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On Shimurian Generalizations of the Stack $\mathrm{BT}_1 \otimes \mathbb{F}_p$

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Let G be a smooth group scheme over $\mathbb{F}_p$ equipped with a $\mathbb{G}_m$-action such that all weights of $\mathbb{G}_m$ on $\mathrm{Lie}(G)$ are ≤ 1. Let Disp_n^G be Eike Lau's stack of n-truncated G-displays (this is an algebraic $\mathbb{F}_p$-stack). In the case $n = 1$ we introduce an algebraic stack equipped with a morphism to Disp_1^G. We conjecture that if $G = GL(d)$ then the new stack is canonically isomorphic to the reduction modulo p of the stack of 1-truncated Barsotti–Tate groups of height d and dimension d', where d' depends on the action of $\mathbb{G}_m$ on $GL(d)$.

We also discuss how to define an analog of the new stack for $n > 1$ and how to replace $\mathbb{F}_p$ by $\mathbb{Z}/p^m\mathbb{Z}$.

Key words: Barsotti–Tate group, Shimura variety, display, F-zip, connection, p-curvature, syntomification

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1. Introduction

Once and for all, we fix a prime p. The words "algebraic stack" are understood in the sense of [20] unless stated otherwise.

1.1. The goal. The notion of n-truncated Barsotti-Tate group was introduced by Grothendieck [13]. It is reviewed in [14, 15, 22]. n-truncated Barsotti-Tate groups of height d and dimension $d' \leq d$ form an algebraic stack over $\mathbb{Z}$, denoted by $\mathrm{BT}_n^{d,d'}$. This stack is smooth by a deep theorem of Grothendieck, whose proof is given in Illusie's article [14].

The stack $\mathrm{BT}_n^{d,d'}$ is related to the group $G = GL(d)$; e.g., $\mathrm{BT}_n^{d,d'} \otimes \mathbb{Z}[p^{-1}]$ is the classifying stack of $GL(d, \mathbb{Z}/p^n\mathbb{Z})$. Since there exist Shimura varieties of type E_7, it is natural to expect that the stacks $\mathrm{BT}_n^{d,d'} \otimes \mathbb{Z}/p^m\mathbb{Z}$ (and maybe the stacks $\mathrm{BT}_n^{d,d'}$ themselves) have interesting E_7-analogs.

In the main body of this article we focus on the case $n = m = 1$. In this case we give a rather elementary definition of the stack $\mathrm{BT}^G_{1,\mathbb{F}_p} = \mathrm{BT}^G_1 \otimes \mathbb{F}_p$, where G is any smooth affine group scheme over $\mathbb{F}_p$ equipped with a 1-bounded $\mathbb{G}_m$-action (1-boundedness means that the weights of $\mathbb{G}_m$ on $\mathrm{Lie}(G)$ are ≤ 1). Note that a semisimple group of type E_7 can be equipped with a nontrivial 1-bounded $\mathbb{G}_m$-action.

In Appendix C we suggest a definition of $\mathrm{BT}^G_{n,\mathbb{Z}/p^m\mathbb{Z}}$ for arbitrary n and m, which uses the prismatic theory.[1]

[1]In the case $m = 1$ one only needs the crystalline theory, and in the case $n = m = 1$ the de Rham theory is enough. This is why in the case $m = n = 1$ an elementary approach is possible.

We prove that the stack $\mathrm{BT}^G_{1,\mathbb{F}_p}$ defined in §4.2 is algebraic (this is not an immediate consequence of the definition). We conjecture that in the case $G = GL(d)$ one has $\mathrm{BT}^G_{1,\mathbb{F}_p} = \mathrm{BT}_1^{d,d'} \otimes \mathbb{F}_p$, where d' depends on the action of $\mathbb{G}_m$ on $GL(d)$. The precise formulations of this result and conjecture are given in §4.4-4.5.

1.2. Structure of the article. In §2 we recall some facts about group schemes equipped with a $\mathbb{G}_m$-action. We also introduce a (2,1)-category $\mathbf{Shim}_n$, whose objects are smooth affine group schemes over $\mathbb{Z}/p^n\mathbb{Z}$ equipped with a 1-bounded $\mathbb{G}_m$-action.

In §3 we recall a certain $\mathbb{F}_p$-stack Disp^G_1, $G \in \mathbf{Shim}_1$, which is called the stack of 1-truncated displays (or the stack of F-zips). We also introduce in §3.3 a commutative group scheme over Disp^G_1 denoted by Lau^G_1.

In §4 we define an $\mathbb{F}_p$-stack $\mathrm{BT}^G_{1,\mathbb{F}_p}$, which depends functorially on $G \in \mathbf{Shim}_1$. We also formulate the main theorems and a conjecture, see §4.4-4.5.

The proof of the main theorems is given in §5-6.

In Appendices A-B we discuss some material on algebraic stacks.

In Appendix C we suggest a definition of $\mathrm{BT}^G_{n,\mathbb{Z}/p^m\mathbb{Z}}$ for arbitrary n and m, which uses the prismatic theory. We also formulate Conjecture C.3.1; it is motivated by Corollary 4.4.3 and by Grothendieck's smoothness theorem mentioned at the beginning of §1.1.

In Appendix D we explain why the definition of $\mathrm{BT}^G_{1,\mathbb{F}_p}$ given in Appendix C is equivalent to the one from §4.

1.3. Status of the conjectures. As far as I understand, both conjectures formulated in this article are proved in [12].

2. The (2,1)-category $\mathbf{Shim}_n$

2.1. 1-bounded $\mathbb{G}_m$-actions. Let G be a smooth group scheme over $\mathbb{Z}/p^n\mathbb{Z}$ and $\mathfrak{g} := \mathrm{Lie}(G)$. An action of $\mathbb{G}_m$ on G induces its action on $\mathfrak{g}$ and therefore a grading

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i \,. \tag{2.1}$$

Following E. Lau [19], we say that an action of $\mathbb{G}_m$ on G is *1-bounded* if $\mathfrak{g}_i = 0$ for $i > 1$.

Example 2.1.1. If G is reductive and connected then an action of $\mathbb{G}_m$ on G is given by a cocharacter $\mu : \mathbb{G}_m \to G_{\mathrm{ad}}$. In this case 1-boundedness implies that all weights of $\mathbb{G}_m$ on $\mathrm{Lie}(G)$ belong to $\{-1, 0, 1\}$. Cocharacters μ with this property are called *minuscule*[2].

[2] This notion of minuscule cocharacter is slightly more general than that of Bourbaki (e.g., Bourbaki requires a minuscule cocharacter to be nonzero).

2.2. The (2,1)-category $\mathbf{Shim}_n$. Let $\mathbf{Shim}_n$ be the following (2,1)-category: its objects are smooth affine group schemes over $\mathbb{Z}/p^n\mathbb{Z}$ equipped with a 1-bounded $\mathbb{G}_m$-action, and for $G_1, G_2 \in \operatorname{Ob}\mathbf{Shim}_n$, the groupoid of morphisms $G_1 \to G_2$ is the quotient groupoid[3] of $\operatorname{Hom}^{\mathbb{G}_m}(G_1, G_2)$ by the conjugation action of $G_2^{\mathbb{G}_m}(\mathbb{Z}/p^n\mathbb{Z})$. Here $\operatorname{Hom}^{\mathbb{G}_m}$ is the set of $\mathbb{G}_m$-equivariant homomorphisms, and $G_2^{\mathbb{G}_m} \subset G_2$ is the group subscheme of $\mathbb{G}_m$-fixed points.

The above (2,1)-category $\mathbf{Shim}_n$ is not quite good from a certain viewpoint[4], but we work with it in this article.

2.3. The subgroups M, $P^{\pm}$, $U^{\pm}$

2.3.1. Recollections from [8]. Let k be a ring and G a smooth affine group k-scheme equipped with a $\mathbb{G}_m$-action, i.e., a homomorphism $\mu : \mathbb{G}_m \to \underline{\operatorname{Aut}}\, G$. Let $M := G^{\mathbb{G}_m}$, i.e., $M \subset G$ is the subgroup of $\mathbb{G}_m$-fixed points. Let P^+ be the attractor for μ, i.e., $P^+ \subset G$ is the maximal closed subscheme such that the action map $\mathbb{G}_m \times P^+ \to G$ extends to a morphism of schemes $\mathbb{A}^1 \times P^+ \to G$. Restricting the latter to $\{0\} \times P^+ \subset \mathbb{A}^1 \times P^+$, we get a retraction $P^+ \to M \subset P^+$. Let $P^- \subset G$ be the attractor for μ^{-1}; we have a retraction $P^- \to M \subset P^-$. Let $U^{\pm}$ be the preimage of 1 with respect to the map $P^{\pm} \to M$. The next lemma is well known if G is reductive and connected (in this case P^+ and P^- are parabolics, and M is a Levi).

Lemma 2.3.2.

(i) *$P^{\pm}$ and $U^{\pm}$ are group subschemes of G. The maps $P^{\pm} \to M$ are group homomorphisms. One has $P^{\pm} = M \ltimes U^{\pm}$.*

(ii) *$P^+ \cap P^- = M$.*

(iii) *$P^{\pm}$, $U^{\pm}$, and M are smooth. In terms of the grading (2.1), one has*

$$\operatorname{Lie}(M) = \mathfrak{g}_0, \quad \operatorname{Lie}(P^+) = \mathfrak{g}_{\geq 0}, \quad \operatorname{Lie}(P^-) = \mathfrak{g}_{\leq 0},$$
$$\operatorname{Lie}(U^+) = \mathfrak{g}_{>0}, \quad \operatorname{Lie}(U^-) = \mathfrak{g}_{<0},$$

where $\mathfrak{g}_{\geq 0} := \bigoplus\limits_{i \geq 0} \mathfrak{g}_i$ and $\mathfrak{g}_{\leq 0}, \mathfrak{g}_{>0}, \mathfrak{g}_{<0}$ are defined similarly.

(iv) *The multiplication map $U^- \times M \times U^+ \to G$ is an open immersion.*

(v) *The fibers of $U^{\pm}$ over $\operatorname{Spec} k$ are connected. If the fibers of G over $\operatorname{Spec} k$ are connected then the same holds for $P^{\pm}$ and M.*

Proof. Apply the results from [8, p. 48-56] (especially [8, Prop. 2.1.8]) to the semidirect product $\mathbb{G}_m \ltimes G$. □

2.3.3. The 1-bounded case. Now assume that $\mathfrak{g}_i = 0$ for $i > 1$. Then $\mathfrak{g}_1$ is abelian. Moreover, by [19, Lemma 6.3.2], there is a unique $\mathbb{G}_m$-equivariant isomorphism of group schemes

$$f : U^+ \xrightarrow{\sim} \mathfrak{g}_1 \tag{2.2}$$

[3] If a group Γ acts on a set X then the *quotient groupoid* is defined as follows: the set of objects is X, a morphism $x \to x'$ is an element $\gamma \in \Gamma$ such that $\gamma x = x'$, and the composition of morphisms is given by multiplication in G.

[4] See §9.1.4 of version 1 of [12].

such that $\mathrm{Lie}(f) = \mathrm{id}_{\mathfrak{g}_1}$. If the ring k is local then the projective k-module $\mathfrak{g}_1$ is free, so the isomorphism (2.2) implies that U^+ is isomorphic to a product of several copies of $\mathbb{G}_a$.

If k is an $\mathbb{F}_p$-algebra then $\mathfrak{g}$ is equipped with a p-operation $x \mapsto x^{(p)}$. By (2.2), if $\mathfrak{g}$ is 1-bounded then

$$x^{(p)} = 0 \text{ for } x \in \mathfrak{g}_1 \,. \tag{2.3}$$

Another way to prove (2.3) is to note that more generally, the p-operation takes $\mathfrak{g}_i$ to $\mathfrak{g}_{pi}$. This follows from the fact that for any k-algebra $\tilde{k}$ and any $a \in \tilde{k}^\times$ the action of a on $\mathfrak{g} \otimes_k \tilde{k}$ preserves the structure of restricted Lie algebra on $\mathfrak{g} \otimes_k \tilde{k}$.

3. Recollections on the stacks Disp_1^G

3.1. Plan of this section. For any $n \in \mathbb{N}$ and a smooth group scheme G over $\mathbb{Z}/p^n\mathbb{Z}$ equipped with a $\mathbb{G}_m$-action, one has the *$\mathbb{F}_p$-stack of n-truncated displays* Disp_n^G defined[5] in [19].

In §3.2 we recall the stack Disp_1^G. Assuming that $G \in \mathbf{Shim}_1$, we define in §3.3 a commutative group scheme on Disp_1^G; we denote it by Lau_1^G (because it is conjecturally related to E. Lau's work [18], see §4.5 below).

