

Sub-Linear Growth of a Special Class of C_0 -Groups on Dense Subsets

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We consider a special class of linearly growing C_0 -groups from [20, 24], whose generators are essentially nonselfadjoint unbounded operators. More precisely, these generators have pure point imaginary spectrum, clustering at $i\infty$, and corresponding dense and minimal, but not uniformly minimal family of eigenvectors, hence this family do not form a Schauder basis. We obtain sharp two-sided estimates for the norms of C_0 -groups from this class on dense subsets of a phase space, namely, on $D(A^k)$ for any $k \in \mathbb{N}$, where A is the unbounded generator of the corresponding C_0 -group. Thereby we prove that these C_0 -groups have sub-linear growth on $D(A^k)$. This yields the sub-linear growth of classical and all more regular solutions of the Cauchy problems for the corresponding abstract linear evolution equations.

Key words: C_0 -group, sub-linear growth, essentially nonselfadjoint unbounded operator, two-sided estimates, spectral XYZ theorem, dense subsets, classical solution

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1. Introduction

The study of the asymptotic behaviour of the C_0 -semigroups $\{T(t)\}_{t \geq 0}$ on dense subsets, e.g. on $D(A)$, where A is the generator of the C_0 -semigroup $\{T(t)\}_{t \geq 0}$, is an important second step in asymptotic theory of C_0 -semigroups on Banach spaces and is quite young and active nowadays area of research. Among the pioneering works in this direction, we note articles of V. Phong [12, 13], C.J.K. Batty and T. Duyckaerts [2, 3], A. Borichev and Y. Tomilov [4], and the first author [19], see also the monograph of J. van Neerven [29]. Most of works in this direction consider the question of decay of C_0 -semigroup $\{T(t)\}_{t \geq 0}$ on $D(A)$, see, e.g. [14, 30] among the newest developments. In [28], one sufficient condition of the decay (or relative growth) of C_0 -semigroup $\{T(t)\}_{t \geq 0}$ on $D(A)$ was obtained in terms of the decay (or relative growth) of the semigroup on the images of Riesz spectral projections, corresponding to different parts of imaginary spectrum of the generator A .

In 2017, G. Sklyar and V. Marchenko [20] constructed classes of C_0 -groups, whose generators have pure point imaginary spectrum, clustering at $i\infty$, and dense minimal family of corresponding eigenvectors, which is, however, not uniformly minimal, hence this family is not a Schauder basis. It means that these generators are essentially nonselfadjoint unbounded operators, i.e. operators which are not similar to any selfadjoint one. For definitions and various properties

of Schauder bases and Schauder decompositions, we refer to [9, 15, 16]. In [21], G. Sklyar and V. Marchenko used these classes of C_0 -groups with non-basis family of eigenvectors from [20] to prove that the spectral XYZ Theorem is sharp in a sense that none of its conditions can be weakened or removed, see Section 2 in [21]. For the formulation, proofs and discussions around the spectral XYZ theorem, we refer to [20, 21, 23, 31, 32].

Throughout the paper, we use notations from [20] and [23], see also [24]. Consider any separable Hilbert space H with norm $\|\cdot\|$ and take arbitrary Riesz basis $\{e_n\}_{n=1}^\infty$ in H . For equivalent definitions and various properties of Riesz bases, we refer to [5, 9, 10]. Then

$$H_1(\{e_n\}) = \left\{ x = (f) \sum_{n=1}^{\infty} c_n e_n : \{c_n - c_{n-1}\}_{n=1}^{\infty} \in \ell_2, c_0 = 0 \right\},$$

is a Hilbert space of formal series $(f) \sum_{n=1}^{\infty} c_n e_n$ with norm

$$\|x\|_1 = \left\| (f) \sum_{n=1}^{\infty} c_n e_n \right\|_1 = \left\| \sum_{n=1}^{\infty} (c_n - c_{n-1}) e_n \right\|.$$

Let $\mathcal{S}_1$ denotes the following class of real sequences:

$$\mathcal{S}_1 = \left\{ \{f(n)\}_{n=1}^{\infty} \subset \mathbb{R} : \lim_{n \rightarrow \infty} f(n) = +\infty; \{n(f(n) - f(n-1))\}_{n=1}^{\infty} \in \ell_{\infty} \right\},$$

where $f(0) = 0$. The construction of C_0 -groups with non-basis family of eigenvectors from [20] on space $H_1(\{e_n\})$ is given then by the following theorem.

