Journal of Mathematical Physics, Analysis, Geometry, **21** (2025), No. 3, 319–345
doi:

Nonlinear Partial Differential Equations in Module of Copolynomials over a Commutative Ring

S.L. Gefter and A.L. Piven'

Let K be an arbitrary commutative integral domain with identity of characteristic 0. We study the copolynomials of n variables, i.e., K-linear mappings from the ring of polynomials $K[x_1, \ldots, x_n]$ into K. We consider copolynomials as algebraic analogues of distributions. With the help of the Cauchy–Stieltjes transform of a copolynomial, we introduce and study a multiplication of copolynomials. We prove the existence and uniqueness theorem of the Cauchy problem for some nonlinear partial differential equations in the ring of formal power series with copolynomial coefficients. We study a connection between some classical nonlinear partial differential equations and integer sequences. In particular, for the Cauchy problem for the Burgers equation, we obtain the representation of the unique solution to this problem in the form of the series in powers of δ-function with integer coefficients.

Key words: copolynomial, δ-function, differential operator of infinite order, Cauchy problem, Cauchy–Stieltjes transform, multiplication of copolynomials

Mathematical Subject Classification 2020: 35R50, 13B25, 35G20, 11Y55

1. Introduction

This paper is a continuation of the paper [9], where linear partial differential equations in the module of copolynomials over a commutative integral domain were studied. Moreover, this paper can be considered as the continuation of the paper [12], where a special class of nonlinear PDE's was studied in the case of one space variable. In the present paper, we investigate the ring of copolynomials of several variables and consider general evolution nonlinear partial differential equations in this ring.

Let K be a commutative integral domain with identity of characteristic 0 [20, Section 1.43], and let $K[x_1, \ldots, x_n]$ be the ring of polynomials of n variables. The most interesting case for us is $K = \mathbb{Z}$. A K-linear functional on the ring $K[x_1, \ldots, x_n]$ is said to be a copolynomial. The module of copolynomials is denoted by $K[x_1, \ldots, x_n]'$. We consider the module $K[x_1, \ldots, x_n]'$ as an algebraic analogue of the space of distributions (see [9, 10, 12]). General properties of copolynomials of n variables are considered in Section 2. In particular, we introduce the notion of a homogeneous copolynomial and obtain an analogue of the Euler differential equation, which completely characterizes these copolynomials (see Theorem 2.8). The properties of copolynomials of one variable were partially studied in [10–13,16]. We notice that in papers [10,11,13,16] copolynomials were called formal generalized functions. In Section 3, we introduce and study the Cauchy–Stieltjes transform of copolynomials. In Section 4, the Cauchy–Stieltjes transform is used to define a multiplication of copolynomials (see Definition 4.1). The main results of this section are the Leibnitz formula for a linear differential operator of infinite order (Theorem 4.8) and the analogue of the formula for partial derivatives of composition (see Theorem 4.10). It should be noticed that several non-equivalent constructions of a multiplication are considered in the classical theories of distributions. For example, in the Colombeau theory [5,6], the square of the δ-function is well defined, but in some other theories it is not defined (see, for example, [1, Section 12.5]). The main results are obtained in Section 5. In Subsection 5.1, we prove the existence and uniqueness theorem to the Cauchy problem for some nonlinear partial differential equations (see Theorem 5.1). In Subsection 5.2, we consider the Cauchy problem for the Burgers equation with the initial condition $u_0\delta(x)$, where $u_0 \in K$ and $\delta(x)$ is the copolynomial δ-function (see Example 2.3). Unlike the general Theorem 5.1, we do not impose additional restrictions on the ring K and obtain the representation of the unique solution to this problem as a series in powers of the δ-function (see Theorem 5.2). Thus, if $K = \mathbb{Z}$, then we obtain the series with integer coefficients. Such results were obtained in [12] for some nonlinear PDE's of the first order only. In Subsection 5.3, we present other meaningful examples which illustrate the constructed theory. In particular, in this section we consider the Cauchy problem for the Harry Dym type equation and for the Korteweg-de Vries equation. In Examples 5.10 and 5.11, nonlinear PDE's with several variables are considered. Notice that in Example 5.11 it happens that we are able to construct a solution over the ring K without the assumption $K \supset \mathbb{Q}$. As in [12, Examples 4.1, 4.2 and 4.4], here the problem of an integrality of coefficients of expansions of solutions in series in powers of the δ-function has appeared. It is noticeable that the unique solution to the Cauchy problem for the Korteweg–de Vries equation with the initial condition $\delta'(x)$ can be represented as the series in powers of the δ-function (see Example 5.8). But the integrality of the coefficients of this representation has yet to be proven. Numerical calculations show that the first 1000 of these coefficients are integers.

Linear functionals in the space of polynomials were studied from different points of view in a number of works on algebra, combinatorics, and the theory of orthogonal polynomials (cf., for example, [7, 8, 23, 24]). In the classical case ($K = \mathbb{R}$ or $K = \mathbb{C}$), the series with respect to the derivatives of the δ-function are intensively studied because of their applications in differential and functional-differential equations and the theory of orthogonal polynomials (see, for example, [7, 17, 19]).

In the 1970s, the study of the Korteweg-de Vries equation and other nonlinear partial differential equations developed remarkable techniques that connected these equations to inverse problems of the spectral theory, infinite-dimensional differential geometry, special problems of the algebraic geometry and the theory of Lie algebras (see, for example, [3, 14, 22]. In particular, in [14], Gel'fand and Dikii described compatible Hamiltonian structures on rings of differential and pseudo-differential operators for the Korteweg-de Vries type equation. In this paper, we do not use these methods. In addition, the main object for us is the Cauchy problem, where the initial condition is an algebraic analogue of the δ-function. For us, the most important problem is when the coefficients of the expansion for the unique solution of the Cauchy problem in powers of the δ-function belong to the ring K and, in particular, to the ring of integers. This problem may possibly be connected to the classical constructs of Gel'fand and Dikii [14], but this connection requires a separate study.

2. Preliminaries

Definition 2.1. By a copolynomial over the ring K, we mean a K-linear functional defined on the ring $K[x_1,\dots,x_n]$, i.e., a homomorphism from the module $K[x_1,\dots,x_n]$ into the ring K.

We denote the module of copolynomials over K by $K[x_1,\dots,x_n]'$. Thus, $T \in K[x_1,\dots,x_n]'$ if and only if $T : K[x_1,\dots,x_n] \to K$ and T has the property of K-linearity: $T(ap+bq) = aT(p)+bT(q)$ for all $p,q \in K[x_1,\dots,x_n]$ and $a,b \in K$. If $T \in K[x_1,\dots,x_n]'$ and $p \in K[x_1,\dots,x_n]$, then for the value of T on p, we use the notation (T,p). We also write the copolynomial $T \in K[x_1,\dots,x_n]'$ in the form $T(x)$, where $x = (x_1,\dots,x_n)$ is considered as the argument of polynomials $p(x) \in K[x_1,\dots,x_n]$ subjected to the action of the K-linear mapping T. In this case, the result of action of T upon p can be represented in the form $(T(x),p(x))$. We define the product $qT \in K[x_1,\dots,x_n]'$ of $q \in K[x_1,\dots,x_n]$ and $T \in K[x_1,\dots,x_n]'$ by $(qT,p) = (T,qp)$ for $p \in K[x_1,\dots,x_n]$.

Let $\mathbb{N}_0$ be the set of nonnegative integers. For a multi-index $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n$, we put (see [26, Chapter 1, §1–2]):

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}, \quad |\alpha| = \sum_{j=1}^n \alpha_j,$$
$$x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}, \quad \alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!.$$

For multi-indexes $\alpha,\beta \in \mathbb{N}_0^n$, the relation $\alpha \le \beta$ means that $\alpha_j \le \beta_j$ for all $j = 1,\dots,n$. If $\alpha \le \beta$, then we use the notation $\binom{\beta}{\alpha} = \prod_{j=1}^n \binom{\beta_j}{\alpha_j}$.

Let $p(x) = \sum_{|\alpha|\le m} a_\alpha x^\alpha \in K[x_1,\dots,x_n]$. If $h = (h_1,\dots,h_n)$, then the polynomial $p(x+h) \in K[x_1,\dots,x_n][h_1,\dots,h_n]$ can be represented in the form

$$p(x+h) = \sum_{|\alpha|\le m} p_\alpha(x)h^\alpha,$$

where $p_\alpha(x) \in K[x_1,\dots,x_n]$. Since in the case of a field with zero characteristic $p_\alpha(x) = \frac{D^\alpha p(x)}{\alpha!}$, we also assume that, by definition, $\frac{D^\alpha p(x)}{\alpha!} = p_\alpha(x), \quad |\alpha| \le m$, is true for any commutative ring K. For $m < |\alpha|$, we assume that $\frac{D^\alpha p(x)}{\alpha!} = 0$.

Definition 2.2. The partial derivative $\frac{\partial T}{\partial x_j}$ of a copolynomial $T \in K[x_1, \ldots, x_n]'$ with respect to the variable x_j $(j = 1, \ldots, n)$ is defined by the formula

$$\left(\frac{\partial T}{\partial x_j}, p\right) = -\left(T, \frac{\partial p}{\partial x_j}\right), \quad p \in K[x_1, \ldots, x_n]. \tag{2.1}$$

By using this formula, we arrive at the following expression for the derivative $D^\alpha T$:

$$(D^\alpha T, p) = (-1)^{|\alpha|}(T, D^\alpha p), \quad p \in K[x_1, \ldots, x_n].$$

Therefore,

$$(D^\alpha T, p) = 0, \quad \text{where } p \in K[x_1, \ldots, x_n] \text{ and } |\alpha| > \deg p.$$

By virtue of the equality

$$\left(\frac{D^\alpha T}{\alpha!}, p\right) = (-1)^{|\alpha|}\left(T, \frac{D^\alpha p}{\alpha!}\right), \quad p \in K[x_1, \ldots, x_n],$$

the copolynomials $\frac{D^\alpha T}{\alpha!}$ are well defined for any $T \in K[x_1, \ldots, x_n]'$ and $\alpha \in \mathbb{N}_0^n$.

