

Finding a Positive Constrained Control for a Linear System to Reach a Given Point within a Finite Time

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In this paper, we consider a linear system with the control $u \in \Omega$, where Ω is a certain domain which does not contain the origin as an interior point. In particular, the origin may not belong to the set Ω . The synthesis problem is solved, i.e. the control $u(x) \in \Omega$ which transfers a point x that belongs to a neighbourhood $V(0)$ to 0 in a finite time is constructed by using the controllability function method. Moreover, this function can be found as the time of motion from a point $x \in V(0)$ to the origin. The case of the linear control system with a non-autonomous term is also considered.

Key words: control system, synthesis problem, controllability function, feedback control, positive constrained control

Mathematical Subject Classification 2020: 93B05, 93B50, 93B52, 93C28, 93D05, 93D40

1. Introduction

Consider the control system

$$\dot{x} = f(x, u), \quad x \in E_n, \quad u \in E_r,$$

and suppose it is completely controllable.

The admissible local positional synthesis problem consists in finding a feedback control $u = u(x)$ satisfying the preassigned constraint ($u(x) \in \Omega$) such that the trajectory of the system

$$\dot{x} = f(x, u(x)) \tag{1.1}$$

starting at an arbitrary point x_0 of some neighbourhood of the origin ends in the origin in a time $T = T(x_0)$. If the neighbourhood V coincides with entire space E_n , then the problem is called global synthesis problem. In [8, 9], the solution to the problem is considered for the set Ω containing origin as an interior point. In this work, we consider the case of $0 \notin \text{int } \Omega$, in particular, $0 \notin \text{co } \Omega$. We assume that the set Ω is in the positive octant of the set E_r . As an example, we consider the stopping problem for the mathematical pendulum by using a one-sided force applied to the pendulum. The case of an additional force $F(t)$ is also considered.

The solution to the admissible synthesis problem is based on construction of a controllability function [8] $\Theta(x)$ and a control $u(x) \in \Omega$ such that for $x \neq 0$ and

some $\alpha \geq 1$ and $\beta > 0$ the following inequality holds:

$$\sum_{i=1}^n \frac{\partial \Theta}{\partial x_i} f_i(x, u(x)) \leq -\beta \Theta^{1-\frac{1}{\alpha}}(x), \quad (1.2)$$

where $f_i(x)$ is the i -th coordinate of the vector function $f(x)$.

In this case, the time of motion from a point x_0 to the origin with respect to system (1.1) satisfies the inequality $T(x) \leq \frac{\alpha}{\beta} \Theta^{\frac{1}{\alpha}}(x_0)$. If $\alpha = \beta = 1$ and

$$\sum_{i=1}^n \frac{\partial \Theta}{\partial x_i} f_i(x, u(x)) \equiv -1, \quad (1.3)$$

then the function $\Theta(x)$ is the time of motion from the point x_0 to the origin, $\Theta(x_0) = T(x_0)$ [4–9, 14]. If, in addition,

$$\min_{u \in \Omega} \sum_{i=1}^n \frac{\partial \Theta}{\partial x_i} f_i(x, u) = -1, \quad (1.4)$$

then the function Θ is the smallest time of motion from the point x_0 to the origin with respect to system (1.1). In this case, the time-optimal control problem in the neighbourhood V is solved. Equation (1.4) is called the Bellman equation. In the case of $\alpha = \infty$, $\beta = 0$, and $f(0, 0) = 0$, inequality (1.2) has the form

$$\sum_{i=1}^n \frac{\partial \Theta}{\partial x_i} f_i(x, u(x)) \leq 0.$$

In this case, the function $\Theta(x)$ is the Lyapunov function and the zero-solution to system (1.1) with this $u(x)$ is stable (asymptotically stable if $\sum_{i=1}^n \frac{\partial \Theta}{\partial x_i} f_i(x, u(x)) < 0$, $x \neq 0$).

The controllability function method is a development of the Lyapunov method for controllable systems.

The asymptotic stability in finite time for robust systems was considered in [5, 12, 13, 15]. The controllability of chaotic motion was considered in [2] using the controllability function, and it was considered for some nonlinear systems in [1, 4, 9].