Theorem 1.1 (The case $k = 1$ in Theorem 11 of [20], see also Theorem 1.1 in [24]). *Assume that $\{e_n\}_{n=1}^{\infty}$ is a Riesz basis of H . Then $\{e_n\}_{n=1}^{\infty}$ is complete and minimal sequence in $H_1(\{e_n\})$ but does not form a Schauder basis of $H_1(\{e_n\})$, and for each $\{f(n)\}_{n=1}^{\infty} \in \mathcal{S}_1$ the operator $A_1 : H_1(\{e_n\}) \supset D(A_1) \mapsto H_1(\{e_n\})$, defined by*

$$A_1 x = A_1 \left((f) \sum_{n=1}^{\infty} c_n e_n \right) = (f) \sum_{n=1}^{\infty} i f(n) c_n e_n,$$

with domain

$$D(A_1) = \left\{ x = (f) \sum_{n=1}^{\infty} c_n e_n \in H_1(\{e_n\}) : \{f(n) c_n - f(n-1) c_{n-1}\}_{n=1}^{\infty} \in \ell_2 \right\},$$

generates the C_0 -group on $H_1(\{e_n\})$, which acts for every $t \in \mathbb{R}$ by the formula

$$e^{A_1 t} x = e^{A_1 t} (f) \sum_{n=1}^{\infty} c_n e_n = (f) \sum_{n=1}^{\infty} e^{it f(n)} c_n e_n. \quad (1.1)$$

In [21], it was proved that for the spectrum $\sigma(A_1)$ of operator A_1 we have

$$\sigma(A_1) = \sigma_p(A_1) = \{if(n)\}_{n=1}^{\infty}.$$

It was shown in [24] that C_0 -semigroups $\{e^{\pm A_1 t}\}_{t \geq 0}$ do not have a maximal asymptotics. It means that the fastest growing orbits do not exist, for details see [24].

Assume that there exists a constant $K > 0$ such that $\forall n \in \mathbb{N}$ we have

$$n |\Delta f(n)| \geq K. \quad (1.2)$$

Then, as it was proved in [22, 23] (the case $f(n) = \ln n$, $n \in \mathbb{N}$), [24] (the general case), the C_0 -group $\{e^{A_1 t}\}_{t \in \mathbb{R}}$ from Theorem 1.1 has an exact linear growth, i.e. there exists a linear function $\mathfrak{l}$, with positive coefficients, and constant $C > 0$ such that for all $t \in \mathbb{R}$ we have

$$C|t| \leq \|e^{A_1 t}\| \leq \mathfrak{l}(|t|). \quad (1.3)$$

Note that C_0 -groups of linear growth naturally appear in theory and applications of evolution equations, see, e.g. [1, 6, 25, 27].

It was shown in [28] for the particular case of eigenvalues $if(n) = i \ln(n+1)$, $n \in \mathbb{N}$, and the particular class of Hilbert phase spaces $H_1(\{e_n\})$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of the initial Hilbert space H , that

$$\frac{\|e^{A_1 t} A_1^{-1}\|}{t} \rightarrow 0 \quad (1.4)$$

as $t \rightarrow +\infty$, see the Corollary 3.1 and the discussion that preceded it. Moreover, in [28] the detailed computations of the rate of decay of the expression above were carried out and it was shown that for $t > e$ we have

$$\frac{\|e^{A_1 t} A_1^{-1}\|}{t} \sim \frac{1}{\ln t}.$$

Equivalently we have the two-sided estimate of the norm of the corresponding semigroup on $D(A_1)$. In this paper we, first, extend this result to a more general behaviour of spectrum of the generator A_1 from Theorem 1.1, i.e. for the case when points of spectrum $if(n)$ satisfy (1.2). Second, we obtain exact two-sided estimates for the norms of C_0 -groups $\{e^{A_1 t}\}_{t \in \mathbb{R}}$ on dense subsets $D(A_1^k) \subset H_1(\{e_n\})$ for any $k \in \mathbb{N}$, where $\{e_n\}_{n=1}^{\infty}$ is an arbitrary Riesz basis of the initial Hilbert space H . More precisely, we prove that there exist two constants $M \geq m > 0$ such that

$$m \frac{|t|}{(f(t))^k} \leq \|e^{A_1 t} A_1^{-k}\| \leq M \frac{|t|}{(f(t))^k} \quad (1.5)$$

for $|t| > t_0$, where $f(t)$ is any continuous and even extension of the monotonic sequence $\{f(n)\}_{n \in \mathbb{N}}$ to $\mathbb{R}_+$. It is worth mentioning that the proof uses the discrete form of the classical Hardy inequality for $p = 2$ several times. This means that these C_0 -groups $\{e^{A_1 t}\}_{t \in \mathbb{R}}$ have sub-linear growth on $D(A^k)$ and we know an

exact hierarchy of growth, i.e. as k increases it turns out that the rate of growth strictly decreases. And this fact, in its turn, implies the sub-linear growth of the classical (starting from $D(A_1)$) and all more regular (starting from $D(A_1^k)$, $k \geq 2$) solutions of the corresponding Cauchy problems for the abstract linear evolution equations. Note that in the case $k = 0$ estimate (1.5) turns into (1.3).