The stack $\mathrm{BT}_{1,\mathbb{F}_p}^G$, which will be defined in §4 assuming that $G \in \mathbf{Shim}_1$, will be equipped with a map to Disp_1^G, which makes it into a gerbe over Disp_1^G banded by Lau_1^G.

Let us make some historical remarks related to Disp_1^G. First, as noted in Example 3.7.5 of [19], Disp_1^G is the same as the stack of *F-zips* in the sense of [23, 25]. Second, the projective limit of Disp_n^G is the reduction modulo p of a stack over $\mathrm{Spf}\, \mathbb{Z}_p$, which was defined in [5] (at least, assuming that G is reductive and the cocharacter $\mu : \mathbb{G}_m \to G_{\mathrm{ad}}$ is minuscule) following the ideas of Thomas Zink. The articles [5, 23, 25] preceded [19].

3.2. The $\mathbb{F}_p$-stack Disp_1^G.

3.2.1. Definition of the stack Disp_1^G. Let G be a smooth group scheme over $\mathbb{F}_p$ equipped with a $\mathbb{G}_m$-action. Let M and $P^\pm$ be as in §2.3. Let S be an $\mathbb{F}_p$-scheme. Then $\mathrm{Disp}_1^G(S)$ is the groupoid of the following data:

(i) a $P^\pm$-torsor $\mathcal{F}^\pm$ on S;

(ii) an isomorphism between the G-torsors corresponding to $\mathcal{F}^+$ and $\mathcal{F}^-$;

(iii) an isomorphism $\mathcal{F}_M^+ \xrightarrow{\sim} \mathrm{Fr}_S^* \mathcal{F}_M^-$, where $\mathcal{F}_M^\pm$ is the M-torsor corresponding to $\mathcal{F}^\pm$.

3.2.2. Disp_1^G as a quotient stack. As noted in [25], the stack Disp_1^G has an explicit realization as a quotient, which shows that Disp_1^G is a quasi-compact smooth alegbraic stack over $\mathbb{F}_p$ of pure dimension 0 with affine diagonal. To get this realization, note that the combination of data (i) and (iii) is the same as a principal K-bundle $\mathcal{E} \to S$, where $K \subset G \times G$ is the following subgroup:

$$K := \{(g,h) \in P_+ \times P_- \,|\, g_M = \mathrm{Fr}(h_M)\}, \tag{3.1}$$

[5]We are talking about the particular case of a general construction of [19] corresponding to the "n-truncated Witt frame" in the sense of [19, Example 2.1.6].

(here g_M, h_M are the images of g, h in M). Moreover, data (ii) is the same as a K-equivariant morphism $\mathcal{E} \to G$, where $(g,h) \in K$ acts on G by

$$x \mapsto hxg^{-1}. \tag{3.2}$$

Thus Disp_1^G identifies with the quotient of the scheme G by the action of K given by (3.2).

3.2.3. The generic locus of Disp_1^G**.** By (3.1), the stabilizer of $1 \in G$ under the K-action given by (3.2) equals the finite group $M(\mathbb{F}_p)$. Moreover, the K-orbit of 1 is open.

By §3.2.2, Disp_1^G is a quotient of the scheme G. The composite morphism

$$\operatorname{Spec} \mathbb{F}_p \xrightarrow{1} G \twoheadrightarrow \mathrm{Disp}_1^G \tag{3.3}$$

will be called the *generic point* of the stack Disp_1^G. The group scheme of automorphisms of this point equals $M(\mathbb{F}_p)$, and the image of (3.3) is an open substack of Disp_1^G, which identifies with the classifying stack of $M(\mathbb{F}_p)$. This open substack will be called the *generic locus* of Disp_1^G and denoted by $\mathrm{Disp}_{1,\mathrm{gen}}^G$. The word "generic" is justified at least if G is connected (which implies that Disp_1^G is irreducible).

3.2.4. Motivation. A pair $(\mathcal{F}^+, \mathcal{F}^-)$ equipped with data (ii) from §3.2.1 is the same as a G-torsor $\mathcal{F}$ equipped with a P^+-structure and a P^--structure. If $G = GL(d)$ this means that the vector bundle ξ corresponding to $\mathcal{F}$ is equipped with two filtrations. As explained at the beginning of [23], the typical example is when ξ is the m-th relative de Rham cohomology of a smooth proper scheme X over S satisfying certain conditions[6]. In this situation the P^--structure corresponds to the Hodge filtration, and the P^+-structure corresponds to the conjugate one.

Note that the Gauss–Manin connection on ξ is not included into the definition of Disp_1^G. However, it is included into the definition of $\mathrm{BT}_{1,\mathbb{F}_p}^G$, which will be given in §4.2.

3.2.5. Reductive case. If G is reductive it is known that the set of points of Disp_1^G is finite; moreover, Theorem 1.6 of [25] gives a complete description of this set (together with the topology on it), and Theorem 1.7 of [25] describes the automorphism group scheme of each point.

3.3. The group scheme Lau_1^G

3.3.1. Definition of Lau_1^G**.** From now on, we assume[7] that $G \in \mathbf{Shim}_1$ (i.e., the action of $\mathbb{G}_m$ on G is 1-bounded). Given an $\mathbb{F}_p$-scheme S and an S-point of Disp_1^G, we will construct a group scheme over S. Let us use the notation

[6] The conditions for the smooth proper morphism $f : X \to S$ are as follows: the sheaves $R^b f_* \Omega^a_{X/S}$ should be locally free, and the Hodge-de Rham spectral sequence for f should degenerate at the E_1 page.

[7] Without this assumption, the definition given below formally makes sense (but is probably useless) if one replaces $\mathfrak{g}_1$ by $\mathfrak{g}_{>0}$.

of §3.2.1. Note that both P^+ and P^- act on $\mathfrak{g}_1$ via M; we have a P^+-equivariant monomorphism $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}$ and a P^--equivariant epimorphism $\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{g}_{\le 0} = \mathfrak{g}_1$. Let $(\mathfrak{g}_1)_{\mathcal{F}^\pm}$ be the vector bundle on S corresponding to $\mathcal{F}^\pm$ and the $P^\pm$-module $\mathfrak{g}_1$. We have the diagram

$$\mathrm{Fr}_S^*(\mathfrak{g}_1)_{\mathcal{F}^-} \xrightarrow{\sim} (\mathfrak{g}_1)_{\mathcal{F}^+} \hookrightarrow \mathfrak{g}_{\mathcal{F}^+} \xrightarrow{\sim} \mathfrak{g}_{\mathcal{F}^-} \twoheadrightarrow (\mathfrak{g}_1)_{\mathcal{F}^-} \tag{3.4}$$

in which the first isomorphism comes from data (iii) of §3.2.1 and the second one from (ii). Consider $(\mathfrak{g}_1)_{\mathcal{F}^-}$ as a commutative restricted Lie $\mathcal{O}_S$-algebra with p-operation[8] (3.4). This restricted Lie $\mathcal{O}_S$-algebra yields a finite locally free group S-scheme of height 1 in the usual way (see [26, Exp.VIIA] or [15, §2]). We denote it by $\mathrm{Lau}^G_{1,S}$. The formation of $\mathrm{Lau}^G_{1,S}$ commutes with base change $S' \to S$. As S varies, the group schemes $\mathrm{Lau}^G_{1,S}$ define a commutative finite flat group scheme over Disp^G_1 of height 1, which we denote by Lau^G_1.

The stack $\mathrm{BT}^G_{1,\mathbb{F}_p}$, which will be defined in §4, will turn out to be a gerbe over Disp^G_1 banded by Lau^G_1, see Theorem 4.4.2.

3.3.2. Restriction of Lau^G_1 to the generic locus of Disp^G_1. Let $\mathrm{Lau}^G_{1,\mathrm{gen}}$ be the fiber of Lau^G_1 over the generic point of Disp^G_1 in the sense of §3.2.3, so $\mathrm{Lau}^G_{1,\mathrm{gen}}$ is a finite group scheme over $\mathbb{F}_p$ equipped with an action of $M(\mathbb{F}_p)$. Then one has a canonical $M(\mathbb{F}_p)$-equivariant isomorphism

$$\mathrm{Lau}^G_{1,\mathrm{gen}} \xrightarrow{\sim} \mathfrak{g}_1 \otimes_{\mathbb{F}_p} \mu_p \tag{3.5}$$

(indeed, in the case of the generic point of Disp^G_1 the torsors $\mathcal{F}^\pm$ are canonically trivial, and the composite map (3.4) is $\mathrm{id}_{\mathfrak{g}_1}$). Less canonically, $\mathrm{Lau}^G_{1,\mathrm{gen}}$ is a direct sum of $\dim \mathfrak{g}_1$ copies of μ_p.

4. Definition of $\mathrm{BT}^G_{1,\mathbb{F}_p}$

4.1. General remarks on smooth algebraic stacks

4.1.1. An easy lemma. Let $\mathrm{Sch}_{\mathbb{F}_p}$ be the category of $\mathbb{F}_p$-schemes and $\mathrm{Sm}_{\mathbb{F}_p}$ the full subcategory of smooth $\mathbb{F}_p$-schemes; unless specified otherwise, we will specify each of these categories with the etale topology. Let Grpds be the (2,1)-category of groupoids.

Let $\mathscr{X}$ be an $\mathbb{F}_p$-stack, i.e., a functor $\mathrm{Sch}^{\mathrm{op}}_{\mathbb{F}_p} \to \mathrm{Grpds}$ satisfying the sheaf property. Then the restriction of $\mathscr{X} : \mathrm{Sch}^{\mathrm{op}}_{\mathbb{F}_p} \to \mathrm{Grpds}$ to $\mathrm{Sm}^{\mathrm{op}}_{\mathbb{F}_p}$ will be denoted by $\mathscr{X}_{\mathrm{Sm}}$. The following variant of Yoneda's lemma is easy and well known.

Lemma 4.1.2. *The functor $\mathscr{X} \mapsto \mathscr{X}_{\mathrm{Sm}}$ becomes fully faithful when restricted to the* (2.1)*-category of* **smooth** *algebraic stacks over* $\mathbb{F}_p$.

This variant of Yoneda's lemma is probably well known. For completeness, we give a proof of a more general statement in Proposition A.0.1 of Appendix A.

[8] This p-operation is *different* from the one induced by the p-operation on $\mathfrak{g}_1$; the latter is zero by formula (2.3).

Remark 4.1.3. One can ask how to reconstruct a smooth algebraic stack $\mathscr{X}$ over $\mathbb{F}_p$ in terms of $\mathscr{X}_{\mathrm{Sm}}$. Abstract nonsense gives the following "answer" (see Proposition A.0.2 and the sentence after it): $\mathscr{X}$ is the sheafified left Kan extension of $\mathscr{X}_{\mathrm{Sm}} : \mathrm{Sm}_{\mathbb{F}_p}^{\mathrm{op}} \to \mathrm{Grpds}$ to $\mathrm{Sch}_{\mathbb{F}_p}^{\mathrm{op}}$.

4.1.4. Example. In this article we will be describing smooth algebraic $\mathbb{F}_p$-stacks by specifying their restriction to $\mathrm{Sm}_{\mathbb{F}_p}^{\mathrm{op}}$. Here is a baby example.

Let $\mathscr{X}$ be the classifying stack of the group scheme

$$\alpha_p := \mathrm{Ker}(\mathrm{Fr} : \mathbb{G}_a \to \mathbb{G}_a), \text{ where } \mathbb{G}_a := (\mathbb{G}_a)_{\mathbb{F}_p}; \tag{4.1}$$

in other words, $\mathscr{X}(S)$ is the groupoid of α_p-torsors on S for the fppf topology. The stack $\mathscr{X}$ is smooth: indeed, the map

$$\mathbb{G}_a \times \mathbb{A}^1 \to \mathbb{A}^1, \quad (a, x) \mapsto x + a^p.$$

defines an action of $\mathbb{G}_a$ on $\mathbb{A}^1$ such that the quotient stack is $\mathscr{X}$. We claim that $\mathscr{X}_{\mathrm{Sm}}$ is the following sheaf of *sets* (rather than groupoids):

$$S \mapsto H^0(S, \Omega^1_{S,\mathrm{exact}}), \text{ where } S \in \mathrm{Sm}_{\mathbb{F}_p} \text{ and } \Omega^1_{S,\mathrm{exact}} := \mathrm{Im}(\mathcal{O}_{S,\mathrm{et}} \xrightarrow{d} \Omega^1_{S,\mathrm{et}}).$$

Indeed, by (4.1), the derived direct image of α_p under the morphism $S_{\mathrm{fppf}} \to S_{\mathrm{et}}$ equals $\mathcal{A}[-1]$, where $\mathcal{A} := \mathrm{Cone}(\mathrm{Fr} : \mathcal{O}_{S,\mathrm{et}} \to \mathcal{O}_{S,\mathrm{et}})$. Finally, smoothness of S implies that $\mathcal{A} = \Omega^1_{S,\mathrm{exact}}$.