2. Construction of a controllability function as the time of motion

Consider the linear control system

$$\dot{x} = Ax + Bu, \quad u \in \Omega, \quad (2.1)$$

where

$$\text{rang}(B, AB, \dots, A^{(n-1)}B) = n.$$

First, we introduce the method used in [9, 14] for constructing a controllability function and a control u solving the synthesis problem in the case of $\|u\| \leq d$. Then we modify it for the case when $0 \notin \text{int } \Omega$, in particular, $0 \notin \text{co } \Omega$.

Let $f(t)$ be a non-increasing non-negative function on the half-axis $[0, \infty)$ with at least m decreasing points such that

$$\int_0^\infty s^{2m-1} e^{-2\lambda'_0 s \Theta} f(s) ds < \infty \quad \text{for } 0 \leq \Theta \leq c,$$

where m is a degree of the minimal polynomial of the matrix A , $\lambda'_0 = \min(0, \lambda_0)$, λ_0 is a minimal real part of eigenvalues of the matrix A . Put

$$N = N(\Theta) = \int_0^\infty f\left(\frac{t}{\Theta}\right) e^{-At} B B^* e^{-A^* t} dt.$$

This matrix is positive definite. For $x \neq 0$, define the positive solution to the equation

$$2a_0\Theta = (N^{-1}x, x) \quad (2.2)$$

by $\Theta(x)$ (note that this equation has a unique positive solution). For $x = 0$, we put $\Theta(0) = 0$. The control

$$u(x) = -\frac{1}{2} B^* N^{-1}(\Theta(x))x \quad (2.3)$$

solves the synthesis problem. Put

$$f(t) = \begin{cases} 1-t & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t > 1. \end{cases}$$

Then

$$N = \int_0^\Theta \left(1 - \frac{t}{\Theta}\right) e^{-At} B B^* e^{-A^* t} dt, \quad (2.4)$$

and the controllability function defined by equation (2.2) is the time of motion from a point x_0 to the origin. In addition, equation (1.3) holds. For some $c > 0$, if

$$a_0 \leq \frac{2}{\sup_{0 \leq \Theta \leq c} \Theta \sum_{k=1}^r (N^{-1}b_k, b_k)},$$

then the control $u = -B^* N^{-1}x$ is bounded ($\|u\| \leq d$) in the domain $Q = \{x : \Theta(x) \leq c\}$ [5, 9, 14], where b_k is the k -th column of the matrix B . The trajectory $x(t)$ transferring the initial point to the origin can be found in the following way. For a given x_0 , we find the positive solution Θ_0 to equation (2.2), which is unique. Then we find the solution to the Cauchy problem for the $(n+1)$ -dimensional system:

$$\begin{cases} \dot{x} = Ax - \frac{1}{2} f(0) B B^* N^{-1}x, \\ \dot{\Theta} = -\frac{\Theta(\hat{N} N^{-1}x, N^{-1}x)}{(N^{-1}x, x) + \Theta(\hat{N} N^{-1}x, N^{-1}x)}, \\ x(0) = x_0, \\ \Theta(0) = \Theta_0. \end{cases}$$

We have $x(T(x_0)) = 0$ and $\Theta(T(x_0)) = 0$ for some $t = T(x_0)$. Here,

$$\begin{aligned}\hat{N} &= \int_0^\infty \frac{t}{\Theta} e^{-At} B B^* e^{-A^*t} d \left(-f \left(\frac{t}{\Theta} \right) \right), \\ \tilde{N} &= \int_0^\infty e^{-At} B B^* e^{-A^*t} d \left(-f \left(\frac{t}{\Theta} \right) \right).\end{aligned}$$

If the function $\Theta(x)$ is a time of motion, we have $\dot{\Theta} = -1$.

We also note that the given above method is used in the synthesis problem for the class of systems which can be mapped to linear ones by a change of variables [19,20]. The detailed description of this class is contained in [17,18,21].

System (2.1) must be locally controllable. In the case of $0 \notin \text{int } \Omega$, the Kalman criterion is not sufficient. Starting from the works [3,11] various controllability criteria were considered for different kinds of constraints. For constraints of the general form under the condition $0 \in \Omega$, the following criterion is true [11].