The construction of C_0 -groups with non-basis family of eigenvectors was presented in [20] on certain Banach spaces $\ell_{p,1}(\{e_n\})$, $p > 1$. Space $\ell_{p,1}(\{e_n\})$, $p > 1$, is a Banach space of formal series (f) $\sum_{n=1}^{\infty} c_n e_n$,

$$\ell_{p,1}(\{e_n\}) = \left\{ x = (\text{f}) \sum_{n=1}^{\infty} c_n e_n : \{c_n - c_{n-1}\}_{n=1}^{\infty} \in \ell_p \right\}, \quad p > 1,$$

where $c_0 = 0$ and $\{e_n\}_{n=1}^{\infty}$ is symmetric basis of corresponding ℓ_p , $p > 1$, with appropriate norm, defined similar to the case of $H_1(\{e_n\})$. The concept of symmetric basis was first introduced and studied by I. Singer [17] in connection with the one Banach problem from isomorphic theory of Banach spaces. For definition and various properties of symmetric bases see, e.g. [7, 11, 15, 17]. Since the construction of C_0 -groups with non-basis family of eigenvectors on Banach space $\ell_{p,1}(\{e_n\})$, $p > 1$, is similar to the construction on $H_1(\{e_n\})$, the third purpose of the paper is to extend the obtained results for the case of the corresponding C_0 -groups defined on Banach space $\ell_{p,1}(\{e_n\})$, $p > 1$. Here, we use several times the discrete form of the classical Hardy inequality for $p > 1$:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n a_k \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p,$$

where $\{a_k\}_{k=1}^{\infty}$ is a sequence of nonnegative real numbers.

2. Sub-linear growth of C_0 -groups on dense subsets on $H_1(\{e_n\})$

In what follows by Δ we denote the backward difference operator,

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & \dots \\ 0 & -1 & 1 & 0 & \dots \\ 0 & 0 & -1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

by $\Delta\{\alpha_n\}_{n=1}^{\infty}$ — the sequence $\{\alpha_n - \alpha_{n-1}\}_{n=1}^{\infty}$ and by $\Delta\alpha_n$ — the n -th element of the sequence $\{\alpha_n - \alpha_{n-1}\}_{n=1}^{\infty}$, i.e. $\Delta\alpha_n = \alpha_n - \alpha_{n-1}$, $n \in \mathbb{N}$. We will often use the following properties of difference operator:

$$\begin{aligned} \Delta a_1 &= a_1, \quad \Delta a_n = a_n - a_{n-1}, \quad n > 1, \\ \Delta(a_n b_n) &= \Delta a_n b_n + a_{n-1} \Delta b_n, \quad n > 1, \end{aligned}$$

$$\Delta \left(\frac{a_n}{b_n} \right) = \frac{\Delta a_n b_{n-1} - a_{n-1} \Delta b_n}{b_{n-1} b_n}, \quad n > 1,$$

$$\Delta a_n^k = \Delta a_n \sum_{j=1}^k a_{n-1}^{j-1} a_n^{k-j}, \quad n > 1, \quad k \in \mathbb{N}. \quad (2.1)$$

We formulate the main result of the paper as the generalization of Theorem 2.1 of [24] as follows.

Theorem 2.1. *Let $\{e^{A_1 t}\}_{t \in \mathbb{R}}$ be the C_0 -group from Theorem 1.1, defined on $H_1(\{e_n\})$, where $\{f(n)\}_{n=1}^\infty \in \mathcal{S}_1$ and is monotonic. Let $k \in \mathbb{N} \cup \{0\}$ and assume that there exists a constant $K > 0$ such that $\forall n \in \mathbb{N}$ we have*

$$n |\Delta f(n)| \geq K. \quad (2.2)$$

Then there exist constants $M_k \geq m_k > 0$ and $t_0 > 0$ such that for any $|t| > t_0$,

$$m_k \frac{|t|}{(f(t))^k} \leq \|e^{A_1 t} A_1^{-k}\| \leq M_k \frac{|t|}{(f(t))^k} \quad (2.3)$$

where $f(t)$ is any continuous and even extension of the sequence $\{f(n)\}_{n \in \mathbb{N}}$ to $\mathbb{R}$.

Proof. The case $k = 0$ in (2.3) follows from the Theorem 2.1 of [24].

Consider $k \in \mathbb{N}$. We start from the representation of any $x \in H_1(\{e_n\})$ by the formal series

$$x = (f) \sum_{n=1}^{\infty} c_n e_n,$$

where $c_0 = 0$, $(c_n) \subset \ell_2(\Delta)$ (see subsection 3.1) is a complex sequence and $(e_n)_{n \in \mathbb{N}}$ is a Riesz basis of the initial Hilbert space H [20]. For brevity denote $T(t) = e^{A_1 t}$, $t \in \mathbb{R}$, $A = A_1$ and denote $f_n = f(n)$, $n \in \mathbb{N}$, after a possible permutation, such that f_n 's are increasing. Since the sequence $(e_n)_{n \in \mathbb{N}}$ constitutes a Riesz basis of the initial Hilbert space H , there exist two constants $C \geq c > 0$ such that for each

$$y = \sum_{n=1}^{\infty} c_n e_n \in H$$

we have

$$c \sum_{n=1}^{\infty} |c_n|^2 \leq \|y\|^2 \leq C \sum_{n=1}^{\infty} |c_n|^2, \quad (2.4)$$

see, e.g. [5, 9]. Therefore we consider the case when $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis of the initial Hilbert space H , the case when $(e_n)_{n \in \mathbb{N}}$ is not orthonormal Riesz basis goes similarly.

