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Some Results on Tangent Bundles with Berger Type Deformed Sasaki Metric over Kählerian Manifolds

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Let M be a Kählerian manifold equipped with an almost complex structure J and a Riemannian metric g, and let TM be its tangent bundle with the Berger type deformed Sasaki metric. In this paper, firstly, we find all forms of Riemannian curvature tensors of TM. Secondly, we search the conditions under which a vector field is harmonic with respect to the Berger type deformed Sasaki metric and give some examples of harmonic vector fields. Finally, we study the harmonicity of maps between the Riemannian manifold and the tangent bundle of another Riemannian manifold and vice versa.

Key words: Berger type deformed Sasaki metric, harmonicity, Kählerian manifold, Riemannian curvature tensor, tangent bundle

Mathematical Subject Classification 2020: 53C07, 53C55, 53A45

1. Introduction

The theory of tangent bundles began to be studied with the paper of Sasaki [16] which studies the differential geometry of tangent bundles of Riemannian manifolds and introduces the Sasaki metric on the tangent bundles as a Riemannian metric. Later, the geometric properties of Sasaki metric have been extensively searched by researchers. But, in most cases, they met the flatness of the base manifold (see, for example, [13,14]). This situation has led many researchers to studying different deformations of the Sasaki metric. In this direction, Abbassi and Sarih [1] proposed general $g-$natural metrics on the tangent bundle and the unit tangent bundles which include the Sasaki metric, the Cheeger-Gromoll metric, the Kaluza–Klein type metric and some others as partial cases. Inspired by the Berger deformation of metric on a unit sphere, Yampolsky [17] gave another natural way of deforming the Sasaki metric on slashed and unit tangent bundles over a Kählerian manifold with the help of an almost complex structure J. He called this metric as a Berger type deformed Sasaki metric and studied geodesics of this metric. In [2], the first author defined the Berger type deformed Sasaki metric on the tangent bundle over an anti-paraKähler manifold. They compute all Riemannian curvature tensors of this metric and presented some geometric results. Also, they defined some almost anti-paraHermitian structures on the tangent bundle and searched the conditions for these structures to be anti-paraKähler and quasi-anti-paraKähler. The deformations of the Sasaki metric on the tangent bundle are not limited to those mentioned above. Also, we refer to [6–8, 19, 22].

The paper deals with the Berger type deformed Sasaki metric on the tangent bundle over a Kählerian manifold. We compute all Riemannian curvature tensors of the tangent bundle with this metric and study harmonicity problems on this setting.

2. Basic definitions and results

Let M_n be an n-dimensional differentiable manifold. An almost complex structure J on M_n is a $(1,1)-$tensor field on M_n such that $J^2 = -I$, (I is the identity tensor field of type $(1,1)$). The pair (M_n, J) is called an almost complex manifold. Since every almost complex manifold is even dimensional, we will take $n = 2k$. Also, note that every complex manifold (topological space endowed with a holomorphic atlas) carries a natural almost complex structure [11].

The integrability of J on M_{2k} is equivalent to the vanishing of the Nijenhuis tensor N_J:

$$N_J(X,Y) = [JX, JY] - J\,[JX,Y] - J\,[X,JY] - [X,Y] \tag{2.1}$$

for all vector fields X, Y on M_{2k}. If the almost complex structure J is integrable, then we call it a complex structure.

On an almost complex manifold (M_{2k}, J), a Hermitian metric is a Riemannian metric g on M_{2k} such that

$$g(JX,Y) = -g(X,JY) \tag{2.2}$$

or, equivalently,

$$g\,(JX,JY) = g\,(X,Y) \tag{2.3}$$

for any vector fields X, Y on M_{2k}. The almost complex manifold (M_{2k}, J) having the Hermitian metric g is called an almost Hermitian manifold. Let (M_{2k}, J, g) be an almost Hermitian manifold. We define the fundamental or Kähler 2-form Ω on M_{2k} by

$$\Omega(X,Y) = g(X,JY) \tag{2.4}$$

for any vector fields X and Y on M_{2k}. A Hermitian metric g on an almost Hermitian manifold M_{2k} is called a Kählerian metric if the fundamental 2-form Ω is closed, i.e., $d\Omega = 0$. In the case, the triple (M_{2k}, J, g) is called an almost Kählerian manifold. If the almost complex structure is integrable, then the triple (M_{2k}, J, g) is called a Kählerian manifold. Moreover, the following conditions are equivalent:

1. $\nabla J = 0$, (∇ is the Levi-Civita connection of g),

2. $\nabla\Omega = 0$,
3. $N_J = 0$ and $d\Omega = 0$ [11].

As a result, the almost Hermitian manifold (M_{2k}, J, g) is a Kählerian manifold if and only if $\nabla J = 0$. The Riemannian curvature tensor R of a Kählerian manifold possess the following properties [11]:

$$\begin{aligned} R(Y,Z)J &= JR(Y,Z), \\ R(JY,JZ) &= R(Y,Z), \\ R(JY,Z) &= -R(Y,JZ) \end{aligned}$$

for all vector fields Y, Z on M_{2k}.

3. Lifts to tangent bundles

Let M be an n-dimensional Riemannian manifold with a Riemannian metric g, and let TM be its tangent bundle denoted by $\pi : TM \to M$. A system of local coordinates (U, x^i) in M induces on TM a system of local coordinates $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = u^i\right)$, $\bar{i} = n + i = n+1, \ldots, 2n$, where (u^i) is the Cartesian coordinates in each tangent space $T_P M$ at $P \in M$ with respect to the natural base $\left\{ \frac{\partial}{\partial x^i}\big|_P \right\}$, P being an arbitrary point in U whose coordinates are (x^i).

Given a vector field $X = X^i \frac{\partial}{\partial x^i}$ on M, the vertical lift ${}^V X$ and the horizontal lift ${}^H X$ of X are given, with respect to the induced coordinates, by

$$\begin{aligned} {}^V X &= X^i \partial_{\bar{i}}, \\ {}^H X &= X^i \partial_i - u^s \Gamma^i_{sk} X^k \partial_{\bar{i}}, \end{aligned}$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial u^i}$ and Γ^i_{sk} are the coefficients of the Levi-Civita connection ∇ of g [18].

In particular, we have the vertical spray ${}^V u$ and the horizontal spray ${}^H u$ on TM defined by

$$ {}^V u = u^i\,{}^V(\partial_i) = u^i \partial_{\bar{i}},\ {}^H u = u^i\,{}^H(\partial_i). $$

${}^V u$ is also called the canonical or Liouville vector field on TM.

Now, let r be the norm of a vector $u \in TM$. Then, for any smooth function f of $\mathbb{R}$ to $\mathbb{R}$, we have [1]:

$$\begin{aligned} {}^H X\left(f(r^2)\right) &= 0, \\ {}^V X\left(f(r^2)\right) &= 2f'(r^2) g(X,u) \end{aligned}$$

and, in particular, we get

$$\begin{aligned} {}^H X(r^2) &= 0, \\ {}^V X(r^2) &= 2g(X,u). \end{aligned}$$

Let X, Y and Z be any vector fields on M. Then we have [1]:

$$\begin{aligned} {}^{H}X(g(Y,u)) &= g\left((\nabla_X Y),u\right), \\ {}^{V}X(g(Y,u)) &= g\left(X,Y\right), \\ {}^{H}X\left({}^{V}(g(Y,Z))\right) &= X\left(g(Y,Z)\right), \\ {}^{V}X\left({}^{V}(g(Y,Z))\right) &= 0. \end{aligned}$$

The bracket operation of vertical and horizontal vector fields is given by the formulas [3, 18]:

$$\begin{aligned} \left[{}^{H}X, {}^{H}Y\right] &= {}^{H}[X,Y] - {}^{V}\left(R(X,Y)u\right) \\ \left[{}^{H}X, {}^{V}Y\right] &= {}^{V}(\nabla_X Y) \\ \left[{}^{V}X, {}^{V}Y\right] &= 0 \end{aligned} \tag{3.1}$$

for all vector fields X and Y on M, where R is the Riemannian curvature tensor of g.

4. The Berger type deformed Sasaki metric on the tangent bundle

Let (M_{2k}, J, g) be an almost Hermitian manifold and TM be its tangent bundle. A fiber-wise Berger type deformation of the Sasaki metric on TM is defined by

$$\begin{aligned} g_{BS}({}^{H}X, {}^{H}Y) &= g(X,Y), \\ g_{BS}({}^{V}X, {}^{H}Y) &= {}^{S}g({}^{H}X, {}^{V}Y) = 0, \\ g_{BS}({}^{V}X, {}^{V}Y) &= g(X,Y) + \delta^2 g(X,Ju)g(Y,Ju) \end{aligned} \tag{4.1}$$

for all vector fields X, Y on M_{2k}, where δ is some constant [17]. Also, it is called a Berger type deformed Sasaki metric.

In the following, we put $\lambda = 1 + \delta^2 g(u,u) = 1 + \delta^2\|u\|^2$. where $\|.\|$ denotes the norm with respect to g.