4.2. Definition of $\mathrm{BT}_1^G(S)$, where $S \in \mathrm{Sm}_{\mathbb{F}_p}$. We will use the notion of p-curvature of an integrable connection (see [16, §5]). We say that a connection is *p-integrable* if it is integrable and its p-curvature is zero.

4.2.1. Definition. Let $G \in \mathbf{Shim}_1$. Let M and $P^\pm$ be as in §2.3. Let $S \in \mathrm{Sm}_{\mathbb{F}_p}$. Then $\mathrm{BT}_1^G(S)$ is the groupoid of the following data:

(i) a $P^\pm$-torsor $\mathcal{F}^\pm$ on S;

(ii) an isomorphism $\mathcal{F}_G^+ \xrightarrow{\sim} \mathcal{F}_G^-$, where $\mathcal{F}_G^\pm$ is the G-torsor corresponding to $\mathcal{F}^\pm$;

(iii) an isomorphism

$$\mathcal{F}_M^+ \xrightarrow{\sim} \mathrm{Fr}_S^* \mathcal{F}_M^-, \tag{4.2}$$

where $\mathcal{F}_M^\pm$ is the M-torsor corresponding to $\mathcal{F}^\pm$;

(iv) an integrable connection ∇ on $\mathcal{F}^+$ satisfying the following conditions: first, the corresponding connection on $\mathcal{F}_M^+$ should equal the one that comes from (4.2) and the usual connection on a Fr_S-pullback; second, the following *Katz condition* should hold:

$$\text{p-Curv}_\nabla = -\mathrm{KS}_\nabla, \tag{4.3}$$

where p-Curv_∇ is the p-curvature of ∇ and $\mathrm{KS}_\nabla \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega^1_S)$ is the *Kodaira–Spencer*[9] 1-form defined in §4.2.2 below.

[9]The terminology is motivated by the picture from §3.2.4. One can think of data (i)-(ii) as a G-bundle on S equipped with a P^+-structure and a P^--structure. Informally, these are the conjugate filtration and Hodge filtration, respectively. Also informally, we think of ∇ as a Gauss-Manin connection.

Let us explain why (4.3) makes sense. The right-hand side of (4.3) is a section of the sheaf $(\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega^1_S$. By the first condition from (iv), the connection on $\mathcal{F}^+_M$ induced by ∇ is p-integrable, so p-Curv_∇ is a section of $(\mathrm{Fr}_*(\mathfrak{g}_1)_{\mathcal{F}^+})^\nabla \otimes \Omega^1_S$, where $\mathrm{Fr} := \mathrm{Fr}_S$ and $(\mathrm{Fr}_*(\mathfrak{g}_1)_{\mathcal{F}^+})^\nabla$ is the horizontal part of $\mathrm{Fr}_*(\mathfrak{g}_1)_{\mathcal{F}^+}$. But (4.2) induces an isomorphism $\mathrm{Fr}^*(\mathfrak{g}_1)_{\mathcal{F}^-} \xrightarrow{\sim} (\mathfrak{g}_1)_{\mathcal{F}^+}$, so $(\mathrm{Fr}_*(\mathfrak{g}_1)_{\mathcal{F}^+})^\nabla = (\mathfrak{g}_1)_{\mathcal{F}^-}$ and (4.3) makes sense.

4.2.2. The Kodaira–Spencer 1-form. By §4.2.1(ii), $\mathcal{F}^+$ and $\mathcal{F}^-$ induce the same G-torsor, which we denote by $\mathcal{F}$. The connection ∇ on $\mathcal{F}^+$ induces a connection ∇_G on $\mathcal{F}$, and KS_∇ "measures" the failure of ∇_G to preserve the P^--structure on $\mathcal{F}$. More precisely, ∇_G is a section of the Atiyah extension[10]

$$0 \to \mathfrak{g}_{\mathcal{F}} \to \mathcal{A} \to \Theta_S \to 0, \quad \Theta_S := (\Omega^1_S)^*, \tag{4.4}$$

and $\mathrm{KS}_\nabla : \Theta_S \to (\mathfrak{g}/\mathfrak{g}_{\le 0})_{\mathcal{F}^-}$ is the composition of $\nabla_G : \Theta_S \to \mathcal{A}$ and the map

$$\mathcal{A} \to (\mathfrak{g}/\operatorname{Lie}(P^-))_{\mathcal{F}^-} = (\mathfrak{g}/\mathfrak{g}_{\le 0})_{\mathcal{F}^-}$$

that comes from the P^--structure on $\mathcal{F}$. Note that $\mathfrak{g}/\mathfrak{g}_{\le 0} = \mathfrak{g}_1$ by the 1-boundedness assumption.

4.2.3. Remark. The above definition of $\mathrm{BT}_1^G(S)$ makes sense without the 1-boundedness assumption if (4.3) is understood as an equality in $H^0(S, (\mathfrak{g}/\mathfrak{g}_{\le 0})_{\mathcal{F}^-} \otimes \Omega^1_S)$; this equality implies the Griffiths transversality condition $\mathrm{KS}_\nabla \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega^1_S)$ because p-Curv_∇ is in $H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega^1_S)$. However, the 1-boundedness assumption is used in the proof of the theorems formulated in §4.4 below.

4.2.4. On the Katz condition. Condition (4.3) is inspired by Theorem 3.2 of [17]. As far as I understand, the minus sign in (4.3) does not agree[11] with [17, Thm. 3.2], but it agrees with Remark 3.20 of [24], which explains a modern point of view on Theorem 3.2 of [17]. (In [24, Remark 3.20] the sign appears when one computes explicitly the functor $C^\bullet_{Y/S}$, which occurs in the l.h.s of (3.19.2) and (3.19.3).)

4.2.5. Remark. Forgetting the connection ∇ from §4.2.1(iv), one gets for each $S \in \mathrm{Sm}_{\mathbb{F}_p}$ a functor $\mathrm{BT}_1^G(S) \to \mathrm{Disp}_1^G(S)$, where Disp_1^G is as in §3.2.1.

4.3. Functoriality in $G \in \mathbf{Shim}_1$. It is clear that the functor

$$\mathrm{Sm}^{\mathrm{op}}_{\mathbb{F}_p} \to \mathrm{Grpds}, \quad S \mapsto \mathrm{BT}_1^G(S). \tag{4.5}$$

from §4.2.1 depends functorially on $G \in \mathbf{Shim}_1$, where $\mathbf{Shim}_1$ is the (2,1)-category from §2.2.

[10] Recall that in terms of the principal G-bundle $E \to S$ corresponding to $\mathcal{F}$, the sheaf $\mathcal{A}$ from (4.4) is the sheaf of G-equivariant vector fields on E.

[11] Theorem 3.2 of [17] involves $(-1)^{b+1}$, where b is the number of the cohomology group. The case relevant for us is $b = 1$.

4.4. Formulation of the main theorems

Theorem 4.4.1. *There exists a smooth algebraic stack* $\mathrm{BT}^G_{1,\mathbb{F}_p}$ *over* $\mathbb{F}_p$ *whose restriction to* $\mathrm{Sm}^{\mathrm{op}}_{\mathbb{F}_p}$ *is the functor* (4.5).

By Lemma 4.1.2, $\mathrm{BT}^G_{1,\mathbb{F}_p}$ is unique. Combining §4.2.5 and Lemma 4.1.2, we get a morphism

$$\mathrm{BT}^G_{1,\mathbb{F}_p} \to \mathrm{Disp}^G_1 \tag{4.6}$$

Theorem 4.4.2. *The morphism* (4.6) *is an fppf gerbe banded by the group scheme* Lau^G_1 *from* §3.3.1.

Theorems 4.4.1 and 4.4.2 will be proved in §6.3.

Corollary 4.4.3. $\mathrm{BT}^G_{1,\mathbb{F}_p}$ *is a quasi-compact smooth algebraic stack over* $\mathbb{F}_p$ *of pure dimension* 0 *with affine diagonal.*

Proof. By Theorems 4.4.1–4.4.2, this follows from similar properties of Disp^G_1 (see §3.2.2). $\square$

4.4.4. Remark. As explained in §B.0.2 of Appendix B, Theorem 4.4.2 implies that the morphism (4.6) is smooth.

4.4.5. A simple example. Let G be $\mathbb{G}_a$ equipped with the usual action of $\mathbb{G}_m$. In this case $\mathrm{Disp}^G_1 = \mathrm{Spec}\,\mathbb{F}_p$ and $\mathrm{Lau}^G_1 = \mu_p$. Since $H^i_{\mathrm{fppf}}(\mathrm{Spec}\,\mathbb{F}_p, \mu_p) = 0$ for all i, Theorem 4.4.2 implies that $\mathrm{BT}^G_{1,\mathbb{F}_p}$ is canonically isomorphic to the classifying stack of μ_p over $\mathbb{F}_p$.

4.5. A conjecture about $\mathrm{BT}^G_{1,\mathbb{F}_p}$ in the case $G = GL(d)$

4.5.1. Some results of E. Lau. Let d, d' be integers such that $0 \le d' \le d$. Let $n \in \mathbb{N}$. Let G be the group scheme $GL(d)$ equipped with the $\mathbb{G}_m$-action corresponding to the composite map

$$\mathbb{G}_m \to GL(d) \to PGL(d), \tag{4.7}$$

where the first map takes t to a diagonal matrix with d' diagonal entries equal to t and $d - d'$ diagonal entries equal to 1.

Let $\mathrm{BT}^{d,d'}_n$ be the stack of n-truncated Barsotti-Tate groups of height d and dimension d'. Using a covariant version of Dieudonné theory[12], E. Lau defined in [18] a canonical morphism $\mathrm{BT}^{d,d'}_n \otimes \mathbb{F}_p \to \mathrm{Disp}^G_n$. Moreover, according to Theorem B of [18], this morphism is a gerbe banded by a commutative locally free finite group scheme over Disp^G_n. We denote this group scheme by $\mathrm{Lau}^{G,\mathrm{true}}_n$.

According to [18], the group scheme $\mathrm{Lau}^{G,\mathrm{true}}_n$ is infinitesimal and has order $p^{nd'(d-d')}$ (see Theorem B of [18] and the paragraph after it). Moreover, Remark 4.8 of [18] describes the restriction of $\mathrm{Lau}^{G,\mathrm{true}}_n$ to the generic locus of Disp^G_n; in the case $n = 1$, it is canonically isomorphic to the restriction of Lau^G_1 (which was described in §3.3.2).

[12] Those who prefer *contra*variant Dieudonné theory should replace d' by $d-d'$ in the definition of (4.7).

Proposition 4.5.2. *The isomorphism between the restrictions of* $\mathrm{Lau}_1^{G,\mathrm{true}}$ *and* Lau_1^G *to the generic locus of* Disp_1^G *extends*[13] *to an isomorphism over the whole* Disp_1^G*.*

Proof. Follows from [10, Thm. 4.4.2(ii)] and [10, § 9.2.2]. □

Combining Proposition 4.5.2 with Theorem 4.4.1, we see that $\mathrm{BT}^G_{1,\mathbb{F}_p}$ and $\mathrm{BT}_1^{d,d'} \otimes \mathbb{F}_p$ are gerbes over Disp_1^G banded by the same group scheme Lau_1^G.

Conjecture 4.5.3. *These two gerbes are isomorphic.*

As far as I understand, Conjecture 4.5.3 has already been proved in [12].

5. Analyzing the conditions for the connection ∇

The definition of $\mathrm{BT}_1^G(S)$ from §4.2.1 involves a connection ∇, which has to satisfy certain conditions. In this section we show that these conditions are affine-$\mathbb{F}_p$-linear; for a precise statement, see Theorem 5.2.6 below. In §6 we will deduce Theorems 4.4.1 and 4.4.2 from Theorem 5.2.6.

5.1. The sheaves $\mathcal{T}$ and $\mathcal{T}'$

5.1.1. The category $\mathrm{Sm}\,/\,\mathrm{Disp}_1^G$**.** Let $\mathrm{Sm}\,/\,\mathrm{Disp}_1^G$ be the category of pairs (S, f), where $S \in \mathrm{Sm}_{\mathbb{F}_p}$ and $f \in \mathrm{Disp}_1^G(S)$. We equip $\mathrm{Sm}\,/\,\mathrm{Disp}_1^G$ with the etale topology.

