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Existence of a Renormalized Solution for a Class of Parabolic Problems

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In the paper, we prove the existence of the renormalized solution for the nonlinear degenerate parabolic equation $\frac{\partial b(u)}{\partial t} - \operatorname{div}(A(t,x,u)Du) = f$, where the matrix $A(t,x,s) = (a_{ij}(t,x,s))_{\substack{1\le i\le N\\ 1\le j\le N}}$ is not controlled with respect to u, $f \in L^1(Q)$, and b is a strictly increasing C^1-function.

Key words: renormalized solutions, blow-up, L^1-data

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1. Introduction

In this paper, we study the existence of a solution for a class of parabolic-type problems of the form

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} - \operatorname{div}(A(t,x,u)Du) = f & \text{in } \Omega \times (0,T), \\ b(u)(t=0) = b(u_0) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times]0,T[\,, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \ge 1)$, T is a positive number. Set $Q = \Omega \times (0,T)$. In the problem (1.1), b is a strictly increasing C^1-function with $b(0) = 0$, $f \in L^1(Q)$, and $b_0(u) \in L^1(\Omega)$. Moreover, the matrix

$$A : (t,x,s) \to A(t,x,s) = (a_{ij}(t,x,s))_{\substack{1\le i\le N\\ 1\le j\le N}} \tag{1.2}$$

is a Carathéodory function from $Q \times (-\infty, m)$ into Sym_n, it blows up when $s \to m^-$ and satisfies the coercivity condition, where Sym_n is the set of $N \times N$ symmetric matrices and $m > 0$.

In the literature, there are other models that contain a blow-up term. We mention

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(A(t,x,u)Du) + g(u) = f & \text{in } \Omega \times (0,T), \\ u(t=0) = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times]0,T[\,, \end{cases} \tag{1.3}$$

where the matrix M is not blowing up and the nonlinearity g has a vertical asymptote. The problem (1.3) was studied in [12] for the case, where $f \in L^1(Q)$.

For the case, where f is replaced by a measure, the problem (1.3) was studied in [1, 2]. On the other hand, we also mention

$$\begin{cases} -\Delta u + g(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where the nonlinearity g has a vertical asymptote at 1 and μ is a bounded measure on Ω. The problem (1.4) was established in [10, 11].

When we look at the problem (1.1), we come up against two kinds of difficulties. First, the assumptions $f \in L^1(Q)$ and $b_0(u) \in L^1(\Omega)$. To overcome this difficulty, we use the framework of renormalized solutions. This concept was presented by DiPerna and Lions [11] to study the Boltzmann equation. See also Lions [13] for a few applications to fluid mechanics models. We refer the reader to [8, 18, 19] for elliptic problems and to [5, 6, 14, 15, 17] for parabolic equations. A similar notion of entropy solution was defined in [16] and it was proven to be the same as the renormalized solution. The second difficulty is due to the matrix $A(t, x, s)$ blowing up when $s \to m^-$, which makes the task of giving meaning to this function on the set $\{x \in \Omega \mid u(x) = m\}$ difficult.

For the case, where the field is a Leray–Lions operator, the existence of renormalized solutions was proved in [6] and for the weighted Sobolev space, in [3]. For the case, where the field is a Leray–Lions operator and b is a maximal monotone graph on $\mathbb{R}$, the existence of renormalized solutions was established in [4]. More precisely, the problem (1.1) is an extension of the work of Blanchard et al. (see [7]).

This paper is organized as follows. In Section 2, we make assumptions and provide the main result. In Section 3, we give the proof of the main result.

2. Basic assumptions and main result

The following assumptions are assumed to be true throughout the paper:

$$b : \mathbb{R} \to \mathbb{R} \text{ is a strictly increasing } C^1\text{-function with } b(0) = 0, \tag{2.1}$$

$$b_0 \in L^1(Q), \tag{2.2}$$

$$f \in L^1(Q), \tag{2.3}$$

$$A : (t, x, s) \to A(t, x, s) = (a_{ij}(t, x, s))_{\substack{1 \le i \le N \\ 1 \le j \le N}} \tag{2.4}$$

is a Caracthéodory function from $Q \times (-\infty, m)$ into Sym_n such that there exist two positive functions β and γ in $C^0((-\infty, m))$ which satisfy

$$\lim_{s \to m^-} \beta(s) = +\infty, \quad \beta(s) \ge \alpha > 0, \quad s \in (-\infty, m), \tag{2.5}$$

$$\int_0^m \gamma(s)\, ds < +\infty, \tag{2.6}$$

$$\beta(s)\, |\xi|^2 \le A(t, x, s)\xi \cdot \xi \le \gamma(s)\, |\xi|^2, \quad \xi \in \mathbb{R}^N. \tag{2.7}$$

For any positive real number ε, we define the function σ_ε by

$$\sigma_\varepsilon(r) = \begin{cases} 1 & \text{if } r \le m - 2\varepsilon, \\ 1 - (r - m + \varepsilon) & \text{if } m - 2\varepsilon \le r \le m - \varepsilon, \\ 0 & \text{if } r \ge m - \varepsilon. \end{cases} \tag{2.8}$$

For a fixed $n \ge 1$ and for all $s \in \mathbb{R}$, we define

$$\theta_n(s) = \frac{1}{n}(T_n(s - T_n(s)) \quad \text{and} \quad h(s) = 1 - |\theta_n(s)| \,. \tag{2.9}$$