Lemma 4.1. *Let (M_{2k}, J, g) be a Kählerian manifold. We have the followings:*

1. ${}^{H}X\left(g(Y,Ju)\right) = g(\nabla_X Y, Ju)$,
2. ${}^{V}X\left(g(Y,Ju)\right) = g(Y,JX)$ *for all vector fields X, Y on M_{2k}.*

Proof. 1. By standard calculations, we have

$$\begin{aligned} {}^{H}X(g(Y,Ju)) &= X^i \partial_i\big(g_{lj} Y^l J^j_t u^t\big) - u^s \Gamma^i_{sk} X^k \partial_{\bar{i}}\big(g_{lj} Y^l J^j_t u^t\big) \\ &= X(g(Y,Ju)) - u^s \Gamma^i_{sk} X^k g_{lj} Y^l J^j_t \delta^t_i \\ &= g\big(\nabla_X Y, Ju\big) + g\big(Y, \nabla_X (Ju)\big) - g_{lj} Y^l J^j_i u^s \Gamma^i_{sk} X^k \\ &= g\big(\nabla_X Y, Ju\big) + g\big(Y, \nabla_X (Ju)\big) - g\big(Y, J(\nabla_X u)\big) \end{aligned}$$

$$= g\big(\nabla_X Y, Ju\big).$$

2. We also have

$${}^V X(g(Y, Ju)) = X^i \partial_{\bar{i}}\big(g_{lj} Y^l J^j_k u^k\big) = X^i g_{lj} Y^l J^j_k \delta^k_i = g_{lj} Y^l J^j_i X^i = g(Y, JX). \quad \square$$

Lemma 4.2. *Let (M_{2k}, g, J) be a Kählerian manifold. We have the following:*

1. ${}^H X\left(g_{BS}({}^H Y, {}^H Z)\right) = X(g(Y, Z))$,
2. ${}^V X\left(g_{BS}({}^H Y, {}^H Z)\right) = 0$,
3. ${}^H X\left(g_{BS}({}^V Y, {}^V Z)\right) = g_{BS}({}^V(\nabla_X Y), {}^V Z) + g_{BS}({}^V Y, {}^V(\nabla_X Z))$,
4. ${}^V X\left(g_{BS}({}^V Y, {}^V Z)\right) = \delta^2\left(g(Y, JX)g(Z, Ju) + g(Y, Ju)g(Z, JX)\right)$

for all vector fields X, Y, Z on M_{2k}.

Proof. The results follow directly from Lemma 4.1. $\square$

Theorem 4.3 ([17])**.** *Let (M_{2k}, J, g) be a Kählerian manifold, and let (TM, g_{BS}) be its tangent bundle equipped with the Berger type deformed Sasaki metric. Then we have the following formulas:*

1. $\widetilde{\nabla}_{{}^H X}{}^H Y = {}^H(\nabla_X Y) - \frac{1}{2}{}^V(R(X, Y)u)$,
2. $\widetilde{\nabla}_{{}^H X}{}^V Y = {}^V(\nabla_X Y) + \dfrac{1}{2}{}^H\left(R(u, Y)X + \delta^2 g(Y, Ju)R(u, Ju)X\right)$,
3. $\widetilde{\nabla}_{{}^V X}{}^H Y = \dfrac{1}{2}{}^H\left(R(u, X)Y + \delta^2 g(X, Ju)R(u, Ju)Y\right)$,
4. $\widetilde{\nabla}_{{}^V X}{}^V Y = \delta^2\left(g(X, Ju){}^V(JY) + g(Y, Ju){}^V(JX)\right) - \dfrac{\delta^4}{\lambda}\big(g(Y, u)g(X, Ju) + g(X, u)g(Y, Ju)\big){}^V(Ju)$

for all vector fields X, Y, where ∇ is the Levi-Civita connection, R is the Riemannian curvature tensor of (M_{2k}, g, J).

Lemma 4.4. *Let (M_{2k}, J, g) be a Kählerian manifold and let (TM, g_{BS}) be its tangent bundle equipped with the Berger type deformed Sasaki metric. Then we have*

$$\begin{aligned}
\widetilde{\nabla}_{{}^H X}{}^V(Ju) &= \frac{\lambda}{2}{}^H(R(u, Ju)X), \\
\widetilde{\nabla}_{{}^H X}{}^V(JY) &= {}^V(\nabla_X JY) + \frac{1}{2}{}^H\left(R(u, JY)X + \delta^2 g(Y, u)R(u, Ju)X\right), \\
\widetilde{\nabla}_{{}^V X}{}^V(Ju) &= \lambda{}^V(JX) - \delta^2 g(X, Ju){}^V u - \frac{\delta^2(\lambda - 1)}{\lambda} g(X, u){}^V(Ju), \\
\widetilde{\nabla}_{{}^V X}{}^V(JY) &= \delta^2\big(g(Y, u){}^V(JX) - g(X, Ju){}^V Y\big) \\
&\quad - \frac{\delta^4}{\lambda}\big(g(X, u)g(Y, u) - g(X, Ju)g(Y, Ju)\big){}^V(Ju)
\end{aligned}$$

for all vector fields X, Y on M_{2k}.

Definition 4.5. Let (M_{2k}, g, J) be a Kählerian manifold and let (TM, g_{BS}) be its tangent bundle equipped with the Berger type deformed Sasaki metric, F :

$TM \to TM$ be a smooth bundle endomorphism of TM and $K : TM \times TM \to TM$ be a differential map preserving the fibers and bilinear on each of them. Then the vertical and horizontal vector fields ${}^VF$, ${}^HF$, ${}^VK$ and ${}^HK$, respectively, are defined on TM by

$$\begin{aligned}
{}^VF : TM &\to TTM \\
(x, u) &\mapsto {}^V(F_x u), \\
{}^HF : TM &\to TTM \\
(x, u) &\mapsto {}^H(F_x u), \\
{}^VK : TM &\to TTM \\
(x, u) &\mapsto {}^V(K_x(u, Ju)), \\
{}^HK : TM &\to TTM \\
(x, u) &\mapsto {}^H(K_x(u, Ju)).
\end{aligned}$$

Locally, we have

$$\begin{aligned}
&{}^V(F(u)) = u^j {}^V(F\partial j), \\
&{}^H(F(u)) = u^j {}^H(F\partial j). \\
&{}^V(K(u)) = u^i u^s J_s^j {}^V(K(\partial i, \partial j)) = u^i u^s {}^V(K(\partial i, J(\partial s))), \\
&{}^H(K(u)) = u^i u^s J_s^j {}^H(K(\partial i, \partial j))) = u^i u^s {}^H(K(\partial i, J(\partial s))).
\end{aligned}$$

Proposition 4.6. *Let (M_{2k}, g, J) be a Kählerian manifold and let (TM, g_{BS}) be its tangent bundle equipped with the Berger type deformed Sasaki metric. Then we have the following formulas:*

1. $\widetilde{\nabla}_{{}^HX} {}^HF_{(x,u)} = {}^H(\nabla_X F)(u) - \frac{1}{2} {}^V(R_x(X, Fu)u),$
2. $\widetilde{\nabla}_{{}^HX} {}^VF_{(x,u)} = {}^V(\nabla_X F)(u) + \dfrac{1}{2} {}^H\big(R_x(u, Fu)X + \delta^2 g(Fu, Ju) R_x(u, Ju)X\big),$
3. $\widetilde{\nabla}_{{}^VX} {}^HF_{(x,u)} = {}^H(FX) + \dfrac{1}{2} {}^H\big(R_x(u, X)Fu + \delta^2 g(X, Ju) R_x(u, Ju)Fu\big),$
4. $\widetilde{\nabla}_{{}^VX} {}^VF_{(x,u)} = {}^V(FX) + \delta^2\big(g(X, Ju){}^V(JFu) + g(Fu, Ju){}^V(JX)\big)$ $\qquad - \dfrac{\delta^4}{\lambda}\big(g(X, Ju)g(Fu, u) + g(X, u)g(Fu, Ju)\big){}^V(Ju),$
5. $(\widetilde{\nabla}_{{}^HX} {}^HK)_{(x,u)} = {}^H\big((\nabla_X K)(u, Ju)\big) - \frac{1}{2} {}^V(R_x(X, K(u, Ju)u),$
6. $(\widetilde{\nabla}_{{}^HX} {}^VK)_{(x,u)} = {}^V\big((\nabla_X K)(u, Ju)\big) + \dfrac{1}{2} {}^H\big(R_x(u, K(u, Ju))X$ $\qquad + \delta^2 g(K(u, Ju), Ju) R_x(u, Ju)X\big),$
7. $(\widetilde{\nabla}_{{}^VX} {}^HK)_{(x,u)} = \dfrac{1}{2} {}^H\big(R_x(u, X)K(u, Ju) + \delta^2 g(X, Ju) R_x(u, Ju) K(u, Ju)\big)$ $\qquad + {}^H(K(X, Ju)) + {}^H(K(u, JX)),$
8. $(\widetilde{\nabla}_{{}^VX} {}^VK)_{(x,u)} = +\delta^2\big[g(X, Ju){}^V(J(K(u, Ju)) + g(K(u, Ju), Ju){}^V(JX)\big]$ $\qquad - \dfrac{\delta^4}{\lambda}\big[g(X, Ju)g(K(u, Ju), u) + g(X, u)g(K(u, Ju), Ju)\big]{}^V(Ju)$ $\qquad + {}^V(K(X, Ju)) + {}^V(K(u, JX))$

for any vector field X on M_{2k}, where ∇ is the Levi-Civita connection, R is its curvature tensor of (M_{2k}, g, J).

Proof. The results come directly from Theorem 4.3. □

5. The Riemannian curvatures of Berger type deformed Sasaki metric

We will calculate the Riemannian curvature tensor $\widetilde{R}$ of TM with the Berger type deformed Sasaki metric g_{BS}. The Riemannian curvature tensor is characterized by the formula

$$\widetilde{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z} - \widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}$$

for all vector fields $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ on TM.