5.1.2. The sheaf $\mathcal{T}$**.** Let $(S, f) \in \mathrm{Sm}\,/\,\mathrm{Disp}_1^G$, so $f \in \mathrm{Disp}_1^G(S)$ is given by data (i)–(iii) from §4.2.1. We will use the notation of §4.2.1 for these data (i.e., $\mathcal{F}^\pm$, $\mathcal{F}^\pm_M$, etc.). Let $\mathcal{T}(S, f)$ be the fiber of the functor $\mathrm{BT}_1^G(S) \to \mathrm{Disp}_1^G(S)$ over $f \in \mathrm{Disp}_1^G(S)$. This fiber is a *set* rather than a groupoid; namely, $\mathcal{T}(S, f)$ is the set of connections ∇ on the P^+-bundle $\mathcal{F}^+$ satisfying certain conditions. The conditions are as follows:

(a) the connection on $\mathcal{F}^+_M$ induced by ∇ is equal to the one that comes from the isomorphism $\mathcal{F}^+_M \xrightarrow{\sim} \mathrm{Fr}_S^* \mathcal{F}^-_M$;
(b) ∇ is integrable;
(c) ∇ satisfies the Katz condition, i.e.,

$$\text{p-Curv}_\nabla = -\,\mathrm{KS}_\nabla, \tag{5.1}$$

where $\mathrm{KS}_\nabla \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega^1_S)$ is the Kodaira–Spencer 1-form defined in §4.2.2 and $\text{p-Curv}_\nabla \in H^0(S, (\mathrm{Fr}_*(\mathfrak{g}_1)_{\mathcal{F}^+})^\nabla \otimes \Omega^1_S) = H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega^1_S)$ is the p-curvature of ∇ (see §4.2.1 for details).

The assignment $(S, f) \mapsto \mathcal{T}(S, f)$ is a sheaf on $\mathrm{Sm}\,/\,\mathrm{Disp}_1^G$ (with respect to the etale topology).

[13] Such an extension is unique. Indeed, by §3.2.2, Disp_1^G is a quotient of the reduced irreducible scheme $G = GL(n)$.

5.1.3. The sheaf $\mathcal{T}' \supset \mathcal{T}$. We keep the notation of §5.1.2. Let $\mathcal{T}'(S, f)$ be the set of connections ∇ on $\mathcal{F}^+$ satisfying condition (a) from §5.1.2. Then $\mathcal{T}'$ is a sheaf of sets on $\mathrm{Sm} / \mathrm{Disp}_1^G$, and $\mathcal{T} \subset \mathcal{T}'$.

5.2. $\mathcal{T}$ and $\mathcal{T}'$ as torsors

5.2.1. The sheaf $\mathcal{A}'$. Define a sheaf of $\mathbb{F}_p$-vector spaces $\mathcal{A}'$ on $\mathrm{Sm} / \mathrm{Disp}_1^G$ as follows:

$$\mathcal{A}'(S, f) := H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^+} \otimes \Omega_S^1). \tag{5.2}$$

Then $\mathcal{A}'$ acts on $\mathcal{T}'$ in the usual way (adding a 1-form to a connection). Moreover, $\mathcal{T}'$ is an $\mathcal{A}'$-torsor.

Using the isomorphism $\mathcal{F}^+ \xrightarrow{\sim} \mathrm{Fr}_S^* \mathcal{F}^-$ (which is a part of the data), we can rewrite (5.2) as

$$\mathcal{A}'(S, f) = H^0(S, \mathrm{Fr}_S^*(\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega_S^1) = H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes (\mathrm{Fr}_S)_* \Omega_S^1). \tag{5.3}$$

Note that $\mathrm{Fr}_S : S \to S$ induces the identity on the underlying set of S, so $(\mathrm{Fr}_S)_* \Omega_S^1$ is just the sheaf Ω_S^1 equipped with the Frobenius-twisted $\mathcal{O}_S$-action.

5.2.2. The next goals. In §5.2.4 we will define a subsheaf of $\mathbb{F}_p$-vector spaces $\mathcal{A} \subset \mathcal{A}'$. Then we will formulate Theorem 5.2.6, which says that $\mathcal{T}$ is an $\mathcal{A}$-torsor.

5.2.3. The maps φ, $\tilde{\varphi}$, C, and $\tilde{C}$. Let $\varphi : (\mathfrak{g}_1)_{\mathcal{F}^-} \to (\mathfrak{g}_1)_{\mathcal{F}^-}$ be the p-linear map corresponidng to the $\mathcal{O}_S$-linear map $\mathrm{Fr}_S^*(\mathfrak{g}_1)_{\mathcal{F}^-} \to (\mathfrak{g}_1)_{\mathcal{F}^-}$ from formula (3.4). The latter is the composition

$$\mathrm{Fr}_S^*(\mathfrak{g}_1)_{\mathcal{F}^-} \xrightarrow{\sim} (\mathfrak{g}_1)_{\mathcal{F}^+} \hookrightarrow \mathfrak{g}_{\mathcal{F}^+} \xrightarrow{\sim} \mathfrak{g}_{\mathcal{F}^-} \twoheadrightarrow (\mathfrak{g}_1)_{\mathcal{F}^-} .$$

The map φ induces a p-linear map

$$\tilde{\varphi} : (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes (\mathrm{Fr}_S)_* \Omega_S^1 \to (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega_S^1 ;$$

namely, $\tilde{\varphi}$ is the tensor product of $\varphi : (\mathfrak{g}_1)_{\mathcal{F}^-} \to (\mathfrak{g}_1)_{\mathcal{F}^-}$ and the identity[14] map

$$(\mathrm{Fr}_S)_* \Omega_S^1 \to \Omega_S^1$$

(the latter is p-linear).

We have the $\mathcal{O}_S$-submodule $((\mathrm{Fr}_S)_* \Omega_S^1)_{\mathrm{closed}} \subset (\mathrm{Fr}_S)_* \Omega_S^1$ and the Cartier operator[15]

$$C : ((\mathrm{Fr}_S)_* \Omega_S^1)_{\mathrm{closed}} \to \Omega_S^1,$$

which is $\mathcal{O}_S$-linear and surjective. C induces a surjective $\mathcal{O}_S$-linear map

$$\tilde{C} : (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes ((\mathrm{Fr}_S)_* \Omega_S^1)_{\mathrm{closed}} \to (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega_S^1, \quad \tilde{C} := \mathrm{id} \otimes C$$

[14] See the end of §5.2.1.

[15] In Lemma 5.4.2 we will recall the explicit description of the Cartier operator assuming that S is equipped with a coordinate system.

5.2.4. The sheaf $\mathcal{A}$. Let us use formula (5.3) for $\mathcal{A}'$. Define a sheaf of $\mathbb{F}_p$-vector spaces $\mathcal{A} \subset \mathcal{A}'$ as follows:

$$\mathcal{A}(S, f) = \{\omega \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes ((\mathrm{Fr}_S)_* \Omega^1_S)_{\mathrm{closed}} \,|\, \tilde{C}(\omega) = \tilde{\varphi}(\omega)\}, \tag{5.4}$$

where $\tilde{C}$ and $\tilde{\varphi}$ were defined in §5.2.3.

5.2.5. Remarks about $\mathcal{A}$

(i) By definition, $\mathcal{A}$ is the kernel of a certain morphism of sheaves. Later we will see that this morphism is surjective, see Lemma 5.3.4.

(ii) The "true nature" of $\mathcal{A}$ will be explained in Proposition 6.2.3.

Theorem 5.2.6. *The subsheaf $\mathcal{T} \subset \mathcal{T}'$ is stable under the action of $\mathcal{A} \subset \mathcal{A}'$. Moreover, $\mathcal{T}$ is an $\mathcal{A}$-torsor.*

The proof will be given in §5.3, 5.4.

Remark 5.2.7. In particular, Theorem 5.2.6 says that for every $S \in \mathrm{Sm}_{\mathbb{F}_p}$, every object of $\mathrm{Disp}_1^G(S)$ admits a lift to $\mathrm{BT}_1^G(S)$ etale-locally on S. As explained in §4.4.4, this is a part of Theorem 4.4.2, which we want to prove.

5.3. Proof of Theorem 5.2.6. Given $(S, f) \in \mathrm{Sm} / \mathrm{Disp}_1^G$, let $\mathcal{T}_S$ be the restriction of $\mathcal{T}$ to S_{et}; similarly, we have $\mathcal{T}'_S$, $\mathcal{A}_S$, $\mathcal{A}'_S$. The problem is to prove that the subsheaf $\mathcal{T}_S \subset \mathcal{T}'_S$ is an $\mathcal{A}_S$-torsor.

Lemma 5.3.1. *Zariski-locally on S, there exists a connection ∇ on $\mathcal{F}^+$ satisfying conditions* (a) *and* (b) *from* §5.1.2.

Proof. Zariski-locally, there exists an isomorphism between $\mathcal{F}^+$ and the P^+-torsor induced from the M-torsor $\mathcal{F}^+_M$ via the inclusion $M \hookrightarrow P^+$. Choose such an isomorphism and take ∇ to be induced by the canonical connection on $\mathcal{F}^+_M = \mathrm{Fr}_S^* \mathcal{F}^-_M$. □

Lemma 5.3.2. *Let ∇ be as in Lemma 5.3.1. A connection $\tilde{\nabla}$ on $\mathcal{F}^+$ satisfies conditions* (a) *and* (b) *from* §5.1.2 *if and only if $\tilde{\nabla} = \nabla + \omega$, where $\omega \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^+} \otimes \Omega^1_S)$ and $d\omega = 0$.*

Proof. Write $\tilde{\nabla} = \nabla + \omega$, where $\omega \in H^0(S, (\mathfrak{g}_{\ge 0})_{\mathcal{F}^+} \otimes \Omega^1_S)$. Condition (a) means that $\omega \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^+} \otimes \Omega^1_S)$. Condition (b) is equivalent to the Maurer-Cartan equation for ω. Since $[\mathfrak{g}_1, \mathfrak{g}_1] = 0$, this is just the equation $d\omega = 0$. □

As noted in §5.2.1, $H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^+} \otimes \Omega^1_S) = H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes (\mathrm{Fr}_S)_* \Omega^1_S)$. So the ω from Lemma 5.3.2 is in $H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes ((\mathrm{Fr}_S)_* \Omega^1_S)_{\mathrm{closed}})$.

Lemma 5.3.3. *In the situation of Lemma 5.3.2, we have*

$$\mathrm{KS}_{\tilde{\nabla}} - \mathrm{KS}_{\nabla} = \tilde{\varphi}(\omega), \tag{5.5}$$

$$\text{p-Curv}_{\tilde{\nabla}} - \text{p-Curv}_{\nabla} = -\tilde{C}(\omega), \tag{5.6}$$

where $\tilde{\varphi} : (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes (\mathrm{Fr}_S)_* \Omega^1_S \to (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega^1_S$ and $\tilde{C} : (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes ((\mathrm{Fr}_S)_* \Omega^1_S)_{\mathrm{closed}} \to (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega^1_S$ are as in §5.2.3.

Formula (5.5) immediately follows from the definition of $\tilde{\varphi}$. A proof of (5.6) will be given in §5.4.

Theorem 5.3 follows from (5.5) and (5.6) and the next lemma.

Lemma 5.3.4. *The map* $(\mathfrak{g}_1)_{\mathcal{F}^-} \otimes ((\mathrm{Fr}_S)_*\Omega^1_S)_{\mathrm{closed}} \xrightarrow{\tilde{C}-\tilde{\varphi}} (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega^1_S$ *is a surjective morphism of etale sheaves of* $\mathbb{F}_p$*-vector spaces.*

Proof. Let $\mathcal{E}_1 := (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes ((\mathrm{Fr}_S)_*\Omega^1_S)_{\mathrm{closed}}$, $\mathcal{E}_2 := (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega^1_S$; these are finitely generated locally free $\mathcal{O}_S$-modules. Let E_1, E_2 be the corresponding vector bundles; these are schemes over S (namely, E_i is the spectrum of the symmetric algebra of $\mathcal{E}_i^*$). Recall that $\tilde{C} : \mathcal{E}_1 \to \mathcal{E}_2$ is a surjective $\mathcal{O}_S$-linear map and $\tilde{\varphi} : \mathcal{E}_1 \to \mathcal{E}_2$ is p-linear. So the morphism of S-schemes $E_1 \xrightarrow{\tilde{C}-\tilde{\varphi}} E_2$ is smooth. It is also surjective: indeed, the image is an open subgroup scheme of E_2, so it has to be equal to E_2. Therefore every section of E_2 etale-locally admits a lift to a section of E_1. □

5.4. Proof of formula (5.6). A proof will be given in §5.4.4; it is parallel to that of Propostion 7.1.2 of [17]. Let us first recall some facts used in the proof.

