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Three-Dimensional Almost Contact Metric Manifolds with a New Approach

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We prove that starting by a global unit vector field on a three-dimensional Riemannian manifold, one can construct an almost contact metric structure. Furthermore, the knowledge of the nature of these structures is achieved through a relationship linking the components of this vector field and the components of the Levi-Civita connection. The illustrative examples are given.

Key words: almost contact metric structure, Sasakian structure, Kenmotsu structure, cosymplectic structure, corner structure

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1. Introduction

The geometrical objects, smooth functions, vector fields, 1-forms, and, in general, tensors on any manifold have an important role in differential geometry, especially in the construction of structures on manifolds.

The notion of almost contact structure was introduced by Boothby and Wang [7]. These manifolds were studied as an odd-dimensional counterpart of almost complex manifolds and warped product are used to give examples of almost contact metric manifolds.

In [12], D. Chinea and C. Gonzalez classified almost contact metric manifolds studying the space that possesses the same symmetries as the covariant derivative of the fundamental 2-form. This space is decomposed into twelve irreducible components $C_1, \dots, C_{12}$.

In dimension 3, the classes C_i reduce to the following classes: C_0 class of cosymplectic manifolds, C_5 class of β-Kenmotsu manifolds, C_6 class of α-Sasakian manifolds, C_9-manifolds and C_{12}-manifolds.

Many works have focused on three-dimensional almost contact metric structures, either by studying and classifying them or by employing them for studying various geometric topics. For example, see [1, 4, 6, 11, 13, 14, 16, 17, 20]

Recently, in [4], the author presented an interesting expression that generalizes the five classes of 3-dimensional almost contact metric structures and introduced a general approach to classify invariant almost contact metric structures on 3-dimensional Lie algebras.

Our aim in this manuscript is to construct all almost contact metric structures on a 3-dimensional Riemannian manifold. We define an interesting method to construct them starting from a unit vector field ξ. That is why ξ is called the characteristic vector field or the structure vector field. By this approach, we discuss the nature of 3-dimensional almost contact metric manifolds. Moreover, these techniques made the classification simpler and more transparent.

First of all, we introduce the basic concepts that we need in this research with new main results.

2. Almost contact metric manifold

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if on M there exists a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η such that

$$\begin{cases} \eta(\xi) = 1, \\ \varphi^2(X) = -X + \eta(X)\xi, \\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \end{cases} \tag{2.1}$$

for any vector fields X, Y on M.

In particular, in an almost contact metric manifold, we also have

$$\varphi\xi = 0 \quad \text{and} \quad \eta \circ \varphi = 0.$$

The fundamental 2-form ϕ is defined by

$$\phi(X,Y) = g(X, \varphi Y).$$

The almost contact structure (φ, ξ, η) is said to be normal if and only if

$$N^{(1)}(X,Y) = N_\varphi(X,Y) + 2\, d\eta(X,Y)\xi = 0 \tag{2.2}$$

for any X, Y on M, where N_φ denotes the Nijenhuis torsion of φ given by

$$N_\varphi(X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \tag{2.3}$$

In [14], the author proved that for an arbitrary 3-dimensional almost contact metric manifold $(M^3, \varphi, \xi, \eta, g)$, we have

$$(\nabla_X \varphi)Y = g(\varphi\nabla_X\xi, Y)\xi - \eta(Y)\varphi\nabla_X\xi, \tag{2.4}$$

$$\mathrm{d}\phi = (\mathrm{div}\xi)\eta \wedge \phi, \tag{2.5}$$

$$\mathrm{d}\eta = \eta \wedge (\nabla_\xi\eta) + \frac{1}{2}\big(tr_g(\varphi\nabla\xi)\big)\phi, \tag{2.6}$$

Moreover, an almost contact metric 3-dimensional manifold M is normal if and only if it satisfies the condition

$$\nabla_{\varphi X}\xi = \varphi\nabla_X\xi. \tag{2.7}$$

The following equation is equivalent to (2.7),

$$\nabla_X\xi = -\alpha\varphi X - \beta\varphi^2 X, \tag{2.8}$$

where $2\alpha = tr_g(\varphi\nabla\xi)$ and $2\beta = \mathrm{div}\xi$.

