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Asymptotics of Correlators of Sparse Bipartite Random Graphs

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We study asymptotic behavior of the correlation functions of bipartite sparse weighted random $N \times N$ matrices. It is shown that the main term of the correlation function of k-th and m-th moments of the integrated density of states is $N^{-1}n_{k,m}$. The closed system of recurrent relations for coefficients $\{n_{k,m}\}_{k,m=1}^{\infty}$ is obtained.

Key words: bipartite sparse random graph, correlator of moments, asymptotics, main term, system of recurrence equations

Mathematical Subject Classification 2020: 15B52

1. Introduction

In the last few years interest in the spectral properties of ensembles of sparse random matrices has sharply increased. It is expected that the spectral properties of sparse random matrices will differ from the properties of the ensembles of most matrices with independent elements (see [1], as well as the survey works [2,3] and literature therein).

Interesting results for sparse random matrices were obtained in a series of physical works [4–7]. In particular, the equation for the Laplace transform of limiting integrated state density was derived, the "density-density correlator" was studied and it was shown that there exists some critical point $p_c > 1$ in the vicinity of which the phase transition in p occurs: for $p < p_c$ all eigenvectors are localized, whereas for $p > p_c$ delocalized eigenvectors appear. All these results were obtained by the replica or supersymmetry method, and therefore need mathematically correct justification.

In mathematical papers [8–10], it was proved that there exists a limit for $N \to \infty$ averaged moments integrated state density in the simplest case, when the matrix elements are equal to 0 with probability $1-p/N$ and 1 with probability p/N. It is shown that limiting moments satisfy the Carleman condition, thereby the existence of a limit of the integrated state density for the ensemble of sparse random matrices is proved. In the papers [11, 12], similar results were obtained for a wider class of sparse random matrix ensembles. In papers [19, 20], the delocalization and the existence of absolutely continuous part of the limiting spectra at 0 were studied. The asymptotic behavior of the correlator of moments as $N \to \infty$ was studied in [14].

In the papers [15, 16], similar results were obtained for the bipartite sparse random matrix ensemble. In this paper, we study asymptotic behavior of the correlator of moments as $N \to \infty$ for the bipartite sparse random matrix ensemble. The speed of approaching to zero and the value of the main term are very important in physical applications. Therefore an extensive literature is devoted to similar studies for various ensembles of random matrices (see, for example, [17,18] and literature therein).

2. Main results

We introduce a randomly weighted adjacency matrix of random bipartite graphs. Let $\Xi = \{a_{ij} : i \le j, \ i,j \in \mathbb{N}\}$ be the set of jointly independent identically distributed random variables determined on the same probability space and possessing the moments

$$\mathbb{E}a_{ij}^k = X_k < \infty, \quad i,j,k \in \mathbb{N}, \tag{2.1}$$

where $\mathbb{E}$ denotes the mathematical expectation corresponding to Ξ. We set $a_{ji} = a_{ij}$ for $i \le j$.

Given $0 < p \le N$, let us define the family $B_N^{(p)} = \{b_{ij}^{(N,p)} : \ i \le j, \ i,j \in \overline{1,N}\}$ of jointly independent random variables

$$b_{ij}^{(N,p)} = \begin{cases} p^{-1/2} & \text{with probability } p/N, \\ 0 & \text{with probability } 1 - p/N. \end{cases} \tag{2.2}$$

We determine $b_{ji}^{(N,p)} = b_{ij}^{(N,p)}$ and assume that $B_N^{(p)}$ is independent from Ξ.

Let $\alpha \in (0,1)$, denote $I_1^{(N,\alpha)} = \overline{1, \lfloor \alpha N \rfloor}$, $I_2^{(N,\alpha)} = \overline{\lfloor \alpha N \rfloor + 1, N}$, where $\lfloor \rfloor$ is a floor function. Now one can consider the real symmetric $N \times N$ matrix $A^{(N,p,\alpha)}(\omega)$:

$$A_{ij}^{(N,p,\alpha)} = a_{ij} b_{ij}^{(N,p)} \xi_{ij}^{(N,\alpha)}, \tag{2.3}$$

where

$$\xi_{i,j}^{(N,\alpha)} = \begin{cases} 1 & \text{if } \left(i \in I_1^{(N,\alpha)} \wedge j \in I_2^{(N,\alpha)}\right) \vee \left(i \in I_2^{(N,\alpha)} \wedge j \in I_1^{(N,\alpha)}\right) \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

that has N real eigenvalues $\lambda_1^{(N,p,\alpha)} \le \lambda_2^{(N,p,\alpha)} \le \ s \le \ \lambda_N^{(N,p,\alpha)}$.

The normalized eigenvalue counting function (or integrated density of states) of $A^{(N,p,\alpha)}$ is determined by the formula

$$\sigma\left(\lambda; A^{(N,p,\alpha)}\right) = \frac{\sharp\{j : \lambda_j^{(N,p,\alpha)} < \lambda\}}{N}.$$

The following denotations are used:

$$\mathcal{M}_k^{(N,p,\alpha)} = \int \lambda^k d\sigma\left(\lambda; A^{(N,p,\alpha)}\right), \quad M_k^{(N,p,\alpha)} = \mathbb{E}\mathcal{M}_k^{(N,p,\alpha)},$$

$$C_{k,m}^{(N,p,\alpha)} = \mathbb{E}\left\{\mathcal{M}_k^{(N,p,\alpha)} \mathcal{M}_m^{(N,p,\alpha)}\right\} - \mathbb{E}\left\{\mathcal{M}_k^{(N,p,\alpha)}\right\} \mathbb{E}\left\{\mathcal{M}_m^{(N,p,\alpha)}\right\}.$$

Theorem 2.1. *Main asymptotic coefficients of correlators* $n^{(p,\alpha)}_{k,m}$,

$$\lim_{N\to\infty} NC^{(N,p,\alpha)}_{k,m} = \begin{cases} n^{(p,\alpha)}_{k/2,m/2} & \text{if } k \text{ and } m \text{ are even,} \\ 0 & \text{otherwise,} \end{cases} \tag{2.5}$$

can be obtained by the system of recurrent relations (3.9)–(3.12), (3.23)–(3.52) *with the initial conditions* (3.54)–(3.62).