Theorem 5.1. *Let (M_{2k}, g, J) be a Kählerian manifold and let (TM, g_{BS}) be its tangent bundle equipped with the Berger type deformed Sasaki metric. Then we have the following formulas:*

$$\begin{aligned}
\widetilde{R}({}^{H}X, {}^{H}Y){}^{H}Z &= \frac{\delta^2}{2} g(R(X,Y)u, Ju)^{H}(R(u,Ju)Z) + \frac{1}{2}{}^{H}(R(u, R(X,Y)u)Z) \\
&+ \frac{\delta^2}{4} g(R(X,Z)u, Ju)^{H}(R(u,Ju)Y) + \frac{1}{4}{}^{H}(R(u, R(X,Z)u)Y) \\
&- \frac{\delta^2}{4} g(R(Y,Z)u, Ju)^{H}(R(u,Ju)X) - \frac{1}{4}{}^{H}(R(u, R(Y,Z)u)X) \\
&+ {}^{H}(R(X,Y)Z) + \frac{1}{2}{}^{V}((\nabla_Z R)(X,Y)u), \qquad (5.1)
\end{aligned}$$

$$\begin{aligned}
\widetilde{R}({}^{H}X, {}^{V}Y){}^{H}Z &= \frac{1}{2}{}^{H}((\nabla_X R)(u,Y)Z) + \frac{\delta^2}{2} g(Y,Ju)^{H}((\nabla_X R)(u,Ju)Z) \\
&+ \frac{\delta^2}{2} g(Y,Ju)^{H}((R(u, J\nabla_X u)Z) - \frac{1}{4}{}^{V}(R(X, R(u,Y)Z)u) \\
&+ \frac{1}{2}{}^{V}(R(X,Z)Y) - \frac{\delta^2}{4} g(Y,Ju)^{V}(R(X, R(u,Ju)Z)u) \\
&+ \frac{\delta^2}{2}\big(g(Y,Ju)^{V}(JR(X,Z)u) + g(R(X,Z)u, Ju)^{V}(JY)\big) \\
&- \frac{\delta^4}{2\lambda} g(Y,u) g(R(X,Z)u, Ju)\big)^{V}(Ju), \qquad (5.2)
\end{aligned}$$

$$\begin{aligned}
\widetilde{R}({}^{H}X, {}^{H}Y){}^{V}Z &= \frac{1}{2}{}^{H}\big((\nabla_X R)(u,Z)Y - \frac{1}{2}(\nabla_Y R)(u,Z)X\big) \\
&+ \frac{\delta^2}{2} g(Z,Ju)^{H}\big((\nabla_X R)(u,Ju)Y - (\nabla_Y R)(u,Ju)X\big) \\
&+ \frac{\delta^2}{2} g(Z,Ju)^{H}\big(R(u, J\nabla_X u)Y - R(u, J\nabla_Y u)X\big) \\
&+ {}^{V}(R(X,Y)Z) - \frac{1}{4}{}^{V}\big(R(X, R(u,Z)Y)u - R(Y, R(u,Z)X)u\big)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\delta^2}{4} g(Z,Ju)^V\big(R(X,R(u,Ju)Y)u - R(Y,R(u,Ju)X)u\big) \\
&+ \delta^2\big(g(R(X,Y)u,Ju)^V(JZ) + g(Z,Ju)^V(JR(X,Y)u)\big) \\
&- \frac{\delta^4}{\lambda} g(Z,u) g(R(X,Y)u,Ju)^V(Ju),
\end{aligned} \tag{5.3}
$$

$$
\begin{aligned}
\widetilde{R}({}^HX,{}^VY)^VZ = &\frac{\delta^4}{2} g(X,Ju)\big(g(Y,Ju)^H(R(u,Ju)JZ) + g(Z,Ju)^H(R(u,Ju)JY)\big) \\
&- \delta^2 g(Z,Ju)^H(R(Y,Ju)X) - \frac{\delta^2}{2} g(Z,JY)^H(R(u,Ju)X) \\
&- \frac{\delta^4}{2}\big(g(Z,u)g(Y,Ju) + g(Y,u)g(Z,Ju)\big)^H(R(u,Ju)X) \\
&+ \frac{\delta^2}{4} g(Y,Ju)^H\big(2R(u,JZ)u - R(u,Ju)R(u,Z)X\big) \\
&+ \frac{\delta^2}{4} g(Z,Ju)^H\big(2R(u,JY)u - R(u,Y)R(u,Ju)X\big) \\
&- \frac{\delta^4}{4} g(Y,Ju)g(Z,Ju)^H(R(u,Ju)R(u,Ju)X) \\
&- \frac{1}{2}{}^H(R(Y,Z)X) - \frac{1}{4}{}^H(R(u,Y)R(u,Z)X),
\end{aligned} \tag{5.4}
$$

$$
\begin{aligned}
\widetilde{R}({}^VX,{}^VY)^HZ = &{}^H(R(X,Y)Z) + \delta^2 g(Y,JX)^H(R(u,Ju)Z) \\
&+ \frac{1}{4}{}^H\big((R(u,X)R(u,Y)Z) - (R(u,Y)R(u,X)Z)\big) \\
&+ \delta^2\big(g(Y,Ju)^H(R(X,Ju)Z) - g(X,Ju)^H(R(Y,Ju)Z)\big) \\
&+ \frac{\delta^2}{4} g(X,Ju)^H\big(R(u,Ju)R(u,Y)Z - R(u,Y)R(u,Ju)Z\big) \\
&- \frac{\delta^2}{4} g(Y,Ju)^H\big(R(u,Ju)R(u,X)Z - R(u,X)R(u,Ju)Z\big),
\end{aligned} \tag{5.5}
$$

$$
\begin{aligned}
\widetilde{R}({}^VX,{}^VY)^VZ = &\delta^4 g(Z,Ju)\big(g(Y,Ju)^VX - g(X,Ju)^VY\big) \\
&+ \delta^2\big(g(Y,JZ)^V(JX) - g(X,JZ)^V(JY)\big) - 2\delta^2 g(X,JY)^V(JZ) \\
&+ \frac{\delta^6}{\lambda} g(Z,Ju)\big(g(X,Ju)g(Y,u) - g(X,u)g(Y,Ju)\big)^Vu \\
&+ \Big(\frac{\delta^6}{\lambda^2} g(Z,u)\big(g(X,u)g(Y,Ju) - g(X,Ju)g(Y,u)\big) \\
&+ \frac{\delta^4}{\lambda}\big(g(X,Ju)g(Y,Z) - g(Y,Ju)g(X,Z)\big) \\
&+ \frac{\delta^4}{\lambda}\big(g(X,JZ)g(Y,u) - g(Y,JZ)g(X,u)\big) \\
&+ \frac{2\delta^4}{\lambda} g(Z,u)g(X,JY)\Big)^V(Ju)
\end{aligned} \tag{5.6}
$$

for all vector fields X, Y, Z on M_{2k}.

Proof. In the proof, we will use Theorem 4.3, Lemma 4.4 and Proposition 4.6.

1. Let $F : TM \to TM$ be the bundle endomorphism given by $F(u) = R(Y,Z)u$. Direct calculations give

$$\begin{aligned}\widetilde{\nabla}_{{}^HX}\widetilde{\nabla}_{{}^HY}{}^HZ &= \widetilde{\nabla}_{{}^HX}\big[{}^H(\nabla_Y Z) - \frac{1}{2}{}^VF\big]\\ &= {}^H(\nabla_X\nabla_Y Z) - \frac{1}{2}{}^V(R(X,\nabla_Y Z)u) - \frac{1}{2}{}^V(\nabla_X(R(Y,Z)u))\\ &\quad + \frac{1}{2}{}^V(R(Y,Z)(\nabla_X u)) - \frac{1}{4}{}^H(R(u,R(Y,Z)u)X)\\ &\quad - \frac{\delta^2}{4}g(R(Y,Z)u,Ju)^H(R(u,Ju)X),\end{aligned}$$

from which, with permutation of X by Y in the formula of $\widetilde{\nabla}_{{}^HX}\widetilde{\nabla}_{{}^HY}{}^HZ$, we get

$$\begin{aligned}\widetilde{\nabla}_{{}^HY}\widetilde{\nabla}_{{}^HX}{}^HZ &= {}^H(\nabla_Y\nabla_X Z) - \frac{1}{2}{}^V(R(Y,\nabla_X Z)u) - \frac{1}{2}{}^V(\nabla_Y(R(X,Z)u))\\ &\quad + \frac{1}{2}{}^V(R(X,Z)(\nabla_Y u)) - \frac{1}{4}{}^H(R(u,R(X,Z)u)Y)\\ &\quad - \frac{\delta^2}{4}g(R(X,Z)u,Ju)^H(R(u,Ju)Y).\end{aligned}$$

Also, we find

$$\begin{aligned}\widetilde{\nabla}_{[{}^HX,{}^HY]}{}^HZ &= \widetilde{\nabla}_{{}^H[X,Y]}{}^HZ - \widetilde{\nabla}_{{}^V(R(X,Y)u)}{}^HZ\\ &= {}^H(\nabla_{[X,Y]}Z) - \frac{1}{2}{}^V(R([X,Y],Z)u) - \frac{1}{2}{}^H(R(u,R(X,Y)u)Z)\\ &\quad - \frac{\delta^2}{2}g(R(X,Y)u,Ju)^H(R(u,Ju)Z).\end{aligned}$$

Using the second Bianchi identity, we obtain formula (5.1).