It is well known that an almost contact metric 3-dimensional manifold is:

(i) Cosymplectic if and only if $\mathrm{d}\eta = \mathrm{d}\phi = 0$ and $N^{(1)} \equiv 0$,
(ii) α-Sasaki if and only if $\mathrm{d}\eta = \alpha\phi$, $\mathrm{d}\phi = 0$ and $N^{(1)} \equiv 0$,
(iii) β-Kenmotsu if and only if $\mathrm{d}\eta = 0$, $\mathrm{d}\phi = 2\beta\eta \wedge \phi$ and $N^{(1)} \equiv 0$,

or, equivalently,

(1) Cosymplectic if and only if $\nabla_X \xi = 0$,
(2) α-Sasaki if and only if $\nabla_X \xi = -\alpha\varphi X$,
(3) β-Kenmotsu if and only if $\nabla_X \xi = -\beta\varphi^2 X$.

Recently, in [3, 8], the authors studied the 3-dimensional C_{12}-structures. They are integrable, non-normal (i.e., $N_\varphi = 0$ and $N^{(1)} \neq 0$) and are characterized by

$$\mathrm{d}\eta = \eta \wedge \omega, \qquad \mathrm{d}\phi = 0, \quad \text{and} \quad N_\varphi \equiv 0,$$

where $\omega = \nabla_\xi \eta$. This is equivalent to

$$\nabla_X \xi = \eta(X)\nabla_\xi \xi. \tag{2.9}$$

Moreover, in [4], it is proved that a 3-dimensional almost contact metric manifold is of class C_9 if and only if

$$\mathrm{d}\eta = 0, \quad \mathrm{d}\phi = 0 \quad N^{(1)} \neq 0, \quad \text{and} \quad N_\varphi \neq 0.$$

These conditions are equivalent to

$$\nabla_X \xi = \varphi \nabla_{\varphi X} \xi. \tag{2.10}$$

We can see that a C_9-manifold is only a proper almost cosymplectic manifold.

We know that in dimension three, there are structures that belong to the sum of two or more classes (the structures in the intersection of two classes are only the cosymplectic ones). We mention, for example, that trans-Sasakian structures are in the class $C_5 \oplus C_6$ [16] and generalized C_{12}-structures are in the class $C_5 \oplus C_{12}$ (see [2, 10]). So, next, we introduce an expression to characterize all almost contact metric structures, i.e., the class $C_5 \oplus C_6 \oplus C_9 \oplus C_{12}$.

First of all, we establish a fundamental formula for an almost contact metric structure with the covariant derivative of structure tensor φ.

Theorem 2.1. *For a 3-dimensional almost contact metric structure (φ, ξ, η, g) on M, the covariant derivative of φ is given by*

$$\begin{aligned}(\nabla_X \varphi)Y = {} & 2\alpha\big(g(X,Y)\xi - \eta(Y)X\big) + 2\beta\big(g(\varphi X, Y)\xi - \eta(Y)\varphi X\big) \\ & - \eta(X)\big((\nabla_\xi \eta)(\varphi Y)\xi + \eta(Y)\varphi\nabla_\xi \xi\big) \\ & - g(\nabla_{\varphi X}\xi, Y)\xi + \eta(Y)\nabla_{\varphi X}\xi.\end{aligned} \tag{2.11}$$

Proof. Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional almost contact metric manifold. As proved in Lemma 6.1 of [5], for any almost contact metric structure one has

$$2g\big((\nabla_X \varphi)Y, Z\big) = 3\mathrm{d}\phi(X, \varphi Y, \varphi Z) - 3\mathrm{d}\phi(X, Y, Z)$$

$$+ g\big(N^{(1)}(Y,Z),\varphi X\big) + N^{(2)}(Y,Z)\eta(X) + 2\mathrm{d}\eta(\varphi Y,X)\eta(Z) - 2\mathrm{d}\eta(\varphi Z,X)\eta(Y). \tag{2.12}$$

We have ([5], p. 81),

$$\begin{aligned} N^{(2)}(Y,Z) &= \big(\mathcal{L}_{\varphi Y}\eta\big)(Z) - \big(\mathcal{L}_{\varphi Z}\eta\big)(Y) \\ &= \varphi Y\big(\eta(Z)\big) - \eta\big([\varphi Y,Z]\big) - \varphi Z\big(\eta(Y)\big) + \eta\big([\varphi Z,Y]\big) \\ &= 2\mathrm{d}\eta(\varphi Y,Z) - 2\mathrm{d}\eta(\varphi Z,Y), \end{aligned} \tag{2.13}$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to the vector field X. Also, we know that

$$N_\varphi(Y,Z) = (\varphi\nabla_Z\varphi - \nabla_{\varphi Z}\varphi)Y - (\varphi\nabla_Y\varphi - \nabla_{\varphi Y}\varphi)Z.$$

Thus, using (2.4), we get

$$N_\varphi(Y,Z) = -2\mathrm{d}\eta(\varphi Y,\varphi Z)\xi + \eta(Y)\big(\nabla_Z\xi + \varphi\nabla_{\varphi Z}\xi\big) - \eta(Z)\big(\nabla_Y\xi + \varphi\nabla_{\varphi Y}\xi\big). \tag{2.14}$$