To estimate the norm of $T(t)A^{-k}$ for $t > 0$ we first note the following. For any $x \in H_1(\{e_n\})$ we have

$$\|T(t)A^{-k}x\|_1^2 - \left| \frac{c_1}{f_1^k} \right|^2 = \sum_{n=2}^{\infty} \left| \Delta \left(\frac{c_n}{(if_n)^k} e^{itf_n} \right) \right|^2$$

$$\begin{aligned}
 &= \sum_{n=2}^{\infty} \left| \frac{e^{itf_{n-1}}}{f_{n-1}^k} \Delta c_n + c_n \Delta \left(\frac{e^{itf_n}}{f_n^k} \right) \right|^2 \\
 &= \sum_{n=2}^{\infty} \left| \frac{e^{itf_{n-1}}}{f_{n-1}^k} \Delta c_n + c_n \frac{\Delta e^{itf_n} f_{n-1}^k - e^{itf_{n-1}} \Delta f_n^k}{f_{n-1}^k f_n^k} \right|^2 \\
 &= \sum_{n=2}^{\infty} \left| \frac{e^{itf_{n-1}}}{f_{n-1}^k} \Delta c_n + \frac{c_n}{f_n^k} \Delta e^{itf_n} - e^{itf_{n-1}} \frac{c_n}{n} \frac{n \Delta f_n^k}{f_{n-1}^k f_n^k} \right|^2.
 \end{aligned}$$

According to the decomposition above we define functional sequences $B(n, t)$, $C(n, t)$, $FD(n, t)$ for $t > 0, n \geq 2$, as follows:

$$\begin{aligned}
 B(n, t) &= \frac{e^{itf_{n-1}}}{f_{n-1}^k} \Delta c_n, \\
 C(n, t) &= \frac{c_n}{f_n^k} \Delta e^{itf_n}, \\
 D(n, t) &= e^{itf_{n-1}} \frac{c_n}{n} \frac{n \Delta f_n^k}{f_{n-1}^k f_n^k}.
 \end{aligned}$$

In the notation above we have

$$\|T(t)A^{-k}x\|_1^2 - \left| \frac{c_1}{f_1^k} \right|^2 = \sum_{n=2}^{\infty} |B(n, t) + C(n, t) - D(n, t)|^2.$$

We recall that by definition of the class $\mathcal{S}_1$ we have $|n \Delta f_n| < M_0$, for $n > n_0$. Next, we estimate three functional sequences $B(n, t), C(n, t), D(n, t)$.

$$|B(n, t)| = \left| \frac{e^{itf_{n-1}}}{f_{n-1}^k} \Delta c_n \right| \leq \frac{1}{|f_{n-1}|^k} |\Delta c_n|, \quad n > 1. \quad (2.5)$$

Hence,

$$\sum_{n=2}^{\infty} |B(n, t)|^2 \leq \frac{1}{f_1^{2k}} \|x\|_1^2, \quad t > 0, \quad (2.6)$$

For the functional sequence $D(n, t)$ the application of (2.1) yields

$$|D(n, t)| = \left| e^{itf_{n-1}} \frac{c_n}{n} \frac{n \Delta f_n^k}{f_{n-1}^k f_n^k} \right| \leq \left| \frac{c_n}{n} \frac{n \Delta f_n^k}{f_{n-1}^k f_n^k} \right| \leq \frac{k M_0}{f_{n-1}^k f_n^k} \left| \frac{c_n}{n} \right|, \quad n > n_0, \quad (2.7)$$

where we have used the monotonicity of f_n , i.e. that $f_{n-1} \leq f_n, n \in \mathbb{N}$. Due to Hardy's inequality we get

$$\sum_{n=2}^{\infty} |D(n, t)|^2 \leq \frac{k^2 M_0^2}{f_1^{2k+2}} \sum_{n=2}^{\infty} \left| \frac{c_n}{n} \right|^2 \leq \frac{4k^2 M_0^2}{f_1^{2k+2}} \|x\|_1^2, \quad n > n_0, \quad t > 0. \quad (2.8)$$

For the functional sequence $C(n, t)$ we have

$$|C(n, t)| = \left| \frac{c_n}{n} \frac{n}{f_n^k} \left(e^{it \Delta f_n} - 1 \right) \right|, \quad n > 1, \quad t > 0. \quad (2.9)$$