Example 2.3. The copolynomial δ-function is given by the formula

$$(\delta, p) = p(0), \quad p \in K[x_1, \ldots, x_n].$$

Therefore,

$$(D^\alpha \delta, p) = (-1)^{|\alpha|}(\delta, D^\alpha p) = (-1)^{|\alpha|} D^\alpha p(0), \quad \alpha \in \mathbb{N}_0^n. \tag{2.2}$$

Example 2.4. Let $K = \mathbb{R}$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue-integrable function such that

$$\int_{\mathbb{R}^n} |x^\alpha f(x)| dx < +\infty, \quad \alpha \in \mathbb{N}_0^n. \tag{2.3}$$

Then f generates the regular copolynomial T_f:

$$(T_f, p) = \int_{\mathbb{R}^n} p(x) f(x) dx, \quad p \in \mathbb{R}[x_1, \ldots, x_n].$$

Notice that in this case, unlike in the classical theory, all copolynomials are regular [7, Theorem 7.3.4], although a nonzero function f can generate the zero copolynomial (see, e.g. [12, Section 2], where examples of functions that satisfy the property (2.3) and generate the δ-function were explicitly presented).

We now consider the problem of convergence in the space $K[x_1, \ldots, x_n]'$. In the ring K, we consider the discrete topology. Further, in the module of copolynomials $K[x_1, \ldots, x_n]'$, we consider the topology of pointwise convergence. The convergence of a sequence $\{T_k\}_{k=1}^\infty$ to T in $K[x_1, \ldots, x_n]'$ means that for every polynomial $p \in K[x_1, \ldots, x_n]$, there exists a number $k_0 \in \mathbb{N}$ such that

$$(T_k, p) = (T, p), \quad k = k_0, k_0 + 1, k_0 + 2, \ldots.$$

We now consider the following linear differential operator of infinite order on $K[x_1, \dots, x_n]'$:

$$\mathcal{F} = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha,$$

where $a_\alpha \in K$. This operator acts upon a copolynomial $T \in K[x_1, \dots, x_n]'$ by the following rule: if $p \in K[x_1, \dots, x_n]$ and $m = \deg p$, then

$$(\mathcal{F}T, p) = \left(\sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha T, p \right) = \sum_{|\alpha|\le m} (-1)^{|\alpha|} a_\alpha (T, D^\alpha p) = \sum_{|\alpha|\le m} a_\alpha (D^\alpha T, p).$$

Thus, the differential operator $\mathcal{F} : K[x_1, \dots, x_n]' \to K[x_1, \dots, x_n]'$ is well defined and for any polynomial p of degree at most m, the equality

$$(\mathcal{F}T, p) = \sum_{|\alpha|\le m} a_\alpha (D^\alpha T, p)$$

is true.

The following lemma shows the possibility of the decomposition of an arbitrary copolynomial in series in terms of the system $\frac{D^\alpha \delta}{\alpha!}$, $\alpha \in \mathbb{N}_0^n$ (see [8, Proposition 2.3] in the case $n = 1$ and $K = \mathbb{C}$).

Lemma 2.5. *Let $T \in K[x_1, \dots, x_n]'$. Then*

$$T = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} (T, x^\alpha) \frac{D^\alpha \delta}{\alpha!}. \tag{2.4}$$

Now, as in the classical theory, we introduce the notion of a homogeneous copolynomial. For any $\lambda \in K$, we define $\lambda x = (\lambda x_1, \dots, \lambda x_n)$ and for any polynomial $p(x) = \sum_{|\alpha|\le m} a_\alpha x^\alpha \in K[x_1, \dots, x_n]$, the polynomial $p(\lambda x) = \sum_{|\alpha|\le m} a_\alpha \lambda^{|\alpha|} x^\alpha \in K[x_1, \dots, x_n]$ is well defined. Now we define the copolynomial $T\left(\frac{x}{\lambda}\right)$ by

$$\left(T\left(\frac{x}{\lambda}\right), p \right) = \lambda^n (T(x), p(\lambda x)). \tag{2.5}$$

Definition 2.6. Let $0 \ne T \in K[x_1, \dots, x_n]'$ and $m \in \mathbb{N}$. We say that the copolynomial T is homogeneous of degree $-m$ if

$$\left(T\left(\frac{x}{\lambda}\right), p \right) = \lambda^m (T, p), \quad \lambda \in K, \quad p \in K[x_1, \dots, x_n]$$

or, by (2.5),

$$\lambda^n (T(x), p(\lambda x)) = \lambda^m (T(x), p(x)), \quad \lambda \in K, \quad p \in K[x_1, \dots, x_n]. \tag{2.6}$$

It follows from (2.6) that δ is a homogeneous copolynomial of degree $-n$. Moreover, equation (2.2) implies that

$$\lambda^n (D^\alpha \delta, p(\lambda x)) = (-1)^{|\alpha|} \lambda^{n+|\alpha|} D^\alpha p(0)$$

$$= \lambda^{n+|\alpha|}(D^\alpha\delta, p), \quad \lambda \in K,\ p \in K[x_1,\dots,x_n],\ \alpha \in \mathbb{N}_0^n,$$

i.e., $D^\alpha\delta$ is a homogeneous copolynomial of degree $-|\alpha|-n$. Lemma 2.5 shows that every copolynomial is a sum of homogeneous copolynomials.

If $T \neq 0$ is a homogeneous copolynomial of degree $-m$, then, due to (2.6), for $p(x) = x^\alpha$, we obtain

$$\lambda^{n+|\alpha|}(T, x^\alpha) = \lambda^m (T, x^\alpha), \quad \alpha \in \mathbb{N}_0^n,\ \lambda \in K.$$

Since K is of characteristic 0, this implies $m \geq n$ and $(T, x^\alpha) = 0, \quad |\alpha| \neq m-n$. Conversely, if $m \geq n$, then every copolynomial such that $(T, x^\alpha) = 0, |\alpha| \neq m-n$ is homogeneous. Thus we obtain the following lemma.

Lemma 2.7. *The copolynomial $0 \neq T \in K[x_1,\dots,x_n]'$ is homogeneous of degree $-m$ if and only if $m \geq n$ and $(T, x^\alpha) = 0$, $|\alpha| \neq m-n$.*

In particular, for $m < n$, there are no homogeneous copolynomials of degree $-m$. Moreover, if $n = 1$ and $m \in \mathbb{N}$, then an arbitrary homogeneous copolynomial of degree $-m$ has the form $\frac{c\delta^{(m-1)}}{(m-1)!}$, where $c \in K$.

Lemma 2.7 implies the following necessary and sufficient conditions for the homogeneity of $T \in K[x_1,\dots,x_n]'$.

Theorem 2.8. *The copolynomial $0 \neq T \in K[x_1,\dots,x_n]'$ is homogeneous of degree $-m$ if and only if $m \geq n$ and T satisfies the Euler differential equation*

$$\sum_{j=1}^n x_j \frac{\partial T}{\partial x_j} = -mT. \tag{2.7}$$

Proof. Sufficiency. Let T satisfies equation (2.7). We can find the solution of this equation in the form (2.4). Substituting (2.4) to (2.7), we have

$$\sum_{j=1}^n x_j \sum_{|\alpha|=0}^\infty (-1)^{|\alpha|} a_\alpha \frac{\partial}{\partial x_j}\frac{D^\alpha\delta}{\alpha!} = -m \sum_{|\alpha|=0}^\infty (-1)^{|\alpha|} a_\alpha \frac{D^\alpha\delta}{\alpha!}, \tag{2.8}$$

where $a_\alpha = (T, x^\alpha)$. Since

$$\sum_{j=1}^n x_j \sum_{|\alpha|=0}^\infty (-1)^{|\alpha|} a_\alpha \left(\frac{\partial}{\partial x_j}\frac{D^\alpha\delta}{\alpha!}, x^\beta\right) = \sum_{|\alpha|=0}^\infty (-1)^{|\alpha|} a_\alpha \sum_{j=1}^n \left(x_j \frac{\partial}{\partial x_j}\frac{D^\alpha\delta}{\alpha!}, x^\beta\right)$$
$$= -\sum_{|\alpha|=0}^\infty a_\alpha \sum_{j=1}^n \left(\delta, \frac{\partial}{\partial x_j}\frac{D^\alpha(x_j x^\beta)}{\alpha!}\right) = -\sum_{j=1}^n (\beta_j+1) a_\beta = -(|\beta|+n) a_\beta,$$

equality (2.8) implies

$$a_\beta(|\beta|+n) = m a_\beta, \quad \beta \in \mathbb{N}_0^n.$$

Therefore, $a_\beta = 0$ for $|\beta| \neq m-n$. By Lemma 2.7, T is homogeneous.

Necessity. Let $0 \neq T \in K[x_1, \ldots, x_n]'$ be homogeneous of degree $-m$. Then, by Lemma 2.7, $m \geq n$ and (2.6) is satisfied for all $\lambda \in K$. Note that the left-hand and right-hand sides of (2.6) are polynomials from $K[\lambda]$. Then, dividing (2.6) by λ^n and differentiating it by λ, we have

$$\frac{d}{d\lambda}(T(x), p(\lambda x)) = \left(T(x), \frac{d}{d\lambda}p(\lambda x)\right) = \left(T, \sum_{j=1}^{n} x_j \frac{\partial p}{\partial x_j}(\lambda x)\right)$$
$$= (m-n)\lambda^{m-n-1}(T, p), \quad p \in K[x_1, \ldots, x_n].$$

Substituting $\lambda = 1$, we have

$$\left(T, \sum_{j=1}^{n} x_j \frac{\partial p(x)}{\partial x_j}\right) = (m-n)(T, p), \quad p \in K[x_1, \ldots, x_n]. \tag{2.9}$$

Since

$$\left(T, \sum_{j=1}^{n} x_j \frac{\partial p}{\partial x_j}\right) = \left(T, \sum_{j=1}^{n} \frac{\partial(x_j p(x))}{\partial x_j}\right) - \left(T, \sum_{j=1}^{n} p(x)\right)$$
$$= -\sum_{j=1}^{n} \left(x_j \frac{\partial T}{\partial x_j}, p\right) - n(T, p), \quad p \in K[x_1, \ldots, x_n],$$

equality (2.9) implies

$$-\sum_{j=1}^{n} \left(x_j \frac{\partial T}{\partial x_j}, p\right) = m(T, p), \quad p \in K[x_1, \ldots, x_n].$$

This proves (2.7). □

3. Cauchy–Stieltjes transform in the module of copolynomials

Let $z = (z_1, \ldots, z_n)$, and let $K\left[\left[z_1, \ldots, z_n, \frac{1}{z_1}, \ldots, \frac{1}{z_n}\right]\right]$ be the module of formal Laurent series with coefficients in K. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, we put $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$. For $g \in K\left[\left[z_1, \ldots, z_n, \frac{1}{z_1}, \ldots, \frac{1}{z_n}\right]\right]$, $g(z) = \sum_{\alpha \in \mathbb{Z}^n} g_\alpha z^\alpha$, we naturally define the formal residue

$$\operatorname{Res}(g(z)) = g_{(-1,\ldots,-1)}.$$

Definition 3.1. Let $T \in K[x_1, \ldots, x_n]'$ and $s = (s_1, \ldots, s_n)$. Consider the following formal Laurent series from the ring $\frac{1}{s_1 s_2 \cdots s_n} K[[\frac{1}{s_1}, \frac{1}{s_2}, \ldots, \frac{1}{s_n}]]$:

$$C(T)(s) = \sum_{|\alpha|=0}^{\infty} \frac{(T, x^\alpha)}{s^{\alpha+\iota}},$$

where $\iota = (1, \ldots, 1) \in \mathbb{N}_0^n$. The Laurent series $C(T)(s)$ is called the *Cauchy–Stieltjes transform* of a copolynomial T.