5.4.1. Recollections. Let $\mathbb{F}_p\langle x, y\rangle$ be the free associative $\mathbb{F}_p$-algebra on x, y. Let us define $J \in \mathbb{F}_p\langle x, y\rangle$ by $J(x, y) := (x + y)^p - x^p - y^p$. We have $J = J_1 + \ldots + J_{p-1}$, where J_i is homogeneous of degree i in y. Jacobson proved that each J_i belongs to the free Lie algebra on x, y (which is a Lie subalgebra of $\mathbb{F}_p\langle x, y\rangle$) and that

$$J_1(x, y) = \mathrm{ad}_x^{p-1}(y). \tag{5.7}$$

Hochschild's proof of these results can be found in [6, p. 199-200]. Since J_i is a Lie polynomial in x, y, which is homogeneous of degree i in y, we see that

$$J_i \in [\mathfrak{a}, \mathfrak{a}] \quad \text{for } i > 1, \tag{5.8}$$

where $\mathfrak{a} \subset \mathbb{F}_p\langle x, y\rangle$ is the Lie subalgebra generated by the elements $\mathrm{ad}_x^j(y)$, $j \geq 0$.

We will also need the following coordinate description of the Cartier operator

$$C : H^0(S, \Omega^1_S)_{\mathrm{closed}} \to H^0(S, \Omega^1_S).$$

Lemma 5.4.2. *Let* S *be a scheme etale over* $\mathrm{Spec}\, \mathbb{F}_p[x_1, \ldots, x_n]$ *and* $\omega \in H^0(S, \Omega^1_S)_{\mathrm{closed}}$. *Write* $\omega = \sum_i f_i \cdot dx_i$, $C(\omega) = \sum_i h_i \cdot dx_i$, *where* $f_i, h_i \in H^0(S, \mathcal{O}_S)$. *Then*

$$h_i^p = -\partial_i^{p-1}(f_i), \quad \textit{where } \partial_i := \frac{\partial}{\partial x_i}. \tag{5.9}$$

This lemma is well known, see formula (7.1.2.6) of [17]; moreover, P. Cartier used (5.9) to define C, see p. 200 and p. 202–203 of [6]. We prove the lemma for completeness.

Proof of Lemma 5.4.2. By Cartier's isomorphism, the space of closed 1-forms is generated by locally exact 1-forms and by

$$g^p \cdot x_i^{p-1} dx_i, \quad g \in H^0(S, \mathcal{O}_S), \quad 1 \leq i \leq n. \tag{5.10}$$

If ω is given by (5.10) then $C(\omega) = g \cdot dx_i$, and (5.9) holds because

$$\partial_i^{p-1}(g^p \cdot x_i^{p-1}) = (p-1)! \cdot g^p = -g^p$$

by Wilson's theorem. On the other hand, if $\omega = \sum\limits_i f_i \cdot dx_i$ is exact then $C(\omega) = 0$, and the problem is to show that $\partial_i^{p-1}(f_i) = 0$. Indeed, if $\omega = du$ then $f_i = \partial_i(u)$ and we have $\partial_i^{p-1}(f_i) = \partial_i^p(u) = 0$. □

Remark 5.4.3. Formula 5.10 tells us that in the situation of Lemma 5.4.2, the function $\partial_i^{p-1}(f_i)$ is a p-th power. Here is a direct proof of this. It suffices to show that $\partial_j \partial_i^{p-1}(f_i) = 0$ for all j. But the equality $d\omega = 0$ means that $\partial_j(f_i) = \partial_i(f_j)$, so $\partial_j \partial_i^{p-1}(f_i) = \partial_i^p(f_j) = 0$.

5.4.4. Proof of formula (5.6)**.** We can assume that S is equipped with an etale morphism to $\operatorname{Spec} \mathbb{F}_p[x_1, \ldots, x_n]$. Write $\omega \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^+} \otimes \Omega_S^1)_{\mathrm{closed}}$ as

$$\omega = \sum_i f_i \otimes dx_i\,, \text{ where } f_i \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^+}).$$

We have $(\mathfrak{g}_1)_{\mathcal{F}^+} = \mathrm{Fr}^* \mathcal{E}$, where $\mathcal{E} = (\mathfrak{g}_1)_{\mathcal{F}^-}$; so $\mathcal{E}$ is the horizontal part of $\mathrm{Fr}_*(\mathfrak{g}_1)_{\mathcal{F}^+}$. The problem is to prove that

$$\text{p-Curv}_{\nabla+\omega} - \text{p-Curv}_{\nabla} = -\tilde{C}(\omega), \tag{5.11}$$

where $\tilde{C} : H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^+} \otimes \Omega_S^1)_{\mathrm{closed}} = H^0(S, \mathcal{E} \otimes (\mathrm{Fr}_* \Omega_S^1)_{\mathrm{closed}}) \to H^0(S, \mathcal{E} \otimes \Omega_S^1)$ is induced by $C : (\mathrm{Fr}_* \Omega_S^1)_{\mathrm{closed}} \to \Omega_S^1$. Write $\tilde{C}(\omega) = \sum\limits_i h_i \otimes dx_i$, where $h_i \in H^0(S, \mathcal{E})$, then Lemma 5.4.2 implies that

$$\mathrm{Fr}^*(h_i) = -\nabla_i^{p-1}(f_i), \quad \text{where } \nabla_i := \nabla_{\frac{\partial}{\partial x_i}}. \tag{5.12}$$

Let us now compute the l.h.s. of (5.11). By §2.3.3, $[\mathfrak{g}_1, \mathfrak{g}_1] = 0$ and the p-operation on $\mathfrak{g}_1$ is zero. So by (5.7), (5.8), we get $(\nabla_i + f_i)^p - \nabla_i^p = \nabla_i^{p-1}(f_i)$. This means that the l.h.s. of (5.11) equals $-\sum\limits_i h_i \otimes dx_i$, where the h_i's are as in (5.12). □

6. Proof of Theorems 4.4.1 and 4.4.2

In §6.3 we will see that Theorems 4.4.1 and 4.4.2 easily follow from Theorem 5.2.6 and a result of Artin–Milne [1], which we are going to recall now.

6.1. A result of Artin–Milne

6.1.1. The setting. Let S be a smooth $\mathbb{F}_p$-scheme and H a finite flat commutative group scheme over S. Assume that H has height 1 (i.e., is killed by Frobenius). We have a morphism of sites $\pi : S_{\mathrm{fppf}} \to S_{\mathrm{et}}$. Artin and Milne described the sheaves $R^q\pi_*H$.

To formulate their result, we need some notation. Let $\mathfrak{h} := \mathrm{Lie}(H)$; this is a finitely generated locally free $\mathcal{O}_S$-module. The p-operation on $\mathfrak{h}$ is a p-linear map $\varphi : \mathfrak{h} \to \mathfrak{h}$. Similarly to §5.2.3, one defines a p-linear map $\tilde{\varphi} : \mathfrak{h} \otimes (\mathrm{Fr}_S)_*\Omega^1_S \to \mathfrak{h} \otimes \Omega^1_S$ and a surjective $\mathcal{O}_S$-linear map $\tilde{C} : \mathfrak{h} \otimes ((\mathrm{Fr}_S)_*\Omega^1_S)_{\mathrm{closed}} \to \mathfrak{h} \otimes \Omega^1_S$ (the definition of $\tilde{\varphi}$ uses φ, and the definition of $\tilde{C}$ uses the Cartier operator C).

Proposition 6.1.2. *In the situation of §6.1.1 one has a canonical exact sequence*

$$0 \to R^1\pi_*H \xrightarrow{f} \mathfrak{h} \otimes ((\mathrm{Fr}_S)_*\Omega^1_S)_{\mathrm{closed}} \xrightarrow{\tilde{C}-\tilde{\varphi}} \mathfrak{h} \otimes \Omega^1_S \to 0 \tag{6.1}$$

of sheaves on S_{et}, which is functorial with respect to H and with respect to base changes $S' \to S$; moreover, $R^q\pi_*H = 0$ if $q > 1$.

This is Proposition 2.4 of [1]. The construction of the map f from (6.1) will be recalled in §6.1.5.

6.1.3. Example. Let $H = (\alpha_p)_S$. In this case Propostion 6.1.2 says that

$$R\pi_*H = ((\mathrm{Fr}_S)_*\Omega^1_S)_{\mathrm{exact}}[-1].$$

Since Fr_S induces the identity functor $S_{\mathrm{et}} \to S_{\mathrm{et}}$, we can rewrite this as $R\pi_*H = \Omega^1_{S,\mathrm{exact}}[-1]$. This is well known (and explained at the end of §4.1.4).

6.1.4. Remarks

(i) The equality $R^0\pi_*H = 0$ is clear because S is reduced and the reduced part of the fiber of H over each point of S is zero.

(ii) The proof of the equality $R^q\pi_*H = 0$ for $q > 1$ given in [1] works for *any* scheme S and *any* finite locally free commutative group scheme over S.

(iii) In a particular situation, surjectivity of the map $\tilde{C} - \tilde{\varphi}$ from (6.1) was proved in Lemma 5.3.4. The same argument works in general.

6.1.5. Interpretation and noncommutative generalization of (6.1). An element of $H^1_{\mathrm{fppf}}(S,H)$ is an isomorphism class of a principal H-bundle $E \to S$. Since H is killed by Frobenius, the geometric Frobenius $F : E \to \mathrm{Fr}^*_S E$ factors as $E \to S \xrightarrow{\sigma} \mathrm{Fr}^*_S E$. The section σ trivializes the bundle $\mathrm{Fr}^*_S E$. On the other hand, any Fr_S-pullback (e.g., $\mathrm{Fr}^*_S E$) is equipped with a p-integrable connection. A connection ∇ on the trivialized $\mathrm{Fr}^*_S H$-bundle $\mathrm{Fr}^*_S E$ is given by an element $\omega \in H^0(S, \mathrm{Fr}^*_S \mathfrak{h} \otimes \Omega^1_S) = H^0(S, \mathfrak{h} \otimes (\mathrm{Fr}_S)_*\Omega^1_S)$. Moreover, integrability of ∇ is equivalent to the Maurer–Cartan equation for ω; since H is commutative, this equation just means that $\omega \in H^0(S, \mathfrak{h} \otimes ((\mathrm{Fr}_S)_*\Omega^1_S))_{\mathrm{closed}})$. Thus we get a map

$$H^1_{\mathrm{fppf}}(S,H) \to H^0(S, \mathfrak{h} \otimes ((\mathrm{Fr}_S)_*\Omega^1_S))_{\mathrm{closed}}).$$

This map and similar maps for all schemes etale over S give rise to the map f from (6.1). (This is essentially the description of f from [1, §2.7] although the word "connection" is not used there).

It is easy to check[16] that the inclusion $\operatorname{Im} f \subset \operatorname{Ker}(\tilde{C} - \tilde{\varphi})$ (which is a part of Proposition 6.1.2) just means that the p-curvature of ∇ is zero. So the exact sequence (6.1) essentially says that $H^1_{\mathrm{fppf}}(S, H)$ *canonically identifies (via* Fr_S*-pullback) with the set of* p*-integrable connections on the trivial* H*-bundle on* S. One can prove that this statement remains valid without assuming H commutative; we will not need this fact.

6.2. The classifying stack of Lau_1^G

6.2.1. The sheaf $\mathcal{B}$. Let $\mathrm{Sch}\,/\,\mathrm{Disp}_1^G$ be the category of pairs (S, f), where S is a scheme and $f : S \to \mathrm{Disp}_1^G$ is a morphism. Let $\mathrm{Sm}\,/\,\mathrm{Disp}_1^G$ be the full subcategory of $\mathrm{Sch}\,/\,\mathrm{Disp}_1^G$ formed by pairs (S, f) such that $S \in \mathrm{Sm}_{\mathbb{F}_p}$. We equip $\mathrm{Sch}\,/\,\mathrm{Disp}_1^G$ and $\mathrm{Sm}\,/\,\mathrm{Disp}_1^G$ with the etale topology.