Note that the sequence $\frac{n}{f_n^k}$ is increasing if $f_n^k > n\Delta f_n^k$, which is satisfied for all but finitely many n 's. Indeed, for positive, increasing, unbounded sequence (f_n) the formula (2.1) yields $\Delta f_n^k \leq k f_n^{k-1} \Delta f_n$, hence, it is clear that

$$n\Delta f_n^k \leq n\Delta f_n k f_n^{k-1} \leq M_0 k f_n^{k-1} < f_n f_n^{k-1} = f_n^k, \quad n > n_0,$$

for n_0 large enough. Denote by M_f maximal value of $\frac{n}{f_n^k}$ for $n \leq n_0$. Since double inequality

$$\tilde{c} + K \ln n < f_n < \tilde{C} + M_0 \ln n, \quad n \in \mathbb{N}, \quad (2.10)$$

by (2.2) holds, the sequence $\frac{n}{f_n^k}$ tends to infinity, hence denote by t_0 the number, starting from which the sequence $\frac{n}{f_n^k}$ exceed the number M_f . Now, we are going back to estimate $C(n, t)$. For $t > \max\{t_0, n_0\}$ we get

$$\begin{aligned} \sum_{n=2}^{\infty} |C(n, t)|^2 &= \sum_{1 < n < t} \left| \frac{c_n}{n} \frac{n}{f_n^k} (e^{it\Delta f_n} - 1) \right|^2 + \sum_{n \geq t} \left| \frac{c_n}{n} \frac{n}{f_n^k} (e^{it\Delta f_n} - 1) \right|^2 \\ &\leq \sum_{1 < n < t} \left| \frac{c_n}{n} \frac{n}{f_n^k} (e^{it\Delta f_n} - 1) \right|^2 + t^2 \sum_{n \geq t} \left| \frac{c_n}{n} \frac{n\Delta f_n}{f_n^k} \frac{e^{it\Delta f_n} - 1}{t\Delta f_n} \right|^2. \end{aligned}$$

Using the properties of exponential function, we obtain

$$\begin{aligned} \sum_{n=2}^{\infty} |C(n, t)|^2 &\leq 4 \max_{1 < n < t} \left\{ \left(\frac{n}{f_n^k} \right)^2 \right\} \sum_{1 < n < t} \left| \frac{c_n}{n} \right|^2 + M_0^2 \left(\frac{t}{f^k(t)} \right)^2 \sum_{n \geq t} \left| \frac{c_n}{n} \right|^2 \\ &\leq 4M_f^2 \sum_{1 < n < t} \left| \frac{c_n}{n} \right|^2 + M_0^2 \left(\frac{t}{f^k(t)} \right)^2 \sum_{n \geq t} \left| \frac{c_n}{n} \right|^2 \\ &\leq 4 \left(\frac{t}{f^k(t)} \right)^2 \sum_{1 < n < t} \left| \frac{c_n}{n} \right|^2 + M_0^2 \left(\frac{t}{f^k(t)} \right)^2 \sum_{n \geq t} \left| \frac{c_n}{n} \right|^2, \end{aligned}$$

where $f(t)$ is any continuous monotonic extension of $\{f(n)\}_{n \in \mathbb{N}}$ to $\mathbb{R}_+$. Then, by virtue of the Hardy inequality and taking $\bar{M} = \max\{2, M_0\}$, we obtain

$$\sum_{n=2}^{\infty} |C(n, t)|^2 \leq 2\bar{M}^2 \left(\frac{t}{f^k(t)} \right)^2 \|x\|_1^2.$$

Since for each $a, b, c \in \mathbb{R}_+$

$$(a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2,$$

we can gather all the estimations of $B(n, t)$, $C(n, t)$, $D(n, t)$ and obtain

$$\begin{aligned} \|T(t)A^{-k}x\|_1^2 - \left| \frac{c_1}{f_1^k} \right|^2 &\leq 3 \sum_{n=2}^{\infty} |C(n, t)|^2 + 3 \sum_{n=2}^{\infty} |B(n, t)|^2 + 3 \sum_{n=2}^{\infty} |D(n, t)|^2 \\ &\leq 6\bar{M}^2 \left(\frac{t}{f^k(t)} \right)^2 \|x\|_1^2 + \frac{3}{f_1^{2k}} \|x\|_1^2 + \frac{12k^2 \bar{M}^2}{f_1^{2k+2}} \|x\|_1^2. \end{aligned}$$

Since the case $t < 0$ is treated analogously, this means that there exist constants $M, t_0 > 0$, such that the following estimation holds

$$\|T(t)A^{-k}\| \leq M \frac{|t|}{f^k(t)}, \quad |t| > t_0,$$

where the function f is an even extension of the previously defined f .