From (2.5) and (2.6), one can get

$$\begin{aligned} 3\mathrm{d}\phi(X,Y,Z) &= 6\beta(\eta\wedge\phi)(X,Y,Z) \\ &= 2\beta\eta(X)g(Y,\varphi Z) + 2\beta\eta(Y)g(Z,\varphi X) + 2\beta\eta(Z)g(X,\varphi Y) \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} 2\mathrm{d}\eta(X,Y) &= 2(\eta\wedge\nabla_\xi\eta + \alpha\phi)(X,Y) \\ &= \eta(X)g(\nabla_\xi\xi,Y) - \eta(Y)g(\nabla_\xi\xi,X) + 2\alpha g(X,\varphi Y). \end{aligned} \tag{2.16}$$

The proof is direct, just replacing formulas (2.2), (2.5), (2.6) and (2.13)-(2.15) in (2.12) with a long but simple computation, we get the formula

$$\begin{aligned} 2(\nabla_X\varphi)Y &= 2\alpha\big(g(X,Y)\xi - \eta(Y)X\big) + 2\beta\big(g(\varphi X,Y)\xi - \eta(Y)\varphi X\big) \\ &\quad - \eta(X)\big((\nabla_\xi\eta)(\varphi Y)\xi + \eta(Y)\varphi\nabla_\xi\xi\big) \\ &\quad + g\big(\varphi(\nabla_X\xi + \varphi\nabla_{\varphi X}\xi),Y\big)\xi - \eta(Y)\varphi(\nabla_X\xi + \varphi\nabla_{\varphi X}\xi). \end{aligned} \tag{2.17}$$

Now, subtracting (2.4) from (2.17), we find our formula. □

In the following, we present a nice and simple equivalent condition to (2.11) in term of $\nabla\xi$.

Theorem 2.2. *Any 3-dimensional almost contact metric structure (φ,ξ,η,g) on M satisfies the following formula:*

$$\nabla_X\xi = -2\alpha\varphi X - 2\beta\varphi^2 X + \eta(X)\nabla_\xi\xi + \varphi\nabla_{\varphi X}\xi. \tag{2.18}$$

Proof. Let (M,φ,ξ,η,g) be a 3-dimensional almost contact metric manifold. Setting $Y=\xi$ in (2.11), we obtain

$$-\varphi\nabla_X\xi = 2\alpha\varphi^2 X - 2\beta\varphi X - \eta(X)\varphi\nabla_\xi\xi + \nabla_{\varphi X}\xi,$$

and hence

$$\nabla_X\xi = -2\alpha\varphi X - 2\beta\varphi^2 X + \eta(X)\nabla_\xi\xi + \varphi\nabla_{\varphi X}\xi. \tag{2.19}$$

Conversely, by substituting (2.18) in (2.4), we get (2.11). This completes the proof. □

3. Construction of 3-dimensional almost contact metric manifolds

Almost contact metric structures in dimension three are quite special. In fact, there are several key facts which make their handling easier.

To begin with, 2-forms and vector fields are in one-to-one correspondence. In fact, let (M^3, g) be a $3D$ oriented Riemannian manifold with volume form Ω and consider a differential 2-form Φ on M. The Hodge star operator $\star$ acts on Φ to produce the 1-form $\theta = \star\Phi$, and the g-equivalent (dual) vector field $V = \theta^\sharp$ is well defined by $g(V, X) = \theta(X)$ for any vector field X on M. The reverse assignation works in the same way: given a vector field V on M, consider its g-dual 1-form $\theta = V^\flat$. Apply the Hodge star operator to get $\star\theta$ which is a 2-form in a $3D$ manifold. Now, the interior contraction i_V (which is defined by means of $(i_V\Omega)(X, Y) = \Omega(V, X, Y)$ allows us to write it as $\star\theta = i_V\Omega = \Phi$. Thus we have a one-to-one map between 2-forms and vector fields.

Based on these facts, our problem can be solved by taking $\xi = V$, $\eta = \theta$, and $\phi = \Phi$, and knowing that $\phi(X, Y) = g(X, \varphi Y)$, the tensor φ can be obtained. Thus, we have constructed an almost contact metric structure (φ, ξ, η, g) on M.

However, in this paper, we are aimed to present another way of constructing an almost contact metric structure. It is simple and practical, based on linear algebra and basic definitions of structural elements.