3. Proof of Theorem 1

3.1. Correlators and double bipartite walks. Let us transform correlator $C^{(N,p,\alpha)}_{k,m}$ to the form convenient for the limiting transition:

$$\begin{aligned}
&C^{(N,p,\alpha)}_{k,m} = \mathbb{E}\left\{\mathcal{M}^{(N,p,\alpha)}_{k}\mathcal{M}^{(N,p,\alpha)}_{m}\right\} - \mathbb{E}\left\{\mathcal{M}^{(N,p,\alpha)}_{k}\right\}\mathbb{E}\left\{\mathcal{M}^{(N,p,\alpha)}_{m}\right\} \\
&= \frac{1}{N^2}\left(\mathbb{E}\left\{\mathrm{Tr}[A^{(N,p,\alpha)}]^k\,\mathrm{Tr}[A^{(N,p,\alpha)}]^m\right\} - \mathbb{E}\left\{\mathrm{Tr}[A^{(N,p,\alpha)}]^k\right\}\mathbb{E}\left\{\mathrm{Tr}[A^{(N,p,\alpha)}]^m\right\}\right) \\
&= \frac{1}{N^2}\sum_{i_1,\dots,i_k=1}^{N}\ \sum_{j_1,\dots,j_m=1}^{N}\Big(\mathbb{E}\Big\{A^{(N,p,\alpha)}_{i_1,i_2}A^{(N,p,\alpha)}_{i_2,i_3}\dots A^{(N,p,\alpha)}_{i_k,i_1}A^{(N,p,\alpha)}_{j_1,j_2}A^{(N,p,\alpha)}_{j_2,j_3}\dots A^{(N,p,\alpha)}_{j_m,j_1}\Big\} \\
&\quad - \mathbb{E}\left\{A^{(N,p,\alpha)}_{i_1,i_2}A^{(N,p,\alpha)}_{i_2,i_3}\dots A^{(N,p,\alpha)}_{i_k,i_1}\right\}\mathbb{E}\left\{A^{(N,p,\alpha)}_{j_1,j_2}A^{(N,p,\alpha)}_{j_2,j_3}\dots A^{(N,p,\alpha)}_{j_m,j_1}\right\}\Big) \\
&= \frac{1}{N^2}\sum_{i_1,\dots,i_k=1}^{N}\ \sum_{j_1,\dots,j_m=1}^{N}\Bigg(\mathbb{E}\left\{a_{i_1,i_2}a_{i_2,i_3}\dots a_{i_k,i_1}a_{j_1,j_2}a_{j_2,j_3}\dots a_{j_m,j_1}\right\} \\
&\quad \times \mathbb{E}\left\{b^{(N,p)}_{i_1,i_2}b^{(N,p)}_{i_2,i_3}\dots b^{(N,p)}_{i_k,i_1}b^{(N,p)}_{j_1,j_2}b^{(N,p}_{j_2,j_3}\dots b^{(N,p)}_{j_m,j_1}\right\} \\
&\quad - \mathbb{E}\left\{a_{i_1,i_2}a_{i_2,i_3}\dots a_{i_k,i_1}\right\}\mathbb{E}\left\{a_{j_1,j_2}a_{j_2,j_3}\dots a_{j_m,j_1}\right\} \\
&\quad \times \mathbb{E}\left\{b^{(N,p)}_{i_1,i_2}b^{(N,p)}_{i_2,i_3}\dots b^{(N,p)}_{i_k,i_1}\right\}\mathbb{E}\left\{b^{(N,p)}_{j_1,j_2}b^{(N,p)}_{j_2,j_3}\dots b^{(N,p)}_{j_m,j_1}\right\}\Bigg) \\
&\quad \times \xi^{(N,\alpha)}_{i_1,i_2}\xi^{(N,\alpha)}_{i_2,i_3}\dots\xi^{(N,\alpha)}_{i_k,i_1}\xi^{(N,\alpha)}_{j_1,j_2}\xi^{(N,\alpha)}_{j_2,j_3}\dots\xi^{(N,\alpha)}_{j_m,j_1}.
\end{aligned} \tag{3.1}$$

Let $W^{(N,\alpha)}_k$ be a set of closed bipartite walks of k steps over the sets $I^{(N,\alpha)}_1 = \overline{1,\lfloor\alpha N\rfloor}$ and $I^{(N,\alpha)}_2 = \overline{\lfloor\alpha N\rfloor+1,N}$: $W^{(N,\alpha)}_k = {}^{(1)}W^{(N,\alpha)}_k \cup {}^{(2)}W^{(N,\alpha)}_k$, where

$$\begin{aligned}
{}^{(1)}W^{(N,\alpha)}_k &= \left\{w = (w_1,w_2,s,w_k,w_{k+1}=w_1) : \forall i\in\overline{1,k+1}\quad w_i\in I^{(N,\alpha)}_{2-(i \bmod 2)}\right\}, \\
{}^{(2)}W^{(N,\alpha)}_k &= \left\{w = (w_1,w_2,s,w_k,w_{k+1}=w_1) : \forall i\in\overline{1,k+1}\quad w_i\in I^{(N,\alpha)}_{1+(i \bmod 2)}\right\}.
\end{aligned}$$

Here, $(a \mod m)$ denotes the residue of a modulo m. Thus, for a walk w from $W^{(N,\alpha)}_k$, either all odd elements are from $I^{(N,\alpha)}_1$ and all even elements are from $I^{(N,\alpha)}_2$, or vice versa, all odd elements are from $I^{(N,\alpha)}_2$ and all even ones are from $I^{(N,\alpha)}_1$. In the first case, w is from ${}^{(1)}W^{(N,\alpha)}_k$, and in the second case, w is from ${}^{(2)}W^{(N,\alpha)}_k$. Here are some examples: $(2,4,1,5,3,4,2,6,2)\in{}^{(1)}W^{(6,1/2)}_8$; $(5,1,4,2,6,3,6,1,5)\in{}^{(2)}W^{(6,1/2)}_8$; $(2,4,1,5,3,4,2,6,1)\notin{}^{(1)}W^{(6,1/2)}_8$

(nonclosed); $(2,4,1,5,3,4,2,3,2) \in {}^{(1)}W_8^{(6,1/2)}$ (nonbipartite). For $w \in W_k^{(N,\alpha)}$, let us denote $a(w) = \prod_{i=1}^k a_{w_i,w_{i+1}}$, $b^{(N,p)}(w) = \prod_{i=1}^k b^{(N,p)}_{w_i,w_{i+1}}$.

