Journal of Mathematical Physics, Analysis, Geometry
2024, Vol. 20, No. 3, pp. 313–331
doi: https://doi.org/10.15407/mag20.03.04

Phase Retrieval for Probability Measures on the Group $\mathbb{Z}_2^3$

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Let $\mathbb{Z}_2$ be the group of the residue classes modulo 2. We give a complete description of the class of probability measures on the group $\mathbb{Z}_2^3$, which are uniquely determined by the modulus of their characteristic functions up to a shift.

Key words: probability measure on a group, characteristic function, trivial equivalence class, phase retrieval

Mathematical Subject Classification 2020: 60B15, 60E10, 62P35

1. Introduction

Phase retrieval problem arose in physics (see [10], [9], and the references therein). It consists in the description of all probability measures on $\mathbb{R}^n$ or, more generally, on a locally compact abelian group, for which the moduli of their characteristic functions are equal to the modulus of the characteristic function of the given measure. The similar question to this one is the following: which probability measures can be restored by the moduli of their characteristic functions up to a shift and central symmetry. To formulate the problem, let us introduce the required notation and definitions. Let X be a locally compact abelian group, Y be its character group, (x, y) be the value of character $y \in Y$ on the element $x \in X$, $M^1(X)$ be the set of probability measures on the group X. Characteristic function of the measure $\mu \in M^1(X)$ is determined by the formula

$$\widehat{\mu}(y) = \int_X (x, y)\mu(dx), \quad y \in Y\,. \tag{1.1}$$

Definition 1.1. We say that the measures $\mu, \nu \in M^1(X)$ are *equivalent* and write $\mu \sim \nu$ if

$$|\widehat{\nu}(y)| = |\widehat{\mu}(y)|, \quad y \in Y\,. \tag{1.2}$$

Let μ_x be the shift of measure μ by element $x \in X$ and let μ^- be the measure obtained from μ by central symmetry:

$$\mu_x(E) = \mu(E + x), \quad \mu^-(E) = \mu(-E),$$

where E is a Borel set in X. It is easy to see that $\mu \sim \mu_x$, $\mu \sim \mu^-$, $\mu \sim \mu_x^-$ for any measure $\mu \in M^1(X)$ and any $x \in X$.

Definition 1.2. We say that a measure $\mu \in M^1(X)$ has *a trivial equivalence class* if only the measures μ_x and μ_x^-, $x \in X$, are equivalent to it.

The set of measures from $M^1(X)$ that have a trivial equivalence class is denoted by $TEC(X)$. Note that the normal distribution on $\mathbb{R}$ belongs to the class $TEC(\mathbb{R})$ that follows from Cramer's decomposition theorem for the normal distribution, and the Poisson distribution does not belong to the class $TEC(\mathbb{R})$.

Papers [1–8] were aimed to give a complete or partial description of the class $TEC(X)$ for some groups X. In papers [2,3], the triviality of the equivalence class for uniform distributions on intervals of the group $\mathbb{Z}_n$ of residue classes modulo n, on the Cartesian product of intervals of the group $\mathbb{Z}^l$, on the unit ball in $\mathbb{R}^l$ was studied. In [8], there was obtained a criterion by which a two-point measure on $\mathbb{Z}_n$ has a trivial equivalence class. In [2], a necessary and sufficient condition was found for the fact that a generalized Poisson distribution on the group $\mathbb{Z}_2^l$, $l \geq 2$, whose spectral measure is proportional to the Haar measure, belongs to the class $TEC(\mathbb{Z}_2^l)$ (Theorem 1.4 below). In [6], a necessary and sufficient condition was obtained for a generalized Poisson distribution with an arbitrary spectral measure to belong to the class $TEC(\mathbb{Z}_2^2)$. In [7], there was set a criterion for a generalized Poisson distribution on the group $\mathbb{Z}_2^3$, whose spectral measure is arbitrarily distributed on any three generators of the group, to belong to the class $TEC(\mathbb{Z}_2^3)$ (Theorem 1.5 below).

Sometimes it is possible to obtain a complete description of the class $TEC(X)$. It is easy to see that $TEC(\mathbb{Z}_2) = M^1(\mathbb{Z}_2)$. The classes $TEC(\mathbb{Z}_3)$ and $TEC(\mathbb{Z}_4)$ are fully described in [8], the class $TEC(\mathbb{Z}_2^2)$ is described in [6] (Theorem 1.6 below). In this paper, we obtain a complete description of the class $TEC(\mathbb{Z}_2^3)$ (Theorem 2.5 below).

Let us formulate some of the results from the papers [2, 6, 7]. We need the following definition.

Definition 1.3. Let ρ be a finite measure on the group X. *The generalized Poisson distribution* with spectral measure ρ is the distribution

$$\Pi_\rho = e^{-\rho(X)} \sum_{k=0}^{\infty} \frac{1}{k!} \rho^{*k}.$$

The characteristic function of the distribution Π_ρ has the form

$$\widehat{\Pi}_\rho(y) = \exp\left\{ \int_X [(x,y) - 1]\rho(dx) \right\}.$$

Let m_X denote the Haar measure on the group X. For a compact group X we put $m_X(X) = 1$. The following theorem gives a necessary and sufficient condition under which the measure $\Pi_{\lambda m}$ (here, m is the Haar measure on the group $\mathbb{Z}_2^l$, $\lambda > 0$) belongs to the class $TEC(\mathbb{Z}_2^l)$, $l = 2, 3, \ldots$.

Theorem 1.4 ([2]). *The measure $\Pi_{\lambda m}$ belongs to the class $TEC(\mathbb{Z}_2^l)$, $l = 2, 3, \ldots,$ if and only if $\lambda < \ln 3$.*

Theorem 1.5 gives a necessary and sufficient condition under which the generalized Poisson distribution on the group $\mathbb{Z}_2^3$, whose spectral measure is arbitrarily distributed on any three generators of the group, belongs to the class $TEC(\mathbb{Z}_2^3)$.

Theorem 1.5 ([7]). *Let π be a generalized Poisson distribution on the group $\mathbb{Z}_2^3$, whose spectral measure is concentrated on any three generators and assigns masses a, b, c to them. A measure π belongs to the class $TEC(\mathbb{Z}_2^3)$ if and only if the system of inequalities*

$$\begin{cases} e^{-2a} + e^{-2b} + e^{-2(a+b)} > 1, \\ e^{-2b} + e^{-2c} + e^{-2(b+c)} > 1, \\ e^{-2a} - e^{-2b} + e^{-2c} + e^{-2(a+b)} + e^{-2(b+c)} - e^{-2(c+a)} + e^{-2(a+b+c)} > 1 \end{cases}$$

is satisfied, or one of the two systems of inequalities obtained from this system by cyclic permutations of the parameters a, b, c is satisfied.

The following theorem gives the full description of the class $TEC(\mathbb{Z}_2^2)$. Let $\mu \in M^1(\mathbb{Z}_2^2)$. We denote $a_{\max} = \max\{\mu(\{x\}) : x \in \mathbb{Z}_2^2\}$. Let $S(\mu)$ be the support of the measure μ, $|C|$ be the number of elements of the set C.

Theorem 1.6 ([6]). *The class $TEC(\mathbb{Z}_2^2)$ contains the following measures and only them:*

1) *all measures μ for which $|S(\mu)| \le 2$;*
2) *all measures μ for which $|S(\mu)| = 3$ and $a_{\max} \ge 1/2$;*
3) *all measures μ for which $|S(\mu)| = 4$ and one of the following two conditions is satisfied:*
   a) $a_{\max} > 1/2$,
   b) $a_{\max} < 1/2$ *and the sum of the masses of some two elements of the group is equal to the sum of the masses of the other two elements.*

We note that $\mu \notin TEC(\mathbb{Z}_2^2)$ if $|S(\mu)| = 4$ and $a_{\max} = 1/2$.

2. Main result

To formulate the main result, we need some notation.