Let us first recall the following well-known standard topological result (see, for example, [15, p. 149] or [18, pp. 11–30, 11–51])

Lemma 3.1. *For a connected orientable manifold M^n the following assertions are equivalent:*

1) *There is a non-vanishing vector field on M^n.*
2) *M^n is either non-compact or compact and has the Euler number $\chi(M^n) = 0$.*

Now, let (M, g) be a 3-dimensional oriented Riemmannian manifold. For every local orthonormal frame $\{e_i\}_{1\leq i\leq 3}$, we define a unit vector field ξ by

$$\xi = \sum_{i=1}^{3} \xi^i e_i, \tag{3.1}$$

where $\xi^i \in \mathcal{C}^\infty(M)$ and $\sum_{i=1}^{3}(\xi^i)^2 = 1$. Consequently, the g-dual of ξ is the differential 1-form η given by

$$\eta = \sum_{i=1}^{3} \xi^i \theta^i, \tag{3.2}$$

where $\{\theta^i\}_{1\leq i\leq 3}$ is the dual coframe.

Note: we will use the convention of Einstein. (Whenever an index is repeated, it is a dummy index), i.e.,

$$\xi = \xi^i e_i \quad \text{and} \quad \eta = \xi^i \theta^i.$$

Now let us start looking for the φ. We put

$$\varphi e_i = \sum_j \varphi_i^j e_j, \tag{3.3}$$

where φ_i^j are functions on M. Using relationships (2.1), i.e.,

$$g(\varphi e_i, e_j) = -g(e_i, \varphi e_j) \quad \text{and} \quad g(\varphi e_i, \varphi e_j) = g(e_i, e_j) - \eta(e_i)\eta(e_j),$$

we get the system

$$\begin{cases} \varphi_i^j = -\varphi_j^i \\ \varphi_i^a \varphi_j^a = \delta_{ij} - \xi^i \xi^j. \end{cases} \tag{3.4}$$

By noting that i and j are fixed, from this system one easily obtains

$$\varphi_i^i = 0, \quad \varphi_i^a \varphi_i^a = 1 - (\xi^i)^2, \quad \text{and} \quad \varphi_i^a \varphi_j^a = -\xi^i \xi^j \quad \text{for } i \neq j.$$

For every $i, j, k \in \{1, 2, 3\}$ with $i \neq j$, $i \neq k$, and $j \neq k$, the first two equations above give the following:

$$\begin{cases} (\varphi_i^j)^2 + (\varphi_i^k)^2 = 1 - (\xi^i)^2 \\ (\varphi_j^i)^2 + (\varphi_j^k)^2 = 1 - (\xi^j)^2 \\ (\varphi_k^i)^2 + (\varphi_k^j)^2 = 1 - (\xi^k)^2. \end{cases}$$

Subtracting the second equation from the first one, taking into account that $\varphi_i^j = -\varphi_j^i$, we find

$$(\varphi_i^k)^2 - (\varphi_j^k)^2 = (\xi^j)^2 - (\xi^i)^2. \tag{3.5}$$

From the third equation in the above system, we have

$$(\varphi_k^j)^2 = (\varphi_j^k)^2 = 1 - (\xi^k)^2 - (\varphi_k^i)^2.$$

So, (3.5) becomes

$$2(\varphi_i^k)^2 = 1 + (\xi^j)^2 - (\xi^k)^2 - (\xi^i)^2 = 1 + (\xi^j)^2 - \big(1 - (\xi^j)^2\big) = 2(\xi^j)^2,$$

which gives

$$\varphi_i^k = \epsilon \xi^j, \tag{3.6}$$

where $\epsilon = \pm 1$. Notice that φ is completely defined with ξ.

Based on these facts, we give the following theorem:

Theorem 3.2 ([8]). *Let (M^3, g) be a 3-dimensional oriented Riemannian manifold. If ξ is a global unit vector field written in the form $\xi = \xi^i e_i$, where $\{e_i\}_{1 \leq i \leq 3}$ is an orthonormal basis on M, then there exist almost contact metric structures (φ, ξ, η, g), where*

$$\varphi = \epsilon \begin{pmatrix} 0 & -\xi^3 & \xi^2 \\ \xi^3 & 0 & -\xi^1 \\ -\xi^2 & \xi^1 & 0 \end{pmatrix},$$

with $\epsilon = \pm 1$ and η being the g-dual of ξ.

Proof. The necessity was observed above. For the sufficiency, it is easy to check the conditions (2.1). □

Regarding this theorem, we can ask the following question: "Can 3-dimensional almost contact metric manifolds be classified according to the components of ξ?" This is what we are to study in the next section.

