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Univalence Criterion and Quasiconformal Extensions of Analytic Mappings

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In the present paper, we study the criterion for univalence and quasiconformal extensions for locally univalent analytic mappings and analytic mappings. For locally univalent analytic functions, we introduce integral operators in Loewner chain and obtain sufficient conditions for univalent and quasiconformal extensions to generalize the results of Becker, Ahlfors and Wang et al. For analytic functions, we use different proof methods to obtain a sufficient condition for univalence, which generalizes the result of Masih et al.

Key words: analytic function, univalence criteria, quasiconformal extensions

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1. Introduction

We denote $\mathbb{D} = \{z : |z| < 1\}$, $\mathbb{D}_r = \{z : |z| < r\}$, where $0 < r \leq 1$, and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the extended complex plane. Let $\mathbb{R}_+$ be the class of all positive real numbers, k be constant in $[0, 1)$, and $\mathbb{C}_+ = \{z \in \mathbb{C} : \Re z > 0\}$. We call Ω a hyperbolic domain if Ω is a domain in the complex plane $\mathbb{C}$ with at least two boundary points. Let f be a locally univalent analytic function. Define the Schwarzian derivative of the function f as

$$S_f = (P_f)' - \frac{1}{2}P_f^2,$$

where $P_f = f''/f'$ is the pre-Schwarzian derivative of the function f. The Schwarzian derivatives norm and pre-Schwarzian derivatives norm of the function f in Ω are defined as

$$\|S_f\|_\Omega = \sup_{z\in\Omega} |S_f(z)|\, \rho_\Omega^{-2}(z) \quad \text{and} \quad \|P_f\|_\Omega = \sup_{z\in\Omega} |P_f(z)|\, \rho_\Omega^{-1}(z),$$

where $\rho_\Omega(z)$ is the Poincaré density with Gaussian curvature -4 in Ω. If $f : \mathbb{D} \to \Omega$ is a covering mapping, then $\rho_\mathbb{D}(z) = \rho_\Omega(f(z))\,|f'(z)| = 1/\big(1 - |z|^2\big)$.

A homeomorphism F on $\mathbb{D}$ is K-quasiconformal if F has locally L^2-derivatives and satisfies

$$|F_{\bar z}| \leq k\,|F_z| \quad \text{for a.a. } z \in \mathbb{D},$$

where $K = (1 + k)/(1 - k) \geq 1$.

The Loewner chain plays an important role in univalent function theory, quasiconformal extension and universal Teichmüller space theory. For locally univalent analytic functions f in $\mathbb{D}$, Becker [1] proved by using the Loewner chain that if f satisfies

$$\|P_f\|_{\mathbb{D}} \le k < 1, \tag{1.1}$$

then f is univalent in $\mathbb{D}$ and has a continuous extension $\tilde{f}$ to the closed unit disk $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le 1\}$. Moreover, Becker proved that f has a quasiconformal extension to $\overline{\mathbb{C}}$ if (1.1) holds. We call (1.1) Becker's extended univalence criterion. In 1974, Ahlfors [2] generalized Becker's extended univalent criterion. It states that

Theorem A ([2]). *If f satisfies*

$$\left|c|z|^2 + \left(1-|z|^2\right) zP_f(z)\right| \le 1, \quad z \in \mathbb{D},$$

where $|c| \le 1$ and $c \ne -1$, then f is univalent in $\mathbb{D}$. Moreover, if

$$\left|c|z|^2 + \left(1-|z|^2\right) zP_f(z)\right| \le k < 1, \quad z \in \mathbb{D},$$

then f has K-quasiconformal extension onto $\mathbb{C}$ of the form

$$F(z) = \begin{cases} f(z) & \text{for } |z| \le 1 \\ f\left(\dfrac{1}{\bar{z}}\right) + \dfrac{1}{1+c}\left(z - \dfrac{1}{\bar{z}}\right) f'\left(\dfrac{1}{\bar{z}}\right) & \text{for } |z| > 1. \end{cases}$$

Recently, Wang et al. [3] constructed different Loewner chains, which generalize the criterion (1.1). It states that

Theorem B ([3]). *Let f and g be locally univalent analytic in $\mathbb{D}$, and α be a constant with $\alpha \in [0,1]$. If the principal branch of $\left(\dfrac{f(z)}{g(z)}\right)^{\alpha}$ is considered and*

$$\alpha \left|P_f(z) - P_g(z)\right| \left(1-|z|^2\right) + \left|1 - f'(z)^{1-\alpha} g'(z)^{\alpha}\right| \le k < 1, \tag{1.2}$$

where $k \in (0,1)$, then $f(z)$ is univalent in $\mathbb{D}$ and has a quasiconformal extension to $\overline{\mathbb{C}}$.

For more generalizations of Becker's univalence criterion and quasiconformal extension, we refer to [4–11].

We use the same methods and techniques as Deniz, Kanas and Orhan (see [12]) for discussing the univalence criteria of integral operators and give a sufficient condition of univalence and quasiconformal extensions. The theorem below generalizes the results of Becker (the criteria (1.1)) and Wang et al. (Theorem B).