Definition 2.1. A measurable function u defined on Ω is a renormalized solution of the problem (1.1) if

$$\forall k \ge 0 \quad T_k(u) \in L^2(0,T;H_0^1(\Omega)) \;, \tag{2.10}$$

$$b(u) \in L^\infty(0,T;L^1(\Omega)), \tag{2.11}$$

$$u \le m \quad \text{a.e. in } Q, \tag{2.12}$$

$$\forall k \ge 0 \quad \chi_{\{-k<u<m\}} A(t,x,u)Du \in \left(L^2(\Omega)\right)^N , \tag{2.13}$$

$$\lim_{p\to+\infty} \frac{1}{p} \int_{\{-2p<u<-p\}} A(t,x,u)Du \cdot Du \, dx\, dt = 0, \tag{2.14}$$

$$\forall \varphi \in C_c^\infty([0,T]) \qquad \lim_{p\to+\infty} p \sum_{i=1}^{N} \int_Q \varphi \chi_{\{m-2/p<u<m-1/p\}} A(t,x,u)Du \cdot Du \, dx\, dt = \int_Q f \varphi \chi_{\{u=m\}} \, dx\, dt, \tag{2.15}$$

and u satisfies

$$\frac{\partial B_s(u)}{\partial t} - \operatorname{div}\left(S'(u)A(t,x,u)Du\right) + S''(u)A(t,x,u)Du \cdot Du = f S'(u) \text{ in } D'(Q), \tag{2.16}$$

where

$$B_s(z) = \int_0^z b'(s)S'(s)\, ds \quad \text{and} \quad B_s(u)(t=0) = B_s(u_0)$$

for every function S in $W^{2,\infty}(\mathbb{R})$ such that $\operatorname{supp}(S')$ is compact and $S'(m) = 0$ and for any $\varphi \in C_c^\infty\left([0,T] \times \overline{\Omega}\right)$ such that $S'(u)\varphi \in L^2(0,T;H_0^1(\Omega))$.

Remark 2.2. Conditions (2.10) and (2.13) show that all terms in (2.16) are well defined. The assumption (2.1) was established in [7] when $b(u) = u$.

Theorem 2.3. *Under the assumptions (2.1)–(2.7) there exists at least a renormalized solution u of the problem (1.1).*

3. Proof of main result

3.1. Step 1: Approximation of the problem. For $\varepsilon > 0$, we consider the field of matrices

$$A^{\varepsilon}(t,x,s) = \sigma_{\varepsilon}(s)A\,(t,x,s) + (1-\sigma_{\varepsilon}(s))\beta(m-\varepsilon)I, \tag{3.1}$$

where σ_{ε} is the function defined by (2.8) and I is the diagonal matrix. Indeed, in (3.1), we use the convention

$$\sigma_{\varepsilon}(s)A\,(t,x,s) = 0 \quad \text{for } s \geq m-\varepsilon.$$

Due to the assumptions (2.5) and (2.7), we have

$$\beta(s)\,|\xi|^2 \leq A^{\varepsilon}(t,x,s)\xi\cdot\xi \leq \left(\gamma\,(s)\,\sigma_{\varepsilon}(s) + \sup_{r\in(0,m-\varepsilon)}\beta(r)\right)|\xi|^2. \tag{3.2}$$

Thus,

$$b_{\varepsilon}(s) = b(T_{1/\varepsilon}(s)) + \varepsilon s \quad \text{for } \varepsilon > 0,$$

and

$$b_{\varepsilon}(s) \to b(s) \quad \text{converges almost everywhere on } Q. \tag{3.3}$$

Finally, there exists $(f_{\varepsilon})_{\varepsilon>0} \in L^{\infty}(Q)$ such that

$$f_{\varepsilon} \to f \quad \text{in } L^1(Q), \tag{3.4}$$

and there exists $(u_0^{\varepsilon})_{\varepsilon>0} \in L^{\infty}(Q)$ such that

$$b_{\varepsilon}(u_0^{\varepsilon}) \to b(u_0) \quad \text{in } L^1(\Omega).$$

The following regularized problem admits a weak solution u^{ε} :

$$\begin{cases} \dfrac{\partial b_{\varepsilon}(u^{\varepsilon})}{\partial t} - \operatorname{div}(A^{\varepsilon}(t,x,u^{\varepsilon})Du^{\varepsilon}) = f_{\varepsilon} & \text{in } Q,\\ b(u^{\varepsilon})(t=0) = b_0(u_0^{\varepsilon}) & \text{in } \Omega,\\ u^{\varepsilon} = 0 & \text{in } \partial\Omega\times\,]0,T[\,. \end{cases} \tag{3.5}$$

As a result, to show the existence of a weak solution $u^{\varepsilon} \in L^2(0,T;H_0^1(\Omega)$ of (3.5) is an easy task (see [4]).

Remark 3.1. Any weak solution is a renormalized solution. Indeed, for any $S \in W^{2,\infty}(\mathbb{R})$ and any $\varphi \in C_c^{\infty}((0,T)\times\Omega)$ such that $S'(u^{\varepsilon})\varphi \in L^2(0,T;H_0^1(\Omega))$, we can choose $S'(u^{\varepsilon})\varphi$ as a test function in (3.5), to deduce, using the integration-by-parts formula (see [4]), that

$$\frac{\partial B_s^{\varepsilon}(u^{\varepsilon})}{\partial t} - \operatorname{div}\left(S'(u)A^{\varepsilon}(t,x,u^{\varepsilon})Du^{\varepsilon}\right) + S''(u)A^{\varepsilon}(t,x,u^{\varepsilon})Du^{\varepsilon}\cdot Du^{\varepsilon} \\ = f_{\varepsilon}S'(u^{\varepsilon}) \text{ in } D'(Q), \tag{3.6}$$

where

$$B_s^{\varepsilon}(z) = \int_0^z b_{\varepsilon}'(s)S'(s)\,ds \quad \text{and} \quad B_s^{\varepsilon}(u^{\varepsilon})(t=0) = B_s(u_0^{\varepsilon})$$

for any $S \in W^{2,\infty}(\mathbb{R})$.