Let $\mathbb{Z}_2 = \{0, 1\}$ be additive group of the residue classes modulo 2,

$$X = \mathbb{Z}_2^3 = \{x = (\alpha, \beta, \gamma) : \alpha, \beta, \gamma \in \{0, 1\}\}.$$

We denote the elements of the group $\mathbb{Z}_2^3$ as follows:

$$x_0 = (0,0,0), \quad x_1 = (1,0,0), \quad x_2 = (0,1,0), \quad x_3 = (0,0,1),$$
$$x_4 = (1,1,0), \quad x_5 = (1,0,1), \quad x_6 = (0,1,1), \quad x_7 = (1,1,1).$$

For the zero element x_0 of the group $\mathbb{Z}_2^3$, we will also use the notation 0.

The group of characters of the group $\mathbb{Z}_2^3$ is isomorphic to $\mathbb{Z}_2^3$. The value of the character $y = (\xi, \eta, \zeta)$, $\xi, \eta, \zeta \in \{0, 1\}$ on the element $x = (\alpha, \beta, \gamma) \in \mathbb{Z}_2^3$ is determined by the formula

$$(x, y) = (-1)^{\alpha\xi+\beta\eta+\gamma\zeta}.$$

Let us denote by $\mathfrak{A}_1$ the set of all subgroups of the group $\mathbb{Z}_2^3$ isomorphic to $\mathbb{Z}_2$:

$$\mathfrak{A}_1 = \{H_i = \{x_0, x_i\} : i = 1, \dots, 7\}.$$

Let us denote by $\mathfrak{A}_2$ the set of all subgroups of the group $\mathbb{Z}_2^3$ isomorphic to $\mathbb{Z}_2^2$:

$$\mathfrak{A}_2 = \{K_i : i = 1, \dots, 7\},$$

$$\begin{aligned} &K_1 = \{x_0, x_2, x_3, x_6\}, \quad K_2 = \{x_0, x_1, x_3, x_5\}, \quad K_3 = \{x_0, x_1, x_2, x_4\}, \\ &K_4 = \{x_0, x_1, x_6, x_7\}, \quad K_5 = \{x_0, x_3, x_4, x_7\}, \quad K_6 = \{x_0, x_2, x_5, x_7\}, \\ &K_7 = \{x_0, x_4, x_5, x_6\}. \end{aligned} \tag{2.1}$$

In what follows, the representation of the group $\mathbb{Z}_2^3$ as a direct sum of a group from $\mathfrak{A}_1$ and a group from $\mathfrak{A}_2$ will play an important role.

Let $\mu \in M^1(\mathbb{Z}_2^3)$. The values of the masses of the measure μ on the elements of the group $\mathbb{Z}_2^3$ are denoted by

$$a_i = \mu(\{x_i\}), \quad i = 0, 1, \dots, 7.$$

Thus we have $a_i \geq 0$, $\sum_{i=0}^{7} a_i = 1$. We also denote

$$a_{\max} := \max\{a_i : i = 0, 1, \dots, 7\}.$$

From (1.1), we find the general form of the characteristic function $\widehat{\mu}$, $\mu \in M^1(\mathbb{Z}_2^3)$,

$$\widehat{\mu}(y) = \sum_{i=0}^{7} a_i (x_i, y), \qquad a_i \geq 0, \quad \sum_{i=0}^{7} a_i = 1. \tag{2.2}$$

Since all non-zero elements of the group $\mathbb{Z}_2^3$ have order 2, the central symmetry is the identity mapping. Therefore Definition 1.2 is simplified: the measure $\mu \in M^1(\mathbb{Z}_2^3)$ has a trivial equivalence class if only the shifts of μ are equivalent to it.

Let E be a subset of $\mathbb{Z}_2^3$. We denote

$$u(E) := \max\{\mu(\{x\}) : x \in E\}, \quad v(E) := \min\{\mu(\{x\}) : x \in E\}.$$

Definition 2.1. Let us denote by $U(E)$ the set of measures $\mu \in M^1(\mathbb{Z}_2^3)$ satisfying the condition

$$2u(E) > \mu(E).$$

In other words, $U(E)$ is the set of such measures for which the maximum mass of the elements of the set E is greater than the sum of the remaining masses of the elements of this set.

Definition 2.2. Let us denote by $V(E)$ the set of measures $\mu \in M^1(\mathbb{Z}_2^3)$ satisfying the condition

$$1/2 + 2v(E) < \mu(E).$$

In other words, $V(E)$ is the set of such measures for which the sum of $1/2$ and minimum mass of the elements of the set E is smaller than the sum of the remaining masses of the elements of this set.

Let $K \in \mathfrak{A}_2$. The coset of the subgroup K in the group $\mathbb{Z}_2^3$, different from K, will be denoted by $\overline{K}$.

For convenience of reference, we formulate the following statement in the form of a lemma. It easily follows from the description of subgroups of $\mathbb{Z}_2^3$.

Lemma 2.3. *Let $H \in \mathfrak{A}_1$, $K \in \mathfrak{A}_2$, and $H \cap K = \{0\}$. Then the following statements are valid*:

1) *there are exactly three subgroups $L_i \in \mathfrak{A}_2$ containing H*;
2) *non-zero elements of the subgroup K belong to the different subgroups L_i, $i = 1, 2, 3$*;
3) *there are exactly four subgroups $K^{(j)} \in \mathfrak{A}_2$, $j = 1, 2, 3, 4$, such that $H \cap K^{(j)} = \{0\}$*;
4) *for each $j_0 = 1, 2, 3, 4$ and for each $j \neq j_0$, there is a unique element $z_j \in \overline{K^{(j)}}$ such that $z_j \notin H$, $z_j \notin K^{(j_0)}$.*

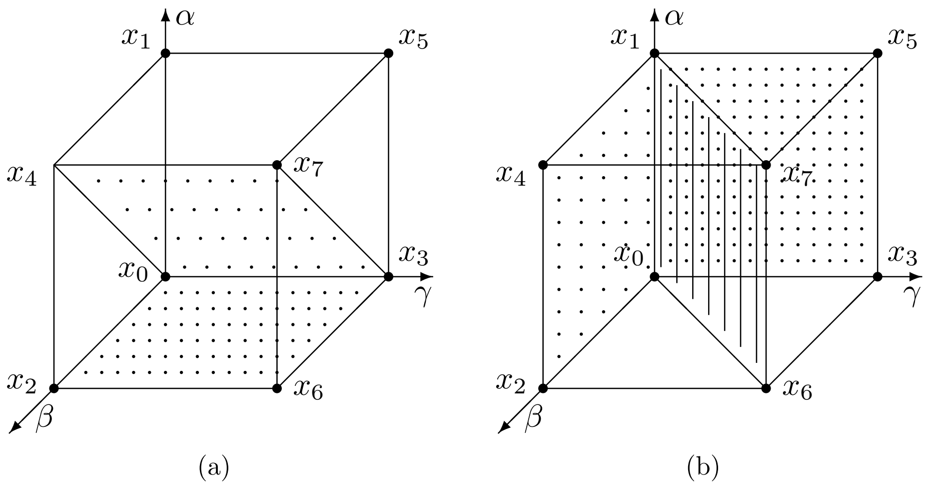

Fig. 2.1

We illustrate Lemma 2.3 with the help of Fig. 2.1. If we take $H = \{0, x_1\}$, then $L_1 = \{0, x_1, x_2, x_4\}$, $L_2 = \{0, x_1, x_3, x_5\}$, $L_3 = \{0, x_1, x_6, x_7\}$ (see Fig. 2.1(b), where subgroups L_1, L_2, L_3 are marked with different shadings). Let $K^{(j_0)} = \{0, x_2, x_3, x_6\}$ and $K^{(j)} = \{0, x_3, x_4, x_7\}$, see Fig. 2.1(a). Then $\overline{K^{(j)}} = \{x_1, x_2, x_5, x_6\}$, and since the conditions $z_j \notin H$, $z_j \notin K^{(j_0)}$ must be satisfied, we see that $z_j \neq x_1$, $z_j \neq x_2$, $z_j \neq x_6$. Therefore $z_j = x_5$.