Let $\mathfrak{D}_{k,m}^{(N,\alpha)} \stackrel{\text{def}}{\equiv} W_k^{(N,\alpha)} \times W_m^{(N,\alpha)}$ be a set of double bipartite walks of k and m steps over the sets $I_1^{(N,\alpha)}$ and $I_2^{(N,\alpha)}$. For $d = (w^{(1)}, w^{(2)}) \in \mathfrak{D}_{k,m}^{(N)}$, let us denote

$$a(d) = a(w^{(1)})a(w^{(2)}), \quad b^{(N,p)}(d) = b^{(N,p)}(w^{(1)})b^{(N,p)}(w^{(2)}).$$

Then we can write equality (3.1) in the following way:

$$C_{k,m}^{(N,p,\alpha)} = \frac{1}{N^2} \sum_{d=(w^{(1)},w^{(2)})\in W_{k,m}^{(N,\alpha)}} \Big\{ \mathbb{E}a(d)\mathbb{E}b^{(N,p)}(d) - \\ - \mathbb{E}a(w^{(1)})\mathbb{E}b^{(N,p)}(w^{(1)})\mathbb{E}a(w^{(2)})\mathbb{E}b^{(N,p)}(w^{(2)}) \Big\}. \qquad (3.2)$$

For $w \in W_k^{(N,\alpha)}$ and $f,g \in \overline{1,N}$, denote by $n_w(f,g)$ the number of steps $f \to g$ and $g \to f$:

$$n_w(f,g) = \# \left\{ i \in \overline{1,k} : \; (w_i = f \;\wedge\; w_{i+1} = g) \vee (w_i = g \;\wedge\; w_{i+1} = f) \right\}.$$

For $w = (w^{(1)}, w^{(2)}) \in \mathfrak{D}_{k,m}^{(N,\alpha)}$, let us introduce a similar denotation

$$n_d(f,g) = n_{w^{(1)}}(f,g) + n_{w^{(2)}}(f,g).$$

Then, for all $w \in W_k^{(N,\alpha)}$ and all $d \in \mathfrak{D}_{k,m}^{(N,\alpha)}$, we have

$$\mathbb{E}a(w) = \prod_{f=1}^N \prod_{g=f}^N V_{n_w(f,g)} \qquad \mathbb{E}a(d) = \prod_{f=1}^N \prod_{g=f}^N V_{n_d(f,g)}.$$

Given $w \in W_k^{(N)}$, let us define the sets $V_w = \cup_{i=1}^k \{w_i\}$ and $E_w = \cup_{i=1}^k \{(w_i, w_{i+1})\}$, where (w_i, w_{i+1}) is a non-ordered pair. It is easy to see that $G_w = (V_w, E_w)$ is a simple connected non-oriented bipartite graph and the walk w covers the graph G_w. Let us call G_w the skeleton of walk w. We denote by $n_w(e)$ the number of passages of the edge e by the walk w in direct and inverse directions. For $(w_j, w_{j+1}) = e_j \in E_w$, let us denote $a_{e_j} = a_{w_j,w_{j+1}} = a_{w_{j+1},w_j}$. Then we obtain

$$\mathbb{E}a(w) = \prod_{e \in E_w} \mathbb{E}a_e^{n_w(e)} = \prod_{e \in E_w} V_{n_w(e)}.$$

In a similar way, we can write

$$\mathbb{E}b^{(N,p)}(w) = \prod_{e \in E_w} \mathbb{E}\left([b_e^{(N,p)}]^{n_w(e)} \right) = \prod_{e \in E_w} \frac{1}{Np^{n_w(e)/2-1}}.$$

Let us introduce similar definitions for a double bipartite walk $d = (w^{(1)}, w^{(2)}) \in \mathfrak{D}_{k,m}^{(N)}$. For $V_d = V_{w^{(1)}} \cup V_{w^{(2)}}$, $E_d = E_{w^{(1)}} \cup E_{w^{(2)}}$, $G_d = (V_d, E_d)$, the following equations hold:

$$\mathbb{E}a(d) = \prod_{e \in E_d} V_{n_d(e)}, \quad \mathbb{E}b^{(N,p)}(d) = \prod_{e \in E_d} \frac{1}{Np^{n_d(e)/2-1}}.$$

Then we can write (3.2) in the form

$$
\begin{aligned}
C_{k,m}^{(N,p,\alpha)} &= \frac{1}{N^2} \sum_{d=(w^{(1)},w^{(2)})\in W_{k,m}^{(N,\alpha)}} \Bigg\{ \prod_{e\in E_d} \mathbb{E} a_e^{n_d(e)} \mathbb{E}\left[b_e^{(N,p)}\right]^{n_d(e)} \\
&\quad - \prod_{e\in E_{w^{(1)}}} \mathbb{E} a_e^{n_{w^{(1)}}(e)} \mathbb{E}\left[b_e^{(N,p)}\right]^{n_{w^{(1)}}(e)} \prod_{e\in E_{w^{(2)}}} \mathbb{E} a_e^{n_{w^{(2)}}(e)} \mathbb{E}\left[b_e^{(N,p)}\right]^{n_{w^{(2)}}(e)} \Bigg\} \\
&= \frac{1}{N^2} \sum_{d=(w^{(1)},w^{(2)})\in W_{k,m}^{(N,\alpha)}} \Bigg\{ N^{-|E_d|} \prod_{e\in E_d} \frac{V_{n_d(e)}}{p^{n_d(e)/2-1}} \\
&\quad - N^{-|E_{w^{(1)}}|-|E_{w^{(2)}}|} \prod_{e\in E_{w^{(1)}}} \frac{V_{n_{w^{(1)}}(e)}}{p^{n_{w^{(1)}}(e)/2-1}} \prod_{e\in E_{w^{(2)}}} \frac{V_{n_{w^{(2)}}(e)}}{p^{n_{w^{(2)}}(e)/2-1}} \Bigg\} \\
&= \frac{1}{N^2} \sum_{d=(w^{(1)},w^{(2)})\in W_{k,m}^{(N,\alpha)}} \Bigg\{ \frac{\prod_{e\in E_d} V_{n_d(e)}}{N^{|E_d|} p^{(k+m)/2-|E_d|}} \\
&\quad - \frac{\prod_{e\in E_{w^{(1)}}} V_{n_{w^{(1)}}(e)} \prod_{e\in E_{w^{(2)}}} V_{n_{w^{(2)}}(e)}}{N^{|E_{w^{(1)}}|+|E_{w^{(2)}}|} p^{(k+m)/2-|E_{w^{(1)}}|-|E_{w^{(2)}}|}} \Bigg\} = \sum_{d\in W_{k,m}^{(N,\alpha)}} \theta(d),
\end{aligned}
\tag{3.3}
$$