Theorem 1.1. *Let f and g be locally univalent analytic in $\mathbb{D}$, and α be a constant with $\alpha \in [0,1]$. Let $m, \beta \in \mathbb{R}_+$ and $c \in \overline{\mathbb{D}} \setminus \{-1\}$. If the principal branch of $\left(\dfrac{f(z)}{g(z)}\right)^{\alpha}$ is considered and*

$$\frac{\alpha}{\beta} \left|P_f(z) - P_g(z)\right| \left(1 - |z|^{(1+m)\beta}\right) + \frac{|m-1|}{2} + \left|(1+c) f'(z)^{1-\alpha} g'(z)^{\alpha} - 1\right|$$

$$\leq \frac{(m+1)k}{2} < \frac{m+1}{2}, \quad (1.3)$$

then the function $\mathcal{F}_\beta(z)$, *defined by*

$$\mathcal{F}_\beta(z) = \left(\beta \int_0^z u^{\beta-1} f'(u)du\right)^{\frac{1}{\beta}}, \quad z \in \mathbb{D}, \quad (1.4)$$

where the principal branch is considered, is analytic and univalent in $\mathbb{D}$. *Moreover,* $\mathcal{F}_\beta(z)$ *has a quasiconformal extension to* $\overline{\mathbb{C}}$.

We get the following corollary, when $\beta = 1$.

Corollary 1.2. *Let* f *and* g *be locally univalent analytic in* $\mathbb{D}$, *and* α *be a constant with* $\alpha \in [0,1]$. *Let* $m \in \mathbb{R}_+$ *and* $c \in \overline{\mathbb{D}} \setminus \{-1\}$. *If the principal branch of* $\left(\frac{f(z)}{g(z)}\right)^\alpha$ *is considered and*

$$\alpha \left|P_f(z) - P_g(z)\right| \left(1 - |z|^{1+m}\right) + \frac{|m-1|}{2} + \left|(1+c)f'(z)^{1-\alpha}g'(z)^\alpha - 1\right|$$
$$\leq \frac{(m+1)k}{2} < \frac{m+1}{2}, \quad (1.5)$$

then the function $f(z)$ *is univalent in* $\mathbb{D}$ *and has a quasiconformal extension to* $\overline{\mathbb{C}}$.

Remark 1.3.

(1) The criterion (1.5) corresponds to the criterion (1.1), when $\alpha = 1$, $g(z) = z$, $c = 0$ and $m = 1$.
(2) The criterion (1.5) corresponds to the criterion (1.2), when $m = 1$ and $c = 0$.

By using the proof method that differs from that of Theorem 1.1, we get the following conclusion, which generalizes Ahlfors's result (Theorem A) as follows.

Theorem 1.4. *Let* f *and* g *be locally univalent analytic in* $\mathbb{D}$, *and* α *be a constant with* $\alpha \in [0,1]$. *Let* $m, \beta \in \mathbb{R}_+$ *and* $c \in \overline{\mathbb{D}} \setminus \{-1\}$. *If the principal branch of* $\left(\frac{f(z)}{g(z)}\right)^\alpha$ *is considered and*

$$\left|\left[(1+c)f'(z)^{1-\alpha}g'(z)^\alpha - 1\right] - \frac{m-1}{2}\right| \leq \frac{k(1+m)}{2} < \frac{1+m}{2} \quad (1.6)$$

and

$$\left||z|^{(1+m)\beta}\left[(1+c)f'(z)^{1-\alpha}g'(z)^\alpha - 1\right] + \frac{\alpha}{\beta}z\left(1 - |z|^{(1+m)\beta}\right)\left(P_f(z) - P_g(z)\right)\right.$$
$$\left. - \frac{|m-1|}{2}\right| \leq \frac{k(1+m)}{2} < \frac{1+m}{2}, \quad (1.7)$$

then the function $\mathcal{F}_\beta(z)$ *defined by* (1.4) *is analytic and univalent in* $\mathbb{D}$. *Moreover, the function* $\mathcal{F}_\beta(z)$ *has a quasiconformal extension to* $\overline{\mathbb{C}}$. *Here the principal branch of* $\mathcal{F}_\beta(z)$ *is considered.*

We get the following corollary, when $\beta = 1$.

Corollary 1.5. *Let f and g be locally univalent analytic in $\mathbb{D}$, and α be a constant with $\alpha \in [0,1]$. Let $m, \beta \in \mathbb{R}_+$ and $c \in \overline{\mathbb{D}} \setminus \{-1\}$. If the principal branch of $\left(\frac{f(z)}{g(z)}\right)^{\alpha}$ is considered and*

$$\left|\left[(1+c)f'(z)^{1-\alpha}g'(z)^{\alpha} - 1\right] - \frac{m-1}{2}\right| \le \frac{k(1+m)}{2} < \frac{1+m}{2} \tag{1.8}$$

and

$$\begin{aligned} &\Big||z|^{1+m}\left[(1+c)f'(z)^{1-\alpha}g'(z)^{\alpha} - 1\right] + \alpha z\left(1-|z|^{1+m}\right)\left(P_f(z) - P_g(z)\right) \\ &\qquad - \frac{|m-1|}{2}\Big| \le \frac{k(1+m)}{2} < \frac{1+m}{2}, \end{aligned} \tag{1.9}$$

then the function $f(z)$ is univalent in $\mathbb{D}$ and has a quasiconformal extension to $\overline{\mathbb{C}}$.

Remark 1.6.

(1) Corollary 1.5 corresponds to the criterion (1.1), when $\alpha = 1$, $g(z) = z$, $c = 0$, and $m = 1$.
(2) Corollary 1.5 corresponds to Theorem A, when $\alpha = 1$, $g(z) = z$, and $m = 1$.