To find the lower bound for $\|T(t)A^{-k}\|$, we have to study the lower bound for the major term $|C(n, t)|$. It is easy to see that if $n > M_0 t$ then $t\Delta f_n < 1 < \frac{\pi}{2}$. Then we use inequality

$$|e^{ix} - 1| \geq \frac{1}{2}x, \quad 0 \leq x \leq \frac{\pi}{2}.$$

to estimate (2.9) and we get

$$|C(n, t)| \geq \frac{t}{2} \left| \frac{c_n}{n} \frac{n\Delta f_n}{f_n^k} \right| \geq \frac{Kt}{2f_n^k} \left| \frac{c_n}{n} \right|, \quad n > [M_0 t], \quad t > 0. \quad (2.11)$$

For each $t > 1$, we consider the element

$$x^{(t)} = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n^{(t)} e_n$$

with the sequence $c_n^{(t)}$ of the following form:

$$c_n^{(t)} = \begin{cases} n & \text{for } 0 < n \leq 2[M_0 t], \\ 4[M_0 t] - n & \text{for } 2[M_0 t] < n < 4[M_0 t], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $x^{(t)} \in H_1(\{e_n\})$ and $A^{-k}x^{(t)} \in D(A^k)$ for any k . Simple calculations yield

$$\|x^{(t)}\|_1^2 = \sum_{n=1}^{\infty} \left| \Delta c_n^{(t)} \right|^2 = 4[M_0 t], \quad t > 1.$$

Now we focus on the lower bound of $\|T(t)A^{-k}x^{(t)}\|_1^2$:

$$\begin{aligned} \|T(t)A^{-k}x^{(t)}\|_1^2 &= \sum_{n=2}^{\infty} |B(n, t) + C(n, t) - D(n, t)|^2 + \left| \frac{c_1}{f_1^k} \right|^2 \\ &\geq \sum_{n=[M_0 t]+1}^{2[M_0 t]} |B(n, t) + C(n, t) - D(n, t)|^2. \end{aligned}$$

Next, we would like to show that the term $|C(n, t)|$ is much bigger than $|B(n, t)| + |D(n, t)|$, for t large enough, which by the reverse triangle inequality will allow us to write

$$|B(n, t) + C(n, t) - D(n, t)| \geq |C(n, t)| - |B(n, t)| - |D(n, t)| > 0$$

for $t > t_0$ and $[M_0t] < n < 2[M_0t]$. Indeed, due to logarithmic character of the function $f(t)$ (see inequality (2.10)), it is easy to check that

$$\frac{f(t)}{f([M_0t])} > \frac{f(t)}{f(2[M_0t])} > \frac{K}{2M_0}, \quad t > t_0,$$

for t_0 large enough. Thus, we continue estimating (2.11) taking into account the above, the fact that $n \leq 2[M_0t]$, $f_n \leq f(2[M_0t])$ and that also $\frac{c_n^{(t)}}{n} \geq 1$ and obtain that

$$|C(n, t)| \geq \frac{Kt}{2f_n^k} \left| \frac{c_n^{(t)}}{n} \right| \geq \frac{t}{f^k(t)} \frac{Kf^k(t)}{2f^k(2[M_0t])} \geq \frac{t}{f^k(t)} \frac{K^{k+1}}{2^{k+1}M_0^k} = m_1 \frac{t}{f^k(t)},$$

for $[M_0t] \leq n \leq 2[M_0t]$, $t > t_0$ and constant $m_1 := \frac{K^{k+1}}{2^{k+1}M_0^k}$. This implies that $|C(n, t)| \rightarrow \infty$ for $t \rightarrow \infty$ while

$$|B(n, t)| + |D(n, t)| \leq \frac{1}{f_{n-1}^k} \left| \Delta c_n^{(t)} \right| + \frac{kM_0}{f_{n-1}^k f_n} \left| \frac{c_n^{(t)}}{n} \right| \leq \frac{1}{f_{n-1}^k} + \frac{kM_0}{f_{n-1}^k f_n} \leq 1$$

for $[M_0t] \leq n \leq 2[M_0t]$ and $t > t_0$ for sufficiently large t_0 .

Finally, we are returning to estimate from below the semigroup operator on elements $A^{-k}x^{(t)} \in D(A^k)$,

$$\begin{aligned} \|T(t)A^{-k}x^{(t)}\|_1^2 &\geq \sum_{n=[M_0t]+1}^{2[M_0t]} |B(n, t) + C(n, t) - D(n, t)|^2 \\ &\geq \sum_{n=[M_0t]+1}^{2[M_0t]} (|C(n, t)| - |B(n, t)| - |D(n, t)|)^2 \\ &\geq \sum_{n=[M_0t]+1}^{2[M_0t]} \left(\frac{t}{f^k(t)} m_1 - 1 \right)^2 \\ &\geq [M_0t] \left(\frac{t}{f^k(t)} \frac{m_1}{2} \right)^2 = \left(\frac{t}{f^k(t)} \frac{m_1}{4} \|x^{(t)}\|_1 \right)^2, \quad t > t_0, \end{aligned}$$

for sufficiently large t_0 . Taking $m = \frac{1}{4}m_1$ and noting the similarity of the proof for negative t we obtain the desired inequality from below,

$$\|T(t)A^{-k}\| \geq m \frac{|t|}{f^k(t)}, \quad t > t_0. \quad \square$$

Remark 2.2. For the case when $f(n) = \ln n$, $n \in \mathbb{N}$, condition (1.2) obviously holds. For this case Theorem 2.1 for $k = 0$ was first obtained in [22] and proved in [23].