3.2. Step 2: Apriori estimate and weak limit of the field. The test function φ is always equal to $\varphi = \min\left(\frac{(T-\delta-t)^+}{\delta}, 1\right)$. Choosing $S'(r) = T_k(u^\varepsilon)$ in (3.6), we have

$$\frac{1}{\delta}\int_{T-2\delta}^{T-\delta}\int_\Omega\int_0^{u^\varepsilon} b'_\varepsilon(s)T_k(s)\,ds\,dx\,dt + \int_0^{T-2\delta}\int_\Omega A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon)\cdot DT_k(u^\varepsilon)\,dx\,dt \le k\left[\|f_\varepsilon\|_{L^1(Q)} + \|b_\varepsilon(u_0^\epsilon)\|_{L^1(\Omega)}\right]. \tag{3.7}$$

Let δ tend to 0. Then we have

$$\int_0^T\int_\Omega A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon)\cdot DT_k(u^\varepsilon)\,dx\,dt \le k\left[\|f_\varepsilon\|_{L^1(Q)} + \|b_\varepsilon(u_0^\epsilon)\|_{L^1(\Omega)}\right].$$

Thanks to (2.7) and $f_\varepsilon \in L^1(Q)$, we have

$$\alpha\int_Q |DT_k(u^\varepsilon)|^2\,dx\,dt \le Ck \tag{3.8}$$

and

$$X^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon) \in (L^2(Q))^N, \tag{3.9}$$

where $X^\varepsilon(x,s) = \left(x_{ij}^\varepsilon(x,s)\right)_{\substack{1\le i\le N\\1\le j\le N}}$ is the square root of the matrix $A^\varepsilon(x,s)$. To establish that $b(u)$ is in $L^\infty(0,T;L^1(\Omega))$, we replace $S'(r) = \chi_{(0,\tau)}T_1(u^\varepsilon)$ in (3.6). Proceeding as above, we get

$$\|b_\varepsilon(u^\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \le C.$$

Then we pass to the limit-inf as ε tends to 0, which gives that $b(u)$ belongs to $L^\infty(0,T;L^1(\Omega))$.

By a classical argument (see, e.g, [3]), for a subsequence still indexed by ε, from (3.8) and (3.3), we have

$$u^\varepsilon \to u \quad \text{a.e. in } Q, \tag{3.10}$$

$$b_\varepsilon(u^\varepsilon) \to b(u) \quad \text{a.e. in } Q, \tag{3.11}$$

$$\forall k>0 \quad T_k(u^\varepsilon) \rightharpoonup T_k(u) \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)). \tag{3.12}$$

Now, using of $S'(s) = T_{2m}^+(s) - T_m^+(s)$ in (3.6), leads to

$$\int_0^T\int_\Omega A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon)\cdot D\left(T_{2m}^+(u^\varepsilon) - T_m^+(u^\varepsilon)\right)\,dx\,dt \le C,$$

where C does not depend on ε. By (3.2), we have

$$\beta(m-\varepsilon)\int_Q \left|T_{2m}^+(u^\varepsilon) - T_m^+(u^\varepsilon)\right|^2\,dx\,dt \le C. \tag{3.13}$$

We pass to the limit in (3.13), as ε tends to 0, to deduce that

$$T_{2m}^{+}(u) - T_{m}^{+}(u) = 0 \quad \text{a.e. in } Q, \qquad (3.14)$$
$$u \leq m \quad \text{a.e. in } Q.$$

We define two sequences of auxiliary functions

$$v^{\varepsilon} = \int_{0}^{(u^{\varepsilon})^{+}} (\gamma(s)\,\sigma_{\varepsilon}(s) + (1 - b_{\varepsilon}(s))\beta(m-\varepsilon)\,ds \qquad (3.15)$$

and

$$d^{\varepsilon} = \int_{0}^{(u^{\varepsilon})^{+}} (\beta(s)\,\sigma_{\varepsilon}(s) + (1 - b_{\varepsilon}(s))\beta(m-\varepsilon)\,ds. \qquad (3.16)$$

For every $k \geq 0$, we have $T_k(v^{\varepsilon}) \in L^2(0,T;H_0^1(\Omega))$ and $T_k(d^{\varepsilon}) \in L^2(0,T;H_0^1(\Omega))$ with

$$\nabla T_k(v^{\varepsilon}) = \chi_{\{v^{\varepsilon}<k\}}\,[(\gamma(u^{\varepsilon})\,\sigma_{\varepsilon}(u^{\varepsilon}) + (1-\sigma_{\varepsilon}(u^{\varepsilon}))\beta(m-\varepsilon)]\,\nabla T_{k/\alpha}(u^{\varepsilon})^{+} \qquad (3.17)$$

and

$$\nabla T_k(d^{\varepsilon}) = \chi_{\{d^{\varepsilon}<k\}}\,[(\beta(u^{\varepsilon})\,\sigma_{\varepsilon}(u^{\varepsilon}) + (1-\sigma_{\varepsilon}(u^{\varepsilon}))\beta(m-\varepsilon)]\,\nabla T_{k/\alpha}(u^{\varepsilon})^{+}. \qquad (3.18)$$

By taking $S'(u^{\varepsilon}) = T_n(d^{\varepsilon} - (u^{\varepsilon})^{-})$ in (3.6), we have

$$\int_Q A^{\varepsilon}(t,x,u^{\varepsilon})Du^{\varepsilon}\cdot T_n(d^{\varepsilon} - (u^{\varepsilon})^{-})\,dx\,dt \leq C. \qquad (3.19)$$

Since the supports of d^{ε} and $(u^{\varepsilon})^{-}$ are disjoint, by using (3.18), we can deduce that

$$\sum_{i=1}^{N}\int_Q \chi_{\{d^{\varepsilon}<k\}}\,[\beta(u^{\varepsilon})\sigma_{\varepsilon}(u^{\varepsilon}) + (1-\sigma_{\varepsilon}(u^{\varepsilon}))]\left(A^{\varepsilon}(t,x,u^{\varepsilon})D\,(u^{\varepsilon})^{+}\right)\cdot DT_{\frac{n}{\alpha}}(u^{\varepsilon})^{+}dx\,dt$$
$$+\sum_{i=1}^{N}\int_Q \chi_{\{(u^{\varepsilon})^{-}<k\}}\sum_{j=1}^{N} A^{\varepsilon}(t,x,u^{\varepsilon})D\,(u^{\varepsilon})^{-}\cdot DT_n(u^{\varepsilon})^{-}dxdt \leq C. \qquad (3.20)$$

Now the definition (3.1) of A^{ε}, together with the assumptions (2.7), shows that

$$(1-\sigma_{\varepsilon}(s))\beta(m-\varepsilon)\,|\xi|^2 + \beta(s)\sigma_{\varepsilon}(s)\,|\xi|^2 \leq A^{\varepsilon}(t,x,u^{\varepsilon})\xi\cdot\xi$$

for any $s \in \mathbb{R}$, any $\xi \in \mathbb{R}^N$, and a.e. in Q.