We may write the result of the Cauchy–Stieltjes transform informally: $C(T)(s) = \Big(T, \frac{1}{(s_1-x_1)(s_2-x_2)\cdots(s_n-x_n)}\Big)$. Obviously, the mapping $C : K[x_1,\dots,x_n]' \to \frac{1}{s_1s_2\cdots s_n}K[[\frac{1}{s_1},\frac{1}{s_2},\dots,\frac{1}{s_n}]]$ is an isomorphism of K-modules.

Proposition 3.2 (The inversion formula)**.** *Let $T \in K[x_1,\dots,x_n]'$ and $p \in K[x_1,\dots,x_n]$. Then*

$$(T,p) = \operatorname{Res}(C(T)(s)p(s)).$$

Proof. It is sufficient to consider the case $p(x) = x^\beta$ for some multi-index $\beta \in \mathbb{N}_0^n$. We have

$$C(T)(s)s^\beta = \sum_{|\alpha|=0}^{\infty} \frac{(T,x^\alpha)s^\beta}{s^{\alpha+\iota}}.$$

Therefore, $\operatorname{Res}(C(T)(s)s^\beta) = (T,x^\beta)$. □

Example 3.3 (The "integral" Cauchy formula). We put $(\delta_z,p) = p(z_1,\dots,z_n)$ for $z = (z_1,\dots,z_n) \in K^n$. Then $C(\delta_z)(s) = \sum_{|\alpha|=0}^{\infty} \frac{z^\alpha}{s^{\alpha+\iota}}$. In particular,

$$C(\delta)(s) = \frac{1}{s_1s_2\cdots s_n}.$$

We may write informally $p(z_1,\dots,z_n) = \operatorname{Res}(\frac{p(s_1,\dots,s_n)}{(s_1-z_1)(s_2-z_2)\cdots(s_n-z_n)})$ if we identify the rational function $\frac{1}{(s_1-z_1)(s_2-z_2)\cdots(s_n-z_n)}$ with the Laurent series $\sum_{|\alpha|=0}^{\infty} \frac{z^\alpha}{s^{\alpha+\iota}}$.

Let $\mathcal{F} = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha$ be a linear differential operator with coefficients $a_\alpha \in K$. Obviously, the operator $\mathcal{F}$ is well defined on the K-module of formal Laurent series $\frac{1}{s_1s_2\cdots s_n}K[[\frac{1}{s_1},\frac{1}{s_2},\dots,\frac{1}{s_n}]]$.

Proposition 3.4. *For any $T \in K[x_1,\dots,x_n]'$, the equality*

$$C(\mathcal{F}T) = \mathcal{F}(C(T))$$

holds.

Proof. We have

$$\begin{aligned} C(\mathcal{F}T) &= \sum_{|\alpha|=0}^{\infty} \frac{(\mathcal{F}T,x^\alpha)}{s^{\alpha+\iota}} = \sum_{|\alpha|=0}^{\infty}\sum_{\beta\le\alpha} a_\beta \frac{(D^\beta T,x^\alpha)}{s^{\alpha+\iota}} \\ &= \sum_{|\alpha|=0}^{\infty}\sum_{\beta\le\alpha} (-1)^{|\beta|}\beta! a_\beta \binom{\alpha}{\beta} \frac{(T,x^{\alpha-\beta})}{s^{\alpha+\iota}} \\ &= \sum_{|\beta|=0}^{\infty}\sum_{\alpha\ge\beta} (-1)^{|\beta|}\beta! a_\beta \binom{\alpha}{\beta} \frac{(T,x^{\alpha-\beta})}{s^{\alpha+\iota}} \\ &= \sum_{|\beta|=0}^{\infty} a_\beta \sum_{|\alpha|=0}^{\infty} (-1)^{|\beta|}\beta! \binom{\alpha+\beta}{\beta} \frac{(T,x^\alpha)}{s^{\alpha+\beta+\iota}} \\ &= \sum_{|\beta|=0}^{\infty} a_\beta D^\beta \sum_{|\alpha|=0}^{\infty} \frac{(T,x^\alpha)}{s^{\alpha+\iota}} = \mathcal{F}(C(T)). \end{aligned}$$

□

Now consider the notion of a homogeneous formal Laurent series. Let F be the quotient field of K. For any

$$g(z) = \sum_{|\alpha|=0}^{\infty} \frac{g_\alpha}{z^{\alpha+\iota}} \in \frac{1}{z_1 z_2 \cdots z_n} K\left[\left[\frac{1}{z_1}, \frac{1}{z_2}, \ldots, \frac{1}{z_n}\right]\right]$$

and $0 \neq \lambda \in K$, we consider the following series:

$$g(\lambda z) = \sum_{|\alpha|=0}^{\infty} \frac{g_\alpha}{\lambda^{\alpha+\iota} z^{\alpha+\iota}} \in \frac{1}{z_1 z_2 \cdots z_n} F\left[\left[\frac{1}{z_1}, \frac{1}{z_2}, \ldots, \frac{1}{z_n}\right]\right].$$

Definition 3.5. Let $m \in \mathbb{N}$. We say that a formal Laurent series $g(z) \in \frac{1}{z_1 z_2 \cdots z_n} K\left[\left[\frac{1}{z_1}, \frac{1}{z_2}, \ldots, \frac{1}{z_n}\right]\right]$ is homogeneous of degree $-m$ if

$$\lambda^m g(\lambda z) = g(z), \quad 0 \neq \lambda \in K.$$

Now we prove the following necessary and sufficient condition for the homogeneity of a copolynomial in terms of its Cauchy–Stieltjes transform.

Theorem 3.6. *The copolynomial $T \in K[x_1, \ldots, x_n]'$ is homogeneous of degree $-m$ if and only if its Cauchy–Stieltjes transform is homogeneous of the same degree.*

Proof. For any $0 \neq \lambda \in K$, we have

$$\lambda^m (C(T)(\lambda s) = \sum_{|\alpha|=0}^{\infty} \frac{\lambda^m (T, x^\alpha)}{(\lambda s)^{\alpha+\iota}} = \sum_{|\alpha|=0}^{\infty} \frac{\lambda^m (T, x^\alpha)}{\lambda^{|\alpha|+n} s^{\alpha+\iota}}$$

and

$$C(T)(s) = \sum_{|\alpha|=0}^{\infty} \frac{(T, x^\alpha)}{s^{\alpha+\iota}} = \sum_{|\alpha|=0}^{\infty} \frac{(T, \lambda^{|\alpha|+n} x^\alpha)}{\lambda^{|\alpha|+n} s^{\alpha+\iota}} = \sum_{|\alpha|=0}^{\infty} \frac{\lambda^n (T, (\lambda x)^\alpha)}{\lambda^{|\alpha|+n} s^{\alpha+\iota}}.$$

Therefore, $\lambda^m (C(T))(\lambda s) = C(T)(s)$, $0 \neq \lambda \in K$, if and only if (2.6) holds for any $\lambda \in K$. ☐

4. Multiplication of copolynomials

The Cauchy–Stieltjes transform and Proposition 3.4 allow us to introduce the multiplication operation on the module of copolynomials such that this operation is consistent with the differentiation.

Definition 4.1. Let $T_1, T_2 \in K[x_1, \ldots, x_n]'$, i.e., T_1, T_2 are copolynomials. Define their *product* by the following equality:

$$C(T_1 T_2) = C(T_1) C(T_2), \tag{4.1}$$

i.e.,

$$T_1 T_2 = C^{-1} \left(C(T_1) C(T_2) \right),$$

where $C : K[x_1, \ldots, x_n]' \to \frac{1}{s_1 s_2 \cdots s_n} K[[\frac{1}{s_1}, \frac{1}{s_2}, \ldots, \frac{1}{s_n}]]$ is a Cauchy–Stieltjes transform (see Section 3).

In the following lemma the action of the product of copolynomials on monomials is expressed through the action of multipliers on monomials.

Lemma 4.2. *Let $T_1, T_2 \in K[x_1, \dots, x_n]'$ and $\alpha \in \mathbb{N}_0^n$. Then*

$$(T_1T_2, x^\alpha) = \begin{cases} \sum\limits_{\beta \le \alpha - \iota} (T_1, x^\beta)(T_2, x^{\alpha-\iota-\beta}), & \alpha \ge \iota, \\ 0, & \textit{otherwise}. \end{cases} \tag{4.2}$$

Proof. By (4.1), we have

$$C(T_1T_2)(s) = C(T_1)(s)C(T_2)(s) = \sum_{|\beta|=0}^{\infty} \sum_{|\gamma|=0}^{\infty} \frac{(T_1, x^\beta)(T_2, x^\gamma)}{s^{\beta+\gamma+2\iota}}$$
$$= \sum_{\alpha \ge \iota} \sum_{\beta \le \alpha-\iota} (T_1, x^\beta)(T_2, x^{\alpha-\iota-\beta}) \frac{1}{s^{\alpha+\iota}}.$$

Applying the inversion formula to both sides of this equality (see Proposition 3.2), we obtain (4.2). □

Formula (4.2) implies the following assertion about the continuity of the multiplication of copolynomials.

Corollary 4.3. *Let $Q_m \to Q$ and $S_m \to S$ as $m \to \infty$ with respect to the topology on $K[x_1, \dots, x_n]'$. Then $Q_mS_m \to QS$ as $m \to \infty$ in the same topology.*

Remark 4.4. Definition 4.1 means that the module of copolynomials $K[x_1, \dots, x_n]'$ with the introduced product is an associative commutative ring, which is isomorphic to the ring of formal Laurent series $\frac{1}{s_1 s_2 \cdots s_n} K[[\frac{1}{s_1}, \frac{1}{s_2}, \dots, \frac{1}{s_n}]]$ with a natural product operation. In particular, the ring of copolynomials is an integral domain and this is a ring without identity.