Denote by $\mathcal{A}$ the class of all analytic functions f in $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$. Masih et al. [13] studied the univalence criterion of integral operators defined by

$$\mathcal{F}_{\gamma}(z) = \left(\gamma \int_0^z u^{\gamma-1} f'(u)\,du\right)^{1/\gamma}, \quad z \in \mathbb{D} \tag{1.10}$$

if $f \in \mathcal{A}$. It states that

Theorem C ([13])**.** *Let $\alpha, \gamma \in \mathbb{C}_+$ with $\Re[\gamma(1+\alpha)] > 0$. Let $h(z) \in \mathcal{A}$ and $g(z)$ be an analytic function in $\mathbb{D}$ such that $g(z) = 1 + b_1 z + b_2 z^2 + \cdots$ with $g(z) \ne 0$. If the inequalities*

$$\left|\left(\frac{h'(z)}{g(z)} - 1\right) + \frac{1-\alpha}{2}\right| < \frac{|1+\alpha|}{2} \quad \text{for } z \in \mathbb{D} \tag{1.11}$$

and

$$\left||z|^{\gamma(1+\alpha)}\left(\frac{h'(z)}{g(z)} - 1\right) + \frac{1-|z|^{\gamma(1+\alpha)}}{\gamma}\frac{zg'(z)}{g(z)} + \frac{1-\alpha}{2}\right| \le \frac{|1+\alpha|}{2} \text{ for } z \in \mathbb{D}\setminus\{0\} \tag{1.12}$$

are satisfied, then the function $\mathcal{F}_{\gamma}(z)$ defined by (1.10) is univalent in $\mathbb{D}$. Here the principal branch of $\mathcal{F}_{\gamma}(z)$ is considered.

Remark 1.7. We point out that the integral operator (1.4) is different from the integral operator (1.10), because the parameter $\beta \in \mathbb{R}_+$ of (1.4) and the parameter $\gamma \in \mathbb{C}_+$ of (1.10).

After some modifications of the proof of Theorem C, the following theorem can be obtained.

Theorem 1.8. *Let $\alpha, \gamma \in \mathbb{C}_+$ with $\Re(\gamma(1+\alpha)) > 0$. Let $h(z) \in \mathcal{A}$ and $g(z)$ be an analytic function in $\mathbb{D}$ such that $g(z) = 1 + b_1 z + b_2 z^2 + \cdots$, $A + B \neq 0$, $|A - B| < 2$, $|A| \leq 1$, and $|B| \leq 1$ with $g(z) \neq 0$. If the inequalities*

$$\left| \left[\left(\frac{h'(z)}{g(z)} - 1 \right) + \frac{1-\alpha}{2} \right] - \frac{1+\alpha}{2} \frac{\overline{(A-B)}(A+B)}{4 - |A-B|^2} \right| < \frac{|1+\alpha||A+B|}{4-|A-B|^2} \quad \text{for } z \in \mathbb{D} \tag{1.13}$$

and

$$\left| \left[|z|^{\gamma(1+\alpha)} \left(\frac{h'(z)}{g(z)} - 1 \right) + \frac{1 - |z|^{\gamma(1+\alpha)}}{\gamma} \frac{zg'(z)}{g(z)} + \frac{1-\alpha}{2} \right] - \frac{1+\alpha}{2} \frac{\overline{(A-B)}(A+B)}{4-|A-B|^2} \right| \leq \frac{|1+\alpha||A+B|}{4-|A-B|^2} \quad \text{for } z \in \mathbb{D} \setminus \{0\} \tag{1.14}$$

are satisfied, then the function $\mathcal{F}_\gamma(z)$ defined by (1.10) is univalent in $\mathbb{D}$. Here the principal branch of $\mathcal{F}_\gamma(z)$ is considered.

Remark 1.9. Theorem 1.8 corresponds to Theorem C, when $A = B = 1$.

We get the following corollary, when $\alpha = 1$.

Corollary 1.10. *Let $\gamma \in \mathbb{C}_+$, $h(z) \in \mathcal{A}$ and $g(z)$ be an analytic function in $\mathbb{D}$ such that $g(z) = 1 + b_1 z + b_2 z^2 + \cdots$, $A + B \neq 0$, $|A - B| < 2$, $|A| \leq 1$, and $|B| \leq 1$ with $g(z) \neq 0$. If the inequalities*

$$\left| \left(\frac{h'(z)}{g(z)} - 1 \right) - \frac{\overline{(A-B)}(A+B)}{4-|A-B|^2} \right| < \frac{2|A+B|}{4-|A-B|^2} \quad \text{for } z \in \mathbb{D} \tag{1.15}$$

and

$$\left| |z|^{2\gamma} \left(\frac{h'(z)}{g(z)} - 1 \right) + \frac{1-|z|^{2\gamma}}{\gamma} \frac{zg'(z)}{g(z)} - \frac{\overline{(A-B)}(A+B)}{4-|A-B|^2} \right| \leq \frac{2|A+B|}{4-|A-B|^2} \quad \text{for } z \in \mathbb{D} \setminus \{0\} \tag{1.16}$$

are satisfied, then the function $\mathcal{F}_\gamma(z)$ defined by (1.10) is univalent in $\mathbb{D}$. Here the principal branch of $\mathcal{F}_\gamma(z)$ is considered.

Using Theorem 1.8 and the method of scaling proof, we can get the following theorem.