Note: Through the rest of this paper, $(M, \varphi, \xi, \eta, g)$ always denotes the 3-dimensional almost contact metric manifolds defined above, where $\xi = \xi^i e_i$ and $\{e_i\}_{1\leq i\leq 3}$ is an orthonormal basis with respect to g.

4. Classification of 3-dimensional almost contact metric manifolds

Based on the classification of D. Chinea and C. Gonzalez [12], the five elementary 3-dimensional almost contact metric structures are characterized by

Class	Condition
C_0	$\nabla_{e_i}\xi = 0.$
C_5	$\nabla_{e_i}\xi = -\beta\varphi^2 e_i.$
C_6	$\nabla_{e_i}\xi = -\alpha\varphi e_i.$
C_9	$\nabla_{e_i}\xi = \varphi\nabla_{\varphi e_i}\xi.$
C_{12}	$\nabla_{e_i}\xi = \eta(e_i)\nabla_\xi\xi.$

Here, $2\alpha = tr_g(\varphi\nabla\xi)$ and $2\beta = \mathrm{div}\xi$. First of all, we prepare

$$\nabla_{e_i}\xi = \nabla_{e_i}(\xi^j e_j) = e_i(\xi^j)e_j + \xi^j\nabla_{e_i}e_j.$$

Putting $\nabla_{e_i}e_j = C^k_{ij}e_k$, where $C^k_{ij} = g\big(\nabla_{e_i}e_j, e_k\big)$, we get

$$\nabla_{e_i}\xi = e_i(\xi^j)e_j + \xi^j C^k_{ij}e_k.$$

With a change of indices, we obtain

$$\nabla_{e_i}\xi = \big(e_i(\xi^k) + \xi^j C^k_{ij}\big)e_k. \tag{4.1}$$

It should be noticed that $C^k_{ij} = -C^j_{ik}$ and $C^j_{ij} = 0$. Consequently,

$$\nabla_\xi\xi = \xi^i\nabla_{e_i}\xi = \xi^i\big(e_i(\xi^k) + \xi^j C^k_{ij}\big)e_k. \tag{4.2}$$

$$\begin{aligned} 2\alpha &= -g\big(\nabla_{e_i}\xi, \varphi e_i\big) \\ &= -g\Big(\big(e_i(\xi^k) + \xi^j C^k_{ij}\big)e_k, \varphi^l_i e_l\Big) = -\varphi^k_i\big(e_i(\xi^k) + \xi^j C^k_{ij}\big). \end{aligned} \tag{4.3}$$

$$2\beta = g\big(\nabla_{e_i}\xi, e_i\big) = g\Big(\big(e_i(\xi^k) + \xi^j C^k_{ij}\big)e_k, e_i\Big) = e_i(\xi^i) + \xi^j C^i_{ij}. \tag{4.4}$$

Based on these facts, we obtain the following results.

4.1. C_0-manifolds. This class is a class of cosymplectic manifolds. It is characterized by the formula

$$\nabla_{e_i}\xi = 0. \tag{4.5}$$

Using (4.1), directly we get

Proposition 4.1. *$(M, \varphi, \xi, \eta, g)$ is a cosymplectic manifold if and only if the following system holds:*

$$e_i(\xi^k) + \xi^j C_{ij}^k = 0. \tag{4.6}$$

4.2. C_5-manifolds. This class is also known as β-Kenmotsu manifolds. It is characterized by the formula

$$\nabla_{e_i}\xi = -\beta\varphi^2 e_i. \tag{4.7}$$

Notice that by(3.4), we have

$$\varphi^2 e_i = \varphi(\varphi e_i) = \varphi(\varphi_i^k e_k) = \varphi_i^k \varphi_k^l e_l = -\varphi_i^k \varphi_l^k e_l = (\xi^i \xi^l - \delta_{il})e_l. \tag{4.8}$$

Substituting (4.1) and (4.8) into (4.7), we find

$$\big(e_i(\xi^k) + \xi^j C_{ij}^k\big)e_k = \beta(\delta_{il} - \xi^i \xi^l)e_l.$$

As a consequence, we have the following proposition.

Proposition 4.2. *$(M, \varphi, \xi, \eta, g)$ is a β-Kenmotsu manifold if and only if*

$$e_i(\xi^k) = \xi^j C_{ik}^j + \beta(\delta_{ik} - \xi^i \xi^k) \tag{4.9}$$

with β given in (4.4).

4.3. C_6-manifolds. This class is a class of α-Sasaki manifolds. It is characterized by the formula

$$\nabla_{e_i}\xi = -\alpha\varphi e_i. \tag{4.10}$$

Using (3.3) and (4.1), we obtain

$$\big(e_i(\xi^k) + \xi^j C_{ij}^k\big)e_k = -\alpha\varphi_i^s e_s.$$

As a consequence, we have the following proposition.