2. Let $F : TM \to TM$ be the bundle endomorphisms given by $F(u) = R(u,Y)Z$ and $K : TM \times TM \to TM$ given by $K(u,v) = R(u,v)Z$. Hence, we obtain

$$\begin{aligned}\widetilde{\nabla}_{{}^HX}\widetilde{\nabla}_{{}^VY}{}^HZ &= \widetilde{\nabla}_{{}^HX}\big[\frac{1}{2}{}^HF + \frac{\delta^2}{2}g(Y,Ju)^HK\big]\\ &= \frac{1}{2}{}^H(\nabla_X(R(u,Y)Z) - R(\nabla_X u,Y)Z) - \frac{1}{4}{}^H(R(X,R(u,Y)Z)u)\\ &\quad + \frac{\delta^2}{2}g(\nabla_X Y,Ju)^H(R(u,Ju)Z)\\ &\quad + \frac{\delta^2}{2}g(Y,Ju)^H(\nabla_X(R(u,Ju)Z) - R(\nabla_X u,Ju)Z)\\ &\quad - \frac{\delta^2}{4}g(Y,Ju)^V(R(X,R(u,Ju)Z)u).\end{aligned}$$

Let $F : TM \to TM$ be the bundle endomorphism given by $F(u) = R(X,Z)u$. We get

$$\widetilde{\nabla}_{{}^VY}\widetilde{\nabla}_{{}^HX}{}^HZ = \widetilde{\nabla}_{{}^VY}\big[{}^H(\nabla_X Z) - \frac{1}{2}{}^VF\big]$$

$$
\begin{aligned}
&= \frac{1}{2}{}^{H}(R(u,Y)\nabla_X Z) + \frac{\delta^2}{2} g(Y,Ju)^{H}(R(u,Ju)(\nabla_X Z)) \\
&- \frac{\delta^2}{2}\big(g(Y,Ju)^{V}(JR(X,Z)u) + g(R(X,Z)u,Ju)^{V}(JY)\big) \\
&- \frac{1}{2}{}^{V}(R(X,Z)Y) + g(R(X,Z)u,Ju)g(Y,u)\big)(Ju)^{V}.
\end{aligned}
$$

Also,

$$
\widetilde{\nabla}_{[{}^HX,{}^VY]}{}^{V}Z = \frac{1}{2}{}^{H}(R(u,\nabla_X Y)Z) + \frac{\delta^2}{2} g(\nabla_X Y,Ju)^{H}(R(u,Ju)Z),
$$

which gives formula (5.2).

3. Applying formula (5.2) and the first Bianchi identity, we find

$$
\begin{aligned}
\widetilde{R}({}^HX,{}^VZ)^{H}Y &= \frac{1}{2}{}^{H}((\nabla_X R)(u,Z)Y) + \frac{\delta^2}{2} g(Z,Ju)^{H}((\nabla_X R)(u,Ju)Y) \\
&+ \frac{\delta^2}{2} g(Z,Ju)^{H}((R(u,J\nabla_X u)Y) - \frac{1}{4}{}^{V}(R(X,R(u,Z)Y)u) \\
&+ \frac{1}{2}{}^{V}(R(X,Y)Z) - \frac{\delta^2}{4} g(Z,Ju)^{V}(R(X,R(u,Ju)Y)u) \\
&+ \frac{\delta^2}{2}\big(g(Z,Ju)^{V}(JR(X,Y)u) + g(R(X,Y)u,Ju)^{V}(JZ)\big) \\
&- \frac{\delta^4}{2\lambda} g(R(X,Y)u,Ju)g(Z,u)(Ju)^{V}
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{R}({}^HY,{}^VZ)^{H}X &= \frac{1}{2}{}^{H}((\nabla_Y R)(u,Z)X) + \frac{\delta^2}{2} g(Z,Ju)^{H}((\nabla_Y R)(u,Ju)X) \\
&+ \frac{\delta^2}{2} g(Z,Ju)^{H}((R(u,J\nabla_Y u)X) - \frac{1}{4}{}^{V}(R(Y,R(u,Z)X)u) \\
&+ \frac{1}{2}{}^{V}(R(Y,X)Z) - \frac{\delta^2}{4} g(Z,Ju)^{V}(R(Y,R(u,Ju)X)u) \\
&+ \frac{\delta^2}{2}\big(g(Z,Ju)^{V}(JR(Y,X)u) + g(R(Y,X)u,Ju)^{V}(JZ)\big) \\
&- \frac{\delta^4}{2\lambda} g(R(Y,X)u,Ju)g(Z,u)(Ju)^{V},
\end{aligned}
$$

which gives formula (5.3).

The other formulas are obtained by a similar calculation. We omit them to avoid repetition. □

Now we consider the sectional curvature $\widetilde{K}$ on (TM, g_{BS}) for P given by

$$
\widetilde{K}(\widetilde{X},\widetilde{Y}) = \frac{g_{BS}(\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Y},\widetilde{Y})}{g_{BS}(\widetilde{X},\widetilde{X})g_{BS}(\widetilde{Y},\widetilde{Y}) - g_{BS}(\widetilde{X},\widetilde{Y})^2}, \tag{5.7}
$$

where $P = P(\widetilde{X},\widetilde{Y})$ denotes the plane spanned by $\{\widetilde{X},\widetilde{Y}\}$ for all linearly independent vector fields $\widetilde{X},\widetilde{Y}$ on TM.

Let $\widetilde{K}({}^HX, {}^HY)$, $\widetilde{K}\big({}^HX, {}^VY\big)$ and $\widetilde{K}\big({}^VX, {}^VY\big)$ denote the sectional curvature of the plane spanned by $\{{}^HX, {}^HY\}$, $\{{}^HX, {}^VY\}$ and $\{{}^VX, {}^VY\}$ on (TM, g_{BS}), respectively, where X, Y are orthonormal vector fields on M_{2k}.

Proposition 5.2. *Let (M_{2k}, J, g) be a Kählerian manifold and let (TM, g_{BS}) be its tangent bundle equipped with the Berger type deformed Sasaki metric. Then we have the following:*

$$\text{i)}\quad g_{BS}(\widetilde{R}({}^HX, {}^HY){}^HY, {}^HX) = g(R(X,Y)Y,X) - \frac{3}{4}\|R(X,Y)u\|^2 - \frac{3\delta^2}{4} g(R(X,Y)u, Ju)^2,$$

$$\begin{aligned}\text{ii)}\quad g_{BS}(\widetilde{R}({}^HX, {}^VY){}^VY, {}^HX) = {} & \frac{1}{4}\|R(u,Y)X\|^2 + \delta^2 g(Y,Ju) g(R(u,JY)u,X) \\ & + \delta^4 g(X,Ju) g(Y,Ju) g(R(u,Ju)JY,X) \\ & + \frac{\delta^2}{2} g(Y,Ju) g(R(u,Ju)X, R(u,Y)X) \\ & + \frac{\delta^4}{4} g(Y,Ju)^2 \|R(u,Ju)X\|^2,\end{aligned}$$

$$\begin{aligned}\text{iii)}\quad g_{BS}(\widetilde{R}({}^VX, {}^VY){}^VY, {}^VX) = {} & \delta^4\big(g(X,Ju)^2 + g(Y,Ju)^2\big) - 3\delta^2 g(X,JY)^2 \\ & - \frac{\delta^6}{\lambda}\big(g(X,u)g(Y,Ju) - g(X,Ju)g(Y,u)\big)^2.\end{aligned}$$

The following result is obtained from Proposition 5.2 and formula (5.7).

Theorem 5.3. *Let (M_{2k}, J, g) be a Kählerian manifold and let (TM, g_{BS}) be its tangent bundle equipped with the Berger type deformed Sasaki metric. Then the sectional curvature $\widetilde{K}$ satisfies the following equations:*

$$(1)\quad \widetilde{K}_p({}^HX, {}^HY) = K_x(X,Y) - \frac{3}{4}\|R_x(X,Y)u\|^2 - \frac{3\delta^2}{4} g_x(R(X,Y)u, Ju)^2,$$

$$\begin{aligned}(2)\quad \widetilde{K}_p({}^HX, {}^VY) = {} & \frac{1}{1+\delta^2 g_x(Y,Ju)^2}\Big(\frac{\delta^4}{4} g_x(Y,Ju)^2 \|R_x(u,Ju)X\|^2 \\ & + \delta^4 g_x(X,Ju) g_x(Y,Ju) g_x(R(u,Ju)JY,X) \\ & + \frac{\delta^2}{2} g_x(Y,Ju) g_x(R(u,Ju)X, R(u,Y)X) \\ & + \frac{1}{4}\|R_x(u,Y)X\|^2 + \delta^2 g_x(Y,Ju) g_x(R(u,JY)u,X)\Big),\end{aligned}$$

$$\begin{aligned}(3)\quad \widetilde{K}_p({}^VX, {}^VY) = {} & \frac{1}{1+\delta^2\big(g_x(X,Ju)^2 + g_x(Y,Ju)^2\big)}\Big(-3\delta^2 g_x(X,JY)^2 \\ & + \delta^4\big(g_x(X,Ju)^2 + g_x(Y,Ju)^2\big) \\ & - \frac{\delta^6}{\lambda}\big(g_x(X,u)g_x(Y,Ju) - g_x(X,Ju)g_x(Y,u)\big)^2\Big),\end{aligned}$$

where $p = (x,u) \in TM$ and K denotes the sectional curvature of (M_{2k}, g, J).

Remark 5.4. Let $p = (x, u) \in TM$ such that $u \in T_xM \backslash \{0\}$ and let $\{E_i\}_{i=\overline{1,2k}}$ be an orthonormal basis of the vector space T_xM such that $E_1 = \frac{Ju}{\|u\|} = \frac{Ju}{\|Ju\|}$. Then

$$\left\{F_i = {}^HE_i, F_{2k+1} = \frac{1}{\sqrt{\lambda}} {}^V(E_1), F_{2k+j} = {}^V(E_j)\right\}_{i=\overline{1,2k}, j=\overline{2,2k}} \tag{5.8}$$

is an orthonormal basis of $T_p(TM)$.