Theorem 1.11. *Let $\alpha,\gamma\in\mathbb{C}_+$ with $\Re[\gamma(1+\alpha)]>0$. Let $h(z)\in\mathcal{A}$ and $g(z)$ be an analytic function in $\mathbb{D}$ such that $g(z)=1+b_1z+b_2z^2+\cdots$, $A+B\neq 0$, $|A-B|<2$, $|A|\leq 1$, and $|B|\leq 1$ with $g(z)\neq 0$. If the inequalities*

$$\left|\left[\left(\frac{h'(z)}{g(z)}-1\right)+\frac{1-\alpha}{2}\right]-\frac{1+\alpha}{2}\frac{\overline{(A-B)}(A+B)}{4-|A-B|^2}\right|<\frac{|1+\alpha||A+B|}{4-|A-B|^2}\ \textit{for } z\in\mathbb{D} \tag{1.17}$$

and

$$|z|^{\Re[\gamma(1+\alpha)]}\left|\frac{h'(z)}{g(z)}-1\right|+\frac{1-|z|^{\Re[\gamma(1+\alpha)]}}{\Re\gamma}\left|\frac{zg'(z)}{g(z)}\right| + \left|\frac{1-\alpha}{2}-\frac{1+\alpha}{2}\frac{\overline{(A-B)}(A+B)}{4-|A-B|^2}\right|\leq\frac{|1+\alpha||A+B|}{4-|A-B|^2}\quad \textit{for } z\in\mathbb{D}\setminus\{0\} \tag{1.18}$$

are satisfied, then the function $\mathcal{F}_\gamma(z)$ defined by (1.10) is univalent in $\mathbb{D}$. Here the principal branch of $\mathcal{F}_\gamma(z)$ is considered.

For more information about the univalence criterion of integral operators, see [14–18].

2. Preliminaries

In this section, we describe the results to be used in the proof. Recall the definition of the Loewner chain.

Definition 2.1. A function $L(z,t):\mathbb{D}\times[0,\infty)\to\mathbb{C}$ is said to be a Loewner chain or a subordination chain if:

(i) $L(z,t)$ is analytic and univalent in $\mathbb{D}$ for all $t\geq 0$.
(ii) $L(z,t)\prec L(z,s)$ for all $0\leq t<\infty$, where "$\prec$" is subordination.

The following result is due to Pommerenke [19].

Lemma 2.2 ([19])**.** *Let $L(z,t)=a_1(t)z+a_2(t)z^2+\cdots$ be an analytic function in $\mathbb{D}_r$ $(0<r\leq 1)$ for all $t\geq 0$. Suppose that*

(i) *$L(z,t)$ is locally absolutely continuous with respect to $t\in[0,\infty)$ and locally uniform with respect to $z\in\mathbb{D}_r$;*
(ii) *$a_1(t)$ is a complex valued continuous function on $[0,\infty)$ such that $a_1(t)\neq 0$, $\lim_{t\to\infty}|a_1(t)|=\infty$ and $\left\{\frac{L(z,t)}{a_1(t)}\right\}_{t\geq 0}$ is a normal family of functions in $\mathbb{D}_r$;*
(iii) *there exists an analytic function $p:\mathbb{D}\times[0,\infty)\to\mathbb{C}$ satisfying $\Re p(z,t)>0$ for all $(z,t)\in\mathbb{D}\times[0,\infty)$ and*

$$z\frac{\partial L(\partial z,t)}{\partial z}=p(z,t)\frac{\partial L(z,t)}{\partial t},\quad z\in\mathbb{D}_r,\ \textit{for a.a. } t\geq 0. \tag{2.1}$$

Then $L(z,t)$ is a Loewner chain.

We call $L(z,t)$ a standard Loewner chain when the function $L(z,t)$ satisfies the above three conditions and $a_1(t) = e^{-t}$. Equation (2.1) is called the generalized Loewner equation.

The following result is due to Becker [20].

Lemma 2.3 ([20]). *Suppose that $L(z,t)$ is a Loewner chain. Consider*

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1}, \quad z \in \mathbb{D}, \quad t \geq 0,$$

where $p(z,t)$ is given in (2.1). If

$$|w(z,t)| \leq k, \quad 0 \leq k < 1$$

for all $z \in \mathbb{D}$ and $t \geq 0$, then $L(z,t)$ admits a continuous extension to $\overline{\mathbb{D}}$ for each $t \geq 0$ and the function $F(z,\bar{z})$ defined by

$$F(z,\bar{z}) = \begin{cases} L(z,0) & \text{if} \quad |z| < 1 \\ L\left(\dfrac{z}{|z|}, \log|z|\right) & \text{if} \quad |z| \geq 1 \end{cases}$$

is a quasiconformal extension of $L(z,0)$ to $\overline{\mathbb{C}}$.

At present, there are many studies of Loewner chain, see [21–25], where [21] established an analogue of Lemma 2.3 in the upper half plane.

3. Proof of the main results

In this section, we state the proof of the main results.

Proof of Theorem 1.1. Without loss of generality, suppose that

$$f(z) = z + a_2 z^2 + \cdots \quad \text{and} \quad g(z) = z + b_2 z^2 + \cdots .$$

Firstly, we prove that there exists a real number $0 < r \leq 1$ such that the function $L(z,t) : \mathbb{D}_r \times [0,\infty) \to \mathbb{C}$ defined by

$$L(z,t) = \left[\beta \int_0^{e^{-t}z} u^{\beta-1} f'(u)\,du + \frac{(e^{mt} - e^{-t})z^\beta}{1+c}\left(\frac{f'(e^{-t}z)}{g'(e^{-t}z)}\right)^\alpha\right]^{1/\beta}$$

is analytic in $\mathbb{D}_r$ for all $t \geq 0$.