Proposition 2.1 ([11]). *System (2.1) is locally controllable if and only if there exists neither an eigenvector v corresponding to a real eigenvalue of the conjugate matrix A^* which is a support vector to the set $B\Omega$ (i.e. $(v, Bu) \geq 0$ for all $u \in \Omega$) nor an eigenvector $v = w_1 + iw_2$ corresponding to a complex eigenvalue of the matrix A^* orthogonal to the set $B\Omega$ (i.e. $(w_1, Bu) = 0$, $(w_2, Bu) = 0$ for all $u \in \Omega$).*

In the case of general constraints on the control without the assumption $0 \notin \Omega$, the necessary and sufficient conditions are given in [10]. In addition to the previous conditions, the return condition is required: the trajectory must return to the origin on some interval $[t_1, t_2]$.

Let us consider the constraints of the form $\Omega = \{u : 0 \leq u_i \leq c_i, i = 1, \dots, r\}$. If the matrix A has a real eigenvalue, then system (2.1) is not locally controllable.

Proposition 2.2. *Let the matrix A of system (2.1) have at least one real eigenvalue λ , and let v be an eigenvector of the matrix A^* corresponding to λ . Then, under the constraints $\Omega = \{u : 0 \leq u_i \leq c_i, i = 1, \dots, r\}$, this system is not locally controllable if $(v, b_i) \geq 0$, $i = 1, \dots, r$, or $(v, b_i) \leq 0$, $i = 1, \dots, r$.*

Proof. The set of points from which we can transfer to the origin in the time T is determined by the equality

$$x_0 = - \int_0^T e^{-A\tau} B u(\tau) d\tau.$$

Let v be an eigenvector of the matrix A^* corresponding to an eigenvalue λ . We have

$$\begin{aligned}(v, x_0) &= - \int_0^T (v, e^{-A\tau} B u(\tau)) d\tau = - \int_0^T (e^{-A\tau} v, B u(\tau)) d\tau \\ &= - \int_0^T e^{-\lambda\tau} (v, \sum_{i=1}^r b_i u_i(\tau)) d\tau = - \sum_{i=1}^r \int_0^T e^{-\lambda\tau} (v, b_i) u_i(\tau) d\tau \leq 0\end{aligned}$$

if $(v, b_j) \geq 0$, $i = 1, \dots, r$. If $(v, b_j) \leq 0$, $i = 1, \dots, r$, we have $(v, x_0) \geq 0$. Therefore, the set of points from which it is possible to transfer to the origin in an arbitrary time T belongs to one of the subspaces $(v, x_0) \geq 0$ or $(v, x_0) \leq 0$. $\square$

By K , denote the convex cone generated by the column vectors $b_1, b_2, \dots, b_r$ of the matrix B . Let K^* be the dual cone

$$K^* = \{y : (y, x) \geq 0\}$$

for any $x \in K$. Then Proposition 2.2 can be reformulated in the following way:

Proposition 2.3. *Let the matrix A have at least one real eigenvalue λ . Let v be an eigenvector of the matrix A^* corresponding to λ and $\Omega = \{u : 0 \leq u_i \leq c_i, i = 1, \dots, r\}$. If the eigenvector v belongs to the dual cone K^* ($v \in K^*$), then the system is not locally controllable.*

3. Construction of a positive control

Consider the solution to the synthesis problem for the linear control system

$$\dot{x} = Ax + Bu, \quad u \in \Omega, \quad (3.1)$$

where $\Omega = \{u : 0 \leq c_i \leq u_i \leq d_i, i = 1, \dots, r\}$, $c_i < d_i, i = 1, \dots, r$.

Let control system (3.1) be globally controllable. The system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + u \end{cases}$$

is an example of the system of such kind (see [10, 16]). The control (2.3) is not appropriate for this case because its components may change their signs. So we have to operate in another way.

Consider a new control $v = (v_1, \dots, v_r)^*$ such that

$$v_i = \frac{1}{d_i - c_i}(2u_i - d_i - c_i), \quad i = 1, \dots, r. \quad (3.2)$$

Then $u_i = ((d_i - c_i)v_i + (d_i + c_i))/2, i = 1, \dots, r$. If $c_i \leq u_i \leq d_i$, then $-1 \leq v_i \leq 1, i = 1, \dots, r$. Put

$$\begin{aligned} B_1 &= \left(\frac{d_1 - c_1}{2}b_1, \dots, \frac{d_r - c_r}{2}b_r \right), \\ B_2 &= \left(\frac{d_1 + c_1}{2}b_1 + \dots + \frac{d_r + c_r}{2}b_r \right), \end{aligned}$$

where B_1 is a matrix, B_2 is a vector, $b_1, \dots, b_r$ are the column vectors of the matrix B . Then we have $Bu = B_1v + B_2$, and system (3.1) takes the form