Then (3.1) and (3.20) yield

$$\int_Q |DT_n(d^{\varepsilon})|^2\,dx\,dt + \alpha\int_Q \left|DT_n((u^{\varepsilon})^{-})\right|^2\,dx\,dt \leq C. \qquad (3.21)$$

Since the supports of d^ε and $(u^\varepsilon)^-$ are disjoint, we deduce that

$$\min(1,\alpha)\int_Q \left|T_n(d^\varepsilon-(u^\varepsilon)^-)\right|^2\,dx\,dt \le C. \tag{3.22}$$

Poincaré's inequality and (3.22) lead to

$$n^2\,\text{meas}\left\{(t,x)\in Q \;\middle|\; \left|d^\varepsilon-(u^\varepsilon)^-\right|>n\right\}=0,$$

where C does not depend on n and ε, and we obtain that

$$\lim_{n\to+\infty}\sup_\varepsilon\left\{(t,x)\in Q \;\middle|\; \left|d^\varepsilon-(u^\varepsilon)^-\right|>n\right\}=0. \tag{3.23}$$

To obtain the analog of (3.23) with

$$v^\varepsilon=d^\varepsilon+\int_0^{(u^\varepsilon)^+}(\gamma(s)-\beta(s)\sigma_\varepsilon(s)\,ds\le d^\varepsilon+\int_0^m(\gamma(s)-\beta(s)\sigma_\varepsilon(s)\,ds, \tag{3.24}$$

where

$$\int_0^m(\gamma(s)-\beta(s))\sigma_\varepsilon(s)ds<+\infty,$$

by (2.6) and (3.23) it follows that

$$\lim_{n\to+\infty}\sup_\varepsilon\left\{(t,x)\in Q \;\middle|\; \left|v^\varepsilon-(u^\varepsilon)^-\right|>n\right\}=0. \tag{3.25}$$

Next, by (3.22), following the same procedures as above, we obtain

$$d^\varepsilon\to d \quad \text{a.e. in } Q, \tag{3.26}$$

where d is a measurable function. Then, by (3.10), (3.24), and (3.26), we have

$$v^\varepsilon\to v \quad \text{a.e. in } Q, \tag{3.27}$$

where

$$v=d+\int_0^{(u)^+}(\gamma(s)-\beta(s)\sigma_\varepsilon(s)\,ds$$

and v is a measurable positive function. Referring to the definition of σ_ε in (2.8) and of v^ε in (3.15), as well as to the convergence (3.12) and (3.27), it is seen that

$$v=\int_0^{(u)^+}\gamma(s)\,ds \quad \text{a.e. in } \{(x,t)\in Q \mid u\,(x,t)<m\}\,. \tag{3.28}$$

However, as far as we know, we cannot expect to have a similar identification on the subset $\{(t,x)\in Q \mid u\,(t,x)=m\}$.

Now we choose $S'(u^\varepsilon)=\theta_n\left(v^\varepsilon-(u^\varepsilon)^-\right)$ in (3.6) which gives us

$$\frac{1}{n}\int_{\{n\le|v^\varepsilon-(u^\varepsilon)^-|\le 2n\}}A^\varepsilon(t,x,u^\varepsilon)D\,(u^\varepsilon)\cdot DT_n(v^\varepsilon-(u^\varepsilon)^-)\,dx\,dt$$

$$\leq \int_{\{|v^{\varepsilon}-(u^{\varepsilon})^{-}|>n\}} |f_{\varepsilon}| \, dx \, dt + \int_{\Omega} \int_{0}^{|u_0|} \left| b'_{\varepsilon}(r) \theta_n \left(G^{\varepsilon}(r) - (r)^{-} \right) \right| \, dr \, dx. \quad (3.29)$$

As for the second term, it should be noticed that the support of $G^{\varepsilon}(r)$ and r^{-} are disjoint. Thus, using $f_{\varepsilon} \in L^1(Q)$ and (3.25), we obtain

$$\lim_{n\to+\infty} \sup_{\varepsilon} \frac{1}{n} \int_{\{n\leq|v^{\varepsilon}-(u^{\varepsilon})^{-}|\leq 2n\}} A^{\varepsilon}(t,x,u^{\varepsilon}) D(u^{\varepsilon}) \cdot DT_n(v^{\varepsilon} - (u^{\varepsilon})^{-}) \, dx \, dt = 0. \quad (3.30)$$

Repeating the above argument with $S'(r) = \theta_n(r)$, we have

$$\lim_{n\to+\infty} \sup_{\varepsilon} \frac{1}{n} \int_{\{n\leq|u^{\varepsilon}|\leq 2n\}} A^{\varepsilon}(t,x,u^{\varepsilon}) D(u^{\varepsilon}) \cdot DT_n(u^{\varepsilon}) \, dx \, dt = 0. \quad (3.31)$$

To prove the weak limit of the field, we need to see that $A^{\varepsilon}(t,x,u^{\varepsilon})Du^{\varepsilon}$ is bounded in $L^2(Q)$ for every $i = 1, \ldots, N$ in the subset, where $v^{\varepsilon} - (u^{\varepsilon})^{-}$ is truncated. Indeed, we plug the test function $T_k(v^{\varepsilon})$ in (3.6) and, by using (3.17), we obtain

$$\int_{\{|v^{\varepsilon}|\leq k\}} A^{\varepsilon}(t,x,u^{\varepsilon}) D(u^{\varepsilon}) \cdot D(u^{\varepsilon})^{+} \left[(\gamma(u^{\varepsilon}) \sigma_{\varepsilon}(u^{\varepsilon}) + (1-\sigma_{\varepsilon}(s))\beta(m-\varepsilon) \right] dx \, dt \leq C.$$