We find that the function $h_1(z,t)$ defined by

$$h_1(z,t) = \left(\frac{f'\left(e^{-t}z\right)}{g'\left(e^{-t}z\right)}\right)^\alpha = 1 + \cdots$$

is analytic in $\mathbb{D}_{r_1}$, $0 < r_1 \leq 1$, for all $t \geq 0$, if we consider the principal branches of the function $h_1(z,t)$.

Define $h_2(z,t)$ as

$$h_2(z,t) = \beta \int_0^{e^{-t}z} u^{\beta-1} f'(u)du = (e^{-t}z)^{\beta} + \sum_{n=2}^{\infty} \frac{n\beta a_n}{\beta+1} \left(e^{-t}z\right)^{n+\beta-1}.$$

We have

$$h_2(z,t) = z^{\beta} h_3(z,t),$$

where

$$h_3(z,t) = \left(e^{-t}\right)^{\beta} + \sum_{n=2}^{\infty} \frac{n\beta a_n}{\beta+1} \left(e^{-t}\right)^{n+\beta-1} z^{n-1}.$$

According to [18], the function $h_3(z,t)$ is analytic in $\mathbb{D}_{r_1}$. Therefore, the function $h_4(z,t)$ defined by

$$h_4(z,t) = h_3(z,t) + \frac{\left(e^{m\beta t} - e^{-\beta t}\right) h_1(z,t)}{1+c}$$

is analytic in $\mathbb{D}_{r_1}$ for all $t \geq 0$, and we observe that

$$h_4(0,t) = h_3(0,t) + \frac{\left(e^{m\beta t} - e^{-\beta t}\right) h_1(0,t)}{1+c} = e^{-\beta t} + \frac{e^{m\beta t} - e^{-\beta t}}{1+c}, \; h_4(0,0) = 1.$$

Further, we have to prove that $h_4(0,t) \neq 0$ for each $t \geq 0$. Suppose $h_4(0,t) = 0$, then there exists a number $t_1 > 0$ such that $h_4(0,t_1) = 0$, namely $-c = e^{\beta(m+1)t_1}$. Since $m, \beta \in \mathbb{R}_+$, then $|c| = \left|e^{\beta(m+1)t_1}\right| > 1$, which contradicts the condition $c \in \overline{\mathbb{D}} \setminus \{-1\}$. Then there exists a disk $\mathbb{D}_{r_2}$, $0 < r_2 \leq r_1$, such that $h_4(0,t) \neq 0$ for all $t \geq 0$, and we can choose an analytic branch of $[h_4(z,t)]^{1/\beta}$ denoted by $h_5(z,t)$.

Thus, the function $L(z,t)$ can be written as

$$L(z,t) = zh_5(z,t) = a_1(t)z + \cdots, \quad z \in \mathbb{D}_{r_2},\; t \geq 0, \tag{3.1}$$

where

$$a_1(t) = \left(\frac{e^{m\beta t} + ce^{-\beta t}}{1+c}\right)^{1/\beta}.$$

Here we consider the uniform branch equal to 1 at the origin. Then we get $\lim\limits_{t\to\infty} |a_1(t)| = \infty$ and $a_1(t) \neq 0$ for all $t \geq 0$. After the above discussion, we can infer that $L(z,t)$ is analytic in $\mathbb{D}_{r_2}$.

Since $L(z,t)$ is analytic in $\mathbb{D}_{r_2}$, then

$$\left|\frac{L(z,t)}{a_1(t)}\right| < K, \quad z \in \mathbb{D}_{r_3},\; t \geq 0,$$

where $0 < r_3 < r_2$ and $K = K(r_3)$.

Thus, by Montel's theorem, $\left\{\frac{L(z,t)}{a_1(t}\right\}_{t\geq 0}$ is a normal family in $\mathbb{D}_{r_3}$. From (3.1), we observe that

$$\frac{\partial L(z,t)}{\partial t} = z\frac{\partial h_5(z,t)}{\partial t},$$

where $\dfrac{\partial h_5(z,t)}{\partial t}$ is analytic in $\mathbb{D}_{r_3}$. It implies that $\dfrac{\partial L(z,t)}{\partial t}$ is also an analytic function in $\mathbb{D}_{r_3}$. From the analyticity of $\dfrac{\partial L(z,t)}{\partial t}$, for any fixed number $T > 0$, we have

$$\left|\frac{\partial L(z,t)}{\partial t}\right| < K_1, \quad z \in \mathbb{D}_{r_4},\ t \in [0,T],$$

where $0 < r_4 < r_3$ and $K_1 > 0$ (that is related to T and r_4).

Hence, the function $L(z,t)$ is locally absolutely continuous in $[0,\infty)$ and locally uniform with respect to $\mathbb{D}_{r_4}$.

From the analyticity of $\dfrac{\partial L(z,t)}{\partial t}$, there exists a disk $D_r,\ 0 < r < r_4$, such that

$$\frac{1}{z}\frac{\partial L(z,t)}{\partial t} \neq 0,$$

and defined

$$p(z,t) = \frac{z\partial L(z,t)}{\partial z} \bigg/ \frac{\partial L(z,t)}{\partial t} \tag{3.2}$$

is analytic in $\mathbb{D}_r$ for each $t \geq 0$.