where $\theta(d)$ is the contribution of the double bipartite walk d to the mathematical expectation of the corresponding correlator. The last expression is not very convenient for the limiting transition. Moreover, the latter formula shows that the contribution of a double bipartite walk d depends only on the sets

$$
\cup_{e\in E_d} \{n_d(e)\}, \quad \cup_{e\in E_{w^{(1)}}} \{n_{w^{(1)}}(e)\}, \quad \cup_{e\in E_{w^{(2)}}} \{n_{w^{(2)}}(e)\}. \tag{3.4}
$$

Therefore, it is natural to introduce an equivalence relation on $\mathfrak{D}_{k,m}^{(N,\alpha)}$. Double bipartite walks $d = (w^{(1)}, w^{(2)}), u = (u^{(1)}, u^{(2)}) \in W_{k,m}^{(N,\alpha)}$ are equivalent $d \sim u$ if and only if there exists a partition preserving bijection ϕ between the sets of vertices V_d and V_u such that

$$
d \sim u \iff \left(\exists \phi : V_d \overset{bij}{\to} V_u \quad \phi(V_d \cap I_1^{(N,\alpha)}) = V_u \cap I_1^{(N,\alpha)}, u = \phi(d)\right).
$$

Let us denote by $[d]$ the class of equivalence of double bipartite walk d and by $\mathfrak{C}_{k,m}^{(N,\alpha)}$ the set of such classes for all $d \in \mathfrak{D}_{k,m}^{(N,\alpha)}$. It is obvious that if two walks d and u are equivalent, then their contributions are equal:

$$
d \sim u \implies \theta(d) = \theta(u).
$$

Cardinality of the equivalence class $[d]$ is equal to the number of all mappings $\phi : V_d \to \overline{1, N}$ such that $\phi(V_{1,d}) \subset I_1^{N,\alpha}$ and $\phi(V_{2,d}) \subset I_2^{N,\alpha}$ (where $V_{1,d} = V_d \cap I_1^{(N,\alpha)}$ and $V_{2,d} = V_d \cap I_2^{(N,\alpha)}$). Therefore, it is equal to the number $\lfloor \alpha N \rfloor (\lfloor \alpha N \rfloor -$

$1)\dots(\lfloor\alpha N\rfloor-|V_{1,d}|+1)(\lceil(1-\alpha)N\rceil)(\lceil(1-\alpha)N\rceil-1)\cdots(\lceil(1-\alpha)N\rceil-|V_{2,d}|+1)$. Then we can write (3.3) in the following form:

$$\begin{aligned}
C_{k,m}^{(N,\alpha)} &= \frac{1}{N^2}\sum_{[d]\in\mathfrak{C}_{k,m}^{(N,\alpha)}}\Bigg\{\frac{\lfloor\alpha_1 N\rfloor(\lfloor\alpha_1 N\rfloor-1)\cdots(\lfloor\alpha_1 N\rfloor-|V_{1,d}|+1)}{N^{|E_d|}p^{(k+m)/2-|E_d|}}\\
&\quad\times(\lceil\alpha_2 N\rceil)(\lceil\alpha_2 N\rceil-1)\cdots(\lceil\alpha_2 N\rceil-|V_{2,d}|+1)\\
&\quad\times\left(\prod_{e\in E_d}V_{n_d(e)}-\frac{p^{|E_{w(1)}|+|E_{w(2)}|-|E_d|}}{N^{|E_{w(1)}|+|E_{w(2)}|-|E_d|}}\prod_{e\in E_{w(1)}}V_{n_{w(1)}(e)}\prod_{e\in E_{w(2)}}V_{n_{w(2)}(e)}\right)\Bigg\}\\
&= \sum_{[d]\in\mathfrak{C}_{k,m}^{(N,\alpha)}}\theta([d]),
\end{aligned}\tag{3.5}$$

where $\alpha_1=\alpha$ and $\alpha_2=1-\alpha$.

But the transition to the limit $N\to\infty$ in the last formula is hindered by the dependence of $\mathfrak{C}_{k,m}^{(N,\alpha)}$ on N. In order to solve this problem, and at the same time for better understanding of $\mathfrak{C}_{k,m}^{(N,\alpha)}$, we introduce the notion of minimal double bipartite walks.

3.2. Minimal and essential walks. It is convenient to deal with $\widetilde{\mathfrak{D}}_{k,m}^{(N,\alpha)}$ instead of $\mathfrak{D}_{k,m}^{(N,\alpha)}$, where $\widetilde{\mathfrak{D}}_{k,m}^{(N,\alpha)}$ is a set of double bipartite closed walks over the sets $I_1^{(N,\alpha)}$ and $\widetilde{I}_2^{(N,\alpha)}=\left\{\widetilde{1},\widetilde{2}\dots,\widetilde{\lceil N(1-\alpha)\rceil}\right\}$. We just renamed the vertices of the second component. Let us consider $\widetilde{\mathfrak{C}}_{k,m}^{(N,\alpha)}$, the set of equivalence classes of $\widetilde{\mathfrak{D}}_{k,m}^{(N,\alpha)}$. As a representative of the equivalence class $[d]\in\widetilde{\mathfrak{C}}_{k,m}^{(N,\alpha)}$, we can take a minimal double walk.

Definition 3.1. A double bipartite closed walk $d\in\widetilde{\mathfrak{C}}_{k,m}^{(N,\alpha)}$ is called minimal if and only if at each stage of the passage a new vertex is the minimum element among the unused vertices of the corresponding component. In this case, we apply the following procedure for passing a double walk: first, we pass the first walk, then we jump over to the initial vertex of the second walk and then pass it.

Let us denote the set of all minimal walks of $\widetilde{\mathfrak{D}}_{k,m}^{(N,\alpha)}$ by $\mathfrak{M}_{k,m}^{(N,\alpha)}$.

