(u_\varepsilon, v_\varepsilon)_{\mathsf{L}^2(T_{j,\varepsilon})} = (\mathbf{f}, \mathbf{v})_{\mathscr{H}^D}. \tag{3.54}$$

Repeating the arguments from the proof of Lemma 3.4 (taking into account (3.53)) we get

$$-\lim_{m\to\infty}(\varepsilon_m)^{-2}\sum_{j\in\mathbb{M}}\left(\int_{T_{j,\varepsilon_m}} \frac{\partial u_{\varepsilon_m}}{\partial x^n}\,\mathrm{d}x\right)\frac{\overline{v_j}}{h_j} = \mathbf{a}^D[\mathbf{u}, \mathbf{v}], \tag{3.55}$$

$$\lim_{m\to\infty}\sum_{j\in\mathbb{M}}\langle u_{\varepsilon_m}\rangle_{B_j}\overline{v_j}|B_j| = (\mathbf{u}, \mathbf{v})_{\mathscr{H}^D}, \qquad \lim_{\varepsilon\to 0}\sum_{j\in\mathbb{M}}(u_\varepsilon, v_\varepsilon)_{\mathsf{L}^2(T_{j,\varepsilon})} = 0. \tag{3.56}$$

From (3.54)–(3.56) we conclude $\mathbf{a}^D[\mathbf{u}, \mathbf{v}] + (\mathbf{u}, \mathbf{v})_{\mathscr{H}^D} = (\mathbf{f}, \mathbf{v})_{\mathscr{H}^D}$ for all $v \in \mathscr{H}^D$, whence

$$\mathbf{u} = \mathscr{R}^D\mathbf{f}. \tag{3.57}$$

The limiting vector $\mathbf{u}$ is independent of a sequence u_{ε_m}, whence the whole family u_ε converges to $\mathbf{u}$, namely,

$$u_\varepsilon \to u_j \text{ strongly in } \mathsf{L}^2(B_j) \text{ as } \varepsilon \to 0. \tag{3.58}$$

Furthermore (cf. (3.38), (3.41)), one has

$$\sum_{j\in\mathbb{M}} \|u_\varepsilon\|^2_{\mathsf{L}^2(T_{j,\varepsilon})} \to 0 \text{ as } \varepsilon \to 0. \tag{3.59}$$

The property (A$_3$) follows immediately from (3.50), (3.53), (3.57), (3.58), (3.59) and the definition of the operator $\mathscr{J}_\varepsilon^D$. The property (A$_4$) is proven similarly (cf. Lemma 3.4).

By Theorem 3.2 and Remark 3.3 the fulfillment of the conditions (A$_1$)–(A$_4$) yield

$$\mu^D_{k,\varepsilon} \to \mu^D_k \text{ as } \varepsilon \to 0, \quad k = 1, \dots, m,$$

whence, using (3.48), we get the desired convergence (3.46). Lemma 3.7 is proven. □

Lemma 3.8. *Let $\theta \notin (0, 0, \dots, 0)$. Then for any $k \in \{1, \dots, m\}$ one has*

$$\lambda_k(\mathbf{A}^\theta_\varepsilon) \to \alpha_k, \quad \varepsilon \to 0.$$

The proof of Lemma 3.8 is similar to the proof of Lemma 3.7. Here we have to take into account that if u_ε satisfies the boundary conditions (3.2), then (3.23)–(3.24) imply that the limiting constant function u_0 satisfies (3.2) too, but for $\theta \notin (0, 0, \dots, 0)$ this is possible only if $u_0 \equiv 0$.

3.6. End of proof. We denote

$$\theta_0 := (0, 0, \dots, 0), \quad \theta_\pi := (\pi, \pi, \dots, \pi).$$

By virtue of (3.4) and (3.5) we get

$$\sigma(\mathbf{A}_\varepsilon) = \bigcup_{k\in\mathbb{N}} L_{k,\varepsilon}, \tag{3.60}$$

where the compact intervals $L_{k,\varepsilon} = \left[\ell^-_{k,\varepsilon}, \ell^+_{k,\varepsilon}\right]$ satisfy

$$\lambda_k\big(\mathbf{A}^N_\varepsilon\big) \le \ell^-_{k,\varepsilon} \le \lambda_k\big(\mathbf{A}^{\theta_0}_\varepsilon\big), \quad \lambda_k\big(\mathbf{A}^{\theta_\pi}_\varepsilon\big) \le \ell^+_{k,\varepsilon} \le \lambda_k\big(\mathbf{A}^D_\varepsilon\big).$$

Due to (3.7) and (3.42) one has

$$\ell^-_{1,\varepsilon} = 0. \tag{3.61}$$

Furthermore, Lemmata 3.4–3.6 yield

$$\lim_{\varepsilon\to 0} \ell^-_{k,\varepsilon} = \beta_{k-1}, \quad k = 2, \dots, m+1, \tag{3.62}$$

and Lemmata 3.7–3.8 give

$$\lim_{\varepsilon\to 0} \ell^+_{k,\varepsilon} = \alpha_k, \quad k = 1,\dots,m. \tag{3.63}$$

Setting $\alpha_{k,\varepsilon} := \ell^+_{k,\varepsilon}$, $\beta_{k,\varepsilon} := \ell^-_{k+1,\varepsilon}$ ($k = 1,\dots,m$) we conclude from (3.60)–(3.63), (2.11) and Lemma 3.1 the desired properties (2.12)–(2.13). Theorem 2.3 is proven.