Proving that $p(z,t)$ has an analytic extension and $\Re p(z,t) > 0$ in $\mathbb{D}$ for each $t \geq 0$ is equivalent to proving that

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1}, \quad z \in \mathbb{D},\ t \geq 0, \tag{3.3}$$

is analytic in $\mathbb{D}$ and

$$|w(z,t)| < 1, \quad z \in \mathbb{D},\ t \geq 0. \tag{3.4}$$

Calculations yield that

$$\begin{aligned} w(z,t) = \frac{2}{m+1}\Big\{ & e^{-(m+1)\beta t}\left[(1+c)f'\left(e^{-t}z\right)^{1-\alpha} g'\left(e^{-t}z\right)^{\alpha} - 1\right] \\ & + \frac{\alpha}{\beta}e^{-t}z\left(1-e^{-(m+1)\beta t}\right)\left(\frac{f''\left(e^{-t}z\right)}{f'\left(e^{-t}z\right)} - \frac{g''\left(e^{-t}z\right)}{g'\left(e^{-t}z\right)}\right) - \frac{m-1}{2}\Big\}. \end{aligned}$$

Since $m, \beta \in \mathbb{R}_+$ and $t \geq 0$, it follows that $\left|e^{-t}z\right|^{(m+1)\beta} < e^{-(m+1)\beta t}$ for $z \in \mathbb{D}$. Using z to represent $e^{-t}z$, by (1.3), we have

$$\begin{aligned} |w(z,t)| \leq & \frac{2}{m+1}\Big[\left|(1+c)f'(e^{-t}z)^{1-\alpha}g'(e^{-t}z)^{\alpha} - 1\right|\left|e^{-(m+1)\beta t}\right| \\ & + \left|\frac{m-1}{2}\right| + \left|\frac{\alpha z e^{-t}}{\beta}\right|\left(1-e^{-(m+1)\beta t}\right)\left|\frac{f''\left(e^{-t}z\right)}{f'\left(e^{-t}z\right)} - \frac{g''\left(e^{-t}z\right)}{g'\left(e^{-t}z\right)}\right|\Big] \\ \leq & \frac{2}{m+1}\Big[\left|(1+c)f'(z)^{1-\alpha}g'(z)^{\alpha} - 1\right| \\ & + \left|\frac{m-1}{2}\right| + \frac{\alpha}{\beta}\left(1-|z|^{(m+1)\beta}\right)\left|\frac{f''(z)}{f'(z)} - \frac{g''(z)}{g'(z)}\right|\Big] \leq k < 1. \end{aligned} \tag{3.5}$$

From (3.5), we obtain $\Re p(z,t)>0$. Since $\left|e^{-t}z\right|<1$ for each $t\geq 0$, $z\in\mathbb{D}$ and $f'(z)^{1-\alpha}g'(z)^{\alpha}$ is analytic in $\mathbb{D}$, it follows that $e^{-t}z\in\mathbb{D}$ and $w(z,t)$ is analytic in $\mathbb{D}$. Hence, $L(z,t)$ is a Loewner chain. It implies that $\mathcal{F}_{\beta}(z)$ is univalent and analytic in $\mathbb{D}$. Furthermore, by Lemma 2.3, we can infer that $\mathcal{F}_{\beta}(z)$ admits a quasiconformal extension onto $\overline{\mathbb{C}}$. □

We change the proof method of Theorem 1.1 as follows.

Proof of Theorem 1.4. Without loss of generality, suppose that

$$f(z)=z+a_2z^2+\cdots \quad \text{and} \quad g(z)=z+b_2z^2+\cdots.$$

Define the function $L(z,t):\mathbb{D}\times[0,\infty)\to\mathbb{C}$ by

$$L(z,t)=\left[\beta\int_0^{e^{-t}z}u^{\beta-1}f'(u)du+\frac{\left(e^{mt}-e^{-t}\right)z^{\beta}}{1+c}\left(\frac{f'\left(e^{-t}z\right)}{g'\left(e^{-t}z\right)}\right)^{\alpha}\right]^{1/\beta}.$$

It can be seen from the proof of Theorem 1.1 that the function $L(z,t)$ satisfies conditions (i) and (ii). Next, we use another method to prove that the function $L(z,t)$ satisfies condition (iii). The functions $p(z,t)$ and $w(z,t)$ are given by (3.2) and (3.3). Notice that

$$w(z,t)=\frac{2}{m+1}\left[G(z,t)-\frac{m-1}{2}\right], \tag{3.6}$$

where

$$\begin{aligned}G(z,t)=&\,e^{-(m+1)\beta t}\left[(1+c)\left(f'(e^{-t}z)^{1-\alpha}g'\left(e^{-t}z\right)^{\alpha}-1\right)\right]\\&+\frac{\alpha ze^{-t}}{\beta}\left(1-e^{-(m+1)\beta t}\right)\left(\frac{f''\left(e^{-t}z\right)}{f'\left(e^{-t}z\right)}-\frac{g''\left(e^{-t}z\right)}{g'\left(e^{-t}z\right)}\right).\end{aligned}$$

It is easy to prove that the condition (3.4) is equivalent to

$$\left|G(z,t)-\frac{m-1}{2}\right|<\frac{m+1}{2},\quad z\in\mathbb{D},\ t\geq 0. \tag{3.7}$$