In §3.3.1 we defined a commutative finite flat group scheme Lau_1^G over Disp_1^G; this group scheme has height 1. By definition, the classifying stack of Lau_1^G is the stack of Picard groupoids on $\mathrm{Sch}\,/\,\mathrm{Disp}_1^G$ whose sections over $(S, f) \in \mathrm{Sch}\,/\,\mathrm{Disp}_1^G$ are $(f^* \mathrm{Lau}_1^G)$-torsors. Let $\mathcal{B}$ be the restriction of this stack to $\mathrm{Sm}\,/\,\mathrm{Disp}_1^G$. For any $(S, f) \in \mathrm{Sm}\,/\,\mathrm{Disp}_1^G$ one has $H^0(S, f^* \mathrm{Lau}_1^G) = 0$ (because S is reduced), so the groupoid of sections of $\mathcal{B}$ over (S, f) is discrete. Therefore $\mathcal{B}$ is just a sheaf of abelian groups.

6.2.2. Remark. Restricting from $\mathrm{Sch}\,/\,\mathrm{Disp}_1^G$ to $\mathrm{Sm}\,/\,\mathrm{Disp}_1^G$ does not lead to loss of information: this follows from Lemma 4.1.2 and the fact that the classifying stack of Lau_1^G is smooth over $\mathbb{F}_p$. The latter follows from §B.0.1 of Appendix B because by §3.2.2, Disp_1^G is smooth over $\mathbb{F}_p$.

Proposition 6.2.3. *The sheaf* $\mathcal{B}$ *from* §6.2.1 *is canonically isomorphic to the sheaf* $\mathcal{A}$ *from* §5.2.4.

Proof. Follows from Proposition 6.1.2 and the definitions of $\mathcal{A}$ and Lau_1^G. $\square$

6.3. Proof of Theorems 4.4.1 and 4.4.2. In §5.1.2 we introduced the notation $\mathcal{T}$ for $\mathrm{BT}^G_{1,\mathbb{F}_p}$ viewed as a sheaf of sets on $\mathrm{Sm}\,/\,\mathrm{Disp}_1^G$. The problem is to show that $\mathcal{T}$ is locally isomorphic to the sheaf $\mathcal{B}$ from §6.2.1. By Proposition 6.2.3, $\mathcal{B} = \mathcal{A}$. By Theorem 5.2.6, $\mathcal{T}$ is locally isomorphic to $\mathcal{A}$. $\square$

A. Generalities on stacks

Let P be a smooth-local property of schemes. Let Sch be the category of all schemes and Sch_P the category of schemes satisfying P; we equip Sch and Sch_P with the etale topology. Let Stacks_P be the (2,1)-category of algebraic stacks satisfying P (this makes sense because P is smooth-local). Let Grpds be the (2,1)-category of groupoids.

[16] The verification is parallel to §5.4.4; the only difference is that the p-operation on $\mathfrak{h}$ is not assumed to be zero, while the restricted Lie algebra $(\mathfrak{g}_1)_{\mathcal{F}^+}$ from §5.4.4 has zero p-operation.

Given a stack $\mathscr{Y} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Grpds}$, let $\mathscr{Y}_P$ denote the restriction of $\mathscr{Y}$ to $\mathrm{Sch}_P^{\mathrm{op}}$.

Proposition A.0.1. *Let* $\mathscr{X} \in \mathrm{Stacks}_P$*. Then the functor* $\mathrm{Mor}(\mathscr{X}, \mathscr{Y}) \to \mathrm{Mor}(\mathscr{X}_P, \mathscr{Y}_P)$ *is an equivalence for any stack* $\mathscr{Y} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Grpds}$.

Proof. Let $\pi : X \to \mathscr{X}$ be a smooth surjective morphism with X being a scheme. Let $X_\bullet$ be the Čech nerve of π. Note that each X_n is an algebraic space with property P. We have

$$\mathrm{Mor}(\mathscr{X}, \mathscr{Y}) = \mathrm{Tot}(\mathrm{Mor}(X_\bullet, \mathscr{Y})), \quad \mathrm{Mor}(\mathscr{X}_P, \mathscr{Y}_P) = \mathrm{Tot}(\mathrm{Mor}((X_\bullet)_P, \mathscr{Y}_P)).$$

Thus we have reduced the problem to the case where $\mathscr{X}$ is an algebraic space.

Running the same argument again, we reduce to the case where $\mathscr{X} \in \mathrm{Sch}_P$, which is covered by Yoneda's lemma. □

Our next goal is to reformulate Proposition A.0.1 in terms of a certain morphism of toposes. Our approach is influenced by §4.10 of Exposé IV of [27] and by the more sophisticated Proposition A.0.4 of [11] (which goes back to A. Mathew and is about *derived* stacks).

Let $\widetilde{\mathrm{Sch}}$ (respectively $\widehat{\mathrm{Sch}}$) be the category of presheaves (respectively sheaves) of groupoids on Sch. Similaly, we have $\widetilde{\mathrm{Sch}_P}$ and $\widehat{\mathrm{Sch}_P}$. Let $\tilde{g}_* : \widetilde{\mathrm{Sch}} \to \widetilde{\mathrm{Sch}_P}$, $\hat{g}_* : \widehat{\mathrm{Sch}} \to \widehat{\mathrm{Sch}_P}$ be the restriction functors. They have left adjoints $\tilde{g}^*$ and $\hat{g}^*$. The functor $\tilde{g}^* : \widetilde{\mathrm{Sch}_P} \to \widetilde{\mathrm{Sch}}$ is the *left Kan extension*. The functor $\hat{g}^* : \widehat{\mathrm{Sch}_P} \to \widehat{\mathrm{Sch}}$ is the *sheafified left Kan extension*, i.e., the composition

$$\widehat{\mathrm{Sch}_P} \hookrightarrow \widetilde{\mathrm{Sch}_P} \xrightarrow{\tilde{g}^*} \widetilde{\mathrm{Sch}} \to \widehat{\mathrm{Sch}}.$$

Each of the adjoint pairs $(\tilde{g}^*, \tilde{g}_*)$ and $(\hat{g}^*, \hat{g}_*)$ defines a morphism of toposes.

Since the functor $\mathrm{Sch}_P \to \mathrm{Sch}$ is fully faithful, so is $\tilde{g}^*$; equivalently, the unit of the adjunction $\mathrm{id} \to \tilde{g}_*\tilde{g}^*$ is an isomorphism. The same is true for $\hat{g}^*$. For sheaves of sets, this is [29, Tag 00XT]. For sheaves of groupoids, one can argue as follows: if $\mathcal{F} \in \widehat{\mathrm{Sch}_P}$ then $\hat{g}_*\hat{g}^*(\mathcal{F})$ is the restriction of the sheafification of $\tilde{g}^*\mathcal{F}$, which equals[17] the sheafification of the restriction of $\tilde{g}^*\mathcal{F}$, i.e., the sheafification of $\mathcal{F}$, i.e., $\mathcal{F}$ itself.

Now we can reformulate Proposition A.0.1 as follows.

Proposition A.0.2. *If* $\mathscr{X} \in \mathrm{Stacks}_P$ *then the canonical morphism*

$$\hat{g}^*\mathscr{X}_P = \hat{g}^*\hat{g}_*\mathscr{X} \to \mathscr{X}$$

is an isomorphism. Equivalently, $\mathscr{X}$ *belongs to the essential image of the fully faithful functor* $\hat{g}^*$. □

Proposition A.0.2 tells us how to reconstruct $\mathscr{X} \in \mathrm{Stacks}_P$ from $\mathscr{X}_P$: namely, $\mathscr{X} = \hat{g}_*\mathscr{X}_P$, i.e., $\mathscr{X}$ is the sheafified left Kan extension of $\mathscr{X}_P$.

[17] Sheafification commutes with restriction by part 1 of [7, Prop. 7.1]. Probably this can also be proved by interpreting $\widehat{\mathrm{Sch}}$ and $\widehat{\mathrm{Sch}_P}$ as explained in part 2 of Exercise 4.10.6 of Exposé IV of [27] (i.e., a sheaf on Sch is just a collection of sheaves $\mathcal{F}_S$ on S_{et} for all $S \in \mathrm{Sch}$ plus certain morphisms relating the sheaves $\mathcal{F}_S$ with each other).

B. Recollections on gerbes

B.0.1. Classifying stacks. Let H be a flat affine group scheme of finite presentation over an algebraic stack $\mathscr{X}$. Then the classifying stack BH is known to be an algebraic stack smooth over $\mathscr{X}$: indeed, we can assume that $\mathscr{X}$ is a scheme, in which case the statement is proved in [29, Tag 0DLS] using [29, Tag 05B5] and a deep theorem of M. Artin, see [29, Tag 06FI]. (On the other hand, if H is finite over $\mathscr{X}$, one can give an elementary constructive proof similar to the one given in §4.1.4 in the case $H = \alpha_p$.)

B.0.2. Gerbes. Let $f : \mathscr{Y} \to \mathscr{X}$ be a morphism of finite presentation between algebraic stacks. If f is an fppf gerbe then f is smooth. Indeed, smoothness can be checked fppf-locally on $\mathscr{X}$, so we can assume that the gerbe is trivial. Then we can apply §B.0.1.

B.0.3. Remark. Let $f : \mathscr{Y} \to \mathscr{X}$ be as in §B.0.2. Since f is smooth and surjective, it admits a section locally for the *smooth* topology of $\mathscr{X}$. This implies that if $\mathscr{X}$ is a scheme then f admits a section etale-locally on $\mathscr{X}$.

C. A definition of BT_n^G via syntomification

Let $n \in \mathbb{N}$ and $G \in \mathbf{Shim}_n$, where $\mathbf{Shim}_n$ is as in §2.2. For every derived p-adic formal scheme S, we define in §C.2 an ∞-groupoid $\mathrm{BT}_n^G(S)$. The assignment $S \mapsto \mathrm{BT}_n^G(S)$ is an etale sheaf. Conjecture C.3.1 says that for each $m \in \mathbb{N}$ the restriction of BT_n^G to the category of derived schemes over $\mathbb{Z}/p^m\mathbb{Z}$ is a smooth algebraic stack over $\mathbb{Z}/p^m\mathbb{Z}$.

C.1. Recollections on syntomification

C.1.1. The stack S^{Δ}. Bhatt and Lurie [4] define a *prismatization* functor $S \mapsto S^{\Delta}$ from the category of derived p-adic formal schemes to the category of fpqc-stacks of ∞-groupoids on the category of p-nilpotent[18] derived schemes. The stacks S^{Δ} are not very far from being algebraic in the sense of Definition 2.3.5 of [9] (whose essential point is that the quotient of a scheme over a ring R by an action of a flat affine group scheme H over R is considered to be an algebraic stack even if H has infinite type).

The above words "not very far" are necessary for two reasons. First, the derived setting is not considered in [9]. Second, already the stacks $(\operatorname{Spf} \mathbb{Z}_p)^{\Delta}$ and $(\operatorname{Spec} \mathbb{Z}/p^n\mathbb{Z})^{\Delta}$ are *formal* (in the sense of [9, §2.9.1]) rather than algebraic; e.g., $(\operatorname{Spec} \mathbb{F}_p)^{\Delta} = \operatorname{Spf} \mathbb{Z}_p$.

C.1.2. The stacks $S^{\mathcal{N}}$ and S^{Syn}. Bhatt defines in his lecture notes [2] the *(Nygaard-)filtered prismatization* functor $S \mapsto S^{\mathcal{N}}$. As before, this is a functor from the category of derived p-adic formal schemes to the category of fpqc-stacks of ∞-groupoids on the category of p-nilpotent derived schemes. However, for pedagogical reasons, Bhatt assumes in [2] that S is a classical[19] scheme, and he

[18] This means that p is Zariski-locally nilpotent.

[19] "Classical" means "not really derived".

defines in [2] only the restriction of the stack $S^{\mathcal{N}}$ to the category of classical p-nilpotent schemes. As before, the stacks $S^{\mathcal{N}}$ are not very far from being algebraic in the sense of [9].

The stack $S^{\mathcal{N}}$ has two open substacks canonically isomorphic to $S^{\mathbb{\Delta}}$. Gluing them together, Bhatt gets a stack which is called the *syntomification* of S and denoted by S^{Syn}. The assignment $S \mapsto S^{\mathrm{Syn}}$ is a functor. By definition, one has a natural morphism

$$S^{\mathcal{N}} \to S^{\mathrm{Syn}}.$$

There is a canonical line bundle on $(\operatorname{Spf} \mathbb{Z}_p)^{\mathrm{Syn}}$ called the Breuil-Kisin twist. It defines a morphism $(\operatorname{Spf} \mathbb{Z}_p)^{\mathrm{Syn}} \to B\mathbb{G}_m$, where $B\mathbb{G}_m$ is the classifying stack of $\mathbb{G}_m$. By functoriality, it induces a morphsim $S^{\mathrm{Syn}} \to B\mathbb{G}_m$ for any S. The corresponding line bundle on S^{Syn} is denoted by $\mathcal{O}_{S^{\mathrm{Syn}}}\{1\}$ or $\mathcal{O}\{1\}$.