To introduce Definition 2.4, we need the following notation. Let $H = \{0, g\} \in \mathfrak{A}_1$, $K \in \mathfrak{A}_2$, and $K \cap H = \{0\}$. Let L_1, L_2, L_3 be all subgroups from $\mathfrak{A}_2$ containing

H (see items 1 and 2 of Lemma 2.3). For all $i = 1, 2, 3$, we denote by t_i an element from L_i that belongs to $\overline{K}$ and $t_i \neq g$. Let $s_i \in L_i$ and $s_i \neq 0, g, t_i$. Thus, $L_i = \{0, g, s_i, t_i\}$, $i = 1, 2, 3$, $K = \{0, s_1, s_2, s_3\}$.

Definition 2.4. We denote by $W(H, K)$ the set of measures $\mu \in M^1(\mathbb{Z}_2^3)$ such that the following three equalities hold:

$$\mu(\{0, t_i\}) = \mu(\{g, s_i\}), \quad i = 1, 2, 3.$$

For example, if $H = \{x_0, x_1\}$, $K = \{x_0, x_2, x_3, x_6\}$ (see Fig. 2.1(b)), then $L_1 = \{x_0, x_1, x_2, x_4\}$, $L_2 = \{x_0, x_1, x_3, x_5\}$, $L_3 = \{x_0, x_1, x_6, x_7\}$, $t_1 = x_4$, $t_2 = x_5$, $t_3 = x_7$, $s_1 = x_2$, $s_2 = x_3$, $s_3 = x_6$. In this case, three equalities from Definition 2.4 have the forms:

$$\mu(\{x_0, x_4\}) = \mu(\{x_1, x_2\}), \mu(\{x_0, x_5\}) = \mu(\{x_1, x_3\}), \mu(\{x_0, x_7\}) = \mu(\{x_1, x_6\}).$$

The main result of the paper is the following theorem. Note that item **I.2**(c) of this theorem uses the items 3) and 4) of Lemma 2.3. Item **II.2** uses item 3) of this lemma. Without loss of generality, we may assume that the condition $a_{\max} = a_0$ is satisfied.

Theorem 2.5. *Let $\mu \in M^1(\mathbb{Z}_2^3)$ and $a_{\max} = a_0$. Then $\mu \in TEC(\mathbb{Z}_2^3)$ if and only if there exists a decomposition of the group $\mathbb{Z}_2^3$:*

$$\mathbb{Z}_2^3 = X_1 \oplus X_2, \quad X_1 \in \mathfrak{A}_1, \quad X_2 \in \mathfrak{A}_2, \tag{2.3}$$

such that conditions **I.1** *and* **I.2** *are satisfied for $a_{\max} \leq 1/4$ and one of the conditions* **II.1**–**II.4** *is satisfied for $a_{\max} > 1/4$:*

I.1. *The projection of the measure μ on X_2 parallel to X_1 is equal to the Haar measure m_{X_2} of the subgroup X_2.*

I.2. *At least one of the following three requirements is true:*

- a) *The projection of the measure μ on X_1 parallel to X_2 is equal to the Haar measure m_{X_1} of the subgroup X_1.*
- b) *The sum of the masses of some two elements of the subgroup X_2 is equal to the sum of the masses of its other two elements.*
- c) *Let $K^{(j)}$, $j = 1, 2, 3, 4$, be all subgroups of $\mathfrak{A}_2$ that do not contain X_1. Then there is j_0 such that the condition $\mu \in U(K^{(j_0)})$ is satisfied and the equality*
$$u\left(\overline{K^{(j)}}\right) = \mu(\{z_j\})$$
holds for any $j \neq j_0$, where $z_j \in \overline{K^{(j)}}$, $z_j \notin X_1$, $z_j \notin K^{(j_0)}$.

II.1. *For any subgroup $K \in \mathfrak{A}_2$ the following conditions are true:*

- a) *$\mu \in U(K)$ or $\mu \in U(\overline{K})$;*
- b) *$\mu \in V(K)$ or $\mu \in V(\overline{K})$.*

II.2. *The following two conditions hold:*

a) *$\mu \in W(X_1, K)$ for at least one subgroup $K \in \mathfrak{A}_2$ that does not contain X_1;*

b) *$\mu \in U(K)$ or $\mu \in U(\overline{K})$ for any subgroup $K \in \mathfrak{A}_2$ that does not contain X_1.*

II.3. *The following two conditions hold:*

a) *There is a two-point subset $E \subset X_2$ such that for both elements $g \in X_1$ the equality*

$$\mu(g + E) = \mu(g + (X_2 \setminus E)) \tag{2.4}$$

is valid;

b) *$\mu \in U(K)$ or $\mu \in U(\overline{K})$ for each subgroup $K \in \mathfrak{A}_2$, $K \neq X_2$.*

II.4. *The following three conditions hold:*

a) *The projection of the measure μ on X_1 parallel to X_2 is equal to the Haar measure m_{X_1} of the group X_1;*

b) *$\mu \in V(K)$ or $\mu \in V(\overline{K})$ for any subgroup $K \in \mathfrak{A}_2$, $K \neq X_2$;*

c) *$\mu \in U(K)$ or $\mu \in U(\overline{K})$ for any subgroup $K \in \mathfrak{A}_2$.*

Notice that in the statement of condition **II.1** of Theorem 2.5, decomposition (2.3) is not used.

Let us indicate a simple sufficient condition under which a measure belongs to the class $TEC(\mathbb{Z}_2^3)$.

Corollary 2.6. *If $a_{\max} > 5/6$, then $\mu \in TEC(\mathbb{Z}_2^3)$.*

Proof. Let K be an arbitrary subgroup from $\mathfrak{A}_2$. Since, without loss of generality, we can assume that $a_{\max} = a_0$, we have that condition $\mu \in U(K)$, and hence condition **II.1**(a), are satisfied. Since $v(K) < 1/6$ and $\mu(K) > 5/6$, we see that

$$1/2 + 2v(K) < 1/2 + 2 \cdot 1/6 = 5/6 < \mu(K).$$

Therefore, condition **II.1**(b) is satisfied. Hence, $\mu \in TEC(\mathbb{Z}_2^3)$. □

3. Examples

In this section, we give the examples of measures that satisfy various conditions of Theorem 2.5. In examples 3.1–3.7, we assume that the subgroups X_1 and X_2, appearing in the formulation of Theorem 2.5, are as follows:

$$X_1 = \{x_0, x_1\}, \quad X_2 = \{x_0, x_2, x_3, x_6\}.$$

In examples 3.8, 3.9, we consider the measures that do not belong to the class $TEC(\mathbb{Z}_2^3)$.

Let us give an example of a distribution that satisfies conditions **I.1** and **I.2**(a).

Example 3.1. We consider the distribution μ with masses

$$a_0 = 1/8 + 4\varepsilon, \quad a_1 = 1/8 - 4\varepsilon, \quad a_2 = 1/8 - 3\varepsilon, \quad a_3 = 1/8 - 2\varepsilon,$$
$$a_4 = 1/8 + 3\varepsilon, \quad a_5 = 1/8 + 2\varepsilon, \quad a_6 = 1/8 + \varepsilon, \quad a_7 = 1/8 - \varepsilon \qquad (3.1)$$

for $0 \le \varepsilon \le 1/32$. It should be noticed that for $\varepsilon = 0$ we obtain the Haar measure on the group $\mathbb{Z}_2^3$. Since $a_0 + a_1 = a_2 + a_4 = a_3 + a_5 = a_6 + a_7 = 1/4$, we see that condition **I.1** is satisfied. Since $a_0 + a_2 + a_3 + a_6 = 1/2$, we see that condition **I.2**(a) is satisfied.

Let us give an example of a distribution that does not satisfy condition **I.2**(a), but satisfies conditions **I.1** and **I.2**(b).