Lemma 5.5. *Let (M_{2k}, J, g) be a Kählerian manifold and let (TM, g_{BS}) be its tangent bundle equipped with the Berger type deformed Sasaki metric. Let also $p = (x, u) \in TM$ and $(F_a)_{a=\overline{1,4k}}$ be an anorthonormal basis of $T_p(TM)$ defined by (5.8). Then the sectional curvatures $\widetilde{K}$ satisfy the following equations:*

$$\begin{aligned}
\widetilde{K}_p(F_i, F_j) &= K_x(E_i, E_j) - \frac{3}{4}\|R_x(E_i, E_j)u\|^2 - \frac{3\delta^2}{4} g_x(R(E_i, E_j)u, Ju)^2, \\
\widetilde{K}_p(F_t, F_{2k+1}) &= \frac{\lambda\delta^2}{4(\lambda-1)}|R_x(u, Ju)E_t\|^2, \\
\widetilde{K}_p(F_1, F_{2k+1}) &= \frac{\lambda\delta^2}{4(\lambda-1)}|R_x(u, Ju)E_1\|^2 + \frac{\delta^4}{\lambda} g_x(R(u, Ju)Ju, u), \\
\widetilde{K}_p(F_i, F_{2k+l}) &= \frac{1}{4}\|R_x(u, E_l)E_i\|^2, \\
\widetilde{K}(F_{2k+t}, F_{2k+1}) &= \frac{\delta^2(\lambda-1)}{\lambda} - \frac{\delta^4(\lambda^2+\lambda+1)}{\lambda^2(\lambda-1)}\big(g_x(E_t, u)\big)^2, \\
\widetilde{K}_p(F_{2k+t}, F_{2k+l}) &= -3\delta^2 g_x(E_t, JE_l)^2
\end{aligned}$$

for $i, j = \overline{1,2k}$ and $t, l = \overline{2,2k}$, where K is a sectional curvature of (M_{2k}, J, g).

Proof. The results come directly from Theorem 5.3 and Remark 5.4. □

We now consider the scalar curvature $\widetilde{\sigma}$ of (TM, g_{BS}). By standard calculations, we have the following result.

Theorem 5.6. *Let (M_{2k}, g, J) be a Kählerian manifold and let (TM, g_{BS}) be its tangent bundle equipped with the Berger type deformed Sasaki metric. If σ (resp., $\widetilde{\sigma}$) denotes the scalar curvature of (M_{2k}, g, J) (resp., (TM, g_{BS})), then we have*

$$\begin{aligned}
\widetilde{\sigma}_p = \sigma_x &- \frac{1}{4}\sum_{i,j=1}^{2k}\|R(E_i, E_j)u\|^2 + \frac{\delta^2(3-\lambda)}{4(\lambda-1)}\sum_{i=1}^{2k}\|R(u, Ju)E_i\|^2 \\
&+ \frac{2\delta^4}{\lambda} g(R(u, Ju)Ju, u) - \frac{\delta^2}{\lambda}[(2k-4)\lambda + 2(2k-1)] - \frac{2\delta^2(\lambda^2+\lambda+1)}{\lambda^2},
\end{aligned}$$

where $p = (x, u) \in TM$ and $(E_i)_{i=\overline{1,2k}}$ is an orthonormal basis of $T_p(TM)$ defined by (5.8).

Proof. From the definition of the scalar curvature, we have

$$\begin{aligned}\widetilde{\sigma}_p &= \sum_{\substack{i,j=1\\ i\neq j}}^{2k} \widetilde{K}(F_i,F_j) + 2\sum_{i,j=1}^{2k} \widetilde{K}(F_i,F_{2k+j}) + \sum_{\substack{i,j=1\\ i\neq j}}^{2k} \widetilde{K}(F_{2k+i},F_{2k+j})\\ &= \sum_{\substack{i,j=1\\ i\neq j}}^{2k} \widetilde{K}(F_i,F_j) + 2\sum_{i=1}^{2k} \widetilde{K}(F_i,F_{2k+1}) + 2\sum_{\substack{i=1\\ j=2}}^{2k} \widetilde{K}(F_i,F_{2k+j})\\ &\quad + 2\sum_{i=2}^{2k} \widetilde{K}(F_{2k+i},F_{2k+1}) + \sum_{\substack{i,j=2\\ i\neq j}}^{2k} \widetilde{K}(F_{2k+i},F_{2k+j}).\end{aligned}$$

Using Lemma 5.5, we obtain

$$\begin{aligned}\widetilde{\sigma}_p &= \sum_{\substack{i,j=1\\ i\neq j}}^{2k} \left(K(E_i,E_j) - \frac{3}{4}\|R(E_i,E_j)u\|^2 - \frac{3\delta^2}{4} g(R(E_i,E_j)u,Ju)^2\right)\\ &\quad + \sum_{i=1}^{2k}\left(\frac{\delta^2\lambda}{2(\lambda-1)}\|R(u,Ju)E_i\|^2\right) + \frac{2\delta^4}{\lambda} g(R(u,Ju)Ju,u)\\ &\quad + \frac{1}{2}\sum_{\substack{i=1\\ j=2}}^{2k} \|R(u,E_j)E_i\|^2 - 3\delta^2 \sum_{\substack{i,j=2\\ i\neq j}}^{2k} g(E_i,JE_j)^2\\ &\quad + 2\sum_{i=2}^{2k}\left(\frac{\delta^2(\lambda-1)}{\lambda} - \frac{\delta^4(\lambda^2+\lambda+1)}{\lambda^2(\lambda-1)}\big(g(E_t,u)\big)^2\right).\end{aligned}$$

In order to simplify this last expression, we use

$$\begin{aligned}\sum_{\substack{i,j=2\\ i\neq j}}^{2k} g(E_i,JE_j)^2 &= \sum_{\substack{i=1\\ j=2\\ i\neq j}}^{2k} g(E_i,JE_j)^2 - \sum_{\substack{i=1\\ i\neq j}}^{2k} g(E_1,JE_j)^2 = 2k-2,\\ \sum_{i=2}^{2k} g(E_t,u)^2 &= \sum_{i=1}^{2k} g(E_t,u)^2 = \|u\|^2,\\ \sum_{i,j=1}^{2k} \|R(u,E_j)E_i\|^2 &= \sum_{i,j=1}^{2k} \|R(E_i,E_j)u\|^2.\end{aligned}$$

From the last equation (see, also [9, 21]), we get

$$\begin{aligned}\widetilde{\sigma}_p &= \sigma_x - \frac{1}{4}\sum_{i,j=1}^{2k} \|R(E_i,E_j)u\|^2 + \frac{\delta^2(3-\lambda)}{4(\lambda-1)}\sum_{i=1}^{2k} \|R(u,Ju)E_i\|^2\\ &\quad + \frac{2\delta^4}{\lambda} g(R(u,Ju)Ju,u) - \frac{\delta^2}{\lambda}[(2k-4)\lambda + 2(2k-1)] - \frac{2\delta^2(\lambda^2+\lambda+1)}{\lambda^2}.\end{aligned}$$

The theorem is proved. □

From Theorem 5.6, we deduce the following theorem.

Theorem 5.7. *Let (M_{2k}, g, J) be a locally flat Kählerian manifold and let (TM, g_{BS}) be its tangent bundle equipped with the Berger type deformed Sasaki metric. If $\widetilde{\sigma}$ denotes the scalar curvature of (TM, g_{BS}), then we have*

$$\widetilde{\sigma}_p = -\frac{\delta^2}{\lambda}[(2k-4)\lambda + 2(2k-1)] - \frac{2\delta^2(\lambda^2+\lambda+1)}{\lambda^2},$$

where $p = (x, u) \in TM$ and $\lambda = 1 + \delta^2\|u\|^2$.

6. The Berger type deformed Sasaki metric and harmonicity

Consider a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds. Then the second fundamental form of ϕ is defined by

$$(\nabla d\phi)(X, Y) = \nabla^{\phi}_X d\phi(Y) - d\phi(\nabla_X Y). \tag{6.1}$$

Here, ∇ is the Riemannian connection on M and ∇^{ϕ} is the pull-back connection on the pull-back bundle $\phi^{-1}TN$, and

$$\tau(\phi) = \operatorname{trace}_g \nabla d\phi \tag{6.2}$$

is the tension field of ϕ.

The energy functional of ϕ is defined by

$$E(\phi) = \int_K e(\phi)\, dv_g \tag{6.3}$$

such that K is any compact of M, where

$$e(\phi) = \frac{1}{2}\operatorname{trace}_g h(d\phi, d\phi) \tag{6.4}$$

is the energy density of ϕ.

A map is called harmonic if it is a critical point of the energy functional E. For any smooth variation $\{\phi_t\}_{t\in I}$ of ϕ with $\phi_0 = \phi$ and $V = \frac{d}{dt}\phi_t\Big|_{t=0}$, we have

$$\frac{d}{dt}E(\phi_t)\Big|_{t=0} = -\int_K h(\tau(\phi), V)dv_g. \tag{6.5}$$

Then ϕ is harmonic if and only if $\tau(\phi) = 0$.

If $\psi : (N^n, g) \to (\overline{N}^n, \overline{h})$ is a smooth map between two Riemannian manifolds, then we have

$$\tau(\psi \circ \phi) = d\psi(\tau(\phi)) + \operatorname{trace}_g \nabla d\psi(d\phi, d\phi). \tag{6.6}$$

One can refer to [4, 5, 10, 15] for background on harmonic maps.