Example 4.5. Let $n = 1$. We find the square of the δ-function:

$$C(\delta^2)(s) = (C(\delta))^2(s) = \frac{1}{s^2} = \left(\frac{-1}{s}\right)' = (-C(\delta))' = C(-\delta'),$$

i.e.,

$$\delta^2 = -\delta'.$$

Since by Proposition 3.4,

$$C\left(\delta^{(k)}\right)(s) = (C(\delta))^{(k)}(s) = \frac{d^k}{ds^k}\frac{1}{s} = (-1)^k k! s^{-k-1}, \quad k = 0, 1, 2, \dots,$$

we have

$$\frac{(-1)^k \delta^{(k)}}{k!} = \delta^{k+1}, \quad k = 0, 1, 2, \dots, \tag{4.3}$$

and, by Lemma 2.5, for any copolynomial $T \in K[x]'$, the decomposition with respect to the degrees of δ-functions holds

$$T = \sum_{k=0}^{\infty} (T, x^k)\delta^{k+1}.$$

The generalization of equalities (4.3) on n variables is given by the following formulas:

$$\frac{\partial^{nk}\delta}{\partial x_1^k \dots \partial x_n^k} = (-1)^{nk}(k!)^n \delta^{k+1}, \quad k = 0, 1, 2, \dots. \tag{4.4}$$

As a consequence,

$$\frac{\partial^n \delta^k}{\partial x_1 \dots \partial x_n} = (-1)^n k^n \delta^{k+1}, \quad k \in \mathbb{N}. \tag{4.5}$$

We show that for the constructed product of copolynomials the natural Leibnitz formula is satisfied. At first, we consider particular cases.

Lemma 4.6. *Let $T_1, T_2, \dots, T_m \in K[x_1, \dots, x_n]'$. Then*

$$\frac{\partial \left(\prod_{k=1}^m T_k\right)}{\partial x_j} = \sum_{k=1}^m \prod_{\substack{l=1\\ l\neq k}}^m T_l \frac{\partial T_k}{\partial x_j}, \quad j = 1, \dots, n. \tag{4.6}$$

In particular,

$$\frac{\partial (T_1 T_2)}{\partial x_j} = \frac{\partial T_1}{\partial x_j} T_2 + T_1 \frac{\partial T_2}{\partial x_j}, \quad j = 1, \dots, n. \tag{4.7}$$

Proof. By Proposition 3.4 and Definition 4.1, we obtain

$$C\left(\frac{\partial \left(\prod_{k=1}^m T_k\right)}{\partial x_j}\right) = \frac{\partial C\left(\prod_{k=1}^m T_k\right)}{\partial s_j} = \frac{\partial \left(\prod_{k=1}^m C(T_k)\right)}{\partial s_j}$$
$$= \sum_{k=1}^m \prod_{\substack{l=1\\ l\neq k}}^m C(T_l) \frac{\partial C(T_k)}{\partial s_j} = C \sum_{k=1}^m \prod_{\substack{l=1\\ l\neq k}}^m T_l \frac{\partial T_k}{\partial x_j}, \quad j = 1, \dots, n. \qquad \square$$

Lemma 4.7. *Let $T_1, T_2 \in K[x_1, \dots, x_n]'$. Then for any $\beta \in \mathbb{N}_0^n$,*

$$D^\beta (T_1 T_2) = \sum_{\alpha \leq \beta} \binom{\beta}{\alpha} D^\alpha T_1 D^{\beta - \alpha} T_2. \tag{4.8}$$

Proof. We prove by induction on $|\beta|$. For $|\beta| = 1$, (4.8) directly follows from (4.7). Suppose that (4.8) holds for some $\beta \in \mathbb{N}_0^n$. For any $j = 1, \dots, n$, consider the multi-index $\beta(j) \in \mathbb{N}_0^n$ which is obtained from the multi-index $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$ by replacing β_j on $\beta_j + 1$. To prove the theorem it is sufficient to prove the equality

$$D^{\beta(j)} (T_1 T_2) = \sum_{\alpha \leq \beta(j)} \binom{\beta(j)}{\alpha} D^\alpha T_1 D^{\beta(j) - \alpha} T_2. \tag{4.9}$$

Indeed, using the induction hypothesis, we obtain

$$D^{\beta(j)} (T_1 T_2) = \frac{\partial}{\partial x_j} D^\beta (T_1 T_2) = \frac{\partial}{\partial x_j} \left(\sum_{\alpha \leq \beta} \binom{\beta}{\alpha} D^\alpha T_1 D^{\beta - \alpha} T_2 \right)$$

$$= \sum_{\alpha \le \beta} \binom{\beta}{\alpha} \frac{\partial}{\partial x_j}(D^\alpha T_1) D^{\beta-\alpha} T_2 + \sum_{\alpha \le \beta} \binom{\beta}{\alpha} D^\alpha T_1 D^{\beta(j)-\alpha} T_2$$

$$= \sum_{\alpha \le \beta(j)} \binom{\beta(j)}{\alpha} D^\alpha T_1 D^{\beta(j)-\alpha} T_2,$$

which proves (4.9). □

Now we establish the general Leibnitz formula.

Theorem 4.8. *Let $\mathcal{F} = \sum_{|\alpha|=0}^\infty a_\alpha D^\alpha$ be a linear differential operator of infinite order on $K[x_1, \dots, x_n]'$ with the symbol $\varphi(z) = \sum_{|\alpha|=0}^\infty a_\alpha z^\alpha$. Then*

$$\mathcal{F}(T_1 T_2) = \sum_{|\alpha|=0}^\infty \frac{D^\alpha T_1}{\alpha!} \mathcal{F}_\alpha(T_2),$$

where $\mathcal{F}_\alpha$ is a differential operator with the symbol $D^\alpha \varphi(z)$.

Proof. Let us multiply equality (4.8) by a_β and add on all multi-indexes $\beta \in \mathbb{N}_0^n$. Then

$$\mathcal{F}(T_1 T_2) = \sum_{|\beta|=0}^\infty \sum_{\alpha \le \beta} a_\beta \binom{\beta}{\alpha} D^\alpha T_1 D^{\beta-\alpha} T_2$$

$$= \sum_{|\alpha|=0}^\infty \sum_{\beta \ge \alpha} a_\beta \binom{\beta}{\alpha} D^\alpha T_1 D^{\beta-\alpha} T_2$$

$$= \sum_{|\alpha|=0}^\infty D^\alpha T_1 \sum_{\beta \ge \alpha} a_\beta \binom{\beta}{\alpha} D^{\beta-\alpha} T_2 = \sum_{|\alpha|=0}^\infty \frac{D^\alpha T_1}{\alpha!} \mathcal{F}_\alpha(T_2),$$

where $\mathcal{F}_\alpha = \sum_{\beta \ge \alpha} a_\beta \alpha! \binom{\beta}{\alpha} D^{\beta-\alpha}$. Since $\varphi(z) = \sum_{|\beta|=0}^\infty a_\beta z^\beta$ is the symbol of the differential operator $\mathcal{F}$, we obtain that $D^\alpha \varphi(z) = \sum_{\beta \ge \alpha} \alpha! \binom{\beta}{\alpha} z^{\beta-\alpha}$ is the symbol of the differential operator $\mathcal{F}_\alpha$. □

Remark 4.9. In [8, 24], the Cauchy product of copolynomials of one variable was considered. It was shown that this multiplication of copolynomials is not consistent with a differentiation. For example, the Leibnitz formula for this multiplication is not valid. Moreover, the δ-function is the identity with respect to this multiplication.

Definition 4.1 implies that for any copolynomials $T_1, \dots, T_m \in K[x_1, \dots, x_n]'$ and for any multi-index $\beta \in \mathbb{N}_0^n$, the formula

$$\left(\prod_{k=1}^m T_k, x^\beta \right) = 0, \quad m > |\beta| + 1 \tag{4.10}$$

holds. Let $T_1, \ldots, T_m \in K[x_1, \ldots, x_n]'$ and $\gamma = (\gamma_1, \ldots, \gamma_m) \in \mathbb{N}_0^m$. We define T^γ by the equality

$$T^\gamma = \prod_{\substack{j=1\\ \gamma_j \neq 0}}^{m} T_j^{\gamma_j}, \quad \gamma \neq 0.$$

Equality (4.10) implies that for any multi-indexes $\beta \in \mathbb{N}_0^n$ and $\gamma \in \mathbb{N}_0^m$, the formula

$$(T^\gamma, x^\beta) = 0, \quad |\gamma| > |\beta| + 1,$$

is satisfied. Then, for any formal power series $g(z) = \sum_{|\gamma|=1}^{\infty} g_\gamma z^\gamma \in K[[z_1, \ldots, z_m]]$, the series $\sum_{|\gamma|=1}^{\infty} g_\gamma T^\gamma$ converges with respect to the topology on $K[x_1, \ldots, x_n]'$. We denote the sum of this series by $g(T_1, \ldots, T_m)$. If $g(z) = \sum_{|\gamma|=0}^{\infty} g_\gamma z^\gamma \in K[[z_1, \ldots, z_m]]$ and $S \in K[x_1, \ldots, x_n]'$, then the product $g(T_1, \ldots, T_m)S$ means the sum of the following copolynomials:

$$g_0 S + \left(\sum_{|\gamma|=1}^{\infty} g_\gamma T^\gamma \right) S = g_0 S + \sum_{|\gamma|=1}^{\infty} g_\gamma T^\gamma S$$

(see Corollary 4.3). The theorem below establishes the rule of the differentiation for a copolynomial $g(T_1, \ldots, T_m)$, which is similar to the rule of the differentiation of a composition.