Example 3.2. We consider the distribution with masses

$$a_0 = a_3 = 1/4 - \varepsilon,\ a_1 = a_5 = \varepsilon,\ a_2 = a_6 = 2\varepsilon,\ a_4 = a_7 = 1/4 - 2\varepsilon \quad (0 < \varepsilon < 1/8).$$

Since $a_0 + a_1 = a_2 + a_4 = a_3 + a_5 = a_6 + a_7 = 1/4$ and $a_0 + a_2 = a_3 + a_6$, we see that **I.1** and **I.2**(b) are true. Since $a_0 + a_2 + a_3 + a_6 = 1/2 + 2\varepsilon$, condition **I.2**(a) is not satisfied.

Next, we give an example of a distribution that satisfies conditions **I.1** and **I.2**(c).

Example 3.3. We consider the distribution μ with masses

$$a_0 = 1/4 - \varepsilon, \quad a_1 = \varepsilon, \quad a_2 = a_3 = a_6 = 2\varepsilon,$$
$$a_4 = a_5 = a_7 = 1/4 - 2\varepsilon \quad (0 < \varepsilon < 1/28).$$

Since $a_0 + a_1 = a_2 + a_4 = a_3 + a_5 = a_6 + a_7 = 1/4$, we obtain that condition **I.1** is satisfied. Let us show that condition **I.2**(c) is satisfied. All subgroups of $\mathfrak{A}_2$ that do not contain subgroup X_1 (see Lemma 2.3, item 3)) are as follows:

$$K^{(1)} = K_1 = \{x_0, x_2, x_3, x_6\}, \qquad K^{(2)} = K_5 = \{x_0, x_3, x_4, x_7\},$$
$$K^{(3)} = K_6 = \{x_0, x_2, x_5, x_7\}, \qquad K^{(4)} = K_7 = \{x_0, x_4, x_5, x_6\}.$$

Let us take $j_0 = 1$. It is easy to see that

$$\max\{a_i : x_i \in K^{(1)}\} = a_0 = 1/4 - \varepsilon > 6\varepsilon = a_2 + a_3 + a_6$$

for $0 < \varepsilon < 1/28$. Therefore, $\mu \in U(K^{(1)})$.

For $j = 2$, we have

$$\max\{a_i : x_i \in \overline{K^{(2)}}\} = \max\{a_1, a_2, a_5, a_6\} = a_5 = 1/4 - 2\varepsilon \quad (0 < \varepsilon \le 1/16),$$

and we can take $z_2 = x_5$ (see Lemma 2.3, item 4)): $z_2 = x_5 \notin X_1$, $z_2 \notin K^{(1)}$.

For $j = 3$, we have

$$\max\{a_i : x_i \in \overline{K^{(3)}}\} = \max\{a_1, a_3, a_4, a_6\} = a_4 = 1/4 - 2\varepsilon \quad (0 < \varepsilon \le 1/16),$$

and we can take $z_3 = x_4$: $z_3 = x_4 \notin X_1$, $z_3 \notin K^{(1)}$.

For $j = 4$, we have

$$\max\{a_i : x_i \in \overline{K^{(4)}}\} = \max\{a_1, a_2, a_3, a_7\} = a_7 = 1/4 - 2\varepsilon \quad (0 < \varepsilon \leq 1/16),$$

and we can take $z_4 = x_7$: $z_4 = x_7 \notin X_1$, $z_4 \notin K^{(1)}$.

Thus, condition **I.2**(c) is satisfied. Therefore, $\mu \in TEC(\mathbb{Z}_2^3)$ for $0 < \varepsilon < 1/28$.

Example 3.4. Condition **II.1** is satisfied for any measure for which $a_{\max} > 5/6$. This follows from Corollary 2.6 of Theorem 2.5.

Let us give an example of a measure that satisfies condition **II.2**.

Example 3.5. We consider the distribution μ with masses

$$a_0 = 8/24 + 3\varepsilon, \qquad a_1 = 7/24 + 3\varepsilon,$$
$$a_2 = a_3 = a_6 = 2/24 - \varepsilon, \qquad a_4 = a_5 = a_7 = 1/24 - \varepsilon,$$

where $0 \leq \varepsilon \leq 1/24$. Let $K = X_2 = \{x_0, x_2, x_3, x_6\}$. The subgroups containing the subgroup X_1 (see Lemma 2.3, item 1)) are as follows:

$$L_1 = \{x_0, x_1, x_2, x_4\}, \quad L_2 = \{x_0, x_1, x_3, x_5\}, \quad L_3 = \{x_0, x_1, x_6, x_7\}.$$

Notice that $x_2 \in L_1, x_3 \in L_2, x_6 \in L_3$. We put $s_1 = x_2$, $s_2 = x_3$, $s_3 = x_6$. Then $t_1 = x_4$, $t_2 = x_5$, $t_3 = x_7$. It is easy to see that the condition $\mu \in W(X_1, K)$ is satisfied since

$$\mu(\{x_0, x_4\}) = \mu(\{x_1, x_2\}) = 9/24 + 2\varepsilon,$$
$$\mu(\{x_0, x_5\}) = \mu(\{x_1, x_3\}) = 9/24 + 2\varepsilon,$$
$$\mu(\{x_0, x_7\}) = \mu(\{x_1, x_6\}) = 9/24 + 2\varepsilon.$$

Therefore, condition **II.2**(a) is satisfied. Let us check the fulfilment of condition **II.2**(b). For the subgroup $\{x_0, x_2, x_3, x_6\} \not\supset X_1$, we have

$$\max\{a_0, a_2, a_3, a_6\} = a_0 = 8/24 + 3\varepsilon > 3 \cdot (2/24 - \varepsilon) = a_2 + a_3 + a_6.$$

For the subgroup $\{x_0, x_3, x_4, x_7\} \not\supset X_1$, we have

$$\max\{a_0, a_3, a_4, a_7\} = a_0 = 8/24 + 3\varepsilon > 4/24 - 3\varepsilon$$
$$= (2/24 - \varepsilon) + 2(1/24 - \varepsilon) = a_3 + a_4 + a_7.$$

Similar inequalities are also valid for subgroups $\{x_0, x_2, x_5, x_7\} \not\supset X_1$, $\{x_0, x_4, x_5, x_6\} \not\supset X_1$. Therefore, condition **II.2**(b) is satisfied, and hence $\mu \in TEC(\mathbb{Z}_2^3)$.

Let us give an example of a distribution that satisfies condition **II.3**.

Example 3.6. We consider the distribution μ with masses

$$a_0 = 1/4 + \varepsilon, \qquad a_1 = a_4 = a_5 = a_7 = 1/8 - \varepsilon,$$
$$a_2 = a_3 = 1/8 + \varepsilon, \qquad a_6 = \varepsilon \quad (1/16 < \varepsilon < 1/8). \tag{3.2}$$

We take $E = \{x_2, x_3\}$. Then $X_2 \setminus E = \{x_0, x_6\}$. We check the fulfilment of condition **II.3**(a).