When $t=0$ and $z=0$, by (1.6), we have

$$\begin{aligned}\left|G(z,0)-\frac{m-1}{2}\right|&=\left|\left[(1+c)f'(z)^{1-\alpha}g'(z)^{\alpha}-1\right]-\frac{m-1}{2}\right|\\&\leq\frac{(m+1)k}{2}<\frac{m+1}{2}\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\left|G(0,t)-\frac{m-1}{2}\right|&=\left|ce^{-(m+1)\beta t}-\frac{m-1}{2}\right|\\&\leq\left|\left(c-\frac{m-1}{2}\right)\right|e^{-(m+1)\beta t}+\left(1-e^{-(m+1)\beta t}\right)\frac{|m-1|}{2}\end{aligned}$$

$$\leq \frac{(m+1)k}{2} e^{-(m+1)\beta t} + \left(1 - e^{-(m+1)\beta t}\right) \frac{(m+1)k}{2}$$
$$< \frac{m+1}{2}. \tag{3.9}$$

Define

$$Q(z,t) = G(z,t) - \frac{m-1}{2}, \quad z \in \mathbb{D},\ t \geq 0.$$

Since $\left|ze^{-t}\right| \leq e^{-t} < 1$ for each $z \in \overline{\mathbb{D}}$, $t > 0$ and $f'(z)^{1-\alpha} g'(z)^{\alpha}$ is analytic in $\mathbb{D}$. It implies that $Q(z,t)$ is analytic in $\overline{\mathbb{D}}$. Using the maximum modulus principle, it follows that

$$|Q(z,t)| < \max_{|z|=1} |Q(z,t)| = \left|Q(e^{\mathrm{i}\theta}, t)\right|, \quad z \in \mathbb{D},\ t > 0, \tag{3.10}$$

where $\theta = \theta(t) \in \mathbb{R}$. Let $\xi = e^{-t} e^{\mathrm{i}\theta}$. Therefore, $|\xi| = e^{-t}$ and $e^{-(m+1)\beta t} = (e^{-t})^{(m+1)\beta} = |\xi|^{(m+1)\beta}$. By (1.7), we have

$$\left|Q(e^{\mathrm{i}\theta}, t)\right| = \Bigg| |\xi|^{(m+1)\beta} \left[(1+c) f'(\xi)^{1-\alpha} g'(\xi)^{\alpha} - 1\right] - \frac{m-1}{2}$$
$$+ \frac{\alpha\xi}{\beta}\left(1 - |\xi|^{(m+1)\beta}\right)\left(\frac{f''(\xi)}{f'(\xi)} - \frac{g''(\xi)}{g'(\xi)}\right)\Bigg|$$
$$\leq \frac{k(m+1)}{2} < \frac{m+1}{2}. \tag{3.11}$$

Combining (3.8)-(3.11), we conclude that inequality (3.7) holds true for all $z \in \mathbb{D}$ and $t \geq 0$. Hence, $L(z,t)$ is a Loewner chain. It implies that the function $\mathcal{F}_{\gamma}(z)$ is univalent and analytic in $\mathbb{D}$. Furthermore, by Lemma 2.3, we can infer that $\mathcal{F}_{\gamma}(z)$ admits a quasiconformal extension onto $\overline{\mathbb{C}}$. □

The proof of Theorem 1.8 is similar to that of [13]. Here we only describe the differences.

Proof of Theorem 1.8. Define the function $F(z,t)$ by

$$F(z,t) = \left[\gamma \int_0^{e^{-t}z} u^{\gamma-1} f'(u) du + \left(e^{\alpha\gamma t} - e^{-\gamma t}\right) z^{\gamma} g\left(e^{-t} z\right)\right]^{1/\gamma}.$$

It can be seen from [13] that the function $F(z,t)$ satisfies the conditions (i) and (ii) of Lemma 2.2. Next, we prove that the function $F(z,t)$ satisfies the condition (iii).

Let the function $p(z,t) : \mathbb{D}_r \times [0,\infty) \to \mathbb{C}$, $0 < r < 1$, defined by

$$p(z,t) = \frac{z\partial L(z,t)}{\partial z} \Bigg/ \frac{\partial L(z,t)}{\partial t},$$

be analytic in $\mathbb{D}_r$ for all $t \geq 0$. Define the function $w_1(z,t)$ by

$$w_1(z,t) = \frac{p(z,t) - 1}{Ap(z,t) + B}, \quad z \in \mathbb{D},\ t \geq 0. \tag{3.12}$$

We can prove that

$$|w_1(z,t)| < 1, \quad z \in \mathbb{D},\ t \geq 0, \tag{3.13}$$

is equivalent to $\Re p(z,t) > 0$ since $A + B \neq 0,\ |A| \leq 1,$ and $|B| \leq 1.$

Make a transformation as follows:

$$w_1(z,t) = \frac{p(z,t)-1}{Ap(z,t)+B} = \frac{2\dfrac{p(z,t)-1}{p(z,t)+1}}{(A-B)\dfrac{p(z,t)-1}{p(z,t)+1} + (A+B)} = \frac{2w(z,t)}{(A-B)w(z,t)+(A+B)}.$$

From [13], we have

$$w(z,t) = \frac{p-1}{p+1} = \frac{2}{\alpha+1}\left[e^{-(\alpha+1)\gamma t}\left(\frac{f'\left(e^{-t}z\right)}{g\left(e^{-t}z\right)} - 1\right) + \frac{1}{\gamma}\left(1 - e^{-(\alpha+1)\gamma t}\right)\frac{e^{-t}zg'\left(e^{-t}z\right)}{g\left(e^{-t}z\right)} + \frac{1-\alpha}{2}\right].$$

It can be proved that (3.13) is equivalent to

$$\left|w(z,t) - \frac{(\overline{A-B})(A+B)}{4-|A-B|^2}\right| < \frac{2|A+B|}{4-|A-B|^2}, \quad z \in \mathbb{D},\ t \geq 0. \tag{3.14}$$