By the definition (3.1) of $A^{\varepsilon}(x,s)$ and (2.7), we get

$$A^{\varepsilon}(t,x,s)\xi \cdot \xi \leq (\gamma(s) \sigma_{\varepsilon}(s) + (1-\sigma_{\varepsilon}(s))\beta(m-\varepsilon) |\xi|^2 \quad (3.32)$$

for any $s \in \mathbb{R}$, any $\xi \in \mathbb{R}^N$, and a.e. in Ω. Use (3.32) with $\xi = X^{\varepsilon}(x,u^{\varepsilon}) D(u^{\varepsilon})^{+}$. Therefore,

$$\int_{\{|v^{\varepsilon}|\leq k\}} \left| A^{\varepsilon}(t,x,s) D(u^{\varepsilon})^{+} \right|^2 dx \, dt \leq C.$$

Then, for any $k \geq 0$,

$$\chi_{\{v^{\varepsilon}<k\}} A^{\varepsilon}(t,x,s) D(u^{\varepsilon})^{+} \quad \text{is bounded in } L^2(Q) \text{ uniformly in } \varepsilon.$$

Now, since $\chi_{\{|v^{\varepsilon}-(u^{\varepsilon})^{-}|<k\}} = \chi_{\{0\leq v^{\varepsilon}<k\}} + \chi_{\{0\leq u^{\varepsilon}<k\}}$ a.e. in Ω, by the continuous character of $A^{\varepsilon}(t,x,s)$ for $s \in (-\infty, 0]$ and the estimate (3.8), we have

$$\chi_{|v^{\varepsilon}-(u^{\varepsilon})^{-}|<k} A^{\varepsilon}(t,x,u^{\varepsilon}) D(u^{\varepsilon})^{+} \quad \text{is bounded in } \left(L^2(Q)\right)^N \text{ uniformly in } \varepsilon. \quad (3.33)$$

Use the estimates (3.33) and (3.9) to extract another subsequence, still indexed by ε, such that

$$\begin{aligned} h_n(v^{\varepsilon} - (u^{\varepsilon})^{-}) A^{\varepsilon}(t,x,u^{\varepsilon}) D(u^{\varepsilon}) \to \psi_n \quad & \text{weakly in } (L^2(Q))^N, \\ X^{\varepsilon}(t,x,u^{\varepsilon}) DT_k(u^{\varepsilon}) \to Y_k \quad & \text{weakly in } L^2(Q) \end{aligned} \quad (3.34)$$

as ε tends to 0, where for any $k \geq 0$ and $n \geq 1$, $\psi_n \in L^2(Q)$ and $Y_k \in L^2(Q)$.

Next, we identify ψ_n on the subset, where $u < m$. Let h be a $C^\infty(\mathbb{R})$-function such that supp(h) is compact in $(-M, l)$ with $l < m$ and $M > 0$. Then, using the fact that $h(s)A^\varepsilon(t,x,u^\varepsilon) = h(s)A(t,x,T_l(s^+) - T_M(s^-))$ for ε small enough and the convergences (3.12), and (3.27), we have

$$h(u^\varepsilon)h_n(v^\varepsilon - (u^\varepsilon)^-)A^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon \to h(u)h_n(v - u^-)A(t,x,u)Du \quad \text{weakly in } \left(L^2(Q)\right)^N \tag{3.35}$$

as ε tends to 0 and Du stands for $DT_l(u^+) - DT_M(u^+)$. It follows from (3.35) and (3.2) that

$$\psi_n = h_n\,(v-u)\,A(t,x,u)Du \quad \text{a.e. in } \{(t,x)\in Q \mid u\,(t,x) < m\} \tag{3.36}$$

since $l < m$ and M are arbitrary.

It should be noticed that on the subset $\{(t,x)\in Q \mid u\,(t,x) < m\}$ we have

$$0 \le v = \int_0^{(u)^+} \gamma(s)\,ds < \int_0^m \gamma(s)\,ds.$$

Then, for

$$n > \int_0^m \gamma(s)ds,$$

we have $h_n\,(v-u) = h_n\,(-u)$ on $\{(t,x)\in Q \mid u\,(t,x) < m\}$. It follows from (3.36) that

$$\psi_n = h_n\,(-u)\,A(t,x,u)Du \quad \text{a.e. in } \{(t,x)\in Q \mid u\,(t,x) < m\}$$

which, in turn, implies that

$$\chi_{\{-k<u<m\}}A(t,x,u)Du \in \left(L^2(Q)\right)^N. \tag{3.37}$$

To identify Y_n, we use ψ_n defined above. For every $k \ge 0$, we have

$$h_n(v^\varepsilon - (u^\varepsilon)^-)A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon) \to \psi_n^k \quad \text{weakly in } \left(L^2(Q)\right)^N.$$

We can write

$$\begin{aligned} &h_n(v^\varepsilon - (u^\varepsilon)^-)X^\varepsilon(t,x,u^\varepsilon)T_k(u^\varepsilon) \\ &\qquad = h_n(v^\varepsilon - (u^\varepsilon)^-)\left(X^\varepsilon(t,x,u^\varepsilon)\right)^{-1} A^\varepsilon(t,x,u^\varepsilon)T_k(u^\varepsilon). \end{aligned}$$

Using some technique developed in ([6]), we can deduce that

$$Y_k = \chi_{\{u<m\}}X(t,x,u)T_k(u) \quad \text{a.e. in } Q.$$

3.3. Step 3: Strong convergence of the field. To show this convergence, several authors use a particular temporal regularization (see, e.g., [10]). In this paper, we use a method developed in [4] by Porretta for the Stefan problem.