Example 3.2. The double walk $\Big((1,\widetilde{1},1,\widetilde{2},1),(\widetilde{3},2,\widetilde{3},1,\widetilde{1},1,\widetilde{3})\Big)$ is the minimal one.

Then (3.5) can be written as

$$\begin{aligned}
C_{k,m}^{(N,\alpha)} &= \frac{1}{N^2}\sum_{d\in\mathfrak{M}_{k,m}^{(N,\alpha)}}\Bigg\{\frac{\lfloor\alpha_1 N\rfloor(\lfloor\alpha_1 N\rfloor-1)\cdots(\lfloor\alpha_1 N\rfloor-|V_{1,d}|+1)}{N^{|E_d|}p^{(k+m)/2-|E_d|}}\\
&\quad\times(\lceil\alpha_2 N\rceil)(\lceil\alpha_2 N\rceil-1)\cdots(\lceil\alpha_2 N\rceil-|V_{2,d}|+1)\\
&\quad\times\left(\prod_{e\in E_d}V_{n_d(e)}-\frac{p^{|E_{w(1)}|+|E_{w(2)}|-|E_d|}}{N^{|E_{w(1)}|+|E_{w(2)}|-|E_d|}}\prod_{e\in E_{w(1)}}V_{n_{w(1)}(e)}\prod_{e\in E_{w(2)}}V_{n_{w(2)}(e)}\right)\Bigg\}
\end{aligned}$$

$$= \sum_{d \in \mathfrak{M}_{k,m}^{(N,\alpha)}} \theta([d]), \tag{3.6}$$

Each double walk $d \in \mathfrak{M}_{k,m}^{(N,\alpha)}$ has at most $k+m$ vertices. Hence, $\mathfrak{M}_{k,m}^{(1,\alpha)} \subset \mathfrak{M}_{k,m}^{(2,\alpha)} \subset \cdots \subset \mathfrak{M}_{k,m}^{(\lceil (k+m)\min(\alpha_1,\alpha_2)^{-1})\rceil,\alpha)} = \mathfrak{M}_{k,m}^{(\lceil (k+m)\min(\alpha_1,\alpha_2)^{-1})\rceil+1,\alpha)} = \cdots$. It is natural to denote $\mathfrak{M}_{k,m}^{(\alpha)} = \mathfrak{M}_{k,m}^{(\lceil (k+m)\min(\alpha_1,\alpha_2)^{-1})\rceil,\alpha)}$. Let us denote the number of common edges of $G_{w(1)}$ and $G_{w(2)}$ by $c(d) = |E_{w^{(1)}}| + |E_{w^{(2)}}| - |E_d|$. Then the following equality for the main asymptotic coefficient of the correlator holds:

$$n_{k/2,m/2}^{(p,\alpha)} = \lim_{N\to\infty} NC_{k,m}^{(N,p,\alpha)} = \sum_{w\in\mathfrak{M}_{k,m}^{(\alpha)}} \lim_{N\to\infty} \left[\frac{N^{|V_d|-|E_d|-1}}{p^{(k+m)/2-|E_d|}} \alpha_1^{|V_{1,d}|}\alpha_2^{|V_{2,d}|} \right.$$
$$\left. \times \left(\prod_{e\in E_d} V_{n_d(e)} - \frac{p^{c(d)}}{N^{c(d)}} \prod_{e\in E_{w^{(1)}}} V_{n_{w^{(1)}}(e)} \prod_{e\in E_{w^{(2)}}} V_{n_{w^{(2)}}(e)} \right) \right]. \tag{3.7}$$

$\mathfrak{M}_{k,m}^{(\alpha)}$ is a finite set. Not all minimal walks make a nonzero contribution to the main asymptotic coefficient of the correlator. The graph G_d has at most 2 connected components because $G_{w(1)}$ and $G_{w(2)}$ are connected graphs. But if the graph G_d has exactly 2 connected components, then

$$V_{w^{(1)}} \cap V_{w^{(2)}} = \varnothing \Rightarrow E_{w^{(1)}} \cap E_{w^{(2)}} = \varnothing \Rightarrow c(d) = 0$$
$$\Rightarrow \left(\prod_{e\in E_d} V_{n_d(e)} - \frac{p^{c(d)}}{N^{c(d)}} \prod_{e\in E_{w^{(1)}}} V_{n_{w^{(1)}}(e)} \prod_{e\in E_{w^{(2)}}} V_{n_{w^{(2)}}(e)} \right) = 0.$$

Consequently, such minimal double bipartite walks make zero contribution to $n_{k/2,m/2}^{(p,\alpha)}$.

This means that only minimal double walks with a connected skeleton G_d can make a nonzero contribution. For any connected graph G_d, the inequality $|V_d| - |E_d| - 1 \le 0$ holds, and the equality holds if and only if G_d is a tree. There are two cases: $E_{w^{(1)}} \cap E_{w^{(2)}} = \varnothing \Rightarrow c(d) = 0$ and $c(d) > 0$. In the first case, the contribution is 0 (see above), and in the second, it is $\alpha_1^{|V_{1,d}|}\alpha_2^{|V_{2,d}|} \prod_{e\in E_d} \frac{V_{n_d(e)}}{p^{n_d(e)/2-1}}$.

Definition 3.3. We call essential a minimal double bipartite walk whose contribution to the main asymptotic coefficient of the corresponding correlator is not equal to 0.

Denote the set of essential double walks by $\mathfrak{S}_{k,m}$. $\mathfrak{S}_{k,m} = \{d \in \mathfrak{M}_{k,m} : G_d \text{ is a tree } \wedge c(d) > 0\}$. These are all minimal double bipartite walks, whose graph is a tree and the graphs of the first and second walks have at least one common edge. Now (3.7) can be written like this:

$$n_{k/2,m/2}^{(p,\alpha)} = \sum_{d\in\mathfrak{S}_{k,m}} \theta(d), \tag{3.8}$$

where

$$\theta(d) = \alpha_1^{|V_{1,d}|}\alpha_2^{|V_{2,d}|}\prod_{e\in E_d}\frac{V_{n_d(e)}}{p^{n_d(e)/2-1}}.$$

Thus the first part of the theorem is proved, namely the existence of $n^{(p)}_{k/2,m/2}$. It remains to derive a system of recurrence relations for $n^{(p)}_{k/2,m/2}$. From the definition it is clear that the weight θ of an essential double bipartite walk is multiplicative along the edges of G_d.