Let us show that equality (2.4) holds for $g = x_0$:

$$\mu(\{x_2, x_3\}) = 2(1/8 + \varepsilon) = 1/4 + 2\varepsilon,$$
$$\mu(\{x_0, x_6\}) = 1/4 + \varepsilon + \varepsilon = 1/4 + 2\varepsilon.$$

Let us show that equality (2.4) holds for $g = x_1$:

$$\mu(\{x_1 + \{x_2, x_3\}\}) = \mu(\{x_4, x_5\}) = 2(1/8 - \varepsilon) = 1/4 - 2\varepsilon,$$
$$\mu(\{x_1 + \{x_0, x_6\}\}) = \mu(\{x_1, x_7\}) = 2(1/8 - \varepsilon) = 1/4 - 2\varepsilon.$$

Let us check that condition **II.3**(b) is satisfied. For $K = \{x_0, x_1, x_3, x_5\}$, condition $\mu \in U(K)$ means that

$$a_1 + a_3 + a_5 = (1/8 - \varepsilon) + (1/8 + \varepsilon) + (1/8 - \varepsilon) = 3/8 - \varepsilon < 1/4 + \varepsilon = a_0,$$

and this inequality is satisfied for any ε such that $1/16 < \varepsilon < 1/8$. The condition $\mu \in U(K)$ for $K = \{x_0, x_1, x_2, x_4\}$, $\{x_0, x_3, x_4, x_7\}$, $\{x_0, x_2, x_5, x_7\}$ can be checked in the same way. For $K = \{x_0, x_4, x_5, x_6\}$, the condition $\mu \in U(K)$ means that

$$a_4 + a_5 + a_6 = 2(1/8 - \varepsilon) + \varepsilon = 1/4 - \varepsilon < 1/4 + \varepsilon = a_0.$$

This inequality is true for any $\varepsilon > 0$. The condition $\mu \in U(K)$ for $K = \{x_0, x_1, x_6, x_7\}$ can be verified in a similar way. Thus, condition **II.3**(b) is verified. Therefore, $\mu \in TEC(\mathbb{Z}_2^3)$.

Let us give an example of a distribution that satisfies condition **II.4**.

Example 3.7. We consider the distribution μ with masses

$$a_0 = 1/2 - \varepsilon, \quad a_2 = a_3 = a_6 = \varepsilon/3, \quad a_1 = a_4 = a_5 = a_7 = 1/8 \quad (0 < \varepsilon < 3/16).$$

Since $a_0 + a_2 + a_3 + a_6 = 1/2$, condition **II.4**(a) is fulfilled. Let us check that condition **II.4**(b) is satisfied. Let $K = K_2 = \{x_0, x_1, x_3, x_5\}$. We have:

$$a_0 + a_1 + a_3 + a_5 = (1/2 - \varepsilon) + 1/8 + \varepsilon/3 + 1/8 = 3/4 - 2\varepsilon/3,$$
$$1/2 + 2\min\{a_0, a_1, a_3, a_5\} = 1/2 + 2\varepsilon/3.$$

The condition $\mu \in V(K)$ takes the form $1/2 + 2\varepsilon/3 < 3/4 - 2\varepsilon/3$ and it is satisfied for $0 < \varepsilon < 3/16$. For other subgroups K_i, $i = 3, \ldots, 7$, the check is made in the same way.

Let us check the fulfilment of condition **II.4**(c). If $K = K_1 = \{x_0, x_2, x_3, x_6\}$, we have

$$a_0 = 1/2 - \varepsilon > 3\varepsilon/3 = \varepsilon = a_2 + a_3 + a_6 \quad (0 < \varepsilon < 1/4).$$

If $K = K_2 = \{x_0, x_1, x_3, x_5\}$, we have

$$a_1 + a_3 + a_5 = 1/8 + \varepsilon/3 + 1/8 = 1/4 + \varepsilon/3 < 1/2 - \varepsilon = a_0 \quad (0 < \varepsilon < 3/16).$$

For other subgroups K_i, $i = 3, \ldots, 7$, the check is the same as that for K_2. Therefore, $\mu \in TEC(\mathbb{Z}_2^3)$ for $0 < \varepsilon < 3/16$.

Let us give two examples of the measures that do not belong to the class $TEC(\mathbb{Z}_2^3)$. Let X_1 and X_2 be the arbitrary subgroups of the group $\mathbb{Z}_2^3$ that satisfy the condition (2.3).

Example 3.8. We consider the distribution μ with masses

$$a_0 = 1/4 + \varepsilon, \quad a_1 = a_2 = \cdots = a_7 = (3/4 - \varepsilon)/7 \quad (0 < \varepsilon \le 1/20).$$

For this distribution, we have $a_{\max} > 1/4$. Let us show that conditions **II.1**(a), **II.2**(b), **II.3**(b), **II.4**(c) are not satisfied. Let K be an arbitrary subgroup of $\mathfrak{A}_2$, $K = \{x_0, x_i, x_j, x_k\}$, where i, j, k are different and nonzero. The condition $\mu \notin U(K)$ means that $a_0 \le a_i + a_j + a_k$, that is,

$$1/4 + \varepsilon \le 3(1/7)(3/4 - \varepsilon).$$

This inequality holds for $\varepsilon \le 1/20$. The condition $\mu \notin U(\overline{K})$ means that

$$(1/7)(3/4 - \varepsilon) \le 3(1/7)(3/4 - \varepsilon).$$

This inequality holds for $\varepsilon \le 3/4$. Thus, $\mu \notin TEC(\mathbb{Z}_2^3)$.

In the previous example, the measure satisfied condition $a_{\max} > 1/4$. In the following example, the condition $a_{\max} \le 1/4$ is valid.

Example 3.9. We consider the distribution with masses

$$a_0 = 1/4 - \varepsilon, \quad a_1 = a_2 = \cdots = a_7 = (3/4 + \varepsilon)/7 \quad (\varepsilon \in [0, 1/8) \cup (1/8, 1/4]).$$

We show that condition **I.1** is not satisfied. We have

$$a_0 + a_i = 5/14 - (6/7)\varepsilon \ne 1/4 \quad \text{if} \quad \varepsilon \ne 1/8, \quad i = 1, 2, \ldots, 7.$$

Therefore, condition **I.1** is not satisfied. Thus, $\mu \notin TEC(\mathbb{Z}_2^3)$. Note that this measure is the Haar measure of the group $\mathbb{Z}_2^3$ if $\varepsilon = 1/8$. Therefore, it belongs to the class $TEC(\mathbb{Z}_2^3)$.

In examples 3.1–3.7 the subgroups X_1 and X_2 are known. Let us show how to find the subgroups X_1 and X_2 for the concrete measure μ. The subgroups X_1 and X_2 for the given measure μ are found by sorting of all possible cases. We can choose one of the subgroups X_1, X_2 in seven ways. Then we can choose another subgroup in four ways in order that

$$\mathbb{Z}_2^3 = X_1 \oplus X_2, \quad X_1 \in \mathfrak{A}_1 \quad X_2 \in \mathfrak{A}_2.$$

Therefore, we have 28 variants of the pair X_1, X_2.

Further, we consider two examples, which show how to find X_1 and X_2 for the concrete measure μ.

Example 3.1′. Let $\mu \in M^1(\mathbb{Z}_2^3)$ be a distribution with masses (3.1), where $0 < \varepsilon \le 1/32$. Since $a_{\max} = a_0 < 1/4$, we need to find a suitable condition among the conditions of part **I** of Theorem 2.5.

Since $a_0 + a_1 = a_2 + a_4 = a_3 + a_5 = a_6 + a_7 = 1/4$, we see that condition **I.1** is valid if $X_1 = \{x_0, x_1\}$. Since $a_0 + a_i \ne 1/4$ for $i = 2, \dots, 7$, condition **I.1** is not valid for $X_1 = \{x_0, x_i\}$, $i = 2, \dots, 7$. Therefore, we take $X_1 = \{x_0, x_1\}$. For $X_1 = \{x_0, x_1\}$, the expansion $\mathbb{Z}_2^3 = X_1 \oplus X_2$ is valid for the following subgroups X_2:

$$\{x_0, x_2, x_3, x_6\}, \quad \{x_0, x_2, x_5, x_7\}, \quad \{x_0, x_3, x_4, x_7\}, \quad \{x_0, x_4, x_5, x_6\}.$$

Condition **I.2**(a) is fulfilled for the subgroup $X_2 = \{x_0, x_2, x_3, x_6\}$ from this list of subgroups because

$$a_0 + a_2 + a_3 + a_6 = 1/2.$$

But it is not fulfilled for the other subgroups from this list because

$$\begin{aligned} a_0 + a_2 + a_5 + a_7 &= 1/2 + 2\varepsilon \ne 1/2, \\ a_0 + a_3 + a_4 + a_7 &= 1/2 + 4\varepsilon \ne 1/2, \\ a_0 + a_4 + a_5 + a_6 &= 1/2 + 10\varepsilon \ne 1/2 \end{aligned}$$

for $\varepsilon > 0$.