Let $\xi \in C_0^\infty(0,T]$ such that $0 \le \xi \le 1$. We choose $S'(r) = h_n(r)T_k(r)$ and $\xi = \varphi$ in (3.6) to obtain

$$\begin{aligned}\int_Q A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon)\cdot DT_k(u^\varepsilon)\,dx\,dt &\le \int_Q \xi_t \int_0^{u^\varepsilon} b'_\varepsilon(s)h_n(s)T_k(s)\,ds\,dx\,dt \\ &+ \int_Q \xi(0)\int_0^{u_0^\varepsilon} b'_\varepsilon(s)h_n(s)T_k(s)\,ds\,dx\,dt + \int_Q \xi f_\varepsilon h_n(u^\varepsilon)T_k(u^\varepsilon)\,ds\,dx\,dt \\ &+ k\frac{1}{n}\int_{\{n<|u^\varepsilon|<2n\}} A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon)\cdot DT_k(u^\varepsilon)\,dt\,dx. \end{aligned} \tag{3.38}$$

We pass to the limit as ε tends to 0 in (3.38) and, by using (3.10) and (3.3), we find that

$$\begin{aligned}\limsup_{\varepsilon\to 0}\int_Q A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon)\cdot DT_k(u^\varepsilon)\,dx\,dt &\le \int_Q \xi_t\int_0^u b'(s)h_n(s)T_k(s)\,ds\,dx\,dt \\ &+ \int_Q \xi(0)\int_0^{u_0} b'(s)h_n(s)T_k(s)\,ds\,dx\,dt + \int_Q \xi f h_n(u)T_k(u)\,ds\,dx\,dt \\ &+ \limsup_{\varepsilon\to 0} k\frac{1}{n}\int_{\{n<|u^\varepsilon|<2n\}} A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon)\cdot DT_k(u^\varepsilon)\,dx\,dt.\end{aligned}$$

Using (3.31), we have

$$\lim_{n\to+\infty}\limsup_{\varepsilon\to 0} k\frac{1}{n}\int_{\{n<|u^\varepsilon|<2n\}} A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon)\cdot DT_k(u^\varepsilon)\,dx\,dt = 0. \tag{3.39}$$

Now, using (3.39), we pass to the limit as n tends to $+\infty$ and obtain

$$\begin{aligned}\lim_{n\to+\infty}\limsup_{\varepsilon\to 0}&\int_Q A^\varepsilon(t,x,u^\varepsilon)DT_k(u^\varepsilon)\cdot DT_k(u^\varepsilon)\,dx\,dt \\ &\le \int_Q \xi_t\int_0^u b'(s)T_k(s)\,ds\,dx\,dt \\ &+ \int_Q \xi(0)\int_0^{u_0} b'(s)T_k(s)\,ds\,dx\,dt + \int_Q \xi f T_k(u)\,dx\,dt.\end{aligned} \tag{3.40}$$

Then, using $S'_n(r) = h_n(G^\varepsilon(r^+) - r^-)$ in (3.6), we have

$$\begin{aligned}&-\|\varphi\|_{L^\infty(\Omega)}\frac{1}{n}\int_{\{n<|v^\varepsilon-(u^\varepsilon)^-|<2n\}} A^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon\cdot D(v^\varepsilon-(u^\varepsilon)^-)\,dx\,dt \\ &\le -\int_Q \varphi_t\int_0^{u^\varepsilon} b'_\varepsilon(s)h_n(G^\varepsilon(s^+)-s^-)\,ds\,dx\,dt \\ &-\int_Q \xi(0)\int_0^{u_0} b'_\varepsilon(s)h_n(G^\varepsilon(s^+)-s^-)\,ds\,dx\,dt - \int_Q \varphi f h_n(v^\varepsilon-(u^\varepsilon)^-)\,dx\,dt\end{aligned}$$

$$+\int_Q A^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon \cdot D\varphi h_n(v^\varepsilon-(u^\varepsilon)^-)\,dx\,dt$$
$$\le \|\varphi\|_{L^\infty(\Omega)}\frac{1}{n}\int_{\{n<|v^\varepsilon-(u^\varepsilon)^-|<2n\}} A^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon\cdot D(v^\varepsilon-(u^\varepsilon)^-)\,dx\,dt. \quad (3.41)$$

We want now to pass to the limit in ε. First, we remark that for $n>\int_0^m \gamma(s)ds$,

$$h_n(G^\varepsilon(s^+)-s^-)\to \chi_{\{s<0\}}h_n(-s^-)+\chi_{\{0<s<m\}}h_n(s^+) \quad (3.42)$$

as ε tends 0. As a consequence of (3.42), it follows that

$$\int_Q \varphi_t\int_0^{u^\varepsilon} b'(s)h_n(G^\varepsilon(s^+)-s^-)\,ds\,dx\,dt$$
$$\to \int_Q \varphi_t\left[\int_0^{-u^-} b'(s)h_n(s)ds+b\left(T_m^+(u)\right)\right]dx\,dt$$

and

$$\int_Q \varphi_t\int_0^{u_0} b'(s)h_n(G^\varepsilon(s^+)-s^-)\,ds\,dx\,dt$$
$$\to \int_Q \varphi_t\left[\int_0^{-u_0^-} b'(s)h_n(s)ds+b\left(T_m^+(u_0)\right)\right]dx\,dt.$$

Secondly, from (3.36), we get

$$\int_Q A^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon\cdot D\varphi h_n(v^\varepsilon-(u^\varepsilon)^-)\,dx\,dt\to\int_Q \psi_n\cdot D\varphi\,dx\,dt,$$