$$\dot{x} = Ax + B_1v + B_2, \quad v \in \Omega_1, \quad (3.3)$$

where $\Omega_1 = \{v : |v_i| \leq 1, i = 1, \dots, r\}$. We have

$$\begin{aligned} x(t) &= e^{At} \left(x_0 + \int_0^t e^{-A\tau} B u(\tau) d\tau \right) \\ &= e^{At} \left(x_0 + \int_0^t e^{-A\tau} B_1 v(\tau) d\tau + \int_0^t e^{-A\tau} B_2 d\tau \right). \end{aligned}$$

Let the control $u(t)$ transfer a point x_0 to the origin in some time $t = T$. Then

$$x_0 + \int_0^T e^{-A\tau} B_2 d\tau = - \int_0^T e^{-A\tau} B_1 v(\tau) d\tau.$$

This equality means that the control $v(t)$ transfers the point

$$y_0 = x_0 + \int_0^T e^{-A\tau} B_2 d\tau$$

to the origin with respect to the system

$$\dot{y} = Ay + B_1 v. \quad (3.4)$$

Since the origin is an interior point of the set Ω_1 , this problem can be solved using the controllability function method if the time T is known.

First, consider the case where the controllability function is the time of motion. The controllability function $\Theta(x)$ is determined by the equation

$$2a_0\Theta = (N_1^{-1}y, y), \quad (3.5)$$

where

$$N_1(\Theta) = \int_0^\Theta \left(1 - \frac{t}{\Theta} \right) e^{-At} B_1 B_1^* e^{-A^*t} dt.$$

The control $v(t)$ is given by the equation

$$v = -\frac{1}{2} B_1^* N_1^{-1}(\Theta) y.$$

The coefficient a_0 is chosen according to the condition

$$\max_{1 \leq i \leq r} |v_i| \leq 1.$$

To this aid, it is sufficient that a_0 satisfy the inequality

$$a_0 \leq \max_{1 \leq i \leq r} \frac{2}{\sup_{\Theta} \Theta \left(N_1^{-1} \tilde{b}_i, \tilde{b}_i \right)},$$

where $\tilde{b}_i$ is the i -th column of the matrix B_1 .

Let us find the equation for determining the time T . Replace Θ by T and y by y_0 in equation (3.5) to obtain

$$2a_0T = \left(N_1^{-1}(T)(x_0 + \int_0^T e^{-A\tau} B_2 d\tau), (x_0 + \int_0^T e^{-A\tau} B_2 d\tau) \right), \quad (3.6)$$

where

$$N_1(T) = \int_0^T \left(1 - \frac{t}{\Theta} \right) e^{-At} B_1 B_1^* e^{-A^*t} dt.$$

In general, this equation has a non-unique solution. For each time T , we obtain the control and the trajectory $y(t)$. After finding T , we get the trajectory $y(t)$ as the solution to the following Cauchy problem on the interval $[0, T]$:

$$\begin{cases} \dot{y} = Ay - \frac{1}{2} B_1 B_1^* N_1^{-1}(\Theta(y))y, \\ \dot{\Theta} = -1, \\ y(0) = y_0 = x_0 + \int_0^T e^{-A\tau} B_2 d\tau, \\ \Theta(0) = T. \end{cases}$$

After finding the trajectory $y(t)$, we obtain the trajectory $x(t)$ according to the equality

$$x(t) = y(t) - e^{At} \int_t^T e^{-A\tau} B_2 d\tau. \quad (3.7)$$

Indeed, by subtracting equality (3.3) from equality (3.4), we get

$$\dot{x} - \dot{y} = A(x - y) + B_2.$$

Hence,

$$\begin{aligned} x - y &= e^{At} \left(x_0 - y_0 + \int_0^t e^{-A\tau} B_2 d\tau \right) = \\ &= e^{At} \left(- \int_0^T e^{-A\tau} B_2 d\tau + \int_0^t e^{-A\tau} B_2 d\tau \right), \end{aligned}$$

therefore, $x(t)$ is given by equality (3.7).