Therefore, we put $X_1 = \{x_0, x_1\}$, $X_2 = \{x_0, x_2, x_3, x_6\}$.

Example 3.6′. Let $\mu \in M^1(\mathbb{Z}_2^3)$ be a distribution with masses (3.2). Since $a_{\max} = a_0 > 1/4$, we have to find a suitable condition among the conditions of part **II** of Theorem 2.5. Let us show that condition **II.1**(a) is not valid. Indeed, this condition must be valid for any subgroup $K \in \mathfrak{A}_2$. But for $K = \{x_0, x_2, x_3, x_6\}$, we have $\overline{K} = \{x_1, x_4, x_5, x_7\}$ and

$$\begin{aligned} \max\{a_i : x_i \in K\} &= a_0 = 1/4 + \varepsilon < 1/4 + 3\varepsilon = a_2 + a_3 + a_6, \\ \max\{a_i : x_i \in \overline{K}\} &= a_1 = 1/8 - \varepsilon < 3(1/8 - \varepsilon) = a_4 + a_5 + a_7. \end{aligned}$$

Thus, condition **II.1**(a) is not valid.

It can be shown that condition **II.2** is also not valid. (The verification of this fact is easy but somewhat long.)

Let us show that condition **II.3** is valid for a suitable choice of X_1, X_2, E. At first, we choose the subgroup X_2. For this purpose, we verify for which of the seven subgroups (2.1) equality (2.4) is valid with $g = x_0$, i. e.,

$$\mu(E) = \mu(X_2 \setminus E) \tag{3.3}$$

for some two-point subset $E \subset X_2$. If $X_2 = \{x_0, x_2, x_3, x_6\}$ and $E = \{x_2, x_3\}$, then the following equalities hold:

$$\begin{aligned} \mu(E) &= \mu(\{x_2, x_3\}) = 1/4 + 2\varepsilon, \\ \mu(X_2 \setminus E) &= \mu(\{x_0, x_6\}) = 1/4 + 2\varepsilon. \end{aligned}$$

Notice that if we take other subgroup from (2.1) for X_2, then condition (3.3) is not valid. Indeed, if $X_2 = \{x_0, x_1, x_3, x_5\}$, then for $E = \{x_0, x_1\}$, we have

$$\mu(E) = 3/8, \quad \mu(X_2 \setminus E) = 1/4;$$

for $E = \{x_0, x_3\}$, we have

$$\mu(E) = 3/8 + 2\varepsilon, \quad \mu(X_2 \setminus E) = 1/4 - 2\varepsilon;$$

for $E = \{x_0, x_5\}$, we have

$$\mu(E) = 3/8, \quad \mu(X_2 \setminus E) = 1/4.$$

Therefore, $X_2 = \{x_0, x_1, x_3, x_5\}$ is not suitable. In complete analogy with this case, other subgroups from (2.1) are not suitable. Thus, we have to put $X_2 = \{x_0, x_2, x_3, x_6\}$.

If $X_2 = \{x_0, x_2, x_3, x_6\}$, we must choose X_1 from the subgroups

$$\{x_0, x_1\}, \quad \{x_0, x_4\}, \quad \{x_0, x_5\}, \quad \{x_0, x_7\}.$$

Let us verify for which subgroup from this list equality (2.4),

$$\mu(g + E) = \mu(g + (X_2 \setminus E)),$$

is valid for $g \neq x_0$, $E = \{x_2, x_3\}$. While considering Example 3.6, we showed that this equality is valid for $g = x_1$. Therefore, we take $X_1 = \{x_0, x_1\}$.

If $g = x_4$, we have

$$\mu(x_4 + \{x_2, x_3\}) = \mu(\{x_1, x_7\}) = 1/4 - 2\varepsilon,$$
$$\mu(x_4 + \{x_0, x_6\}) = \mu(\{x_4, x_5\}) = 1/4 - 2\varepsilon.$$

Therefore, we may take $X_1 = \{x_0, x_4\}$.

If $g = x_5$, we have

$$\mu(x_5 + \{x_2, x_3\}) = \mu(\{x_7, x_1\}) = 1/4 - 2\varepsilon,$$
$$\mu(x_5 + \{x_0, x_6\}) = \mu(\{x_5, x_4\}) = 1/4 - 2\varepsilon,$$

and we may take $X_1 = \{x_0, x_5\}$.

If $g = x_7$, we have

$$\mu(x_7 + \{x_2, x_3\}) = \mu(\{x_5, x_4\}) = 1/4 - 2\varepsilon,$$
$$\mu(x_7 + \{x_0, x_6\}) = \mu(\{x_7, x_1\}) = 1/4 - 2\varepsilon,$$

and we may take $X_1 = \{x_0, x_7\}$.

Therefore, we put $X_2 = \{x_0, x_2, x_3, x_6\}$ and $X_1 = \{x_0, x_i\}$, where $i = 1, 4, 5, 7$.

The fulfilment of condition **II.3**(b) for $X_2 = \{x_0, x_2, x_3, x_6\}$ was shown in Example 3.6.

4. Derivation of Theorems 1.4–1.6 from Theorem 2.5

Let us show that Theorem 2.5 implies Theorem 1.4 in the case of $l = 3$ and Theorems 1.5 and 1.6.

We show that the sufficiency in Theorem 1.4 for the case $l = 3$ follows from item **II.1** of Theorem 2.5. Indeed, if m is the Haar measure of the group $\mathbb{Z}_2^3$, we have $\widehat{\Pi}_{\lambda m}(y) = 1$ for $y = 0$ and $\widehat{\Pi}_{\lambda m}(y) = e^{-\lambda}$ in other cases. Therefore, as it is easy to check, the masses of the measure $\Pi_{\lambda m}$ are as follows: the mass of the zero element of the group $\mathbb{Z}_2^3$ is equal to $(1+7e^{-\lambda})/8$, and the masses of the remaining elements of the group are equal to $(1 - e^{-\lambda})/8$. Therefore, condition II.1(a) of Theorem 4 takes the form $(1+7e^{-\lambda})/8 > 3(1-e^{-\lambda})/8$ and is satisfied for $e^{-\lambda} > 1/5$, while condition **II.1**(b) takes the form $1/2 < (1 + 7e^{-\lambda})/8 + (1 - e^{-\lambda})/8$ and is satisfied for $e^{-\lambda} > 1/3$. Thus, the conditions of item **II.1** of Theorem 2.5 are satisfied for the measure $\Pi_{\lambda m}$ for $\lambda < \ln 3$, which proves the sufficiency in Theorem 1.4.

The sufficiency in Theorem 1.5 also follows from item **II.1** of Theorem 2.5. If one of the three systems of inequalities from the formulation of Theorem 1.5 is true, the conditions of item **II.1** of Theorem 2.5 are satisfied. (The proof of this fact is lengthy, so we omit it.)

Let us show how the sufficiency in Theorem 1.6 follows from Theorem 2.5. To do this, we consider several cases:

1) $a_{\max} > 1/2$, $|S(\mu)| = 2, 3, 4$;
2) $a_{\max} = 1/2$, $|S(\mu)| = 2$;
3) $a_{\max} = 1/2$, $|S(\mu)| = 3$;
4) $a_{\max} < 1/2$, $|S(\mu)| = 4$ and the sum of some two masses of elements from $S(\mu)$ is equal to the sum of two other masses.

Item 4) is divided into three sub-items, differing in the number of coinciding masses:

(i) all masses of elements are equal (the Haar measure on $\mathbb{Z}_2^2$);
(ii) among the masses, there are two pairs of the same mass, but not all of these masses are equal;
(iii) all masses are different or there are exactly two identical masses among them.

(The case when there are exactly three identical masses among them is impossible, since in this case the condition that the sum of some two masses is equal to the sum of two other masses cannot be satisfied.)