6.1. Harmonicity of a vector field $X : (M, g) \longrightarrow (TM, g_{BS})$

Lemma 6.1 ([12, 20])**.** *Let (M, g) be a Riemannian manifold. If X, Y are vector fields on M and (x, u) on TM such that $Y_x = u$, then we have*

$$d_xY(X_x) = {}^HX_{(x,u)} + {}^V(\nabla_X Y)_{(x,u)}.$$

Lemma 6.2. *Let (M_{2k}, J, g) be a Kählerian manifold and let (TM, g_{BS}) be its tangent bundle equipped with the Berger type deformed Sasaki metric. If X is a vector field on M_{2k}, then the energy density associated to X is given by*

$$e(X) = k + \frac{1}{2}\operatorname{trace}_g\left(g(\nabla X, \nabla X) + \delta^2 g(\nabla X, JX)^2\right).$$

Proof. Let $(x, u) \in TM$, X be a vector field on M_{2k}, $X_x = u$ and let $(E_1, \dots, E_{2k})$ be a local orthonormal frame on M_{2k}. Then

$$e(X)_x = \frac{1}{2}\operatorname{trace}_g g_{BS}(dX, dX)_{(x,u)} = \frac{1}{2}\sum_{i=1}^{2k} g_{BS}(dX(E_i), dX(E_i))_{(x,u)}.$$

Using Lemma 6.1, we obtain

$$\begin{aligned}
e(X) &= \frac{1}{2}\sum_{i=1}^{2k} g_{BS}({}^HE_i + {}^V(\nabla_{E_i}X), {}^HE_i + {}^V(\nabla_{E_i}X)) \\
&= \frac{1}{2}\sum_{i=1}^{2k}\left(g_{BS}({}^HE_i, {}^HE_i) + g_{BS}({}^V(\nabla_{E_i}X), {}^V(\nabla_{E_i}X))\right) \\
&= \frac{1}{2}\sum_{i=1}^{2k}\left(g(E_i, E_i) + g(\nabla_{E_i}X, \nabla_{E_i}X) + \delta^2 g(\nabla_{E_i}X, JX)^2\right) \\
&= k + \frac{1}{2}\operatorname{trace}_g\left(g(\nabla X, \nabla X) + \delta^2 g(\nabla X, JX)^2\right).
\end{aligned}$$

The lemma is proved. □

Theorem 6.3. *Let (M_{2k}, J, g) be a Kählerian manifold and let (TM, g_{BS}) be its tangent bundle equipped with the Berger type deformed Sasaki metric. If X is a vector field on M_{2k}, then the tension field associated to X is given by*

$$\tau(X) = {}^H\big(\operatorname{trace}_g A(X)\big) + {}^V\big(\operatorname{trace}_g B(X)\big),$$

where $A(X)$ and $B(X)$ are the bilinear maps defined by

$$\begin{aligned}
A(X) &= R(X, \nabla X) * + \delta^2 g(\nabla X, JX) R(X, JX) *, \\
B(X) &= \nabla^2 X + 2\delta^2 g(\nabla X, JX)\left(J\nabla X - \frac{\delta^2}{\lambda} g(\nabla X, X) JX\right),
\end{aligned}$$

where $\lambda = 1 + \delta^2\|X\|^2$ and $\|X\|^2 = g(X, X)$.

Proof. Let $(x,u) \in TM$, X be a vector field on M_{2k}, $X_x = u$ and let $\{E_i\}_{i=\overline{1,2k}}$ be a local orthonormal frame on M_{2k} such that $(\nabla^M_{E_i} E_i)_x = 0$. Then

$$\begin{aligned}
\tau(X)_x &= \sum_{i=1}^{2k} \{(\nabla^X_{E_i} dX(E_i))_x - dX(\nabla^M_{E_i} E_i)_x\} = \sum_{i=1}^{2k} \left\{\widetilde{\nabla}_{dX(E_i)} dX(E_i)\right\}_{(x,u)} \\
&= \sum_{i=1}^{2k} \left\{\widetilde{\nabla}_{({}^H E_i + {}^V(\nabla_{E_i} X))}({}^H E_i + {}^V(\nabla_{E_i} X))\right\}_{(x,u)} \\
&= \sum_{i=1}^{2k} \Big\{\widetilde{\nabla}_{{}^H E_i}{}^H E_i + \widetilde{\nabla}_{{}^H E_i}{}^V(\nabla_{E_i} X) + \widetilde{\nabla}_{{}^V(\nabla_{E_i} X)}{}^H E_i \\
&\qquad\qquad + \widetilde{\nabla}_{{}^V(\nabla_{E_i} X)}{}^V(\nabla_{E_i} X)\Big\}_{(x,u)}.
\end{aligned}$$

Using Theorem 4.3, we obtain

$$\begin{aligned}
\tau(X) &= \sum_{i=1}^{2k} \Big({}^H(\nabla_{E_i} E_i) - \frac{1}{2}{}^V(R(E_i,E_i)X) + {}^V(\nabla_{E_i}\nabla_{E_i} X) \\
&\quad + \frac{1}{2}{}^H\big(R(X,\nabla_{E_i} X)E_i + \delta^2 g(\nabla_{E_i} X, JX) R(X,JX)E_i\big) \\
&\quad + \frac{1}{2}{}^H\big(R(X,\nabla_{E_i} X)E_i + \delta^2 g(\nabla_{E_i} X, JX) R(X,JX)E_i\big) \\
&\quad + \delta^2\big(g(\nabla_{E_i} X, JX){}^V(J\nabla_{E_i} X) + g(\nabla_{E_i} X, JX){}^V(J\nabla_{E_i} X)\big) \\
&\quad - \frac{\delta^4}{\lambda}\big(g(\nabla_{E_i} X, X) g(\nabla_{E_i} X, JX) + g(\nabla_{E_i} X, X) g(\nabla_{E_i} X, JX)\big){}^V(JX)\Big) \\
&= \sum_{i=1}^{2k} \Big({}^H\big(R(X,\nabla_{E_i} X)E_i + \delta^2 g(\nabla_{E_i} X, JX) R(X,JX)E_i\big) + {}^V(\nabla_{E_i}\nabla_{E_i} X) \\
&\quad + 2\delta^2 g(\nabla_{E_i} X, JX){}^V(J\nabla_{E_i} X) - \frac{2\delta^4}{\lambda} g(\nabla_{E_i} X, X) g(\nabla_{E_i} X, JX){}^V(JX)\Big) \\
&= {}^H\Big(\operatorname{trace}_g \big(R(X,\nabla X) * + \delta^2 g(\nabla X, JX) R(X,JX) *\big)\Big) \\
&\quad + {}^V\Big(\operatorname{trace}_g \big(\nabla^2 X + 2\delta^2 g(\nabla X, JX)\big(J\nabla X - \frac{\delta^2}{\lambda} g(\nabla X, X) JX\big)\big)\Big).
\end{aligned}$$

The theorem is proved. □

Theorem 6.4. *Let (M_{2k}, J, g) be a Kählerian manifold and let (TM, g_{BS}) be its tangent bundle equipped with the Berger type deformed Sasaki metric. If X is a vector field on M_{2k}, then X is harmonic if and only if the following conditions are verified:*

$$\operatorname{trace}_g \Big(R(X,\nabla X) * + \delta^2 g(\nabla X, JX) R(X,JX) *\Big) = 0$$

and

$$\operatorname{trace}_g \left(\nabla^2 X + 2\delta^2 g(\nabla X, JX)\left(J\nabla X - \frac{\delta^2}{\lambda} g(\nabla X, X) JX\right)\right) = 0.$$

Proof. The statement is a direct consequence of Theorem 6.3. □

Corollary 6.5. *Let (M_{2k}, J, g) be a Kählerian manifold and let (TM, g_{BS}) be its tangent bundle equipped with the Berger type deformed Sasaki metric. Any parallel vector field is harmonic.*

Example 6.6. Let $(\mathbb{R}^2, J, g)$ be a Kählerian manifold such that

$$g = e^{2x}dx^2 + e^{2y}dy^2$$

and

$$J\partial_x = -\frac{e^x}{e^y}\partial_y, \quad J\partial_y = \frac{e^y}{e^x}\partial_x,$$

where $\partial_x = \frac{\partial}{\partial x}$. The vector field $X = e^{-x}\partial_x - e^{-y}\partial_y$ is harmonic. Indeed, it is enough to set $u = e^x$ and $v = e^y$ for getting the Euclidean metric $g = du^2 + dv^2$ and $X = \partial_u - \partial_v$, which is trivially parallel.

Example 6.7. Let $\mathbb{R}^2$ be endowed with the Kählerian structure (J, g) in polar coordinate defined by

$$g = dr^2 + r^2 d\theta^2$$

and

$$J\partial_r = -\frac{1}{r}\partial_\theta, \quad J\partial_\theta = r\partial_r.$$

The vector field $X = \sin\theta\partial_r + \dfrac{1}{r}\cos\theta\partial_\theta$ is harmonic. Indeed,
the non-null Christoffel symbols of the Riemannian connection are

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^1_{22} = -r$$

and

$$\nabla_{\partial_r}\partial_r = 0, \quad \nabla_{\partial_r}\partial_\theta = \nabla_{\partial_\theta}\partial_r = \frac{1}{r}\partial_\theta, \quad \nabla_{\partial_\theta}\partial_\theta = -r\partial_r.$$

Hence we have

$$\begin{aligned}
\nabla_{\partial_r}X &= \sin\theta\nabla_{\partial_r}\partial_r - \frac{1}{r^2}\cos\theta\frac{\partial}{\partial\theta} + \frac{1}{r}\cos\theta\nabla_{\partial_r}\partial_\theta = 0,\\
\nabla_{\partial_\theta}X &= \cos\theta\partial_r + \sin\theta\nabla_{\partial_\theta}\partial_r - \frac{1}{r}\sin\theta\partial_\theta + \frac{1}{r}\cos\theta\nabla_{\partial_\theta}\partial_\theta = 0,
\end{aligned}$$

i.e., $\nabla X = 0$.