The trajectory $x(t)$ can also be found as a solution to the Cauchy problem on the segment $[0, T]$ for the system

$$\begin{cases} \dot{y} = Ay - \frac{1}{2} B_1 B_1^* N_1^{-1}(\Theta(y))y, \\ \dot{\Theta} = -1, \\ \dot{x} = Ax - \frac{1}{2} B_1 B_1^* N_1^{-1}(\Theta(y))y + B_2, \\ y(0) = x_0 + \int_0^T e^{-A\tau} B_2 d\tau, \\ x(0) = x_0, \\ \Theta(0) = T, \end{cases} \quad (3.8)$$

where T is the found time of motion from the point x_0 to the origin.

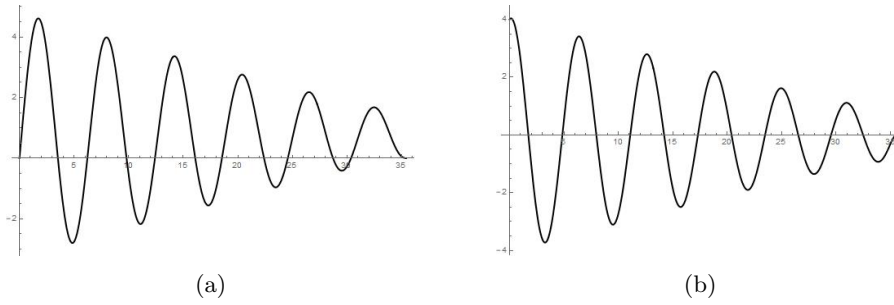


Fig. 3.1: Plots of (a) trajectory $x_1(t)$ and (b) trajectory $x_2(t)$. Horizontal axis means t , and vertical ones are $x_1(t)$ and $x_2(t)$.

3.1. The stopping problem for the mathematical pendulum. Let system (3.1) has the form

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + u, & \frac{1}{2} \leq u \leq 1, \\ x_1(0) = 0, \\ x_2(0) = 4. \end{cases} \quad (3.9)$$

Let us construct a control $u(t)$ transferring the point $x_0 = (0, 4)$ to the point $(0, 0)$. To this aid, change the control $u = v/4 + 3/4$. Then this system takes the form (3.3):

$$\dot{x} = Ax + B_1 v + B_2,$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 1/4 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 3/4 \end{pmatrix}.$$

System (3.4) has the following form:

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = -y_1 + \frac{1}{4}v, & -1 \leq v \leq 1. \end{cases} \quad (3.10)$$

Consider equation (3.5), where

$$\begin{aligned} N_1 &= \frac{1}{16} \int_0^\Theta \left(1 - \frac{t}{\Theta}\right) \begin{pmatrix} \sin^2 t & -\sin t \cos t \\ -\sin t \cos t & \cos^2 t \end{pmatrix} dt \\ &= \begin{pmatrix} \frac{-1 + 2\Theta^2 + \cos 2\Theta}{128\Theta} & \frac{-2\Theta + \sin 2\Theta}{128\Theta} \\ \frac{-2\Theta + \sin 2\Theta}{128\Theta} & \frac{1 + 2\Theta^2 - \cos 2\Theta}{128\Theta} \end{pmatrix}. \end{aligned}$$

Restriction on a_0 has the form $a_0 \leq 1/3$. Further, we will assume $a_0 = 1/3$. Equation (3.5) for Θ has the form

$$\frac{\text{num}}{\text{den}} = 0, \quad (3.11)$$

where

$$\begin{aligned} \text{num} &= \frac{2}{3}\Theta(-1 - 2\Theta^2 + 2\Theta^4 + 2\Theta \sin 2\Theta + \cos 2\Theta) + 64\Theta(-1 \\ &\quad - 2\Theta + \cos 2\Theta)y_1^2 + 128(-2\Theta + \sin 2\Theta)y_1y_2 + \\ &\quad + 64\Theta(1 - 2\Theta^2 + \cos 2\Theta)y_2^2, \\ \text{den} &= -1 - 2\Theta^2 + 2\Theta^4 + \cos 2\Theta + 2\Theta \sin 2\Theta. \end{aligned}$$

The initial state y_0 for system (3.10) is

$$\begin{aligned} y_0 &= x_0 + \int_0^T e^{-At} B_2 dt \\ &= \left(\frac{3}{4}(-1 + \cos T), 4 + \frac{3 \sin T}{4} \right) = (y_{10}, y_{20}). \end{aligned}$$