Proposition 4.3. *$(M, \varphi, \xi, \eta, g)$ is a C_6-manifold if and only if*

$$e_i(\xi^k) = \xi^j C_{ik}^j - \alpha\varphi_i^k \tag{4.11}$$

with α given in (4.3).

4.4. C_9-manifolds. This class is characterized by the formula

$$\nabla_{e_i}\xi = \varphi\nabla_{\varphi e_i}\xi. \tag{4.12}$$

Using (3.3) and (4.1), we obtain

$$\varphi\nabla_{\varphi e_i}\xi = \varphi\nabla_{\varphi_i^j e_j}\xi = \varphi_i^j\big(e_j(\xi^k) + \xi^s C_{js}^k\big)\varphi e_k = \varphi_i^j\big(e_j(\xi^k) + \xi^s C_{js}^k\big)\varphi_k^l e_l.$$

So, (4.12) gives us

$$\big(e_i(\xi^k) + \xi^j C_{ij}^k\big)e_k = \big(e_j(\xi^k) + \xi^s C_{js}^k\big)\varphi_i^j\varphi_k^l e_l.$$

Hence we have the following proposition.

Proposition 4.4. *(M,φ,ξ,η,g) is a C_9-manifold if and only if*

$$e_i(\xi^t) + \xi^j C_{ij}^t = \big(e_j(\xi^k) + \xi^s C_{js}^k\big)\varphi_i^j\varphi_k^t. \tag{4.13}$$

4.5. C_{12}-manifolds. This class is characterized by

$$\nabla_{e_i}\xi = \eta(e_i)\nabla_\xi\xi. \tag{4.14}$$

Using (4.1) and (4.2), we obtain

$$\big(e_i(\xi^k) + \xi^j C_{ij}^k\big)e_k = \xi^i\xi^l\big(e_l(\xi^k) + \xi^j C_{lj}^k\big)e_k.$$

Thus, we state the following proposition.

Proposition 4.5. *(M,φ,ξ,η,g) is a C_{12}-manifold if and only if*

$$e_i(\xi^k) + \xi^j C_{ij}^k = \xi^i\xi^l\big(e_l(\xi^k) + \xi^j C_{lj}^k\big).$$

5. A class of examples

We denote the Cartesian coordinates in a 3-dimensional Euclidean space $\mathbb{R}^3$ by (x_1, x_2, x_3) and define a symmetric tensor field g by

$$g = \mathrm{e}^{2\sigma}\begin{pmatrix} \rho^2+\tau^2 & 0 & -\tau \\ 0 & \rho^2 & 0 \\ -\tau & 0 & 1 \end{pmatrix},$$

where σ, τ and ρ are functions on $\mathbb{R}^3$, where $\rho \neq 0$ everywhere. One can check that

$$\left\{ e_1 = \frac{\mathrm{e}^{-\sigma}}{\rho}\left(\frac{\partial}{\partial x_1} + \tau\frac{\partial}{\partial x_3}\right),\ e_2 = \frac{\mathrm{e}^{-\sigma}}{\rho}\frac{\partial}{\partial x_2},\ e_3 = \mathrm{e}^{-\sigma}\frac{\partial}{\partial x_3} \right\}$$

is an orthonormal basis with respect to g. For the non-zero Lie brackets of e_i, we have

$$[e_1, e_2] = \frac{\mathrm{e}^{-\sigma}}{\rho^2}\big((\rho\sigma_2 + \rho_2)e_1 - (\rho\sigma_1 + \rho_1 + \tau\rho\sigma_3 + \tau\rho_3)e_2 - \tau_2 e_3\big),$$

$$[e_1, e_3] = \frac{\mathrm{e}^{-\sigma}}{\rho}\big((\rho\sigma_3 + \rho_3)e_1 - (\sigma_1 + \tau\sigma_3 + \tau_3)e_3\big),$$
$$[e_2, e_3] = \frac{\mathrm{e}^{-\sigma}}{\rho}\big(\sigma_2 e_1 - (\rho\sigma_3 + \rho_3)e_2 - \sigma e_3\big),$$

where $\sigma_i = \frac{\partial\sigma}{\partial x_i}$, $\rho_i = \frac{\partial\rho}{\partial x_i}$ and $\tau_i = \frac{\partial\tau}{\partial x_i}$. Using Koszul's formula for the metric g,

$$\begin{aligned}2g(\nabla_X Y, Z) = {}& Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)\\ &- g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y],\end{aligned}$$

we get

$$2C_{ij}^k = 2g\big(\nabla_{e_i} e_j, e_k\big) = -g\big(e_i, [e_j, e_k]\big) + g\big(e_j, [e_k, e_i]\big) + g\big(e_k, [e_i, e_j]\big).$$