It is clear that $\mathfrak{S}_{k,m} = \varnothing$ if k or m is odd. Indeed, from the definition of the essential double walk d it follows that G_d is a tree. Hence, $G_{w^{(1)}}$ and $G_{w^{(2)}}$ are trees. Each edge of any tree is a bridge. Since $w^{(1)}$ and $w^{(2)}$ are closed walks, then their lengths are even numbers. Thus, $n^{(p)}_{k/2,m/2} = 0$ for such k and m.

3.3. First edge decomposition of essential walks. The idea of derivation of the recurrent system is the same as that of Wigner ([1]), however its implementation is more complicated. Remove from the graph G_d the first edge (r, v) of the first walk $w^{(1)}$. Since G_d is a tree, the graph splits into two pieces: the upper graph G_u, which contains the vertex v, and the right graph G_r, which contains the vertex r. Then the bipartite walk $w^{(1)}$ (respectively, $w^{(2)}$) is divided into the upper bipartite walk $w^{(1,u)}$ (respectively, $w^{(2,u)}$) on G_u and the right bipartite walk $w^{(1,r)}$ (respectively, $w^{(2,r)}$) on G_r. Similarly, let us call $d^{(u)} = (w^{(1,u)}, w^{(2,u)})$ an upper double walk and $d^{(r)} = (w^{(1,r)}, w^{(2,r)})$ a right double walk. But for an unambiguous restoration of the minimal double walk d it is not enough to know these pieces. It is also necessary to know the multiplicity of the edge (r, v) and the behavior of the double walk d at the vertices r and v (that is, after what moments of passing r and v the edge (r, v) is passed). Let us call this information a code of the double walk d. Thus, after removing the edge (r, v) from G, instead of one set of double walks we get a set of upper double walks, a set of right double walks and a set of codes. We divide the original set of double walks into such non-intersecting subsets for which the corresponding set of upper double walks, the set of right double walks and the set of codes are independent (that is, there is a bijection between the original set of double walks and the Cartesian product of the set of upper double walks, the set of right double walks and the set of codes). Then, using the weight multiplicity, we can write a total weight of double walks from original set as a product of a total weight of upper double walks by a total weight of right double walks by some number specified below. Then we do the same for new sets of double walks until the system of recurrence relations closes. Each such step is carried out in two stages:

(i) cut the graph G_d along the root r and call it the first cutting;

(ii) the resulting piece of G_d, containing the edge (r, v), cut along the vertex v and call it the second cutting.

Let us introduce some notations. The first walk of a minimal double walk is called a gray walk, and the second, a blue one. The first vertex of the gray (blue) walk is called a gray (blue) root. One can see that r is a gray root. We also denote the left graph by $G_l = (r, v) \cup G_u$. G_l is a tree with the root r and exactly one edge extending from the root is (r, v). Half of the length of the gray (blue) walk we denote by l_g (l_b). Let also u_g (u_b) denote a half of the length of a gray (blue) walk along the upper graph, and f_g (f_b) denote the number of gray (blue) steps from the gray root r to the vertex v. We also denote by r_g (r_b) the number of all gray (blue) steps leaving the gray root r, and denote by v_g (v_b) the number of steps leaving v vertex, other than $\overrightarrow{(v, r)}$.

Let $\mathrm{Set}(l_g, l_b)$ denote the set of essential (l_g, l_b)-walks, and $\mathrm{S}(l_g, l_b)$ denote their total weight. The following table describes the used denotations. The same denotations are also used for total weight S. If several designations are used simultaneously, then the corresponding set is the intersection of sets with only one designation, that is, all requirements are met simultaneously (see Table 3.1 below).

$\mathrm{Set}^{(a)}$	the parameters $r_g(d)$, $r_b(d)$ in this class can take any valid values
the absence of $^{(a)}$	the parameters $r_g(d)$, $r_b(d)$ are fixed
$\mathrm{Set}_{(=)}$	the gray root matches the blue root
$\mathrm{Set}_{(\neq)}$	the gray root does not match the blue root
$\mathrm{Set}_{(c)}$	the skeleton of the gray walk and the skeleton of the blue walk have at least one common edge
$\mathrm{Set}_{(\not c)}$	the skeleton of the gray walk and the skeleton of the blue walk have at least zero common edge
$\mathrm{Set}^{(r)}$	the skeleton of the blue walk has the edge (r, v)
$\mathrm{Set}^{(g)}$	the skeleton of the blue walk does not have the edge (r, v)
$\mathrm{Set}^{(u)}$	the blue root is in the upper tree
$\mathrm{Set}^{(d)}$	the blue root which does not coincide with the gray root is in the right tree
$\mathrm{Set}^{(v)}$	the parameters $v_g(d)$, $v_b(d)$ are fixed
$\mathrm{Set}^{(1)}$	the gray or blue walk is lacking
$\mathrm{Set}_{(1)}$	the skeleton of the double walk G_d has only one edge with the gray root r
$\mathrm{Set}^{(l)}$	the lengths of the gray walk and the blue walk on the upper (left) graph are fixed
$\mathrm{Set}^{(f)}$	gray multiplicity and blue multiplicity of the edge (r, v) are fixed
$\mathrm{Set}^{(\varnothing)}$	top graph is empty
$\mathrm{Set}^{(s)}$	the blue walk passes the gray root
$\mathrm{Set}^{(n)}$	the gray walk does not have any steps
$^{(1)}\mathrm{Set}$	the gray root is in the first component $(V_{1,d})$
$^{(2)}\mathrm{Set}$	the gray root is in the second component $(V_{2,d})$

Table 3.1

Schematically, the system of recurrence relations is presented in Fig. 3.1. Each element of the scheme is expressed through those elements which are indicated by arrows coming from it. The dotted arrow means that the total length is necessarily reduced.