Further, using (3.36) and the inequalities

$$\int_Q f_\varepsilon\varphi\,dx\,dt-\|\varphi\|_{L^\infty(\Omega)}\int_{\{|v^\varepsilon-(u^\varepsilon)^-|>n\}}|f_\varepsilon|\,dx\,dt\le\int_Q \varphi f_\varepsilon h_n(v^\varepsilon-(u^\varepsilon)^-)\,dx\,dt$$
$$\le\int_Q f_\varepsilon\varphi\,dx\,dt+\|\varphi\|_{L^\infty(\Omega)}\int_{\{|v^\varepsilon-(u^\varepsilon)^-|>n\}}|f_\varepsilon|\,dx\,dt$$

setting

$$\kappa_1(n)=\frac{1}{n}\sup_\varepsilon\frac{1}{n}\int_{\{n<|v^\varepsilon-(u^\varepsilon)^-|<2n\}} A^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon\cdot D(v^\varepsilon-(u^\varepsilon)^-)\,dx\,dt$$

and

$$\kappa_2(n)=\sup_\varepsilon\int_{\{|v^\varepsilon-(u^\varepsilon)^-|>n\}}|f_\varepsilon|\,dx\,dt,$$

we pass to the limit

$$-\|\varphi\|_{L^\infty(\Omega)}(\kappa_1(n)+\kappa_2(n))\le-\int_{\{u(t,x)=m\}}\psi_n\cdot D\varphi\,dx\,dt$$

$$
\begin{aligned}
& - \int_Q \varphi_t \left[\int_0^{-u_0^-} b'(s) h_n(s) ds + b\left(T_m^+(u_0)\right) \right] dx\, dt \\
& + \int_Q \varphi_t \left[\int_0^{-u^-} b'(s) h_n(s) ds + b\left(T_m^+(u)\right) \right] dx\, dt \\
& + \int_{u(t,x)<m} A(t,x,u) Du \cdot D\varphi h_n(v-(u)^-)\, dx\, dt \\
& - \int_Q f_\varepsilon \varphi dx dt \leq \|\varphi\|_{L^\infty(\Omega)} (\kappa_1(n) + \kappa_2(n)).
\end{aligned}
$$

Now, let u_{0j} be a sequence of the class $C_0^\infty(\Omega)$ such that

$$u_{0j} \to u_0 \quad \text{strongly in } L^1(\Omega)$$

and

$$u(t) = u_{0j} \quad \text{for } t < 0.$$

We choose

$$\varphi = \xi \frac{1}{h} \int_{t-h}^t T_k(u(\tau)) d\tau$$

as a test function in (3.6), which gives us

$$
\begin{aligned}
-k(\kappa_1(n) + \kappa_2(n)) \leq & \int_{\{u(t,x)=m\}} \psi_n \cdot D\left(\frac{1}{h} \int_{t-h}^t T_k(u(\tau))\, d\tau \right) dx\, dt \\
& - \int_Q \varphi(0) \left[\int_0^{-u_0^-} b'(s) h_n(s) ds + b(T_m^+(u_0)) \right] dx\, dt \\
& - \int_Q \frac{\partial}{\partial t} \left(\xi \frac{1}{h} \int_{t-h}^t T_k(u(\tau))\, d\tau \right) \left[\int_0^{-u^-} b'(s) h_n(s)\, ds + b(T_m^+(u)) \right] dx\, dt \\
& - \int_{u(t,x)<m} A(t,x,u) Du \cdot D\left(\frac{1}{h} \int_{t-h}^t T_k(u(\tau)) d\tau \right) h_n(v-(u)^-) \omega_i\, dx\, dt \\
& - \int_Q f\varphi\, dx\, dt \leq k(\kappa_1(n) + \kappa_2(n)).
\end{aligned}
$$

To control the parabolic term in the previous inequality, we apply Lemma 2.3 from [4] with $w = u$, $F(u) = u$,

$$B(r) = \int_0^{-r} h_n(s) b'(s) ds + T_m^+(r),$$

and we can easily prove that when h tends to 0

$$\frac{1}{h} \int_{t-h}^t T_k(u(\tau)\, d\tau \to T_k(u) \quad \text{strongly in } L^2(0,T; H_0^1(\Omega)),$$

we get

$$-k(\kappa_1(n) + \kappa_2(n)) \leq - \int_Q \xi_t \left[\left(\int_0^{-u^-} b'(s) h_n(s) ds + b(T_m^+(u)) \right) T_k(u)\, dr \right.$$

$$- \int_0^u T_k'(r)\left(b(-r^-)+b(T_m^+(u))\right)dr\Bigg]\,dx\,dt$$
$$- \int_\Omega \xi(0)\left[\left(\left(\int_0^{u_0^-} b'(s)h_n(s)\,ds + b(T_m^+(u_0))\right]T_k(u_{0j})\,dr\right.\right.$$
$$- \int_0^{u_{0j}} T_k'(r)\left(b(-r^-)+b(T_m^+(u))\right)dr\Bigg]\,dx$$
$$+ \int_{\{u(t,x)=m\}} \xi\psi_n \cdot DT_k(u_0)\,dx\,dt$$
$$+ \int_{u(t,x)<m} \xi A(t,x,u)Du\cdot DT_k(u)h_n(v-(u)^-)\,dx\,dt$$
$$- \int_Q f\xi T_k(u)\;dx\,dt. \tag{3.43}$$

Finally, let n go to infinity. Observe first that, by the definition of $T_k(s)$, we have

$$\chi_{\{u=m\}}\psi_n \cdot DT_k(u) = 0\ .$$

Thanks to (3.31) and (3.25), we have

$$\kappa_1(n)\to 0 \quad \text{and} \quad \kappa_2(n)\to 0.$$

Since $h_n(s)\to 1$ and for every $n > \int_0^m \gamma(s)ds$, the inequality (3.43) yields

$$0 \le -\int_Q \xi_t\left[\left(\int_0^{-u^-} b'(s)h_n(s)ds + b(T_m^+(u))\right)T_k(u)\,dr\right.$$
$$- \int_0^u T_k'(r)\left(b(-r^-)+b(T_m^+(u))\right)dr\Bigg]\,dx\,dt$$
$$- \int_\Omega \xi(0)\left[\left(\left(\int_0^{u_0^-} b'(s)h_n(s)\,ds + b(T_m^+(u_0))\right]T_k(u_{0j})\,dr\right.\right.$$
$$- \int_0^{u_{0j}} T_k'(r)\left(b(-r^-)+b(T_m^+(u))\right)dr\Bigg]\,dx$$
$$- \int_{u(t,x)<m} \xi A(t,x,u)Du\cdot DT_k(u)\,dx\,dt - \int_Q f\xi T_k(u)\,dx\,dt. \tag{3.44}$$