Theorem 4.10. *Let $T_1, \ldots, T_m \in K[x_1, \ldots, x_n]'$ and $g(z) = \sum_{|\gamma|=1}^{\infty} g_\gamma z^\gamma \in K[[z_1, \ldots, z_m]]$. Then*

$$\frac{\partial g(T_1, \ldots, T_m)}{\partial x_j} = \sum_{q=1}^{m} \frac{\partial g}{\partial z_q}(T_1, \ldots, T_m) \frac{\partial T_q}{\partial x_j}, \quad j = 1, \ldots, n.$$

Proof. To prove the theorem, it is sufficient to show that

$$\left(\frac{\partial g(T_1, \ldots, T_m)}{\partial x_j}, x^\beta \right) = \left(\sum_{q=1}^{m} \frac{\partial g}{\partial z_q}(T_1, \ldots, T_m) \frac{\partial T_q}{\partial x_j}, x^\beta \right), \ j = 1, \ldots, n, \ \beta \in \mathbb{N}_0^n. \tag{4.11}$$

For $\beta_j = 0$, equality (4.11) follows from (4.2) and (2.1). In what follows, we suppose $\beta_j \neq 0$. Denote by e_k the multi-index of an arbitrary dimension whose kth coordinate is equal to 1 and other coordinates vanish. Let $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$ and $\beta(j) = \beta - e_j \in \mathbb{N}_0^n$. Then

$$\left(\frac{\partial g(T_1, \ldots, T_m)}{\partial x_j}, x^\beta \right) = \left(\sum_{q=1}^{m} g_{e_q} \frac{\partial T_q}{\partial x_j}, x^\beta \right) - \beta_j \sum_{|\gamma|=2}^{\infty} g_\gamma \left(T^\gamma, x^{\beta(j)} \right). \tag{4.12}$$

On the other hand,

$$\sum_{q=1}^{m} \left(\frac{\partial g}{\partial z_q}(T_1, \ldots, T_m) \frac{\partial T_q}{\partial x_j}, x^\beta \right)$$

$$= \left(\sum_{q=1}^{m} g_{e_q} \frac{\partial T_q}{\partial x_j}, x^\beta\right) + \sum_{q=1}^{m} \sum_{|\gamma|=2}^{\infty} g_\gamma \gamma_q \left(T^{\gamma - e_q} \frac{\partial T_q}{\partial x_j}, x^\beta\right)$$

$$= \left(\sum_{q=1}^{m} g_{e_q} \frac{\partial T_q}{\partial x_j}, x^\beta\right) + \sum_{|\gamma|=2}^{\infty} g_\gamma \sum_{q=1}^{m} \gamma_q \left(T^{\gamma - e_q} \frac{\partial T_q}{\partial x_j}, x^\beta\right). \quad (4.13)$$

In the sum $\sum_{q=1}^{m} \sum_{|\gamma|=2}^{\infty} g_\gamma \gamma_q \left(T^{\gamma-e_q} \frac{\partial T_q}{\partial x_j}, x^\beta\right)$, the product $\gamma_q \left(T^{\gamma-e_q} \frac{\partial T_q}{\partial x_j}, x^\beta\right)$ is well defined, because either $\gamma_q = 0$ or $\gamma \geq e_q$. By the Leibnitz formula (see (4.6)),

$$\sum_{q=1}^{m} \gamma_q \left(T^{\gamma - e_q} \frac{\partial T_q}{\partial x_j}, x^\beta\right) = \left(\frac{\partial T^\gamma}{\partial x_j}, x^\beta\right), \quad |\gamma| \geq 2. \quad (4.14)$$

Substituting (4.14) into (4.13) and taking into account (4.12), we obtain

$$\sum_{q=1}^{m} \left(\frac{\partial g}{\partial z_q}(T_1, \ldots, T_m) \frac{\partial T_q}{\partial x_j}, x^\beta\right)$$

$$= \left(\sum_{q=1}^{m} g_{e_q} \frac{\partial T_q}{\partial x_j}, x^\beta\right) + \sum_{|\gamma|=2}^{\infty} g_\gamma \sum_{q=1}^{m} \gamma_q \left(T^{\gamma - e_q} \frac{\partial T_q}{\partial x_j}, x^\beta\right)$$

$$= \left(\sum_{q=1}^{m} g_{e_q} \frac{\partial T_q}{\partial x_j}, x^\beta\right) + \left(\sum_{|\gamma|=2}^{\infty} g_\gamma \frac{\partial T^\gamma}{\partial x_j}, x^\beta\right)$$

$$= \left(\sum_{q=1}^{m} g_{e_q} \frac{\partial T_q}{\partial x_j}, x^\beta\right) - \beta_j \sum_{|\gamma|=2}^{\infty} g_\gamma \left(T^\gamma, x^{\beta(j)}\right) = \left(\frac{\partial g(T_1, \ldots, T_m)}{\partial x_j}, x^\beta\right),$$

which proves (4.11). □

5. Nonlinear partial differential equations in the ring of copolynomials

5.1. The Cauchy problem for some nonlinear differential equations in the ring of copolynomials. At first, we give some notions from [9]. The module of formal power series of the form $u(t, x) = \sum_{k=0}^{\infty} u_k(x) t^k$ with coefficients $u_k(x) \in K[x_1, \ldots, x_n]'$ is denoted by $K[x_1, \ldots, x_n]'[[t]]$.

The partial derivative with respect to t of the series $u(t, x) \in K[x_1, \ldots, x_n]'[[t]]$ is defined by the formula

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{\infty} k u_k(x) t^{k-1}.$$

The partial derivative D^α with respect to variables $x_1, \ldots, x_n$ of the series $u(t, x) \in K[x_1, \ldots, x_n]'[[t]]$ is defined as follows:

$$D^\alpha u(t, x) = \sum_{k=0}^{\infty} (D^\alpha u_k)(x) t^k.$$

The action of the K-linear operator $\mathcal{A} : K[x_1, \dots, x_n]' \to K[x_1, \dots, x_n]'$ on a formal power series $u(t, x) = \sum_{k=0}^{\infty} u_k(x)t^k \in K[x_1, \dots, x_n]'[[t]]$ is defined coefficient-wisely:

$$(\mathcal{A}u)(t, x) = \sum_{k=0}^{\infty} (\mathcal{A}u_k)(x)t^k.$$

Now, let $P \in K[z_1, \dots, z_m]$, $P(0) = 0$, and let $\mathcal{F}_j = \sum_{|\alpha|=0}^{\infty} a_{j,\alpha} D^\alpha$ $(j = 1, \dots, m)$ be linear differential operators of infinite order with coefficients $a_{j,\alpha} \in K$ which act on the module of copolynomials $K[x_1, \dots, x_n]'$. Now we consider the module $K[x_1, \dots, x_n]'$ as the commutative ring in which the multiplication was introduced in Section 4. Consider the following Cauchy problem in the ring $K[x_1, \dots, x_n]'[[t]]$:

$$\frac{\partial u(t, x)}{\partial t} = P\left((\mathcal{F}_1 u)(t, x), \dots, (\mathcal{F}_m u)(t, x)\right), \tag{5.1}$$

$$u(0, x) = Q(x) \in K[x_1, \dots, x_n]'. \tag{5.2}$$

Now we are to prove the existence and uniqueness theorem for the Cauchy problem (5.1)–(5.2).

Theorem 5.1. *Let $K \supset \mathbb{Q}$. Then for any copolynomial $Q \in K[x_1, \dots, x_n]'$, the Cauchy problem (5.1)–(5.2) has a unique solution.*

Proof. We are looking for a solution of the Cauchy problem (5.1)–(5.2) in the form

$$u(t, x) = \sum_{k=0}^{\infty} u_k(x)t^k, \tag{5.3}$$

where $u_k(x) \in K[x_1, \dots, x_n]'$. Then, by the initial condition (5.2), we have $u_0(x) = Q(x)$. Substituting (5.3) into equation (5.1) and equating coefficients of t^k, we obtain that there exist polynomials $p_k \in K[z_1, \dots, z_{m(k+1)}]$ $(k = 0, 1, 2, \dots)$ such that

$$(k+1)u_{k+1}(x) = p_k\left(\mathcal{F}_1 u_0, \mathcal{F}_2 u_0, \dots, \mathcal{F}_m u_0, \mathcal{F}_1 u_1, \mathcal{F}_2 u_1, \dots, \mathcal{F}_m u_1, \dots, \mathcal{F}_1 u_k, \mathcal{F}_2 u_k, \dots, \mathcal{F}_m u_k\right).$$

Since the ring K contains the field of rational numbers, we uniquely find $u_k(x)$, $k \in \mathbb{N}$:

$$u_k = k^{-1} p_{k-1}(\mathcal{F}_1 u_0, \dots, \mathcal{F}_m u_0, \mathcal{F}_1 u_1, \dots, \mathcal{F}_m u_1, \dots, \mathcal{F}_1 u_{k-1}, \dots, \mathcal{F}_m u_{k-1}). \quad \square$$

5.2. The Cauchy problem for the Burgers equation. We denote the quotient field of K by F. Let $a, b, u_0 \in K$. Consider the Cauchy problem for the Burgers equation in the ring $K[x]'[[t]]$:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + bu\frac{\partial u}{\partial x} \tag{5.4}$$

$$u(0, x) = u_0 \delta(x). \tag{5.5}$$

We prove the following existence and uniqueness theorem for this Cauchy problem without the additional assumption $K \supset \mathbb{Q}$.

Theorem 5.2. *The Cauchy problem (5.4)–(5.5) has a unique solution. This solution has the form*

$$u(t,x)=\sum_{k=0}^{\infty}u_k\delta^{2k+1}t^k, \tag{5.6}$$

where $u_k \in K$. Moreover, for every $t\in K$, the series (5.6) converges with respect to the topology on the module $K[x]'$.

Proof. Differentiating (5.6) on t and x, we have

$$\frac{\partial u}{\partial t}=\sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{2k+3}t^k, \tag{5.7}$$

$$\frac{\partial u}{\partial x}=-\sum_{k=0}^{\infty}(2k+1)u_k\delta^{2k+2}t^k, \tag{5.8}$$

$$\frac{\partial^2 u}{\partial x^2}=\sum_{k=0}^{\infty}(2k+1)(2k+2)u_k\delta^{2k+3}t^k. \tag{5.9}$$

Substituting (5.6)–(5.9) into (5.4), we get

$$\sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{2k+3}t^k = a\sum_{k=0}^{\infty}(2k+1)(2k+2)u_k\delta^{2k+3}t^k - b\sum_{k=0}^{\infty}\sum_{j=0}^{k}(2j+1)u_ju_{k-j}\delta^{2k+3}t^k.$$

Equating the coefficients at t^k, we obtain

$$(k+1)u_{k+1}=a(2k+1)(2k+2)u_k-b\sum_{j=0}^{k}(2j+1)u_ju_{k-j}, \quad k=0,1,2,\dots. \tag{5.10}$$

Due to

$$2\sum_{j=0}^{k}ju_ju_{k-j}=k\sum_{j=0}^{k}u_ju_{k-j}, \quad k=0,1,2,\dots,$$

equation (5.10) implies

$$(k+1)u_{k+1}=a(2k+1)(2k+2)u_k-b(k+1)\sum_{j=0}^{k}u_ju_{k-j}, \quad k=0,1,2,\dots. \tag{5.11}$$

Since K is of characteristic 0, from (5.11), we obtain

$$u_{k+1}=a(4k+2)u_k-b\sum_{j=0}^{k}u_ju_{k-j}, \quad k=0,1,2,\dots, \tag{5.12}$$

hence $u_k \in K$ for all $k \in \mathbb{N}$. In particular, if $u_0 \in \mathbb{Z}$, then $u_k \in \mathbb{Z}$, $k \in \mathbb{N}$. By Theorem 5.1, the formal power series (5.6) is the unique solution of the Cauchy problem (5.4)–(5.5) in the ring $F[x]'[[t]]$. Since $u(t,x) \in K[x]'[[t]]$, it is the unique solution of the Cauchy problem (5.4)–(5.5) in the ring $K[x]'[[t]]$. Moreover, for every $t \in K$, the series (5.6) converges with respect to the topology on the module $K[x]'$. □

Remark 5.3. Now, let $bu_0 = 2a$. Then (5.12) implies $u_k = 0$, $k \in \mathbb{N}$, and the corresponding solution of the Cauchy problem (5.4)–(5.5) has the form $u(t,x) = u_0\delta(x)$. For $a = 1, b = 2$ and $u_0 \in \mathbb{N}$, the solution of (5.4) with the initial condition $u(0,x) = \frac{u_0}{x}$ was obtained in a classic situation by using the Cole–Hopf transform in [21].