When $t = 0$ and $z = 0$, by (1.13), we have

$$|w_1(z,0)| = \left|w(z,0) - \frac{(\overline{A-B})(A+B)}{4-|A-B|^2}\right| = \frac{2}{|\alpha+1|}\left|\left[\left(\frac{f'(z)}{g(z)} - 1\right) + \frac{1-\alpha}{2}\right] - \frac{\alpha+1}{2}\frac{(\overline{A-B})(A+B)}{4-|A-B|^2}\right| < \frac{2|A+B|}{4-|A-B|^2} \tag{3.15}$$

and

$$|w_1(0,t)| = \left|w(0,t) - \frac{(\overline{A-B})(A+B)}{4-|A-B|^2}\right| = \frac{2}{|\alpha+1|}\left|\frac{1-\alpha}{2} - \frac{\alpha+1}{2}\frac{(\overline{A-B})(A+B)}{4-|A-B|^2}\right| < \frac{2|A+B|}{4-|A-B|^2}, \tag{3.16}$$

respectively. Define

$$Q_1(z,t) = w(z,t) - \frac{(\overline{A-B})(A+B)}{4-|A-B|^2}, \quad z \in \mathbb{D},\ t \geq 0.$$

Since $\left|ze^{-t}\right| \leq e^{-t} < 1$, for each $z \in \overline{\mathbb{D}}$, $t > 0$, $w(z,t)$ is analytic in $\mathbb{D}$. It implies that $Q_1(z,t)$ is analytic in $\overline{\mathbb{D}}$. Using the maximum modulus principle, we obtain

$$|Q_1(z,t)| < \max_{|z|=1}|Q_1(z,t)| = \left|Q_1(e^{i\theta},t)\right|, \quad z \in \mathbb{D},\ t > 0, \tag{3.17}$$

where $\theta = \theta(t) \in \mathbb{R}$.

Let $\zeta = e^{-t}e^{i\theta}$. Then $|\zeta| = e^{-t}$ and $e^{-(\alpha+1)\gamma t} = (e^{-t})^{(\alpha+1)\gamma} = |\zeta|^{(\alpha+1)\gamma}$. By (1.14), we have

$$|Q_1(e^{i\theta}, t)| = \left| \left[|\zeta|^{(1+\alpha)\gamma} \left(\frac{f'(\zeta)}{g(\zeta)} - 1 \right) + \frac{1 - |\zeta|^{(1+\alpha)\gamma}}{\gamma} \frac{\zeta g'(\zeta)}{g(\zeta)} + \frac{1-\alpha}{2} \right] - \frac{1+\alpha}{2} \frac{\overline{(A-B)}(A+B)}{4 - |A-B|^2} \right| \leq \frac{|1+\alpha||A+B|}{4-|A-B|^2}. \tag{3.18}$$

Combining (3.15)-(3.18), we conclude that inequality (3.14) holds true for all $z \in \mathbb{D}$ and $t \geq 0$. Hence, $F(z,t)$ is a Loewner chain. It implies that $\mathcal{F}_\gamma(z)$ is univalent in $\mathbb{D}$. □

Proof of Theorem 1.11. According to [26], we have

$$\left| \frac{1 - |z|^{\gamma(1+\alpha)}}{\gamma} \right| \leq \frac{1 - |z|^{\Re[\gamma(1+\alpha)]}}{\Re\gamma}, \quad z \in \mathbb{D} \setminus \{0\}. \tag{3.19}$$

Using the relation (3.19), we obtain

$$\begin{aligned} &\left| \left[|z|^{\gamma(1+\alpha)} \left(\frac{h'(z)}{g(z)} - 1 \right) + \frac{1 - |z|^{\gamma(1+\alpha)}}{\gamma} \frac{zg'(z)}{g(z)} + \frac{1-\alpha}{2} \right] \right. \\ &\qquad \left. - \frac{1+\alpha}{2} \frac{\overline{(A-B)}(A+B)}{4-|A-B|^2} \right| \leq |z|^{\Re[\gamma(1+\alpha)]} \left| \frac{h'(z)}{g(z)} - 1 \right| \\ &\qquad + \frac{1 - |z|^{\Re[\gamma(1+\alpha)]}}{\Re\gamma} \left| \frac{zg'(z)}{g(z)} \right| + \left| \frac{1-\alpha}{2} - \frac{1+\alpha}{2} \frac{\overline{(A-B)}(A+B)}{4-|A-B|^2} \right|. \end{aligned} \tag{3.20}$$

Combining (1.18) and (3.20), we have (1.14). Therefore, from Theorem 1.8, we can infer that the function $\mathcal{F}_\gamma(z)$ defined by (1.10) is univalent in $\mathbb{D}$. □

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Критерій однолистості та квазіконформні продовження аналітичних відображень

Chaochuan Wang and Min Yang

У цій статті ми вивчаємо критерій однолистості та квазіконформні продовження для локально однолистих аналітичних відображень та аналітичних відображень. Для локально однолистих аналітичних функцій ми вводимо інтегральні оператори в ланцюгу Левнера та отримуємо достатні умови для однолистих і квазіконформних продовжень, щоб узагальнити результати Беккера, Альфорса, Ванга та ін. Для аналітичних функцій ми використовуємо інші методи доведення, щоб одержати достатню умову однолистості, яка узагальнює результат Масіха та ін.

Ключові слова: аналітична функція, критерій однолистості, квазіконформні продовження