It is easy to see that condition **II.1** of Theorem 2.5 is satisfied under condition 1) of Theorem 1.6.

Let us show that, under condition 4(i), conditions **I.1** and **I.2**(a) of Theorem 2.5 are satisfied. Without loss of generality, we can assume that $S(\mu) = \{x_0, x_1, x_3, x_5\}$. Then $a_0 = a_1 = a_3 = a_5 = 1/4$, $a_2 = a_4 = a_6 = a_7 = 0$. Let us put $X_1 = \{x_0, x_2\}$, $X_2 = \{x_0, x_3, x_4, x_7\}$. Since $a_0 + a_2 = a_1 + a_4 = a_3 + a_6 = a_5 + a_7 = 1/4$, condition **I.1** of Theorem 2.5 is satisfied. Since $a_0 + a_3 + a_4 + a_7 =$

$a_1 + a_2 + a_5 + a_6 = 1/2$, condition **I.2**(a) is satisfied. Therefore $\mu \in TEC(\mathbb{Z}_2^3)$.

Let us show that, under condition 4(ii), condition **II.2** of Theorem 2.5 is satisfied. Without loss of generality, we can assume that $S(\mu) = \{x_0, x_1, x_3, x_5\}$, $a_{\max} = a_0 < 1/2$, $a_0 + a_5 = a_1 + a_3$, $a_0 = a_1$, $a_3 = a_5$, $a_0 \neq a_3$. Let $X_1 = \{x_0, x_1\}$, $X_2 = \{x_0, x_2, x_3, x_6\}$ (see Fig. 2.1(b)). We put $K = \{x_0, x_2, x_3, x_6\} \not\supset X_1$. Let us show that $\mu \in W(X_1, K)$. We denote

$$L_1 = \{x_0, x_1, x_2, x_4\}, \quad L_2 = \{x_0, x_1, x_3, x_5\}, \quad L_3 = \{x_0, x_1, x_6, x_7\}$$

(see item 1) of Lemma 2.3). We have $L_i \supset X_1$ for $i = 1, 2, 3$. It follows from the conditions on the masses a_i that the equalities

$$\mu(\{x_0, x_4\}) = \mu(\{x_1, x_2\}), \quad \mu(\{x_0, x_5\}) = \mu(\{x_1, x_3\}),$$
$$\mu(\{x_0, x_7\}) = \mu(\{x_1, x_6\})$$

are valid. Therefore, condition **II.2**(a) is satisfied. It is easy to see that the condition $\mu \in U(K)$ is satisfied: $a_0 > a_2 + a_3 + a_6 = a_3$. The fulfilment of condition **II.2**(b) for the remaining subgroups from $\mathfrak{A}_2$ that do not contain X_1 is verified in a similar way. So, $\mu \in TEC(\mathbb{Z}_2^3)$.

Let us show that, under condition 2), condition **II.2** of Theorem 2.5 is satisfied. Without loss of generality, we can assume that $S(\mu) = \{x_0, x_1\}$, $a_0 = a_1 = 1/2$. The fulfilment of condition **II.2** is checked in exactly the same way as for the case 4(ii) if we put $a_3 = a_5 = 0$ in the previous argument.

Let us show that, under condition 4(iii), condition **II.3** of Theorem 2.5 is satisfied. We can assume that $S(\mu) = \{x_0, x_1, x_3, x_5\}$, $a_0 > a_1 \geq a_3 > a_5$, $a_0 + a_5 = a_1 + a_3$. Let us put $X_1 = \{x_0, x_2\}$, $X_2 = \{x_0, x_1, x_3, x_5\}$, $E = \{x_0, x_5\}$. Then $X_2 \setminus E = \{x_1, x_3\}$. It is clear that

$$\mu(E) = a_0 + a_5 = a_1 + a_3 = \mu(X_2 \setminus E),$$
$$\mu(x_2 + E) = \mu(\{x_2, x_7\}) = 0, \quad \mu(x_2 + (X_2 \setminus E)) = \mu(\{x_4, x_6\}) = 0.$$

Therefore, condition **II.3**(a) is satisfied. Let us check the fulfilment of condition **II.3**(b). If $K = \{x_0, x_2, x_3, x_6\}$, we have $\mu \in U(K)$ since $a_0 > a_2 + a_3 + a_6 = a_3$. Analogously, the condition $\mu \in U(K)$ is satisfied for all other subgroups $K \in \mathfrak{A}_2$, $K \neq X_2$. So, condition **II.3**(b) is satisfied. Therefore, $\mu \in TEC(\mathbb{Z}_2^3)$.

Let us show that, under condition 3), condition **II.3** of Theorem 2.5 is also satisfied. We can assume that $S(\mu) = \{x_0, x_1, x_3\}$, $a_0 = a_1 + a_3 = 1/2$, $a_1 > 0$, $a_3 > 0$. Condition **II.3** is verified in exactly the same way as in the case 4(iii) if we put $a_5 = 0$ in the previous reasoning.

To verify the validity of Theorem 1.4 (for $l = 3$) and Theorems 1.5, 1.6 in the direction of necessity, we have to show that if the conditions of one of them are not satisfied, then condition **I.1** or condition **I.2** of Theorem 2.5 is not satisfied, and also all four conditions **II.1**–**II.4** are not satisfied. To check it is easy, but lengthy. Therefore, we omit it.

Notice that the derivation of Theorem 1.6 from Theorem 2.5 turned out to be more difficult than the proof of Theorem 1.6 itself.

5. Scheme of the proof of Theorem 2.5

The proof of Theorem 1.6 from [6] is easy. The proof of Theorem 1.5 from [7] is rather laborious. The proof of Theorem 2.5, based on the results of computer calculations, is much more complicated, multi-step and lengthy. Let us briefly describe the scheme of the proof of Theorem 2.5.

Let $\mu \in M^1(\mathbb{Z}_2^3)$ be a given measure, $\nu \in M^1(\mathbb{Z}_2^3)$ be a measure equivalent to it, that is, condition (1.2) is satisfied. There arises the question: under which conditions on the measure μ does the equality $\nu = \mu_x$ hold for some x? In other words, we look for the conditions on the measure μ under which the equality

$$\widehat{\nu}(y) = (x_j, y)\widehat{\mu}(y) \tag{5.1}$$

holds for some $j = 0, 1, \ldots, 7$. Let the characteristic function of the measure μ be of the form (2.2) and the characteristic function of the measure ν be equal to

$$\widehat{\nu}(y) = \sum_{i=0}^{7} b_i(x_i, y).$$

If in equality (1.2), which is valid for all $y = (\xi, \eta, \zeta)$, we assume $\xi, \eta, \zeta = 0, 1$, then it can be written as a system of eight equalities:

$$\begin{aligned}
&b_0 + b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 = 1,\\
&|b_0 - b_1 + b_2 + b_3 - b_4 - b_5 + b_6 - b_7| = |a_0 - a_1 + a_2 + a_3 - a_4 - a_5 + a_6 - a_7|,\\
&|b_0 + b_1 - b_2 + b_3 - b_4 + b_5 - b_6 - b_7| = |a_0 + a_1 - a_2 + a_3 - a_4 + a_5 - a_6 - a_7|,\\
&|b_0 + b_1 + b_2 - b_3 + b_4 - b_5 - b_6 - b_7| = |a_0 + a_1 + a_2 - a_3 + a_4 - a_5 - a_6 - a_7|,\\
&|b_0 - b_1 - b_2 + b_3 + b_4 - b_5 - b_6 + b_7| = |a_0 - a_1 - a_2 + a_3 + a_4 - a_5 - a_6 + a_7|,\\
&|b_0 - b_1 + b_2 - b_3 - b_4 + b_5 - b_6 + b_7| = |a_0 - a_1 + a_2 - a_3 - a_4 + a_5 - a_6 + a_7|,\\
&|b_0 + b_1 - b_2 - b_3 - b_4 - b_5 + b_6 + b_7| = |a_0 + a_1 - a_2 - a_3 - a_4 - a_5 + a_6 + a_7|,\\
&|b_0 - b_1 - b_2 - b_3 + b_4 + b_5 + b_6 - b_7| = |a_0 - a_1 - a_2 - a_3 + a_4 + a_5 + a_6 - a_7|.
\end{aligned}$$

Expanding the modulus in the seven equalities of this system, we see that (1.2) is equivalent to a set of $2^7 = 128$ systems of linear equations with eight unknowns b_i and eight given parameters a_i. Computer calculations of the paper [7] give the solution of each system. Careful analysis of the solutions shows that 128 systems are divided into four sets according to the type of solutions; we denote them by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$.