C.1.3. The canonical morphism $S \times B\mathbb{G}_m \to S^{\mathcal{N}}$. There is a canonical morphism

$$S \times B\mathbb{G}_m \to S^{\mathcal{N}} \tag{C.1}$$

over $B\mathbb{G}_m$; namely, (C.1) is the composite map

$$S \times \{0\}/\mathbb{G}_m \hookrightarrow S \times \mathbb{A}^1/\mathbb{G}_m \to S^{dR,+} \to S^{\mathcal{N}},$$

where $S^{dR,+}$ and the maps $S \times \mathbb{A}^1/\mathbb{G}_m \to S^{dR,+} \to S^{\mathcal{N}}$ are defined in [2, §5.3.13].

We need the morphism (C.1) only if $S = \operatorname{Spec} k$, where k is a perfect field. In this case it can be described as follows. It is known[20] that

$$(\operatorname{Spec} k)^{\mathcal{N}} \otimes \mathbb{F}_p = (\operatorname{Spec} k[u,t]/(ut))/\mathbb{G}_m\,,$$

where the $\mathbb{G}_m$-action is such that $\deg t = 1$, $\deg u = -1$; so the closed substack of the stack $(\operatorname{Spec} k)^{\mathcal{N}} \otimes \mathbb{F}_p$ given by $t = u = 0$ equals $\operatorname{Spec} k \times B\mathbb{G}_m$. (Let us note that this closed substack is called the *Hodge locus*.)

C.1.4. On notation. The definition of $S^{\mathcal{N}}$ and S^{Syn} is sketched in [9, §1.7], but the notation is different there ($S^{\mathbb{\Delta}'}$ instead of $S^{\mathcal{N}}$ and $S^{\mathbb{\Delta}''}$ instead of S^{Syn}). In [3, 4] the stack $S^{\mathbb{\Delta}}$ was denoted by WCart_S.

C.2. Definition of $\mathrm{BT}_n^G(S)$. Let S be a derived p-adic formal scheme. Let $G \in \mathbf{Shim}_n$.

C.2.1. The group scheme $G_{\mathcal{O}\{1\}}$. Our G is a group scheme over $\mathbb{Z}/p^n\mathbb{Z}$. Twisting G by the canonical $\mathbb{G}_m$-torsor on $B\mathbb{G}_m \otimes \mathbb{Z}/p^n\mathbb{Z}$, one gets a group scheme over $B\mathbb{G}_m \otimes \mathbb{Z}/p^n\mathbb{Z}$, which we denote by $G_{\mathcal{O}\{1\}}$. One has

$$\mathrm{Lie}(G_{\mathcal{O}\{1\}}) = \bigoplus_i \mathfrak{g}_i \otimes \mathcal{O}\{i\},$$

where $\mathfrak{g}_i$ is the i-th graded component of $\mathrm{Lie}(G)$ and $\mathcal{O}\{i\}$ is the i-th tensor power of the canonical line bundle on $B\mathbb{G}_m \otimes \mathbb{Z}/p^n\mathbb{Z}$.

Since $S^{\mathrm{Syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is equipped with a morphism to $B\mathbb{G}_m \otimes \mathbb{Z}/p^n\mathbb{Z}$, we can consider $G_{\mathcal{O}\{1\}}$-torsors on $S^{\mathrm{Syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$.

[20]See [2, §3.3 and §5.4].

C.2.2. Definition. For any derived p-adic formal scheme S of finite type[21] over $\mathrm{Spf}\,\mathbb{Z}_p$, let $\mathrm{BT}_n^G(S)$ be the ∞-groupoid of $G_{\mathcal{O}\{1\}}$-torsors $\mathcal{E}$ on $S^{\mathrm{Syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ such that for every geometric point $\alpha : \mathrm{Spec}\, k \to S$, the pullback of $\mathcal{E}$ via the composite morphism

$$\mathrm{Spec}\, k \times B\mathbb{G}_m \to (\mathrm{Spec}\, k)^{\mathcal{N}} \otimes \mathbb{F}_p \xrightarrow{\alpha_*} S^{\mathcal{N}} \otimes \mathbb{F}_p \hookrightarrow S^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z} \to S^{\mathrm{Syn}} \otimes \mathbb{Z}/p^n\mathbb{Z} \tag{C.2}$$

is trivial. The first arrow in (C.2) was described in §C.1.3.

C.2.3. Remarks

(i) The isomorphism class of the above-mentioned pullback is an element $\nu_\alpha \in H^1((\mathbb{G}_m)_k, G_k)$, where G_k is the base change of G to k, and we want ν_α to be zero for all geometric points α. One can show that if S is connected then it suffices to check the condition $\nu_\alpha = 0$ for a single α.

(ii) $H^1((\mathbb{G}_m)_k, G_k)$ is the set of $G(k)$-conjugacy classes of splittings for the canonical epimorphism $(\mathbb{G}_m)_k \ltimes G_k \twoheadrightarrow (\mathbb{G}_m)_k$. The latter set depends only on the quotient of G_k by its unipotent radical (together with the action of $\mathbb{G}_m$ on this quotient).

(iii) The definition of $\mathrm{BT}_n^G(S)$ from §C.2.2 makes sense without assuming the action of $\mathbb{G}_m$ on G to be 1-bounded. But the conjecture formulated below is *unlikely to hold without this assumption.*

C.3. The conjecture. For each $n \in \mathbb{N}$ and $G \in \mathbf{Shim}_n$ we have defined a contravariant functor $S \mapsto \mathrm{BT}_n^G(S)$, where S is a derived p-adic formal scheme of finite type over $\mathrm{Spf}\,\mathbb{Z}_p$.

Conjecture C.3.1. *For each $m \in \mathbb{N}$ the restriction of this functor to the category of derived schemes over $\mathbb{Z}/p^m\mathbb{Z}$ is a quasicompact smooth algebraic stack*[22] *over $\mathbb{Z}/p^m\mathbb{Z}$ with affine diagonal.*

In particular, the conjecture would imply that the restriction of the functor BT_n^G to the category of classical schemes over $\mathbb{Z}/p^m\mathbb{Z}$ is a smooth algebraic stack in the sense of [20].

As far as I understand, Conjecture C.3.1 has already been proved in [12].

C.3.2. Remarks

(i) If S is a smooth $\mathbb{F}_p$-scheme then the definitions of $\mathrm{BT}_1^G(S)$ given in §4.2.1 and §C.2.2 are equivalent, see [28, Thm. 3.8] or Appendix D below.

(ii) Let G be the group $\mathbb{G}_a \otimes \mathbb{Z}/p^n\mathbb{Z}$ equipped with the usual $\mathbb{G}_m$-action. Then the triviality condition from §C.2.2 is automatic by §C.2.3, so $\mathrm{BT}_n^G(S)$ is the ∞-groupoid of all $\mathcal{O}\{1\}$-torsors on $S^{\mathrm{Syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. Bhatt and Lurie proved[23]

[21] "Finite type" means that the classical truncation of $S \otimes \mathbb{F}_p$ has finite type in the usual sense. For the motivation of the finite type precaution, see §C.3.2(ii) below.

[22] Here the words "algebraic stack" are understood in the derived sense (since S is allowed to be derived); see the definition of 1-stack in [21, Def. 5.1.3] (which goes back to [30]).

[23] Theorem 7.5.6 of [3] identifies the p-adic completion of $R\Gamma(S_{\mathrm{et}}, \mathcal{O}_S^\times)[-1]$ with a certain complex $R\Gamma_{\mathrm{syn}}(S, \mathbb{Z}_p(1))$, whose definition does not involve S^{Syn}. If S is quasisyntomic then it is known that $R\Gamma_{\mathrm{syn}}(S, \mathbb{Z}_p(1)) = R\Gamma(S^{\mathrm{Syn}}, \mathcal{O}\{1\})$.

that

$$R\Gamma(S^{\mathrm{Syn}}\otimes\mathbb{Z}/p^n\mathbb{Z},\mathcal{O}\{1\})=R\Gamma(S_{\mathrm{et}},\mathrm{Cone}(\mathcal{O}_S^\times\xrightarrow{p^n}\mathcal{O}_S^\times)[-1]).\qquad\text{(C.3)}$$

if S is quasisyntomic. Moreover, (C.3) is expected to hold[24] for *any* p-nilpotent derived scheme of finite type over $\operatorname{Spf}\mathbb{Z}_p$ (similarly to [4, Prop. 8.16]). If so then BT_n^G identifies with the classifying stack of μ_{p^n}. This agrees with Conjecture C.3.1; in the case $n=m=1$ this also agrees with §4.4.5.

D. Equivalence between the two definitions of $\mathrm{BT}_1^G(S)$

Let G be a smooth affine group scheme over $\mathbb{F}_p$ equipped with an action of $\mathbb{G}_m$. Given a smooth $\mathbb{F}_p$-scheme S, we have two definitions of the groupoid $\mathrm{BT}_1^G(S)$: an elementary one (see §4.2.1) and a definition via the stack S^{Syn} (see §C.2.2); both definitions make sense without assuming that the $\mathbb{G}_m$-action is 1-bounded, see §4.2.3 and §C.2.3(iii). In this Appendix we sketch a proof of the equivalence between the two definitions.

D.1. Recollections on $S^{\mathrm{Syn}}\otimes\mathbb{F}_p$. We will follow [2, Ch. 2].

D.1.1. As mentioned in §C.1.2, S^{Syn} is obtained from $S^{\mathcal{N}}$ by gluing together the two open substacks canonically isomorphic S^{Δ}. One has $S^{\Delta}\otimes\mathbb{F}_p=S^{\mathrm{dR}}:=S^{\mathrm{dR}/\mathbb{F}_p}$. Following §2.8 of [2], we use the notation $S^C:=S^{\mathcal{N}}\otimes\mathbb{F}_p$. The stack S^C contains two open substacks canonically isomorphic to S^{dR}, and after gluing them together, one gets $S^{\mathrm{Syn}}\otimes\mathbb{F}_p$.

Let us now recall the material on S^C from [2, Ch. 2].

D.1.2. Just as in Construction 2.8.2 from [2], let C denote the quotient of the coordinate cross $\operatorname{Spec}\mathbb{F}_p[u,t]/(ut)$ by the hyperbolic action of $\mathbb{G}_m$. One has $(\operatorname{Spec}\mathbb{F}_p)^C=C$. The two irreducible components of C are denoted by $C_{u=0}$ and $C_{t=0}$.

D.1.3. The preimage of $C_{t=0}$ (respectively $C_{u=0}$) in S^C is denoted in [2] by $S^{\mathrm{dR},c}$ (respectively $S^{\mathrm{dR},+}$); the superscript c stands for "conjugate filtration". Let $S^{\mathrm{Hodge}}:=S^{\mathrm{dR},c}\cap S^{\mathrm{dR},+}$.

In §D.1.1 we mentioned two open substacks of S^C canonically isomorphic S^{dR}; these substacks are $S^{\mathrm{dR},c}\setminus S^{\mathrm{Hodge}}$ and $S^{\mathrm{dR},+}\setminus S^{\mathrm{Hodge}}$.

D.1.4. By §D.1.1 and §D.1.3, we have a pushout diagram of stacks

$$\begin{tikzcd} S^{\mathrm{Syn}}\otimes\mathbb{F}_p & S^{\mathrm{dR},+} \arrow[l] \\ S^{\mathrm{dR},c} \arrow[u] & S^{\mathrm{dR}}\sqcup S^{\mathrm{Hodge}} \arrow[l] \arrow[u] \end{tikzcd}\qquad\text{(D.1)}$$

Chapter 2 of [2] contains a rather explicit description of the whole diagram (D.1) via the procedure of *transmutation* (see [2, Rem. 2.3.8]). The key point is that

[24] Again, the problem is to show that $R\Gamma_{\mathrm{syn}}(S,\mathbb{Z}_p(1))=R\Gamma(S^{\mathrm{Syn}},\mathcal{O}\{1\})$.

it suffices to describe (D.1) in the particular case $S = \mathbb{G}_a$ *as a diagram of ring stacks*. For such a description, see [2, 2.8.3] and references therein, as well as [2, 2.5.1], [2, 2.7.8], and [2, Prop. 2.7.12]. We paraphrase this description in §D.4.