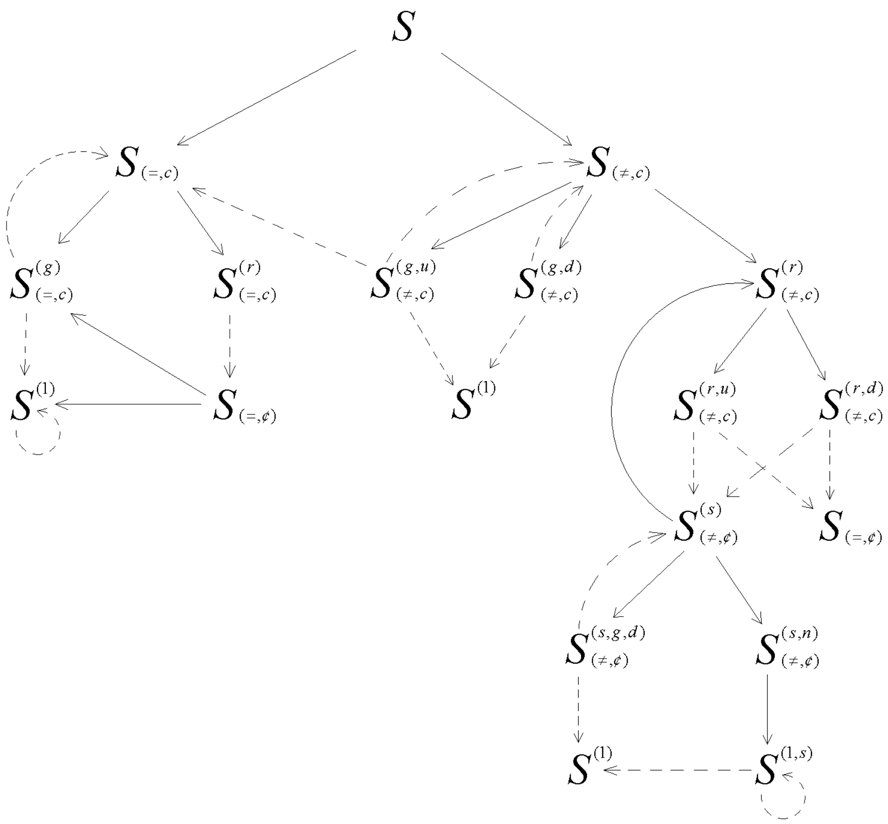

Fig. 3.1: Scheme of the system of recurrence relations

In the figures, the blue root is depicted as a black circle, the gray one is depicted as a white circle, and if the gray and blue roots coincide, the circle is black and white. Two parallel segments indicate gray and blue edges. The case when the blue walk reaches the gray root is depicted as a small black circle inside the gray root.

Since for each essential double walk the gray and blue roots either coincide or they do not coincide, the following equality is true:

$$n_{l_g,l_b}^{(p,\alpha)} = \mathrm{S}(l_g, l_b; p, \alpha) = \mathrm{S}_{(=,\,c)}^{(a)}(l_g, l_b; p, \alpha) + \mathrm{S}_{(\neq,\,c)}^{(a)}(l_g, l_b; p, \alpha). \tag{3.9}$$

Here, $\mathrm{S}(l_g, l_b; p, \alpha)$, $\mathrm{S}_{(=,\,c)}^{(a)}(l_g, l_b; p, \alpha)$, $\mathrm{S}_{(\neq,\,c)}^{(a)}(l_g, l_b; p, \alpha)$ and other S depend on p and α, but in order not to overload the formulas, we will further omit the explicit expression of this dependence. Looking through all possible values of the parameters r_g and r_b, we get the equality

$$\mathrm{S}_{(=,\,c)}^{(a)}(l_g, l_b) = \sum_{r_g=0}^{l_g} \sum_{r_b=0}^{l_b} \mathrm{S}_{(=,\,c)}(l_g, l_b; r_g, r_b). \tag{3.10}$$

Since the edge (r,v) is either in the blue skeleton or it is not, the following equality holds:

$$\mathrm{S}_{(=,\,c)}(l_g,l_b;r_g,r_b)=\mathrm{S}^{(g)}_{(=,\,c)}(l_g,l_b;r_g,r_b)+\mathrm{S}^{(r)}_{(=,\,c)}(l_g,l_b;r_g,r_b). \tag{3.11}$$

Since the gray root is either in the first component or it is in the second one, the following equality holds:

$$\mathrm{S}_{(=,\,c)}(l_g,l_b;r_g,r_b)={}^{(1)}\mathrm{S}_{(=,\,c)}(l_g,l_b;r_g,r_b)+{}^{(2)}\mathrm{S}_{(=,\,c)}(l_g,l_b;r_g,r_b). \tag{3.12}$$

Take an arbitrary double bipartite walk d from $\mathrm{Set}^{(g)}_{(=,\,c)}(l_g,l_b;r_g,r_b)$. We divide its skeleton G_d into the left graph G_l and the right graph G_r. And the double bipartite walk d splits into a left double bipartite walk f and a right one s. At the same time, f is really a single walk since there is no blue walk in f (the edge (r,v) in the blue walk d is not traversed, and the skeleton G_d is a tree). At the root r of the skeleton of f there is only one edge, therefore $f\in\mathrm{Set}^{(1)}_{(1)}(f_g+u_g,f_g)$. Once in G_d there is a blue-gray edge, but in G_l it does not exist, then it is in G_r. The gray and blue roots in s coincide, therefore $s\in\mathrm{Set}_{(=,c)}(l_g-u_g-f_g,l_b,r_g-f_g,r_b)$. The following lemma holds.

Lemma 3.4 (The first cutting lemma)**.** *Let l_g,l_b,r_g,r_b be natural numbers such that $l_g\ge r_g>0$ and $l_b\ge r_b>0$. Then the following equalities are true*:

$${}^{(1)}\mathrm{S}^{(g)}_{(=,\,c)}(l_g,l_b;r_g,r_b)=\sum_{u_g=0}^{l_g-r_g}\sum_{f_g=1}^{r_g}{}^{(1)}\mathrm{S}^{(g,l,f)}_{(=,\,c)}(l_g,l_b;r_g,r_b;u_g,f_g), \tag{3.13}$$

$$\begin{aligned}&{}^{(1)}\mathrm{S}^{(g,l,f)}_{(=,c)}(l_g,l_b;r_g,r_b;u_g,f_g)\\&\quad=\alpha_1^{-1}\binom{r_g-1}{f_g-1}{}^{(1)}\mathrm{S}^{(1)}_{(1)}(f_g+u_g,f_g){}^{(1)}\mathrm{S}_{(=,c)}(l_g-u_g-f_g,l_b,r_g-f_g,r_b),\end{aligned} \tag{3.14}$$