Notice that for every $s\le m$, we have

$$\left(b(-r^-)+b(T_m^+(s))\right)T_k(s) - \int_0^s T_k'(r)\left(b(-r^-)+b(T_m^+(r))\right)dr$$
$$= \int_0^s b'(s)T_k(r)\,dr. \tag{3.45}$$

Thus, from (3.45) and putting together (3.44) and (3.40), we can prove that

$$\limsup_{\varepsilon\to 0}\int_Q \xi A(t,x,u)Du\cdot DT_k(u)dx\,dt$$

$$\leq \int_{\{u(t,x)<m\}} \xi A(t,x,u)Du \cdot DT_k(u)\,dx\,dt.$$

By Minty's trick lemma, we conclude that for any $k \geq 0$ and any $0 < \tau < T$,

$$\chi_{\{u^\varepsilon<m\}} X^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon \cdot DT_k(u^\varepsilon) \to \chi_{\{u<m\}} X(t,x,u^\varepsilon)Du \cdot DT_k(u) \quad \text{strongly in } L^2(0,\tau;H_0^1(\Omega)) \tag{3.46}$$

for every $i = 1, \ldots, N$. Note that (3.46) implies that

$$T_k(u^\varepsilon) \to T_k(u) \quad \text{strongly in } L^2(0,\tau;H_0^1(\Omega)). \tag{3.47}$$

3.4. Step 4: End of the proof. In this step, we prove that u is a renormalized solution in the sense of definition. It is easy to prove that u satisfies (2.10)–(2.13).

Firstly, we prove that u satisfies (2.16). Let $S \in W^{2,\infty}(\mathbb{R})$, with $\operatorname{supp}(S') \subset (-L,m)$ being compact. Then we obtain

$$\frac{\partial B_s^\varepsilon(u^\varepsilon)}{\partial t} - \operatorname{div}(S'(u^\varepsilon)A^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon) + S''(u^\varepsilon)A^\varepsilon(t,x,u^\varepsilon)Du^\varepsilon \cdot Du^\varepsilon = fS'(u^\varepsilon) \quad \text{in } D'(Q), \tag{3.48}$$

where

$$B_s^\varepsilon(z) = \int_0^z b_\varepsilon'(s)S'(s)ds.$$

Taking the limit as ε tends to 0 and n tends to $+\infty$ in (3.48).

Limit of $\frac{\partial B_s^\varepsilon(u^\varepsilon)}{\partial t}$. Since S is bounded and continuous, according to the convergences (3.11) and (3.10), we have that $\frac{\partial B_s^\varepsilon(u^\varepsilon)}{\partial t}$ converges to $\frac{\partial B_s(u)}{\partial t}$ in $D'(Q)$ as n tends to $+\infty$.

Limit of the second and the third terms in (3.48). Since $\operatorname{supp}(S') \subset (-L,L)$, we can replace u^ε by $T_L(u^\varepsilon)$ in the second and the third terms of (3.48). Then, due to (3.10) and (3.46), we have

$$S''(T_L(u^\varepsilon))A^\varepsilon(t,x,u^\varepsilon)DT_L(u^\varepsilon) \cdot DT_L(u^\varepsilon) \rightharpoonup S''(T_L(u))A(t,x,u)DT_L(u) \cdot DT_L(u) \quad \text{weakly in } L^1(Q).$$

In view of (3.10) and (3.47), we have

$$S'(T_L(u^\varepsilon))A^\varepsilon(t,x,u^\varepsilon)DT_L(u^\varepsilon) \rightharpoonup S'(T_L(u))A(t,x,u)DT_L(u) \quad \text{weakly in } L^2(Q)$$

for every $i = 1, \ldots, N$.

Limit of the right-hand side of (3.48). Due to (3.10) and (3.4), we have

$$f_\varepsilon S(u^\varepsilon) \to f S(u) \quad \text{strongly in } L^1(Q).$$

Secondly, we prove that u satisfies (2.14). We choose $S'(r) = \theta_p(-r^-)$ in (3.6) for a fixed integer $p \geq 1$ and we do the same procedure as in Step 2 to obtain

$$\lim_{p\to+\infty} \frac{1}{p} \int_{\{-2p<u<-p\}} A(t,x,u)Du \cdot Du\, dx\, dt = 0.$$

Finally, to establish (2.1), we take $S'(r) = (1-\sigma_{1/p}(r^+))$, where p is a fixed integer ≥ 1, and for any $\varphi \in C_c^\infty([0,T])$ in (3.6), we have

$$-\int_Q \varphi_t \int_0^{u^\varepsilon} (1-\sigma_{1/p}(r^+))b'(r)\, dr\, dt\, dx - \int_\Omega \varphi(0) \int_0^{u^0} b'(r)(1-\sigma_{1/p}(r^+))\, dr\, dx$$
$$+p\int_Q \chi_{\{m-2/p<u<m-1/p\}} A(t,x,u)Du \cdot \varphi\, dx\, dt = \int_Q f_\varepsilon (1-\sigma_{1/p}(u^+))\varphi\, dx\, dt.$$

Now, as p tends to $+\infty$, $(1-\sigma_{1/p}(u^+)) \to \chi_{\{u=m\}}$ a.e. in Q, we have

$$\lim_{p\to+\infty} p\int_Q \chi_{\{m-2/p<u<m-1/p\}} A(t,x,u)Du \cdot Du\varphi\, dx\, dt = \int_Q f\chi_{\{u=m\}}\varphi\, dx\, dt,$$

which is (2.1).

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Існування ренормалізованого розв'язку для класу параболічних задач

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У цій статті ми доводимо існування ренормалізованого розв'язку для нелінійного виродженого параболічного рівняння $\frac{\partial b(u)}{\partial t} - \operatorname{div}(A(t,x,u)Du) = f$, де матриця $A(t,x,s) = (a_{ij}(t,x,s))_{\substack{1\leq i\leq N\\ 1\leq j\leq N}}$ не контролюється за u, $f \in L^1(Q)$, а b є строго зростальною C^1-функцією.

Ключові слова: ренормалізований розв'язок, вибух, L^1-дані