Theorem 6.8. *Let (M_{2k}, J, g) be a Kählerian compact manifold and let (TM, g_{BS}) be its tangent bundle equipped with the Berger type deformed Sasaki metric. If X is a vector field on M_{2k}, then X is harmonic if and only if X is parallel.*

Proof. If X is parallel, from Corollary 6.5, we deduce that X is a harmonic vector field.
Conversely, let X_t be a variation of X defined by

$$\begin{aligned} \mathbb{R}\times M &\longrightarrow T_xM \\ (t,x) &\longmapsto X_t(x) = (t+1)X_x. \end{aligned}$$

From Lemma 6.2, we have

$$\begin{aligned} e(X_t) &= k + \frac{(1+t)^2}{2}\operatorname{trace}_g g(\nabla X,\nabla X) + \frac{(1+t)^4}{2}\delta^2 \operatorname{trace}_g g(\nabla X, JX)^2, \\ E(X_t) &= k\,Vol(M) + \frac{(1+t)^2}{2}\int_M \operatorname{trace}_g g(\nabla X,\nabla X)dv_g \\ &\quad + \frac{(1+t)^4}{2}\delta^2\int_M \operatorname{trace}_g g(\nabla X, JX)^2 dv_g. \end{aligned}$$

If X is a critical point of the energy functional, then we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}E(X_t)|_{t=0} \\ &= \frac{\partial}{\partial t}\Big(k\,Vol(M) + \frac{(1+t)^2}{2}\int_M \operatorname{trace}_g g(\nabla X,\nabla X)dv_g\Big)_{t=0} \\ &\quad + \frac{\partial}{\partial t}\Big(\frac{(1+t)^4}{2}\delta^2\int_M \operatorname{trace}_g g(\nabla X, JX)^2 dv_g\Big)_{t=0} \\ &= \int_M \operatorname{trace}_g g(\nabla X,\nabla X)dv_g + 2\delta^2\int_M \operatorname{trace}_g g(\nabla X, JX)^2 dv_g \\ &= \int_M \operatorname{trace}_g \big(g(\nabla X,\nabla X) + 2\delta^2 g(\nabla X, JX)^2\big)dv_g \end{aligned}$$

which gives

$$g(\nabla X,\nabla X) + 2\delta^2 g(\nabla X, JX)^2 = 0.$$

Hence, it follows that $\nabla X = 0$. □

Example 6.9 (Counterexample). Let $(\mathbb{R}^{2k}, J, <,>)$ be a Kählerian real space (flat and non compact manifold) and let $T\mathbb{R}^{2k}$ be its tangent bundle equipped with the Berger type deformed Sasaki metric such that J is a canonical complex structure on $\mathbb{R}^{2k}$. It is given by the matrix

$$\begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$$

if $X = (X^1, \ldots, X^{2k})$ is a vector field on $\mathbb{R}^{2k}$.
For $\delta = 0$, we have

$$\tau(X) = \operatorname{trace}_g \nabla^2 X = \left(\sum_{i=1}^{2k}\frac{\partial^2 X^1}{\partial x_i^2}, \ldots, \sum_{i=1}^{2k}\frac{\partial^2 X^{2k}}{\partial x_i^2}\right).$$

1) If X is constant, then X is harmonic.
2) If $X_i = a_i x_i$ and $a_i \neq 0$, then X is harmonic ($\tau(X) = 0$) but $\nabla X \neq 0$.
Indeed,

$$\nabla X\left(\frac{\partial}{\partial x_j}\right) = \nabla_{\frac{\partial}{\partial x_j}} X = \sum_i a_i \nabla_{\frac{\partial}{\partial x_j}}\left(x_i \frac{\partial}{\partial x_i}\right) = \sum_i \delta_i^j a_i \frac{\partial}{\partial x_i} = a_j \frac{\partial}{\partial x_j} \neq 0.$$

Remark 6.10. In general, using Corollary 6.5 and Theorem 6.8, we can construct many examples for harmonic vector fields.

6.2. Harmonicity of the map $\sigma : (M, g) \longrightarrow (TN, h_{BS})$

Theorem 6.11. *Let (M_k, g) be a Riemannian manifold, (N_{2n}, J, h) be a Kählerian manifold and let (TN, h_{BS}) be the tangent bundle of N equipped with the Berger type deformed Sasaki metric. Let $\phi : M \to N$ be a smooth map and*

$$\begin{aligned} \sigma : M &\longrightarrow TN \\ x &\longmapsto \sigma(x) = V \circ \phi(x), \end{aligned}$$

where V is a vector field on N. The tension field of σ is given by

$$\begin{aligned} \tau(\sigma) = {}^H\big[&\tau(\phi) + \operatorname{trace}_g \big(R^N(\sigma, \nabla^\phi \sigma) d\phi(*) \\ &+ \delta^2 h(\nabla^\phi \sigma, J\sigma) R^N(\sigma, J\sigma) d\phi(*)\big)\big] \\ &+ {}^V\big[\nabla_{\tau(\phi)}\sigma - \operatorname{trace}_g \nabla_{\nabla_{d\phi(*)} d\phi(*)}\sigma + \operatorname{trace}_g \big((\nabla^\phi)^2 \sigma \\ &+ 2\delta^2 h(\nabla^\phi \sigma, J\sigma)\big(J(\nabla^\phi \sigma) - \frac{\delta^2}{\lambda} h(\nabla^\phi \sigma, \sigma) J\sigma\big)\big)\big], \end{aligned}$$

where $\lambda = 1 + \delta^2 \|\sigma\|^2$ and $\|\sigma\|^2 = h(\sigma, \sigma)$.

Proof. We have

$$\begin{aligned} \tau(\sigma) = \tau(V \circ \phi) &= dV(\tau(\phi)) + \operatorname{trace}_g \nabla\, dV(d\phi(*), d\phi(*)) \\ &= dV(\tau(\phi)) + \operatorname{trace}_g \widetilde{\nabla}_{dV(d\phi(*))}\, dV(d\phi(*)) \\ &\quad - \operatorname{trace}_g dV(\nabla^N_{d\phi(*)}\, d\phi(*)). \end{aligned}$$

The using of Theorem 4.3, Lemma 6.1 and Theorem 6.11 results in the following. □

From Theorem 6.11, we obtain

Theorem 6.12. *Let (M_k, g) be a Riemannian manifold, (N_{2n}, J, h) be a Kählerian manifold and let (TN, h_{BS}) be the tangent bundle of N equipped with the Berger type deformed Sasaki metric. Let $\phi : M \to N$ be a smooth map and*

$$\begin{aligned} \sigma : M &\longrightarrow TN \\ x &\longmapsto \sigma(x) = V \circ \phi(x), \end{aligned}$$

where V is a vector field on N. Then σ is harmonic if and only if the following conditions are verified:

$$\tau(\phi) = -\operatorname{trace}_g \big(R^N(\sigma, \nabla^\phi\sigma)d\phi(*) + \delta^2 h(\nabla^\phi\sigma, J\sigma)R^N(\sigma, J\sigma)d\phi(*)\big),$$

$$\begin{aligned}\nabla_\sigma\tau(\phi) = &-\operatorname{trace}_g \big[2\delta^2 h(\nabla^\phi\sigma, J\sigma)\big(J(\nabla^\phi\sigma) - \frac{\delta^2}{\lambda}h(\nabla^\phi\sigma,\sigma)J\sigma\big)\big] \\ &+ \operatorname{trace}_g \big[\nabla_{\nabla_{d\phi(*)}d\phi(*)}\sigma - (\nabla^\phi)^2\sigma\big].\end{aligned}$$

6.3. Harmonicity of the map $\Phi : (TM, g_{BS}) \longrightarrow (N, h)$

Lemma 6.13. *Let (M_{2k}, J, g) be a Kählerian manifold and let (TM, g_{BS}) be its tangent bundle equipped with the Berger type deformed Sasaki metric. The canonical projection*

$$\begin{aligned}\pi : (TM, g_{BS}) &\longrightarrow (M_{2k}, J, g)\\ (x, u) &\longmapsto x\end{aligned}$$

is harmonic, i.e., $\tau(\pi) = 0$.