Then the components C_{ij}^k are given by

$$C_{ij}^1 = \frac{\mathrm{e}^{-\sigma}}{\rho^2}\begin{pmatrix} 0 & \rho\sigma_2 + \rho_2 & \rho(\rho\sigma_3 + \rho_3) \\ 0 & -(\rho\sigma_1 + \rho_1 + \rho\tau\sigma_3 + \tau\rho_3) & -\frac{\tau_2}{2} \\ 0 & -\frac{\tau_2}{2} & -\rho(\tau\sigma_3 + \tau_3 + \sigma_1)\end{pmatrix},$$
$$C_{ij}^2 = \frac{\mathrm{e}^{-\sigma}}{\rho^2}\begin{pmatrix} -(\rho\sigma_2 + \rho_2) & 0 & \frac{\tau_2}{2} \\ \rho\sigma_1 + \rho_1 + \rho\tau\sigma_3 + \tau\rho_3 & 0 & \rho(\rho\sigma_3 + \rho_3) \\ \frac{\tau_2}{2} & 0 & -\rho\sigma_2\end{pmatrix},$$
$$C_{ij}^3 = \frac{\mathrm{e}^{-\sigma}}{\rho^2}\begin{pmatrix} -\rho(\rho\sigma_3 + \rho_3) & -\frac{\tau_2}{2} & 0 \\ \frac{\tau_2}{2} & -\rho(\rho\sigma_3 + \rho_3) & 0 \\ \rho(\tau\sigma_3 + \tau_3 + \sigma_1) & \rho\sigma_2 & 0\end{pmatrix}.$$

Let us take $\xi = e_3$. From Theorem 3.2, $(\mathbb{R}^3, \varphi, \xi, \eta, g)$ is an almost contact metric manifold with $\varphi e_1 = \epsilon e_2$, $\varphi e_2 = -\epsilon e_1$, $\varphi e_3 = 0$ and $\eta = \mathrm{e}^{-\sigma}(dx_3 - \tau dx_1)$.

We will now define the five classes studied above by discussing the parameters ρ, τ, and σ.

C_0-manifolds: By Proposition 4.1, we have

$$e_i(\xi^k) + \xi^j C_{ij}^k = 0 \Leftrightarrow C_{i3}^k = 0 \Leftrightarrow \begin{cases}\sigma_2 = \tau_2 = 0\\ \rho\sigma_3 + \rho_3 = 0\\ \tau\sigma_3 + \sigma_1 + \tau_3 = 0.\end{cases}$$

C_5-manifolds: By Proposition 4.2, we have

$$e_i(\xi^k) = \xi^j C_{ik}^j + \beta(\delta_{ik} - \xi^i\xi^k) \Leftrightarrow C_{ik}^3 = \beta(\xi^i\xi^k - \delta_{ik}) \Leftrightarrow \begin{cases} C_{i1}^3 = -\beta\delta_{i1}\\ C_{i2}^3 = -\beta\delta_{i2}\end{cases}$$
$$\Leftrightarrow \begin{cases} C_{11}^3 = -\beta\\ C_{21}^3 = C_{31}^3 = 0\\ C_{22}^3 = -\beta\\ C_{12}^3 = C_{32}^3 = 0\end{cases} \Leftrightarrow \begin{cases}\beta = \dfrac{\mathrm{e}^{-\sigma}}{\rho}(\rho\sigma_3 + \rho_3)\\ \sigma_2 = \tau_2 = 0,\\ \tau\sigma_3 + \sigma_1 + \tau_3 = 0.\end{cases}.$$

C_6-manifolds: By Proposition 4.3, we have

$$e_i(\xi^k) = \xi^j C_{ik}^j - \alpha\varphi_i^k \Leftrightarrow C_{i3}^k = \alpha\varphi_i^k$$
$$\Leftrightarrow \begin{cases} C_{i1}^i = 0, \\ C_{13}^2 = -\epsilon\alpha \\ C_{23}^1 = \epsilon\alpha \\ C_{33}^1 = C_{23}^2 = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha = -\epsilon\dfrac{\tau_2}{2\rho^2}\mathrm{e}^{-\sigma} \\ \sigma_2 = 0 \\ \rho\sigma_3 + \rho_3 = 0 \\ \tau\sigma_3 + \sigma_1 + \tau_3 = 0 \end{cases}.$$

C_9-manifolds: By Proposition 4.4, we have

$$e_i(\xi^t) + \xi^j C_{ij}^t = \big(e_j(\xi^k) + \xi^s C_{js}^k\big)\varphi_i^j\varphi_k^t \Leftrightarrow C_{i3}^t = C_{j3}^3\varphi_i^j\varphi_3^t = 0,$$

because $C_{j3}^3 = g(\nabla_{e_j}e_3, e_3) = 0$. Hence, there exists no C_9-structure (see C_0-manifolds).