Substituting T for Θ and y_{10}, y_{20} for y_1, y_2 in (3.11), we obtain equation (3.6) for finding the time of motion from y_0 to the origin:

$$\begin{aligned} &2T^4 - 3290T^2 + 1152T + 1535 - 1152T^2 \sin T \\ &\quad + 216T \sin T + 1152 \sin T - 106T \sin 2T \\ &\quad - 576 \sin 2T + 72T(-16 + 3T) \cos T - 1535 \cos 2T = 0. \end{aligned}$$

This equation has 5 positive roots: $\Theta_1 \approx 35.4201$, $\Theta_2 \approx 37.4601$, $\Theta_3 \approx 40.9275$, $\Theta_4 \approx 44.8576$, $\Theta_5 \approx 46.2312$. The control

$$v = v(t) = -\frac{1}{2}B_1^* N_1^{-1} y = \frac{-16\Theta(\Theta - \cos \Theta \sin \Theta)y_1 - 8\Theta(-1 + 2\Theta^2 + \cos 2\Theta)y_2}{-1 - 2\Theta^2 + 2\Theta^4 + \cos 2\Theta + 2\Theta \sin 2\Theta}$$

solves the problem for system (3.10). Further, for definiteness, we will assume $T = \Theta_1$. But each of the values $\Theta_1, \dots, \Theta_5$ can be taken as T . Then we solve the system

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = -y_1 + \frac{1}{4}v, \\ \dot{\Theta} = -1, \\ y_1(0) = \frac{3}{4}(-1 + \cos T), \\ y_2(0) = 4 + \frac{3 \sin T}{4}, \\ \Theta(0) = T \end{cases}$$

on the segment $[0, T]$. To obtain $x(t)$, we use equation (3.7):

$$\begin{aligned} x_1(t) &= y_1(t) - \frac{3}{4} \sin t(\sin T - \sin t) + \frac{3}{4} \cos t(\cos t - \cos T), \\ x_2(t) &= y_2(t) - \frac{3}{4} \sin t(\cos t - \cos T) - \frac{3}{4} \cos t(\sin T - \sin t). \end{aligned}$$

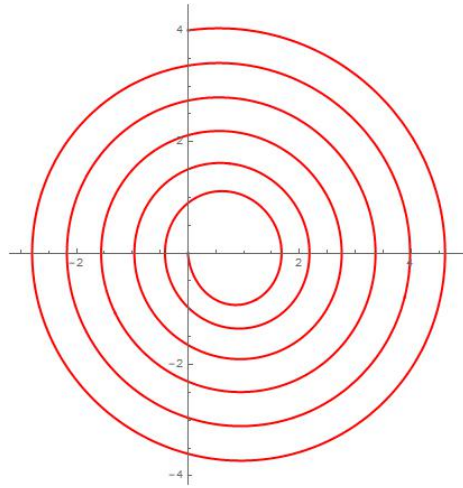


Fig. 3.2: The phase trajectory of trajectory of the system 3.9.

Plots of $x_1(t)$ and $x_2(t)$ are given in Figs. 3.1, the phase trajectory is given in Fig. 3.2.

The controllability problem for a pendulum with restrictions in the form $0 \leq u \leq 1$ was considered in [22].

Trajectories $x(t)$ can be found by solving the Cauchy problem (3.8) on the segment $[0, T]$. In this example, the problem takes the form

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = -y_1 + \frac{1}{4}v, \\ \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + \frac{1}{4}v + \frac{3}{4}, \\ \dot{\Theta} = -1, \\ y_1(0) = \frac{3}{4}(-1 + \cos T), \\ y_2(0) = 4 + \frac{3 \sin T}{4}, \\ x_1(0) = 0, \\ x_2(0) = 4, \\ \Theta(0) = \Theta_1. \end{cases}$$

4. The case of an arbitrary controllability function

Consider the case where the controllability function is not the time of motion. E.g., the function $f(t/\Theta) = e^{-t/\Theta}$ allows us to find the time of motion. In this case, the main difference in finding the trajectory is in determining the time of motion T . This time can be found using an iterative approach, e.g. the half-division method. For a given y_0 , we find Θ_0 as the only positive solution to equation (2.2). Then we solve the Cauchy problem (y, Θ) on some segment

$[0, T]$. We choose the value of T large enough to satisfy the inequality $\Theta(T) < 0$, then we divide the segment in half and solve the Cauchy problem (y, Θ) on the left part of the segment $[0, T_1]$, where $T_1 = T/2$. If it turns out that $\Theta(T_1) < 0$, then we divide the segment $[0, T_1]$ in half and solve the Cauchy problem on this segment. If it turns out that $\Theta(T_1) > 0$, then we divide the segment $[T_1, T]$ in half and solve the Cauchy problem on the segment $[T_1, T_2]$, where $T_2 = 3T/4$ etc. We continue the process until we find $\Theta(T_n)$ with the accuracy close to zero. We find the trajectory $x(t)$ in the same way as in the case where the function $\Theta(x)$ is the time of motion.