Example 5.4. Here we study the Cauchy problem (5.4)–(5.5), where b is an invertible element of the ring K and $u_0 = 4ab^{-1}$. Consider the rational solution

$$w(t,x) = \frac{4ab^{-1}x}{x^2 + 2at}$$

of equation (5.4) (see [28, Section 5.1.5]). Decomposing this solution into the series with respect to negative powers x, we get

$$w(t,x) = \frac{4ab^{-1}x}{x^2 + 2at} = 4b^{-1}\sum_{k=0}^{\infty} \frac{(-2)^k a^{k+1} t^k}{x^{2k+1}}.$$

The inverse Cauchy–Stieltjes transform for this Laurent series gives us the following element of the ring $K[x]'[[t]]$:

$$u(t,x) = 4b^{-1}\sum_{k=0}^{\infty} a^{k+1}(-2)^k \delta(x)^{2k+1} t^k = 4b^{-1}\sum_{k=0}^{\infty} \frac{a^{k+1}(-2)^k \delta^{(2k)}(x)}{(2k)!} t^k \tag{5.13}$$

(see (4.3)). By Theorem 5.2, this formal power series is the unique solution of the Cauchy problem (5.4)–(5.5). Moreover, for every $t \in K$, the series (5.13) converges with respect to the topology on the module $K[x]'$.

Remark 5.5. Theorem 5.2, Example 5.4 and Examples 4.1, 4.2, 4.4 in [12] show that there is an interesting connection between some classical partial differential equations and well-known integer sequences. The recurrent sequence (5.12) is a particular case of a self-convolutive recurrence which was studied in [25]. If $u_0 = 1, a = 1, b = -1$, then the corresponding element u_k ($k \in \mathbb{N}$) of (5.12) is connected with the number of rooted quadrangulations on some darts [4, Example 4.4] (see also the integer sequence A292186 in [29]).

5.3. Some other examples

Example 5.6. Let $K \supset \mathbb{Q}$ and $u_0 \in K$. Consider the Cauchy problem in $K[x]'[[t]]$:

$$\frac{\partial u}{\partial t} = u\frac{\partial^2 u}{\partial x^2}, \tag{5.14}$$

$$u(0,x) = u_0\delta(x). \tag{5.15}$$

By Theorem 5.1, there exists a unique solution of the Cauchy problem (5.14)–(5.15). As in [12, Example 4.4], we are looking for the solution of this problem in the following form:

$$u(t,x) = \sum_{k=0}^{\infty} u_k \delta^{3k+1} t^k, \tag{5.16}$$

where $u_k \in K$. Thus,

$$\frac{\partial u}{\partial t} = \sum_{k=0}^{\infty} (k+1)u_{k+1}\delta^{3k+4}t^k, \tag{5.17}$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{k=0}^{\infty} (3k+1)(3k+2)u_k\delta^{3k+3}t^k,$$

$$u\frac{\partial^2 u}{\partial x^2} = \sum_{k=0}^{\infty}\sum_{j=0}^{k} (3j+1)(3j+2)u_j u_{k-j}\delta^{3k+4}t^k. \tag{5.18}$$

Substituting (5.16)–(5.18) into (5.14), we get

$$\sum_{k=0}^{\infty} (k+1)u_{k+1}\delta^{3k+4}t^k = \sum_{k=0}^{\infty}\sum_{j=0}^{k} (3j+1)(3j+2)u_j u_{k-j}\delta^{3k+4}t^k.$$

Equating the coefficients at $\delta^{3k+4}t^k$, we obtain

$$(k+1)u_{k+1} = \sum_{j=0}^{k} (3j+1)(3j+2)u_j u_{k-j}, \quad k \in \mathbb{N}_0.$$

Since $K \supset \mathbb{Q}$, we have

$$u_{k+1} = \frac{1}{k+1}\sum_{j=0}^{k} (3j+1)(3j+2)u_j u_{k-j}, \quad k \in \mathbb{N}_0. \tag{5.19}$$

If $K = \mathbb{Z}$ and $u_0 = 1$, then (5.19) implies that $u_5 = \frac{2627152}{5} \notin \mathbb{Z}$. This example shows that the condition $K \supset \mathbb{Q}$ in Theorem 5.1 is essential.

Example 5.7. Let $K \supset \mathbb{Q}$, $a, u_0 \in K$ and $m \in \mathbb{N}$. Consider the following Cauchy problem for the Harry Dym type equation in $K[x]'[[t]]$:

$$\frac{\partial u}{\partial t} = au^m \frac{\partial^m u}{\partial x^m}, \tag{5.20}$$

$$u(0,x) = u_0\delta(x). \tag{5.21}$$

By Theorem 5.1, there exists a unique solution of the Cauchy problem (5.20)–(5.21). As in [12, Example 4.4], we are looking for the solution of this problem in the following form:

$$u(t,x) = \sum_{k=0}^{\infty} u_k \delta^{2mk+1} t^k, \tag{5.22}$$

where $u_k \in K$. Thus,

$$\frac{\partial u}{\partial t} = \sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{2mk+2m+1}t^k, \tag{5.23}$$

$$\frac{\partial^m u}{\partial x^m} = (-1)^m \sum_{k=0}^{\infty}(2mk+1)(2mk+2)\cdots(2mk+m)u_k\delta^{2mk+m+1}t^k,$$

$$au^m\frac{\partial^m u}{\partial x^m} = (-1)^m a\sum_{k=0}^{\infty}\sum_{|\alpha|=k}(2m\alpha_1+1)\cdots(2m\alpha_1+m)u_{\alpha_1}\cdots u_{\alpha_{m+1}}\delta^{2mk+2m+1}t^k. \tag{5.24}$$

Substituting (5.23) and (5.24) into (5.20), we get

$$\sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{2mk+2m+1}t^k$$
$$= (-1)^m a\sum_{k=0}^{\infty}\sum_{|\alpha|=k}(2m\alpha_1+1)\ldots(2m\alpha_1+m)u_{\alpha_1}\ldots u_{\alpha_{m+1}}\delta^{2mk+2m+1}t^k.$$

Equating the coefficients at $\delta^{2mk+2m+1}t^k$, we obtain

$$(k+1)u_{k+1} = (-1)^m a\sum_{|\alpha|=k}(2m\alpha_1+1)\cdots(2m\alpha_1+m)u_{\alpha_1}\cdots u_{\alpha_{m+1}},\ k=0,1,2,\ldots.$$

Since $K \supset \mathbb{Q}$, this difference equation is equivalent to the equation

$$u_{k+1} = \frac{(-1)^m}{k+1}a\sum_{|\alpha|=k}(2m\alpha_1+1)\cdots(2m\alpha_1+m)u_{\alpha_1}\cdots u_{\alpha_{m+1}}, \quad k=0,1,2,\ldots,$$

i.e., u_k ($k \in \mathbb{N}$) is uniquely determined by u_0. Hence, the formal power series (5.22) is a unique solution of the Cauchy problem (5.20)–(5.21) in the ring $K[x]'[[t]]$. Moreover, for every $t \in K$, the series (5.22) converges with respect to the topology on the module $K[x]'$.

If $a = -1$ and $m = 1$, then equation (5.20) is an Euler–Hopf equation which was studied in [12, Example 4.1]. It was shown that u_k ($k = 0, 1, 2, \ldots$) is the well-known sequence of Catalan numbers. If $a = 1$ and $m = 3$, then equation (5.20) is the Harry Dym equation which was also considered in [12].

Example 5.8. Let $K \supset \mathbb{Q}$. Consider the following Cauchy problem for the Korteweg-de Vries equation in $K[x]'[[t]]$:

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x}, \tag{5.25}$$

$$u(0,x) = \delta'(x). \tag{5.26}$$

By Theorem 5.1, there exists a unique solution of the Cauchy problem (5.25)–(5.26). As for the Burgers equation (see Subsection 5.2), we are looking for a solution of this problem in the following form:

$$u(t,x)=\sum_{k=0}^{\infty} u_k\delta^{3k+2}t^k, \tag{5.27}$$

where $u_k \in K$. Thus,

$$\frac{\partial u}{\partial t}=\sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{3k+5}t^k, \tag{5.28}$$

$$\frac{\partial u}{\partial x}=-\sum_{k=0}^{\infty}(3k+2)u_k\delta^{3k+3}t^k,$$

$$\frac{\partial^3 u}{\partial x^3}=-\sum_{k=0}^{\infty}(3k+2)(3k+3)(3k+4)u_k\delta^{3k+5}t^k, \tag{5.29}$$

$$6u\frac{\partial u}{\partial x}=-6\sum_{k=0}^{\infty}\sum_{j=0}^{k}(3j+2)u_ju_{k-j}\delta^{3k+5}t^k. \tag{5.30}$$

Substituting (5.28)–(5.30) into (5.25) and (5.26), we get $u_0=-1$ (see Example 4.5) and

$$\sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{3k+5}t^k=\sum_{k=0}^{\infty}(3k+2)(3k+3)(3k+4)u_k\delta^{3k+5}t^k - 6\sum_{k=0}^{\infty}\sum_{j=0}^{k}(3j+2)u_ju_{k-j}\delta^{3k+5}t^k.$$

Equating the coefficients at $\delta^{3k+5}t^k$, we obtain

$$(k+1)u_{k+1}=(3k+2)(3k+3)(3k+4)u_k-6\sum_{j=0}^{k}(3j+2)u_ju_{k-j}, \quad k=0,1,2,\dots. \tag{5.31}$$

Since $K\supset\mathbb{Q}$, the difference equation (5.31) is equivalent to the equation

$$u_{k+1}=3(3k+2)(3k+4)u_k-6(k+1)^{-1}\sum_{j=0}^{k}(3j+2)u_ju_{k-j}, \quad k=0,1,2,\dots,$$

i.e., u_k ($k\in\mathbb{N}$) is uniquely determined by u_0. Hence, the formal power series (5.27) is the unique solution of the Cauchy problem (5.25)–(5.26) in the ring $K[x]'[[t]]$. If $K=\mathbb{Z}$, then computer experiments show that the first 1000 terms of the sequence u_k are integers. Although this sequence is absent in the online encyclopedia of integer sequences [29], we formulate the conjecture that $u_k\in\mathbb{Z}$ for all $k\in\mathbb{N}_0$.