Remark 2.3. Note that for any $k \in \mathbb{N}$ the function $\frac{|t|}{(f(t))^k}$ from Theorem 2.1, which controls the growth of $\|e^{A_1 t} A_1^{-k}\|$, is sub-linear. Moreover, for any $k \in$

$\mathbb{N}$, the function $\frac{|t|}{(f(t))^k}$ grow as $t \rightarrow +\infty$ faster than t^α for any $\alpha \in (0, 1)$, but slower than βt for any $\beta > 0$. In other words, for any $k \in \mathbb{N}$ the function $\frac{|t|}{(f(t))^k}$ is asymptotically below any linear function and above any root function.

Corollary 2.4. *Let A_1 be the operator from Theorem 2.1 and consider the Cauchy problem for the abstract differential equation*

$$\begin{cases} \dot{x}(t) = A_1 x(t), & t \in \mathbb{R}, \\ x(0) = x_0, \end{cases} \quad (2.12)$$

on the space $H_1(\{e_n\})$. Then all classical and more regular solutions of (2.12) have sub-linear growth (2.3).

Proof. It is known that the classical solution of the problem (2.12) has the form

$$x_c(t) = e^{A_1 t} x_0,$$

where $x_0 \in D(A_1)$, see, e.g. [29]. Then the Theorem (2.1) (the case $k = 1$) yields that there exist constants $M \geq m > 0$ and $t_0 > 0$ such that for any $|t| > t_0$,

$$m \frac{|t|}{f(t)} \leq \|x_c(t)\| \leq M \frac{|t|}{f(t)}.$$

More generally, k -regular solution of the problem (2.12) has the form $x_r(t) = e^{A_1 t} x_0$, where $x_0 \in D(A_1^k)$, see, e.g. [29]. So in this general case Theorem (2.1) yields that there exist constants $M \geq m > 0$ and $t_0 > 0$ such that for any $|t| > t_0$ we have

$$m \frac{|t|}{(f(t))^k} \leq \|x_r(t)\| \leq M \frac{|t|}{(f(t))^k}.$$

It follows that k -regular solution $x_r(t)$ has sub-linear growth for any k . $\square$

3. Sub-linear growth of C_0 -groups on dense subsets on Banach spaces $\ell_{p,1}(\{e_n\})$, $p > 1$

3.1. Preliminary constructions. First, we remind the reader the construction of C_0 -groups on special class of Banach spaces $\ell_{p,1}(\{e_n\})$, $p > 1$. Let $\{e_n\}_{n=1}^\infty$ be arbitrary symmetric basis of ℓ_p , $p > 1$. Then, $\ell_{p,1}(\{e_n\})$, $p > 1$, is a Banach space of formal series (f) $\sum_{n=1}^\infty c_n e_n$,

$$\ell_{p,1}(\{e_n\}) = \left\{ x = (\text{f}) \sum_{n=1}^\infty c_n e_n : \{c_n\}_{n=1}^\infty \in \ell_p(\Delta) \right\},$$

where

$$\ell_p(\Delta) = \{x = \{\alpha_n\}_{n=1}^\infty \subset \mathbb{C} : \Delta x \in \ell_p\}, \quad p > 1.$$

By Proposition 5 in [20], we have that $\overline{\text{Lin}}\{e_n\}_{n=1}^\infty = \ell_{p,1}(\{e_n\})$, the sequence $\{e_n\}_{n=1}^\infty$ is minimal but $\{e_n\}_{n=1}^\infty$ is not uniformly minimal in $\ell_{p,1}(\{e_n\})$, hence it

does not form a Schauder basis of $\ell_{p,1}(\{e_n\})$. We refer to Section 2.2 in [20] for more details.

Consider the operator

$$\widetilde{A}_1 : \ell_{p,1}(\{e_n\}) \supset D(\widetilde{A}_1) \mapsto \ell_{p,1}(\{e_n\})$$

defined on a Banach space $\ell_{p,1}(\{e_n\})$, $p > 1$, as follows:

$$\widetilde{A}_1 x = \widetilde{A}_1(f) \sum_{n=1}^{\infty} c_n e_n = (f) \sum_{n=1}^{\infty} i f(n) c_n e_n, \quad (3.1)$$

where $x \in D(\widetilde{A}_1)$, $\{f(n)\}_{n=1}^{\infty} \in \mathcal{S}_1$ and $\{e_n\}_{n=1}^{\infty}$ is a symmetric basis of the initial Banach space ℓ_p , $p > 1$, with domain

$$D(\widetilde{A}_1) = \left\{ x = (f) \sum_{n=1}^{\infty} c_n e_n \in \ell_{p,1}(\{e_n\}) : \{f(n) c_n\}_{n=1}^{\infty} \in \ell_p(\Delta) \right\}. \quad (3.2)$$