Remark 3.9. The first step in proof of the convergence result in [25], where "zero-thickness" resonators were treated, is the same to the current problem: to enclose the left (respectively right) endpoint of each spectral band between the Neumann and periodic (resp. antiperiodic and Dirichlet) eigenvalues. However, further investigation of the asymptotic beghaviour of these eigenvalues was carried out in a different way: the limiting operators $\mathbf{A}^N$ and $\mathbf{A}^D$ was not identified at all, instead, to to obtain the asymptotics of eigenvalues, a sutable convenient approximations for the corresponding eigenfunctions were constructed.

4. Proof of Theorem 2.4

Let $(\widetilde\alpha_j)_{j\in\mathbb{M}}$ and $(\widetilde\beta_j)_{j\in\mathbb{M}}$ be positive numbers satisfying (2.14). We consider the following system of m linear equations with unknowns ϱ_j, $j = 1,\dots,m$:

$$1 + \sum_{j=1}^m \varrho_j \frac{\widetilde\alpha_j}{\widetilde\alpha_j - \widetilde\beta_k} = 0, \quad k = 1,\dots,m. \tag{4.1}$$

It was shown in [22, Lemma 4.1] that the system (4.1) has the unique solution $\varrho_1,\dots,\varrho_m$ given by

$$\varrho_j = \frac{\widetilde\beta_j - \widetilde\alpha_j}{\widetilde\alpha_j} \prod_{i=\overline{1,m}\,|\,i\neq j} \left(\frac{\widetilde\beta_i - \widetilde\alpha_j}{\widetilde\alpha_i - \widetilde\alpha_j}\right), \quad j \in \mathbb{M}. \tag{4.2}$$

Note that due to (2.14) we have

$$\forall j:\ \widetilde\alpha_j < \widetilde\beta_j, \quad \forall i\neq j:\ \mathrm{sign}(\widetilde\beta_i - \widetilde\alpha_j) = \mathrm{sign}(\widetilde\alpha_i - \widetilde\alpha_j) \neq 0.$$

Consequently, the numbers ϱ_j are positive.

Let $\gamma > 0$. We set

$$\tau_j := \frac{\varrho_j}{\gamma^n + \sum_{i\in\mathbb{M}} \varrho_i}, \quad j \in \mathbb{M},$$

where ϱ_j are given by (4.2). One has

$$\tau_j > 0 \text{ and } \sum_{i\in\mathbb{M}} \tau_j < 1. \tag{4.3}$$

Now, let F_j, $j = 1,\dots,m$ be hyperrectangles with axes being parallel to the coordinate ones and satisfying

$$\cup_{j\in\mathbb{M}}\overline{F_j} \subset Y, \quad \overline{F_i}\cap\overline{F_j} = \varnothing,\ i\neq j, \quad |F_j| = \tau_j.$$

It is easy to see that such a choice is always possible due to (4.3). Let $y_j \in F_j$ be the center of the hyperrectangle F_j; then we define

$$B_j := \gamma(F_j - y_j) + y_j$$

(i.e., B_j is obtained from F_j by a homothety with the center at y_j and the ratio γ). In the following we choose $\gamma < 1$, which implies $\overline{B_j} \subset F_j$. Further, we remove from $F_j \setminus \overline{B_j}$ the set $T_{j,\varepsilon}$ of the form (2.5); due to the fact that F_j and B_j are two homothetic hyperrectangle with axes being parallel to the coordinate ones, one can always do this in such a way that the assumptions (2.2)–(2.4) are fulfilled.

We choose an arbitrary cross-section profile D_j, while the constant η_j in (2.6) are chosen as follows:

$$\eta_j = \left(\frac{\widetilde{\alpha}_j h_j |B_j|}{|D_j|}\right)^{1/(n-1)}.$$

With such a choice of η_j we immediately obtain

$$\alpha_j = \widetilde{\alpha}_j, \quad j \in \mathbb{M}, \tag{4.4}$$

where α_j are defined by (2.7). Moreover, we have

$$\frac{|B_j|}{|B_0|} = \frac{\gamma^n |F_j|}{|Y| - \sum_{i\in\mathbb{M}} |F_i|} = \frac{\gamma^n \tau_j}{1 - \sum_{i\in\mathbb{M}} \tau_j} = \varrho_j. \tag{4.5}$$

Since $(\varrho_1, \ldots, \varrho_m)$ is a solution to the system (4.1), we conclude from (4.4) and (4.5) that

$$1 + \sum_{j\in\mathbb{M}} \frac{\alpha_j |B_j|}{|B_0|(\alpha_j - \widetilde{\beta}_k)} = 0, \quad k = 1, \ldots, m.$$

Hence $\widetilde{\beta}_j$ are zeros of the function $F(\lambda)$ (see (2.10)), and consequently (taking into account (2.11), (2.14)), we obtain

$$\beta_j = \widetilde{\beta}_j, \quad j \in \mathbb{M}.$$

Theorem 2.4 is proven.

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Створення та контроль спектральних лакун у періодичних середовищах із малими резонаторами

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Досліджуються спектральні властивості лапласіана Неймана $\mathscr{A}_\varepsilon$ на періодичній необмеженій області Ω_ε, яка залежить від малого параметра $\varepsilon > 0$. Область Ω_ε отримується шляхом видалення з $\mathbb{R}^n$ $m \in \mathbb{N}$ сімейств ε-періодично розташованих малих резонаторів. Доведено, що спектр $\mathscr{A}_\varepsilon$ має принаймні m лакун. Перші m лакун прямують при $\varepsilon \to 0$ до деяких інтервалів, розташуванням і довжиною яких можна керувати шляхом певного вибору резонаторів; інші лакуни (якщо вони є) прямують до нескінченності. Обговорюються застосування до теорії фотонних кристалів.

Ключові слова: періодичні середовища, резонатори, лапласіан Неймана, спектральні лакуни