Consider three examples for the case $n \geq 2$.

Example 5.9. Let $a \in K$. Consider the following Cauchy problem in $K[x_1, \ldots, x_n]'[[t]]$:

$$\frac{\partial u}{\partial t} = a\frac{\partial^n u}{\partial x_1 \cdots \partial x_n}, \tag{5.32}$$

$$u(0, x) = \delta(x). \tag{5.33}$$

Since (5.32) is a linear equation, we can apply Theorem 6.3 from [9]. According to this theorem, the Cauchy problem (5.32)–(5.33) has a unique solution and this solution has the form

$$u(t, x) = \sum_{k=0}^{\infty} a^k \frac{1}{k!} \frac{\partial^{nk}\delta}{\partial x_1^k \cdots \partial x_n^k} t^k = \sum_{k=0}^{\infty} (-1)^{nk} a^k (k!)^{n-1} \delta^{k+1} t^k$$

(see Formula (4.4)). For the case $n = 3$, S.L. Sobolev considered equation (5.32) in the classical situation (see, for example, [27, Section 2.3]).

Example 5.10. Let $K \supset \mathbb{Q}$, $a, u_0 \in K$. Consider the following Cauchy problem in $K[x_1, \ldots, x_n]'[[t]]$:

$$\frac{\partial u}{\partial t} = au\frac{\partial^n u}{\partial x_1 \cdots \partial x_n}, \tag{5.34}$$

$$u(0, x) = u_0\delta(x) \tag{5.35}$$

By Theorem 5.1, there exists a unique solution of the Cauchy problem (5.34)–(5.35). As in [12, Example 4.4], we are looking for a solution of this problem in the following form:

$$u(t, x) = \sum_{k=0}^{\infty} u_k \delta^{2k+1} t^k, \tag{5.36}$$

where $u_k \in K$. Due to (4.5), we have

$$\frac{\partial^n u}{\partial x_1 \cdots \partial x_n} = (-1)^n \sum_{k=0}^{\infty} (2k+1)^n u_k \delta^{2k+2} t^k.$$

Thus,

$$\frac{\partial u}{\partial t} = \sum_{k=0}^{\infty} (k+1) u_{k+1} \delta^{2k+3} t^k, \tag{5.37}$$

$$au\frac{\partial^n u}{\partial x_1 \ldots \partial x_n} = (-1)^n a \sum_{k=0}^{\infty} \sum_{j=0}^{k} (2j+1)^n u_j u_{k-j} \delta^{2k+3} t^k. \tag{5.38}$$

Substituting (5.37),(5.38) into (5.34), we get

$$\sum_{k=0}^{\infty} (k+1) u_{k+1} \delta^{2k+3} t^k = (-1)^n a \sum_{k=0}^{\infty} \sum_{j=0}^{k} (2j+1)^n u_j u_{k-j} \delta^{2k+3} t^k.$$

Equating the coefficients at $\delta^{2k+3}t^k$, we obtain

$$(k+1)u_{k+1} = (-1)^n a \sum_{j=0}^{k} (2j+1)^n u_j u_{k-j}. \tag{5.39}$$

Since $K \supset \mathbb{Q}$, the difference equation (5.39) is equivalent to the equation

$$u_{k+1} = (k+1)^{-1}(-1)^n a \sum_{j=0}^{k} (2j+1)^n u_j u_{k-j}, \quad k = 0, 1, 2, \dots,$$

i.e., u_k ($k \in \mathbb{N}$) is uniquely determined by u_0. Hence, the formal power series (5.36) is the unique solution of the Cauchy problem (5.20)–(5.21) in the ring $K[x_1, \dots, x_n]'[[t]]$. Moreover, for every $t \in K$, the series (5.36) converges with respect to the topology on the module $K[x_1, \dots, x_n]'$.

If $K = \mathbb{Z}$, $a = 1$, $n = 2$ and $u_0 = 1$, then (5.39) implies that $u_3 = \frac{139}{3} \notin \mathbb{Z}$. But, if $n = 3$, then computer experiments show that the first 1000 terms of the sequence u_k are integers. Although this sequence is absent in the online encyclopedia of integer sequences [29], we formulate the conjecture that $u_k \in \mathbb{Z}$ for all $k \in \mathbb{N}_0$.

Example 5.11. Let $a, u_0 \in K$. Consider the following Cauchy problem in $K[x_1, \dots, x_n]'[[t]]$:

$$\frac{\partial u}{\partial t} = (-1)^n a \prod_{j=1}^{n} \frac{\partial u}{\partial x_j}, \tag{5.40}$$

$$u(0, x) = u_0 \delta(x) \tag{5.41}$$

By Theorem 5.1, there exists a unique solution of the Cauchy problem (5.40)–(5.41) in the ring $F[x_1, \dots, x_n]'[[t]]$. We are looking for the solution of this problem in the following form:

$$u(t, x) = \sum_{k=0}^{\infty} u_k \delta^{nk+1} t^k, \tag{5.42}$$

where $u_k \in K$. By Theorem 4.10,

$$\begin{aligned} \frac{\partial u}{\partial x_j} &= \sum_{\alpha_j=0}^{\infty} u_{\alpha_j} \frac{\partial \delta^{n\alpha_j+1}}{\partial x_j} t^j \\ &= \sum_{\alpha_j=0}^{\infty} u_{\alpha_j} (n\alpha_j + 1) \delta^{n\alpha_j} \frac{\partial \delta}{\partial x_j} t^j, \quad j = 1, \dots, n. \end{aligned} \tag{5.43}$$

Due to Proposition 3.4 and Example 3.3,

$$C\left(\frac{\partial \delta}{\partial x_j}\right) = \frac{\partial}{\partial s_j} C(\delta) = \frac{\partial}{\partial s_j} \frac{1}{s_1 \cdots s_n} = -\frac{1}{s_1 \cdots s_n s_j}, \quad j = 1, \dots, n.$$

Therefore,

$$C\left(\prod_{j=1}^{n}\frac{\partial\delta}{\partial x_j}\right)=\prod_{j=1}^{n}C\left(\frac{\partial\delta}{\partial x_j}\right)$$

$$=(-1)^n\frac{1}{(s_1\cdots s_n)^n}\prod_{j=1}^{n}\frac{1}{s_j}=(-1)^n\frac{1}{s_1^{n+1}\cdots s_n^{n+1}}=(-1)^nC(\delta^{n+1}).$$

Hence,

$$\prod_{j=1}^{n}\frac{\partial\delta}{\partial x_j}=(-1)^n\delta^{n+1} \tag{5.44}$$

By (5.44) and (5.43),

$$(-1)^n a\prod_{j=1}^{n}\frac{\partial u}{\partial x_j}=a\sum_{\alpha_1=0}^{\infty}\cdots\sum_{\alpha_n=0}^{\infty}(n\alpha_1+1)\cdots(n\alpha_n+1)u_{\alpha_1}\cdots u_{\alpha_n}\delta^{n|\alpha|+n+1}t^{|\alpha|}$$

$$=a\sum_{k=0}^{\infty}\sum_{|\alpha|=k}(n\alpha_1+1)\cdots(n\alpha_n+1)u_{\alpha_1}\cdots u_{\alpha_n}\delta^{nk+n+1}t^k. \tag{5.45}$$

Now,

$$\frac{\partial u}{\partial t}=\sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{nk+n+1}t^k, \tag{5.46}$$

Substituting (5.46),(5.45) into (5.40), we get

$$\sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{nk+n+1}t^k=a\sum_{k=0}^{\infty}\sum_{|\alpha|=k}(n\alpha_1+1)\cdots(n\alpha_n+1)u_{\alpha_1}\cdots u_{\alpha_n}\delta^{nk+n+1}t^k.$$

Equating the coefficients at $\delta^{nk+n+1}t^k$, we obtain the difference equation

$$(k+1)u_{k+1}=a\sum_{|\alpha|=k}(n\alpha_1+1)\cdots(n\alpha_n+1)u_{\alpha_1}\cdots u_{\alpha_n}\delta^{nk+n+1},\quad k=0,1,2,\ldots. \tag{5.47}$$

We show that for any given $u_0\in K$, the sequence

$$u_k=\frac{A_k(n+1,n^2)}{nk+1}a^k u_0^{(n-1)k+1},\quad k=0,1,2,\ldots,$$

is the unique solution of (5.47), where $A_k(r,m)=\frac{r}{r+mk}\binom{r+mk}{k}$ are Fuss–Catalan–Raney numbers [30] and $r,m\in\mathbb{N}$. Indeed, by using combinatorial identity, which was proved in [15], we obtain for $k=0,1,2,\ldots$,

$$a\sum_{|\alpha|=k}(n\alpha_1+1)\cdots(n\alpha_n+1)u_{\alpha_1}\cdots u_{\alpha_n}\delta^{nk+n+1}$$

$$= a^{k+1}u_0^{(n-1)k+n}\sum_{|\alpha|=k}\frac{A_{\alpha_1}(n+1,n^2)}{n\alpha_1+1}\cdots\frac{A_{\alpha_n}(n+1,n^2)}{n\alpha_n+1}(n\alpha_1+1)\cdots(n\alpha_n+1)$$

$$= a^{k+1}u_0^{(n-1)(k+1)+1}\sum_{|\alpha|=k}A_{\alpha_1}(n+1,n^2)\cdots A_{\alpha_n}(n+1,n^2)$$

$$= a^{k+1}u_0^{(n-1)(k+1)+1}A_k(n(n+1),n^2)$$

$$= a^{k+1}u_0^{(n-1)(k+1)+1}\binom{n(n+1)+kn^2}{k}\frac{n+1}{n+1+kn}$$

$$= a^{k+1}u_0^{(n-1)(k+1)+1}\frac{(n+(k+1)n^2)!}{k!(n+(k+1)n^2-k)!}\,\frac{n+1}{n+1+kn}$$

$$= a^{k+1}u_0^{(n-1)(k+1)+1}\frac{(n+(k+1)n^2)!}{(k+1)!(n+(k+1)n^2-k)!}\,\frac{(k+1)(n+1)}{nk+n+1}$$

$$= a^{k+1}u_0^{(n-1)(k+1)+1}\frac{k+1}{nk+n+1}\binom{n+1+(k+1)n^2}{k+1}\frac{n+1}{n+1+(k+1)n^2}$$

$$= a^{k+1}u_0^{(n-1)(k+1)+1}\frac{(k+1)A_{k+1}(n+1,n^2)}{n(k+1)+1} = (k+1)u_{k+1}.$$