5. The case of a linear system with an additional term

The method of constructing the positive control given in Section 3 can be also applied to the system

$$\dot{x} = Ax + Bu + F(t). \quad (5.1)$$

Assume that

$$\left\| \int_0^T e^{-A\tau} F(\tau) d\tau \right\| < \infty \quad \text{for any } T > 0. \quad (5.2)$$

Changing the control u by the control v determined by (3.2) in control system (5.1), we obtain an analogue of system (3.3):

$$\dot{x} = Ax + B_1v + B_2 + F(t).$$

Then we consider the problem of constructing a control transferring the trajectory of system (3.4) with initial condition

$$y(0) = x_0 + \int_0^T e^{-A\tau} (B_2 + F(\tau)) d\tau \quad (5.3)$$

to the origin.

Example 5.1. Consider the system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + u + \sin t, & \frac{1}{2} \leq u \leq 1, \\ x_1(0) = 0, \\ x_2(0) = 4. \end{cases}$$

For this system, it is not possible to construct a control using the controllability function which is the time of motion because

$$\int_0^T e^{-A\tau} F(\tau) d\tau = \left(\frac{1}{4}(-2T + \sin T), \frac{\sin^2 T}{2} \right)$$

and condition (5.2) is true, however the equation for determining the time T has no positive roots.

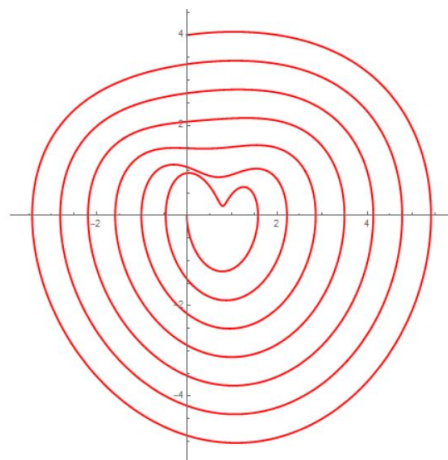


Fig. 5.1: The phase trajectory of example 5.2

Example 5.2. In the case of a system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + u + \sin 2t, & \frac{1}{2} \leq u \leq 1, \\ x_1(0) = 0, \\ x_2(0) = 4, \end{cases}$$

the initial condition (5.3) for system (3.4) has the form

$$\begin{aligned} y_1(0) &= \frac{3}{4}(-1 + \cos T) - \frac{2}{3} \sin^3 T, \\ y_2(0) &= 4 - \frac{2}{3}(-1 + \cos^3 T) + \frac{3}{4} \sin T \end{aligned}$$

and condition (5.2) is true. The equation for determining the time of motion has positive solutions: $T_1 \approx 41.7791$, $T_2 \approx 44.4547$, $T_3 \approx 47.6571$, $T_4 \approx 51.2663$, $T_5 \approx 53.4776$. The type of control remains the same as in the case of system (3.9). The phase trajectory of the system is given in Fig. 5.1.

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Пошук позитивного обмеженого керування для лінійної системи для досягнення заданої точки за кінцевий час

Valerii Korobov and Katerina Sklyar

У цій роботі розглядається лінійна система з керуванням $u \in \Omega$, де Ω — деяка область, яка не містить початку координат, як внутрішньої точки. Зокрема, початок координат може не належати множині ω .

Розв'язано задачу синтезу, тобто за допомогою методу функції керованості побудоване керування $u(x) \in \Omega$, яке переводить точку x , що належить околу $V(0)$, до 0 за скінченний час. Крім того, цю функцію можна знайти, як час руху від точки $x \in V(0)$ до початку координат.

Також розглянуто задачу синтезу для керованої лінійної системи з неавтономним членом.

Ключові слова: керована система, задача синтезу, функція керованості, позиційне керування, додатне обмежене керування