The set $\mathcal{A}$ consists of eight systems whose solutions $b_0, b_1, \ldots, b_7$ are such that the function $\widehat{\nu}(y)$ has the form (5.1). The set $\mathcal{B}$ consists of eight systems whose solutions are such that

$$\widehat{\nu}(y) = (x_j, y)\sum_{i=0}^{7}(1/4 - a_i)(x_i, y), \quad j = 0, 1, \ldots, 7.$$

To describe systems from the sets $\mathcal{C}$ and $\mathcal{D}$, we will need the following substitutions of the indices $0, 1, \ldots, 7$ (e is the unit substitution):

$$\sigma_1 = e, \qquad \sigma_2 = (12)(56), \qquad \sigma_3 = (13)(46), \qquad \sigma_4 = (12)(37),$$

$$\sigma_5 = (24)(67), \qquad \sigma_6 = (35)(67), \qquad \sigma_7 = (24)(35). \tag{5.2}$$

The substitution σ_k, $k = 2, 3, \dots, 7$, transforms the subgroup $K_1 \in \mathfrak{A}_2$ into the subgroup K_k (see (2.1)).

The set $\mathcal{C}$ consists of 56 systems, which are subdivided into 7 subsets $\mathcal{C}_k$, $k = 1, 2, \dots, 7$, of 8 systems each. For solutions $b_0, b_1, \dots, b_7$ of the systems from the subset $\mathcal{C}_k$, the function $\widehat{\nu}(y)$ has the form

$$\widehat{\nu}(y) = (x_j, y)\psi_k(y), \quad j = 0, 1, \dots, 7,$$

where

$$\psi_1(y) = \sum_{i=0}^{7} c_i(x_i, y),$$

$$\begin{aligned} c_0 &= (-a_0 + a_2 + a_3 + a_6)/2, & c_4 &= (a_1 - a_4 + a_5 + a_7)/2, \\ c_1 &= (-a_1 + a_4 + a_5 + a_7)/2, & c_5 &= (a_1 + a_4 - a_5 + a_7)/2, \\ c_2 &= (a_0 - a_2 + a_3 + a_6)/2, & c_6 &= (a_0 + a_2 + a_3 - a_6)/2, \\ c_3 &= (a_0 + a_2 - a_3 + a_6)/2, & c_7 &= (a_1 + a_4 + a_5 - a_7)/2. \end{aligned} \tag{5.3}$$

To obtain the coefficients of the function $\psi_k(y)$, $k = 2, 3, \dots, 7$, it is necessary to apply in formulas (5.3) the same substitution σ_k from (5.2) to the coefficients a_i and c_i.

The set $\mathcal{D}$ consists of 56 systems, which are subdivided into 7 subsets $\mathcal{D}_k$, $k = 1, 2, \dots, 7$, of 8 systems each. For solutions of systems from the subset $\mathcal{D}_k$, the function $\widehat{\nu}(y)$ has the form

$$\widehat{\nu}(y) = (x_j, y)\varphi_k(y), \quad j = 0, 1, \dots, 7,$$

where

$$\varphi_1(y) = \sum_{i=0}^{7} d_i(x_i, y), \quad d_i = 1/4 - c_i, \quad i = 0, 1, \dots, 7.$$

Coefficients of the function $\varphi_k(y)$, $k = 2, 3, \dots, 7$, are obtained from coefficients of the function $\varphi_1(y)$ by using the substitution σ_k applied to coefficients a_i and d_i.

A solution $b_0, b_1, \dots, b_7$ of any of the systems is called *trivial* if the function $\widehat{\nu}(y)$ satisfies equality (5.1) (this means that the measure ν is a shift of the measure μ). We call a solution *non-trivial* if $b_i \geq 0$ for all i, and equality (5.1) is not satisfied for any j. Since we are only interested in non-negative b_i, we will say that the system *has no solution* if $b_i < 0$ for some i.

Since all systems of the set A have trivial solutions, for proving Theorem 2.5, it is necessary to find conditions on a_i under which each system from the sets $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ either has a trivial solution or has no solutions.

The finding of a condition under which a particular system has no solutions is not difficult. For example, at least one of the coefficients of the function $\psi_1(y)$ is negative if and only if one of the two inequalities holds

$$2\max\{a_0, a_2, a_3, a_6\} > a_0 + a_2 + a_3 + a_6,$$

$$2\max\{a_1, a_4, a_5, a_7\} > a_1 + a_4 + a_5 + a_7.$$

It is much more difficult to find out under what conditions a particular system or group of systems has a trivial solution. To do this, one needs to find out when the solution of this system $b_0, b_1, \dots, b_7$ coincides with the set of coefficients of some of the 8 functions $(x_j, y)\widehat{\mu}(y)$, $j = 0, 1, 2, \dots, 7$. And this is the hardest part of the proof.

Let us explain what the conditions of Theorem 2.5 mean. The inequality $a_{\max} \leq 1/4$ is a necessary and sufficient condition for the fact that all systems of the set $\mathcal{B}$ have solutions. Condition **I.1** is a necessary and sufficient condition that all systems of the set $\mathcal{B}$ have trivial solutions. In order for all systems of the sets $\mathcal{C}$ and $\mathcal{D}$ to have trivial solutions, it is necessary and sufficient that one of the conditions **I.2**(a) and **I.2**(b) be satisfied. Condition **II.1**(a) is a necessary and sufficient condition that all systems of the set $\mathcal{C}$ have no solutions. Condition **II.1**(b) is a necessary and sufficient condition that all systems of the set $\mathcal{D}$ have no solutions. Conditions **I.1** and **I.2**(c) are necessary and sufficient conditions that all systems of the set $\mathcal{B}$, part of the systems of the set $\mathcal{C}$, and part of the systems of the set $\mathcal{D}$ have trivial solutions, and the remaining systems of the sets $\mathcal{C}$ and $\mathcal{D}$ have no solutions. Condition $a_{\max} > 1/4$ and fulfilment of one of the conditions **II.2** and **II.3** — this is a necessary and sufficient condition that all systems of the set $\mathcal{B}$, part of the systems of the set $\mathcal{C}$ and part of the systems of the set $\mathcal{D}$ have no solutions, and the remaining systems of the sets $\mathcal{C}$ and $\mathcal{D}$ have trivial solutions. Conditions $a_{\max} > 1/4$ and **II.4** — this is a necessary and sufficient condition that all systems of the sets $\mathcal{B}, \mathcal{C}$ and part of the systems of the set $\mathcal{D}$ have no solutions, and the remaining systems of the set $\mathcal{D}$ have trivial solutions. Several cases that are not described here (for example, when all systems of the set $\mathcal{C}$ have trivial solutions, and all systems of the set $\mathcal{D}$ have no solutions) are impossible.

Acknowledgments. The author would like to thank an anonymous reviewer for useful discussions of the manuscript.

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Received June 15, 2022, revised December 1, 2022.

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Відновлення фази для ймовірнісних мір на групі $\mathbb{Z}_2^3$

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Нехай $\mathbb{Z}_2$ є групою класів лишків за модулем 2. Надано повний опис класу ймовірнісних мір на групі $\mathbb{Z}_2^3$, які визначаються модулем своєї характеристичної функції однозначно з точністю до зсуву.

Ключові слова: ймовірнісна міра на групі, характеристична функція, тривіальний клас еквівалентності, відновлення фази