Hence, u_k $(k = 0,1,2\ldots)$ satisfy equation (5.47) and therefore $u(t,x)$ defined by (5.42) is the unique solution of the Cauchy problem (5.40)–(5.41) in the ring $F[x_1,\ldots,x_n]'[[t]]$. By using combinatorial transformations, we can prove

$$\frac{A_k(n+1,n^2)}{nk+1} = \frac{n}{k(k(n-1)+1)}\binom{n^2k+n-1}{k-1}, \quad k\in\mathbb{N}.$$

Since the right-hand side of this equality is integer [2], we have $u_k \in K$. Therefore, $u(t,x)\in K[x_1,\ldots,x_n]'[[t]]$ and it is the unique solution of the Cauchy problem (5.40)–(5.41) in the ring $K[x_1,\ldots,x_n]'[[t]]$.

In Examples 5.6, 5.7, 5.9–5.11 and in Subsection 5.2 the unique solution of the Cauchy problem (5.1)–(5.2) with $Q(x)=\delta(x)$ has the form

$$u(t,x) = \sum_{k=0}^{\infty}u_k\delta^{m_k}(x)t^k, \quad u_k\in K,\ m_k\in\mathbb{N}. \tag{5.48}$$

The following example shows that the solution of the corresponding Cauchy problem for $Q(x)=\delta(x)$ not always has the form (5.48).

Example 5.12. Let $K\supset\mathbb{Q}$. Consider the following Cauchy problem for the Korteweg-de Vries equation in $K[x]'[[t]]$:

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x}, \tag{5.49}$$

$$u(0,x) = \delta(x). \tag{5.50}$$

By Theorem 5.1, there exists a unique solution of this Cauchy problem, which has the form

$$u(t,x) = \sum_{k=0}^{\infty}u_k(x)t^k, \tag{5.51}$$

where $u_k(x) \in K[x]'$. Substituting (5.51) into (5.49), (5.50) and taking into account (4.3), we have

$$u_1(x) = -\delta''' + 6\delta\delta' = 6\delta^4 - 6\delta^3.$$

This shows that the solution is not of the form (5.48).

Acknowledgments. This research was partially supported by the Akhiezer Foundation grant, 2024. The authors are grateful to Eugene Karolinsky and Sergey Poslavsky for useful discussions of the paper results. The authors also would like to thank the referee for helpful comments and suggestions that improved the text of this paper.

References

[1] P. Antosik, J. Mikusinski, and R. Sikorski, *Theory of Distributions. The Sequential Approach*, Elsevier Science Publ. Co., Amsterdam, PWN, Warsaw, 1973.

[2] M. Bousquet-Mélou, É. Fusy, and L.-F. Préville-Ratelle, *The number of intervals in the m-Tamari lattices*, Electron. J. Combin., **18** (2011), No. 2, Paper No. 31.

[3] F.Calogero and A.Degasperis, *Spectral Transform and Solitons: Tolls to Solve and Investigate Nonlinear Evolution Equations,* North-Holland Publishing Company, Amsterdam, 1982.

[4] L. Ciobanu and A. Kolpakov, *Free subgroups of free products and combinatorial hypermaps,* Discrete Math. **342** (2019), No. 5, 1415–1433.

[5] J.F. Colombeau, *Multiplication of Distributions. A Tool in Mathematics, Numerical Engineering and Theoretical Physics*, Springer-Verlag, Berlin, 1992.

[6] Yu. V. Egorov, *A contribution to the theory of generalized functions,* Russian Math. Surveys **45** (1990), No. 5, 1–49.

[7] R. Estrada and R.P. Kanwal, *A Distributional Approach to Asymptotics. Theory and Applications*, Birkhäuser Advanced Texts, Birkhäuser Boston, Inc., Boston, MA, 2002.

[8] J.C. Garcia-Ardila, F. Marcellán, and M.E. Marriaga, *Orthogonal Polynomials and Linear Functionals. An Algebraic Approach and Applications,* EMS Series of Lectures in Mathematics, EMS Press, Berlin, 2001.

[9] S.L. Gefter and A.L. Piven', *Partial differential equations in module of copolynomials over a commutative ring*, J. Math. Phys. Anal. Geom. **21** (2025), No. 1, 56–83.

[10] S.L. Gefter and A.L. Piven', *Linear partial differential equations in module of formal generalized functions over commutative ring*, J. Math. Sci. **257** (2021), No. 5, 579–596.

[11] S.L. Gefter and A.L. Piven', *Implicit linear differential-difference equations in the module of formal generalized functions over a commutative ring*, J. Math. Sci., **255** (2021), No. 4, 409–422.

[12] S.L. Gefter and A.L.Piven', *Some class of nonlinear partial differential equations in the ring of copolynomials over a commutative ring*, Front. Appl. Math. Stat. **10** (2024), Paper No. 1466569.

[13] S.L. Gefter and T.E. Stulova, *Fundamental solution of the simplest implicit linear differential equation in a vector space*, J. Math. Sci. **207** (2015), No. 2, 166–175.

[14] I. M. Gel'fand and L. A. Dikii *Asymptotic behaviour of the resolvent of Sturm–Liouville equations and the algebra of the Korteweg–de Vries equations*, Russian Math. Surveys **30** (1975), No. 5, 77–113.

[15] H.W. Gould, *Some generalizations of Vandermonde's convolution*, Amer. Math. Monthly, **63** (1956) No. 2, 84–91.

[16] S.L. Hefter and O.L. Piven', *Infinite-order differential operators in the module of formal generalized functions and in a ring of formal power series*, Ukr. Math. J., **74** (2022), No. 6, 896–915.

[17] L.G. Hernández and R. Estrada, *Solutions of ordinary differential equations by series of delta functions*, J. Math. Anal. Appl. **191** (1995), No. 1, 40–55.

[18] L. Hörmander, *The Analysis of Linear Partial Differential Equations,* **1**, *Distribution Theory and Fourier Analysis*, Springer-Verlag, Berlin, 1983.

[19] A.M. Krall, R.P. Kanwal, and L.L. Littlejohn, *Distributional solutions of ordinary differential equations.* CMS Conf. Proc., 8. Published by the American Mathematical Society, Providence, RI, 1987, 227–246.

[20] R. Lidl and H. Niederreiter, *Finite Fields*, Cambridge University Press, Cambridge, 1996.

[21] G. Lysik, *Borel summable solutions of the Burgers equation*, Ann. Polon. Math. **95** (2009), 187–197.

[22] Yu.I. Manin, *Algebraic aspects of nonlinear differential equations*, J. Soviet Math. **11** (1979), No. 1, 1–122.

[23] P. Maroni, *Sur quelques espaces de distributions qui sont des formes linéaires sur léspace vectoriel des polynômes,* Lecture Notes in Math., **1171**, Springer-Verlag, Berlin, 1985, 184–194.

[24] P. Maroni, *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques*, Orthogonal Polynomials and their applications, MACS, Ann. Comput. Appl. Math., **9**, Baltzer, Basel, 1991, 95–130.

[25] R.J. Martin and M.J. Kearney, *An exactly solvable self-convolutive recurrence*, Aequat. Math. **80** (2010), 291–318.

[26] M. Morimoto, *An Introductions to Sato's Hyperfunctions*, Amer. Math. Soc. Providence, RI, 1993.

[27] M. Ortner and P. Wagner, *Fundamental solutions of linear partial differential operators. Theory and Practice*, Springer International Publishing Switzerland, Cham, Switzerland, 2015.

[28] A.D. Polyanin and V.F. Zaitsev, *Handbook of Nonlinear Partial Differential Equations*, Second Edition, Taylor & Francis Group, CRC Press, Boca Raton, FL, 2012.

[29] N.J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org/.

[30] R.D.P. Zhou and S.H.F. Yan, *The Raney numbers and* $(s, s+1)$*-core partitions*, European J. Combin. **59** (2017), 114–121.

Received June 6, 2024, revised May 12, 2025.

S.L. Gefter,
V.N. Karazin Kharkiv National University, 4 Svobody Sq., Kharkiv, 61022, Ukraine,
B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv, 61103, Ukraine,
E-mail: gefter@karazin.ua

A.L. Piven',
V.N. Karazin Kharkiv National University, 4 Svobody Sq., Kharkiv, 61022, Ukraine,
E-mail: aleksei.piven@karazin.ua

Нелінійні диференціальні рівняння з частинними похідними у модулі кополіномів над комутативним кільцем

S.L. Gefter and A.L. Piven'

Нехай K — довільне комутативне цілісне кільце з одиницею характеристики 0. Досліджуються кополіноми n змінних, тобто, K-лінійні відображення з кільця поліномів $K[x_1, \dots, x_n]$ у K. Ми розглядаємо кополіноми як алгебраїчні аналоги розподілів. За допомоги перетворення Коші–Стілтьєса введено та досліджено множення кополіномів. Доведено теорему існування та єдиності розв'язку задачі Коші для деяких нелінійних диференціальних рівнянь з частинними похідними в кільці формальних степеневих рядів з кополіноміальними коефіцієнтами. Встановлено зв'язок між деякими класичними нелінійними диференціальними рівняннями з частинними похідними та цілочисельними послідовностями. Зокрема, одержано зображення єдиного розв'язку задачі Коші для рівняння Бюргерса у вигляді ряду за степенями δ-функції з цілими коефіцієнтами.

Ключові слова: кополіном, δ-функція, диференціальний оператор нескінченного порядку, задача Коші, перетворення Коші–Стілтьєса, множення кополіномів