By virtue of Theorem 16 in [20], operator $\widetilde{A}_1$ generates the C_0 -group $\{e^{\widetilde{A}_1 t}\}_{t \in \mathbb{R}}$ on $\ell_{p,1}(\{e_n\})$, $p > 1$, which acts on $\ell_{p,1}(\{e_n\})$ for every $t \in \mathbb{R}$ by the following formula

$$e^{\widetilde{A}_1 t} x = e^{\widetilde{A}_1 t}(f) \sum_{n=1}^{\infty} c_n e_n = (f) \sum_{n=1}^{\infty} e^{itf(n)} c_n e_n. \quad (3.3)$$

3.2. Class of linearly growing C_0 -groups on $\ell_{p,1}(\{e_n\})$, $p > 1$. For the case of C_0 -groups with non-basis family of eigenvectors from [20] on a Banach space $\ell_{p,1}(\{e_n\})$, $p > 1$, we obtain the following theorem on their sub-linear growth on dense subsets of $\ell_{p,1}(\{e_n\})$, similar to the Theorem 2.1.

Theorem 3.1. *Let $\{e^{\widetilde{A}_1 t}\}_{t \in \mathbb{R}}$ be the C_0 -group given by (3.3) defined on a Banach space $\ell_{p,1}(\{e_n\})$, $p > 1$, where $\{f(n)\}_{n=1}^{\infty} \in \mathcal{S}_1$ and is monotonic. Assume that $\exists K > 0$ such that $\forall n \in \mathbb{N}$ we have (1.2), i.e.*

$$n |\Delta f(n)| \geq K.$$

Then, for each $k \in \mathbb{N} \cup \{0\}$, there exist constants $\widetilde{M}_k \geq \widetilde{m}_k > 0$ and $\widetilde{t}_0 > 0$ such that for any $|t| > \widetilde{t}_0$,

$$\widetilde{m}_k \frac{|t|}{(f(t))^k} \leq \|e^{\widetilde{A}_1 t} \widetilde{A}_1^{-k}\| \leq \widetilde{M}_k \frac{|t|}{(f(t))^k}, \quad (3.4)$$

where $f(t)$ is any continuous and even extension of the sequence $\{f(n)\}_{n \in \mathbb{N}}$ to $\mathbb{R}$.

Proof. The case $k = 0$ in (3.4) follows from the Theorem 3.1 of [24].

To prove (3.4) for $k \in \mathbb{N}$, we note that if $\{e_n\}_{n=1}^{\infty}$ is a symmetric basis of Banach space ℓ_p , $p \geq 1$, then there exist constants $C \geq c > 0$ such that for each

$$y = \sum_{n=1}^{\infty} c_n e_n \in \ell_p,$$

we have

$$c \sum_{n=1}^{\infty} |c_n|^p \leq \|y\|^p \leq C \sum_{n=1}^{\infty} |c_n|^p, \quad (3.5)$$

i.e. two-sided estimate, similar to (2.4), see Proposition 4 in [20] and [7] for more details. That means that all the tricks of the proof of the Theorem 2.1 could be done in the proof of this theorem, see also the proof of the Theorem 3.1 from [24]. $\square$

Also we have the following corollary of the Theorem 3.1, similar to the Corollary 2.4.

Corollary 3.2. *Let $\widetilde{A}_1$ be the operator from Theorem (3.1) and consider the Cauchy problem for the abstract differential equation*

$$\begin{cases} \dot{x}(t) = \widetilde{A}_1 x(t), & t \in \mathbb{R}, \\ x(0) = x_0, \end{cases} \quad (3.6)$$

on the space $\ell_{p,1}(\{e_n\})$, $p > 1$. Then all classical and more regular solutions of (3.6) have sub-linear growth (3.4).

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Сублінійне зростання спеціального класу C_0 -груп на щільних підмножинах

Grigory Sklyar, Vitalii Marchenko, and Piotr Polak

Ми розглядаємо спеціальний клас лінійно зростаючих C_0 -груп з [20, 24], генератори яких є суттєво несамопрямими необмеженими операторами. Точніше, ці генератори мають чисто точковий уявний спектр, що згущується в точці $i\infty$, та відповідну повну та мінімальну, але не рівномірно мінімальну, сім'ю власних векторів, яка не утворює базис Шaudера. Ми одержуємо точні двосторонні оцінки норм C_0 -груп з цього класу на щільних підмножинах фазового простору, а саме, на $D(A^k)$ для будь-яких $k \in \mathbb{N}$, де A є необмеженим генератором відповідної C_0 -групи. Тим самим ми доводимо, що ці C_0 -групи мають сублінійне зростання на $D(A^k)$. Це означає сублінійне зростання класичних та всіх більш регулярних розв'язків задачі Коші для відповідних абстрактних лінійних еволюційних рівнянь.

Ключові слова: C_0 -група, сублінійне зростання, суттєво несамопрямий необмежений оператор, двосторонні оцінки, спектральна XYZ теорема, щільні підмножини, класичний розв'язок