Proof. We put $p = 2k$. Let $\{E_i\}_{i=\overline{1,p}}$ be a local orthonormal frame on M and let $\{\widetilde{E}_j\}_{j=\overline{1,2p}}$ be a local frame on TM, where

$$\widetilde{E}_j = \begin{cases} {}^H E_j, & 1 \le j \le p\\ {}^V E_{j-p}, & p+1 \le j \le 2p.\end{cases}$$

The tension field of π is given by

$$\tau(\pi) = \operatorname{trace}_{g_{BS}} \nabla d\pi = \sum_{i,j=1}^{2p} G^{ij}\Big\{\nabla^M_{d\pi(\widetilde{E}_i)} d\pi(\widetilde{E}_j) - d\pi(\nabla^{TM}_{\widetilde{E}_i}\widetilde{E}_j)\Big\},$$

where (G_{ij}) is the matrix of g_{BS} and its inverse matrix is (G^{ij}) such that

$$\begin{aligned} &G_{ij} = \delta_{ij}, && 1 \le i, j \le p,\\ &G_{ij} = 0, && 1 \le i \le p,\ p+1 \le j \le 2p,\\ &G_{ij} = \delta_{ij} + \delta^2 (Ju)^{i-p}(Ju)^{j-p}, && p+1 \le i,\ j \le 2p,\end{aligned}$$

and

$$\begin{aligned} &G^{ij} = \delta_{ij}, && 1 \le i,\ j \le p,\\ &G^{ij} = 0, && 1 \le i \le p,\ p+1 \le j \le 2p,\\ &G^{ij} = \frac{1}{1+\delta^2\|Ju\|^2}\big[\delta_{ij} + \delta^2(\|Ju\|^2\delta_{ij} \\ &\qquad\qquad - (Ju)^{i-p}(Ju)^{j-p})\big], && p+1 \le i,\ j \le 2p,\end{aligned}$$

where $\varphi u = (Ju)^k E_k$. Then

$$\tau(\pi) = \sum_{i,j=1}^{p} G^{ij}\Big\{\nabla^M_{d\pi(^HE_i)} d\pi(^HE_j) - d\pi(\nabla^{TM}_{^HE_i}{}^HE_j)\Big\}$$
$$+ \sum_{i,j=p+1}^{2p} G^{ij}\Big\{\nabla^M_{d\pi(^VE_{i-p})} d\pi(^VE_{j-p}) - d\pi(\nabla^{TM}_{^VE_{i-p}}{}^VE_{j-p})\Big\}.$$

With $d\pi(^VX) = 0$ and $d\pi(^HX) = X \circ \pi$ for any vector field X, we can find

$$\begin{aligned}\tau(\pi) &= \sum_{i,j=1}^{p} G^{ij}\Big\{\nabla^M_{(E_i\circ\pi)}(E_j\circ\pi) - d\pi\big(^H(\nabla^M_{E_i}E_j) - \frac{1}{2}{}^V(R(E_i,E_j)u)\big)\Big\}\\
&\quad - \sum_{i,j=p+1}^{2p} G^{ij}\delta^2\Big\{g(E_{i-p},Ju)d\pi\big(^V(JE_{j-p})\big) + g(E_{j-p},Ju)d\pi\big(^V(JE_{i-p})\big)\\
&\quad - \frac{\delta^2}{\lambda}\left(g(E_{j-p},u)g(E_{i-p},Ju) + g(E_{i-p},u)g(E_{j-p},Ju)\right)d\pi\big(^V(Ju)\big)\Big\}\\
&= \sum_{i,j=1}^{p} \delta_{ij}\Big\{(\nabla^M_{E_i}E_j)\circ\pi - (\nabla^M_{E_i}E_j)\circ\pi\Big\} = 0.\end{aligned}$$

□

Theorem 6.14. *Let (M_{2k}, J, g) be a Kählerian manifold, (N^n, h) be a Riemannian manifold and let (TM, g_{BS}) be the tangent bundle of M equipped with the Berger type deformed Sasaki metric. Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map. The tension field of the map*

$$\Phi : (TM, g_{BS}) \longrightarrow (N, h)$$
$$(x, u) \longmapsto \phi(x)$$

is given by $\tau(\Phi) = \tau(\phi)\circ\pi$.

Proof. We put $p = 2k$. Let $\{E_i\}_{i=\overline{1,p}}$ be a local orthonormal frame on M and let $\{\widetilde{E}_j\}_{j=\overline{1,2p}}$ be a local frame on TM. As the Φ is defined by $\Phi = \phi\circ\pi$, we have

$$\tau(\Phi) = \tau(\phi\circ\pi) = d\phi(\tau(\pi)) + \mathrm{trace}_{g_{BS}}\nabla d\phi(d\pi, d\pi),$$

$$\begin{aligned}\mathrm{trace}_{g_{BS}}\nabla d\phi(d\pi, d\pi) &= \sum_{i=1}^{2p} G^{ij}\Big\{\nabla^N_{d\phi(d\pi(\widetilde{E}_i))} d\phi(d\pi(\widetilde{E}_j)) - d\phi(\nabla^M_{d\pi(\widetilde{E}_i)} d\pi(\widetilde{E}_j))\Big\}\\
&= \sum_{i,j=1}^{p} \delta_{ij}\Big\{\nabla^N_{d\phi(d\pi(^HE_i))} d\phi(d\pi(^HE_j))\\
&\quad - d\phi(\nabla^M_{d\pi(^HE_i)} d\pi(^HE_j))\Big\}\\
&= \sum_{i=1}^{p}\big(\nabla^N_{d\phi(E_i)} d\phi(E_i) - d\phi(\nabla^M_{E_i}E_i)\big)\circ\pi = \tau(\phi)\circ\pi.\end{aligned}$$

Using Lemma 6.13, we obtain $\tau(\Phi) = \tau(\phi)\circ\pi$. □

Theorem 6.15. *Let (M_{2k}, J, g) be a Kählerian manifold, (N^n, h) be a Riemannian manifold and let (TM, g_{BS}) be the tangent bundle of M equipped with Berger type deformed Sasaki metric. Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map. The map*

$$\begin{aligned}\Phi : (TM, g_{BS}) &\longrightarrow (N, h)\\ (x, u) &\longmapsto \phi(x)\end{aligned}$$

is harmonic if and only if ϕ is harmonic.

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References

[1] M.T.K. Abbassi and M. Sarih, *On natural metrics on tangent bundles of Riemannian manifolds*, Arch. Math. **41** (2005), 71–92.

[2] M. Altunbas, R. Simsek, and A. Gezer, *A study concerning Berger type deformed Sasaki metric on the tangent bundle*, J. Math. Phys. Anal. Geom. **15** (2019), 435–447.

[3] P. Dombrowski, *On the geometry of the tangent bundle*, J. Reine Angew. Math. **210** (1962), 73–88.

[4] J. Ells and L. Lemaire, *Another report on harmonic maps*, Bull. Lond. Math. Soc. **20** (1988), 385-524.

[5] J. Eells and J.H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer.J. Math. **86** (1964), 109–160.

[6] A. Gezer, *On the tangent bundle with deformed Sasaki metric*, Int. Electron. J. Geom. **6** (2013), 19–31.

[7] A. Gezer and M. Altunbas, *On the geometry of the rescaled Riemannian metric on tensor bundles of arbitrary type*, Kodai Math. J. **38** (2015), 37–64.

[8] A. Gezer and M. Altunbas, *Notes on the rescaled Sasaki type metric on the cotangent bundle*, Acta Math. Sci. Ser. B (Engl. Ed.) **34** (2014), 162–174.

[9] S. Gudmundsson and E. Kappos, *On the geometry of the tangent bundle with the Cheeger–Gromoll metric*, Tokyo J. Math. **25** (2002), 75–83.

[10] T. Ishihara, *Harmonic sections of tangent bundles*, J. Math. Tokushima Univ. **13** (1979), 23–27.

[11] S. Kobayashi and K. Nomizu, *Fondations of Differential Geometry* , **II**, Intersciense, New York-London 1963.

[12] J.J. Konderak, *On harmonic vector fields*, Publications Mathematiques. **36** (1992), 217–288.

[13] O. Kowalski, *Curvature of the induced Riemannian metric of the tangent bundle of Riemannian manifold*, J. Reine Angew.Math. **250** (1971), 124–129.

[14] E. Musso and F. Tricerri, *Riemannian metrics on tangent bundles*, Ann. Math. Pura Appl. **150** (1988), 1–20.

[15] V. Opriou, *Harmonic maps between tangent bundles*, Rend. Sem. Mat. Univ. Politec. Torino **47** (1989), 47–55.

[16] S. Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds II*, Tokyo J. Math. **14** (1962), 146–155.

[17] A. Yampolsky, *On geodesics of tangent bundle with fiberwise deformed Sasaki metric over Kahlerian manifolds*, J. Math. Phys. Anal. Geom. **8** (2012), 177–189.

[18] K. Yano and S. Ishihara, *Tangent and Cotangent Bundles*, M. Dekker, New York, 1973.

[19] A. Zagane, *Berger type deformed Sasaki metric on the cotangent bundle*, Commun. Korean Math. Soc. **36** (2021), 575–592.

[20] A. Zagane and M. Djaa, *On geodesics of warped Sasaki metric*, Math. Sci. Appl. E-Notes **5** (2017), 85–92.

[21] A. Zagane and M. Djaa, *Geometry of Mus-Sasaki metric*, Commun. Math. **26** (2018), 113–126.

[22] A. Zagane and M. Zagane, *g-natural metric and harmonicity on the cotangent bundle*, Commun. Korean Math. Soc. **36** (2021), 135–147.

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Деякі результати щодо дотичних розшарувань з метрикою Сасакі, деформованою за типом Берже, на многовидах Келера

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Нехай M означає келеровий многовид, оснащений майже комплексною структурою J і рімановою метрикою g, а TM є його дотичним розшаруванням з метрикою Сасакі, деформованою за типом Берже. У цій статті, по-перше, ми знаходимо всі форми тензорів рімановох кривини TM. По-друге, ми шукаємо умови, за яких векторне поле є гармонічним відносно метрики Сасакі, деформованої за типом Берже, та наводимо кілька прикладів гармонічного вектора поля. На завершення, ми досліджуємо гармонійність відображень між рімановим многовидом та дотичним розшаруванням іншого ріманового многовиду і навпаки.

Ключові слова: метрика Сасакі, деформована за типом Берже; гармонійність; тензор ріманової кривини; дотичне розшарування