C_{12}-manifolds: By Proposition 4.5, we have

$$e_i(\xi^k) + \xi^j C_{ij}^k = \xi^i\xi^l\big(e_l(\xi^k) + \xi^j C_{lj}^k\big) \Leftrightarrow C_{i3}^k = \xi^i C_{33}^k$$
$$\Leftrightarrow \begin{cases} C_{13}^k = 0 \\ C_{23}^k = 0 \end{cases} \Leftrightarrow \begin{cases} \rho\sigma_3 + \rho_3 = 0 \\ \tau_2 = 0 \end{cases}.$$

References

[1] B. Bayour and G. Beldjilali, *Ricci solitons on 3-dimensional C_{12}-manifolds*, Balkan J. Geom. Appl. **27** (2022), No. 2, 26–36.

[2] B. Bayour, G. Beldjilali, and M. L. Sinacer, *Almost contact metric manifolds with certain condition*, Research Square, preprint. Available from: https://doi.org/10.21203/rs.3.rs-2622224/v2

[3] G. Beldjilali, *3-dimensional C_{12}-manifolds*, Rev. Un. Mat. Argentina **67** (2024), No. 1, 1–14.

[4] G. Beldjilali, *Classification of almost contact metric structures on 3D Lie algebra*, J. Math. Sci. **271** (2023), 210–222.

[5] D.E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, **203**, Birkhäuser, Boston, 2002.

[6] D.E. Blair, T. Koufogirgos, and R. Sharma, *A classification of 3-dimensional contact metric manifolds with $Q\varphi = \varphi Q$*, Kodai Math. J. **13** (1990), 391–401.

[7] W.M. Boothby and H.C. Wang, *On contact manifolds*, Ann. Math. **68**, (1958), 721–734.

[8] H. Bouzir, G. Beldjilali, and B. Bayour, *On three dimensional C_{12}-manifolds*, Mediterr. J. Math. **18** (2021), Paper No. 239.

[9] C. P. Boyer, K. Galicki, and P. Matzeu, *On eta-Einstein Sasakian geometry*, Comm. Math. Phys. **262** (2006), 177–208.

[10] S. de Candia and M. Falcitelli, *Curvature of $C_5 \oplus C_{12}$-manifolds*, Mediterr. J. Math. 16 (2019), Paper No. 105.

[11] F.M. Cabrera, *On the classification of almost contact metric manifolds*, Differential Geom. Appl. **64** (2019), 13–28.

[12] D. Chinea and C. Gonzalez, *A classification of almost contact metric manifolds*, Ann. Mat. Pura Appl. **156** (1990), 15–36.

[13] I. Hideki and M. Satsuki, *Solutions to the Einstein-Maxwell-current system with Sasakian maifolds*, preprint, https://arxiv.org/abs/2012.02432v1

[14] Z. Olszak, *Normal almost contact manifolds of dimension three*, Annales Pol. Math. **XLVII** (1986), 41–50.

[15] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.

[16] J. A., Oubiña, *New classes of almost contact metric structures*. Publ. Math. Debrecen **32** (1985), 187–193.

[17] A. Sarkar and A. Mondal, *Certain curves on some classes of of three-dimensional almost contact metric manifolds*, Rev. Un. Mat. Argentina **58** (2017), 107–125.

[18] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, **I**, Publisher Perish, Inc., Boston, 1970.

[19] K. Yano and M. Kon, *Structures on Manifolds*, Series in Pure Mathematics, **3**, World Sciences, 1984.

[20] G. Yilmaz and M.A. Akyol, *Conformal slant submersions from cosymplectic manifolds*, Turkish J. Math. **42** (2018), No. 5, 2672–2689.

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Тривимірні майже контактні метричні многовиди з новим підходом

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Ми доводимо, що, виходячи з глобального одиничного векторного поля на тривимірному рімановому многовиді, можна побудувати майже контактну метричну структуру. Крім того, розуміння природи цих структур досягається через співвідношення, що пов'язує компоненти цього векторного поля та компоненти зв'язності Леві-Чівіти. Наведено ілюстративні приклади.

Ключові слова: майже контактна метрична структура, структура Сасакі, структура Кенмоцу, косимплектична структура, кутова структура