D.2. $\mathrm{BT}_1^G(S)$ **as a fiber product.** Let $\mathrm{BT}_1^G(S)$ denote the groupoid defined in §C.2.2. Thus $\mathrm{BT}_1^G(S)$ is the groupoid of $G_{\mathcal{O}\{1\}}$-torsors $\mathcal{E}$ on $S^{\mathrm{Syn}} \otimes \mathbb{F}_p$ such that for every geometric point $\alpha : \operatorname{Spec} k \to S$, the pullback of $\mathcal{E}$ via a certain morphism $f_\alpha : \operatorname{Spec} k \times B\mathbb{G}_m \to S^{\mathrm{Syn}} \otimes \mathbb{F}_p$ is trivial. Let us note that f_α factors as

$$\operatorname{Spec} k \times B\mathbb{G}_m \to S^{\mathrm{Hodge}} \to S^{\mathrm{Syn}} \otimes \mathbb{F}_p, \tag{D.2}$$

see the last paragraph of §C.1.3

Diagram (D.1.4) yields a pullback diagram of groupoids

$$\begin{tikzcd} \mathrm{BT}_1^G(S) \arrow[r] \arrow[d] & \mathscr{X}^{\mathrm{dR},+}(S) \arrow[d] \\ \mathscr{X}^{\mathrm{dR},c}(S) \arrow[r] & \mathscr{X}^{\mathrm{dR}}(S) \times \mathscr{X}^{\mathrm{Hodge}}(S) \end{tikzcd} \tag{D.3}$$

where $\mathscr{X}^{\mathrm{dR}}(S)$ is the groupoid of G-torsors[25] on S^{dR} and $\mathscr{X}^{\mathrm{dR},+}(S)$ (respectively $\mathscr{X}^{\mathrm{dR},c}(S)$ or $\mathscr{X}^{\mathrm{Hodge}}(S)$) is the groupoid of $G_{\mathcal{O}\{1\}}$-torsors on $S^{\mathrm{dR},+}$ (respectively $S^{\mathrm{dR},c}$ or S^{Hodge}) satisfying the triviality condition[26] similar to the one from the definition of $\mathrm{BT}_1^G(S)$.

In the next subsection we will translate the diagram

$$\mathscr{X}^{\mathrm{dR},c}(S) \to \mathscr{X}^{\mathrm{dR}}(S) \times \mathscr{X}^{\mathrm{Hodge}}(S) \leftarrow \mathscr{X}^{\mathrm{dR},+}(S) \tag{D.4}$$

into an elementary language. The isomorphism between $\mathrm{BT}_1^G(S)$ and the groupoid from §4.2.1 will immediately follow from this description because diagram (D.3) is Cartesian.

D.3. Diagram (D.4) in elementary terms. Let $\mathfrak{g} := \mathrm{Lie}(G)$. We will use the subgroups $M, P^{\pm} \subset G$ defined in §2.3 and the grading $\mathfrak{g} = \bigoplus\limits_i \mathfrak{g}_i$ corresponding to the $\mathbb{G}_m$-action.

D.3.1. $\mathscr{X}(S)$ is the groupoid of G-torsors on S equipped with a nilpotent integrable connection.

D.3.2. $\mathscr{X}^{\mathrm{Hodge}}(S)$ is the groupoid of pairs $(\mathcal{F}_M, \sigma)$, where $\mathcal{F}_M$ is an M-torsor on S and $\sigma \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}_M} \otimes \Omega^1_S)$, see [2, Rem. 2.5.9].

D.3.3. $\mathscr{X}^{\mathrm{dR},+}(S)$ is the groupoid of P^--torsors $\mathcal{F}^-$ on S equipped with a nilpotent integrable connection ∇ on the corresponding G-bundle $\mathcal{F}^-_G$ satisfying the Griffiths transversality condition $\mathrm{KS}_\nabla \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}^-} \otimes \Omega^1_S) \subset H^0(S, (\mathfrak{g}/\mathfrak{g}_{\leq 0})_{\mathcal{F}^-} \otimes \Omega^1_S)$ (we are using the notation from §4.2.2). See [2, Rem. 2.5.8].

[25] A $G_{\mathcal{O}\{1\}}$-torsor on S^{dR} is the same as a G-torsor because $(\operatorname{Spec} \mathbb{F}_p)^{dR} = \operatorname{Spec} \mathbb{F}_p$.

[26] The formulation of this condition uses the factorization (D.2).

D.3.4. $\mathscr{X}^{\mathrm{dR},c}(S)$ is the groupoid of P^+-torsors $\mathcal{F}^+$ on S equipped with an integrable connection ∇ inducing a p-integrable connection on the M-torsor $\mathcal{F}^+_M$ corresponding to $\mathcal{F}^+$. See [2, Rem. 2.7.10].

D.3.5. The functor $\mathscr{X}^{\mathrm{dR},c}(S) \to \mathscr{X}^{\mathrm{Hodge}}(S)$ from diagram (D.4) is as follows:

(i) the M-torsor $\mathcal{F}_M$ from §D.3.2 is the Fr_S-descent of the M-torsor $\mathcal{F}^+_M$ from §D.3.4 via the p-integrable connection;

(ii) $\sigma \in H^0(S, (\mathfrak{g}_1)_{\mathcal{F}_M} \otimes \Omega^1_S)$ is the image of $-\operatorname{p-Curv}_\nabla \in H^0(S, (\mathfrak{g}_{\geq 1})_{\mathcal{F}_M} \otimes \Omega^1_S)$.

The other functors from diagram (D.4) are self-explanatory.

D.3.6. Origin of the sign. The "minus" sign in §D.3.5(ii) comes from the "minus" sign in formula (D.7) below.

D.4. The ring stack $\mathbb{G}_a^{\mathrm{Syn}} \otimes \mathbb{F}_p$. Let $\mathbb{G}_a$ denote the additive group over $\mathbb{F}_p$. In this subsection we recall the description of $\mathbb{G}_a^{\mathrm{Syn}} \otimes \mathbb{F}_p$ given in [2, Ch. 2] and explain the origin of the "minus" sign in §D.3.5(ii).

D.4.1. Plan. $\mathbb{G}_a$ is a ring scheme, so $\mathbb{G}_a^{\mathrm{Syn}} \otimes \mathbb{F}_p$ is a ring stack over $(\operatorname{Spec}\mathbb{F}_p)^{\mathrm{Syn}} \otimes \mathbb{F}_p$. One gets $(\operatorname{Spec}\mathbb{F}_p)^{\mathrm{Syn}} \otimes \mathbb{F}_p$ from the stack C considered in §D.1.2 by gluing together the two open points of C. Equivalently, $(\operatorname{Spec}\mathbb{F}_p)^{\mathrm{Syn}} \otimes \mathbb{F}_p$ is obtained from $\mathbb{P}^1/\mathbb{G}_m$ by gluing $\{0\}/\mathbb{G}_m$ with $\{\infty\}/\mathbb{G}_m$.

Let $\mathscr{R}$ be the pullback of $\mathbb{G}_a^{\mathrm{Syn}} \otimes \mathbb{F}_p$ via the map $\mathbb{P}^1 \to (\operatorname{Spec}\mathbb{F}_p)^{\mathrm{Syn}} \otimes \mathbb{F}_p$. To describe $\mathbb{G}_a^{\mathrm{Syn}} \otimes \mathbb{F}_p$, we will describe in §D.4.2-D.4.3 the $\mathbb{G}_m$-equivariant ring stack $\mathscr{R}$ over $\mathbb{P}^1$ and the isomorphism $\mathscr{R}_0 \xrightarrow{\sim} \mathscr{R}_\infty$, where $\mathscr{R}_0, \mathscr{R}_\infty$ are the fibers of $\mathscr{R}$ over $0, \infty \in \mathbb{P}^1$.

D.4.2. The ring stack $\mathscr{R}$. Let $(\mathbb{G}_a)_{\mathbb{P}^1} = \mathbb{G}_a \times \mathbb{P}^1$; this is a ring scheme over $\mathbb{P}^1$. One has

$$\mathscr{R} = \operatorname{Cone}(G \xrightarrow{d} (\mathbb{G}_a)_{\mathbb{P}^1}),$$

where (G, d) is a quasi-ideal[27] in $(\mathbb{G}_a)_{\mathbb{P}^1}$, which we are going to define.

Let $\mathbb{G}_a^\sharp$ denote the the PD-hull of zero in $\mathbb{G}_a$. Then $\mathbb{G}_a^\sharp$ is a $\mathbb{G}_a$-module, and the natural map $f : \mathbb{G}_a^\sharp \to \mathbb{G}_a$ is a $\mathbb{G}_a$-module homomorphism such that

$$\operatorname{Im} f = \alpha_p := \operatorname{Ker}(\mathrm{Fr} : \mathbb{G}_a \to \mathbb{G}_a).$$

One defines G to be a certain $\mathbb{G}_a$-submodule of $\mathbb{G}_a^\sharp \times \mathbb{G}_a \times \mathbb{P}^1$ and $d : G \to \mathbb{G}_a \times \mathbb{P}^1$ to be the projection. The equations defining $G \subset \mathbb{G}_a^\sharp \times \mathbb{G}_a \times \mathbb{P}^1$ are as follows:

$$uy = f(x), \tag{D.5}$$

$$y^p = 0, \tag{D.6}$$

where $x \in \mathbb{G}_a^\sharp$, $y \in \mathbb{G}_a$, $u \in \mathbb{P}^1$. Strictly speaking, (D.5) means that

$$uy = f(x) \text{ if } u \neq \infty, \quad y = u^{-1} f(x) \text{ if } u \neq 0.$$

Note that if $u \neq 0$ then (D.6) follows from (D.5).

Finally, the action of $\lambda \in \mathbb{G}_m$ on G and $\mathbb{G}_a \times \mathbb{P}^1$ is given by $\tilde{x} = \lambda^{-1} x$, $\tilde{u} = \lambda^{-1} u$, $\tilde{y} = y$.

[27] For the language of quasi-ideals and cones, see [9, §1.3.3-1.3.4].

D.4.3. The isomorphism $\mathscr{R}_0 \xrightarrow{\sim} \mathscr{R}_\infty$. Let $K := \operatorname{Ker}(\mathbb{G}_a^\sharp \xrightarrow{f} \mathbb{G}_a)$. Then

$$\mathscr{R}_0 = \operatorname{Cone}(K \oplus \alpha_p \xrightarrow{(0,1)} \mathbb{G}_a) = \operatorname{Cone}(K \xrightarrow{0} \mathbb{G}_a/\alpha_p) = \operatorname{Cone}(K \xrightarrow{0} \mathbb{G}_a)$$

(we have used the ring isomorphism $\mathbb{G}_a/\alpha_p \xrightarrow{\sim} \mathbb{G}_a$ induced by $\mathrm{Fr} : \mathbb{G}_a \to \mathbb{G}_a$). On the other hand,

$$\mathscr{R}_\infty = \operatorname{Cone}(\mathbb{G}_a^\sharp \xrightarrow{0} \mathbb{G}_a).$$

The isomorphism $\mathscr{R}_0 \xrightarrow{\sim} \mathscr{R}_\infty$ comes from the isomorphism

$$-V : \mathbb{G}_a^\sharp \xrightarrow{\sim} K, \tag{D.7}$$

where V is the Verschiebung of $\mathbb{G}_a^\sharp$.

D.4.4. Origin of formula (D.7). One gets (D.7) by comparing [2, §2.8.3] with [2, §2.8.1] and [2, Prop. 2.7.12]. The "minus" sign in (D.7) comes from diagram (2.7.5) of [2].

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Шимурові узагальнення стеку $\mathrm{BT}_1 \otimes \mathbb{F}_p$

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Нехай G є гладкою груповою схемою над $\mathbb{F}_p$, оснащеною дією $\mathbb{G}_m$ так, що всі ваги $\mathbb{G}_m$ на $\mathrm{Lie}(G)$ не перевищують 1. Нехай Disp_n^G є стеком n-зрізаних G-дисплеїв Айке Лау (це алгебраїчний $\mathbb{F}_p$-стек). У випадку $n = 1$ ми вводимо алгебраїчний стек, оснащений морфізмом до Disp_1^G. Ми припускаємо, що якщо $G = GL(d)$, то новий стек канонічно ізоморфний редукції за модулем p стеку 1-зрізаних груп Барсотті–Тейта висоти d та розмірності d', де d' залежить від дії $\mathbb{G}_m$ на $GL(d)$.

Ми також обговорюємо, як визначити аналог нового стеку для $n > 1$ та як замінити $\mathbb{F}_p$ на $\mathbb{Z}/p^m\mathbb{Z}$.

Ключові слова: група Барсотті–Тейта, многовид Шимури, дисплей, F-зіп, зв'язність, p-кривина, синтоміфікація