$${}^{(2)}\mathrm{S}^{(g)}_{(=,\,c)}(l_g,l_b;r_g,r_b)=\sum_{u_g=0}^{l_g-r_g}\sum_{f_g=1}^{r_g}{}^{(2)}\mathrm{S}^{(g,l,f)}_{(=,\,c)}(l_g,l_b;r_g,r_b;u_g,f_g), \tag{3.15}$$

$$\begin{aligned}&{}^{(2)}\mathrm{S}^{(g,l,f)}_{(=,c)}(l_g,l_b;r_g,r_b;u_g,f_g)\\&\quad=\alpha_2^{-1}\binom{r_g-1}{f_g-1}{}^{(2)}\mathrm{S}^{(1)}_{(1)}(f_g+u_g,f_g){}^{(2)}\mathrm{S}_{(=,c)}(l_g-u_g-f_g,l_b,r_g-f_g,r_b).\end{aligned} \tag{3.16}$$

What is ${}^{(1)}\mathrm{Set}^{(g)}_{(=,\,c)}(l_g,l_b;r_g,r_b)$? If we look at the spreadsheet, we can easily understand that ${}^{(1)}\mathrm{Set}^{(g)}_{(=,\,c)}(l_g,l_b;r_g,r_b)$ is a set of essential double bipartite closed walks such that:

1) the length of the first (gray) walk is $2l_g$, the length of the second (blue) walk is $2l_b$;
2) ${}^{(1)}$ means that the root of the first walk is 1 and the first step of the first walk is $\overrightarrow{(1,\tilde{1})}$;

3) $_=$ means that the root of the second walk coincides with the root of the first root, so it is 1;
4) $_c$ means that the skeleton of the first walk and the skeleton of the second walk have at least one common edge;
5) ^{g} means that the skeleton of the second walk does not have edge $(1,\tilde{1})$, so $f_b = 0$, then $u_b = 0$, because the skeleton of the second walk is a tree and it does not have edge $(1,\tilde{1}$, but it has vertex 1; 6) the number of steps of the first walk from vertex 1 is r_g, the number of steps of the second walk from vertex 1 is r_b. So, going over all the possible values u_g, u_b, we get (3.13).

Relation (3.14) follows from the bijection

$$ {}^{(1)}\mathrm{Set}^{(g,l,f)}_{(=,c)}(l_g, l_b; r_g, r_b; u_g, f_g) \to {}^{(1)}\mathrm{Set}^{(1)}_{(1)}(f_g + u_g, f_g) \\ \times {}^{(1)}\mathrm{Set}_{(=,c)}(l_g - u_g - f_g, l_b, r_g - f_g, r_b) \times \mathrm{Code}^{(1)}(r_g, f_g), \quad (3.17) $$

where $\mathrm{Code}^{(1)}(r_g, f_g)$ is a set of sequences of zeros and ones of length r_g, which have exactly f_g ones and the first term is 1. Fig. 3.2 illustrates equality (3.14).

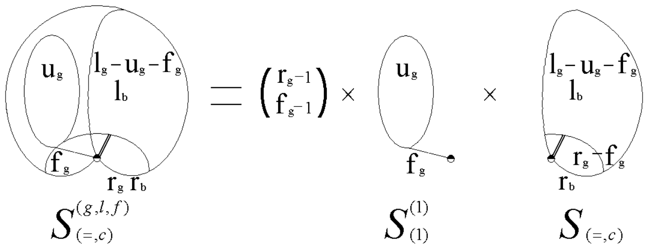

Fig. 3.2: Representation of ${}^{(1)}\mathrm{Set}^{(g,l,f)}_{(=,c)}$

Indeed, since the contribution of essential double bipartite walks is multiplicative along the edges and vertices (see (3.8)), the contribution of an essential walk from ${}^{(1)}\mathrm{Set}^{(g,l,f)}_{(=,c)}(l_g, l_b; r_g, r_b; u_g, f_g)$ is equal to the product of contributions of its parts from ${}^{(1)}\mathrm{Set}^{(g,l,f)}_{(=,c)}(l_g, l_b; r_g, r_b; u_g, f_g)$ and ${}^{(1)}\mathrm{Set}_{(=,c)}(l_g - u_g - f_g, l_b, r_g - f_g, r_b)$ and a factor α_1^{-1}. The multiplier α_1^{-1} arises due to the double use of the root in the first and second double bipartite walks of the partition. Applying this fact and the Cartesian product of the image of the above described bijective map, we obtain the following equality:

$$ {}^{(1)}\mathrm{S}^{(g,l,f)}_{(=,c)}(l_g, l_b; r_g, r_b; u_g, f_g) = \left|\mathrm{Code}^{(1)}(r_g, f_g)\right| {}^{(1)}\mathrm{S}^{(1)}_{(1)}(f_g + u_g, f_g) \\ \times {}^{(1)}\mathrm{S}_{(=,c)}(l_g - u_g - f_g, l_b, r_g - f_g, r_b)\alpha_1^{-1}, $$

where $\left|\mathrm{Code}^{(1)}(r_g, f_g)\right| = \binom{r_g - 1}{f_g - 1}$ is a trivial combinatorial fact. Indeed, if the first element is fixed, then we have to choose $f_g - 1$ positions for the remaining elements among the $r_g - 1$ free places.

It remains to prove the bijectivity of (3.17). First, we are to obtain two numerical double walks and a code and then to minimize both double walks. For splitting, the following splitting algorithm is used. We first split a gray walk.

Further, we go along $d^{(1)}$ and if the next step belongs to G_l, then we add it to the first gray walk $f^{(1)}$, otherwise we add it to the second gray walk $s^{(1)}$. At the same time, if the next step begins with the gray root, then, if it is $(\overline{r,v})$, we assign 1 to the code, otherwise we assign 0 to the code. Obviously, the first element of the code is 1 since the first edge of the gray walk is $(\overline{r,v})$ by definition, and the total number of ones in the code is f_g, i.e., $\#\{i : c_i = 1\} = f_g$, $\#\{i : c_i = 0\} = r_g - f_g \ \wedge c_1 = 1$. It is easy to see that $f^{(1)}$ and $s^{(1)}$ are really bipartite walks (in particular for every adjacent edge, the origin of the next edge coincides with the end of the previous edge), closed walks (in particular both walks have the same root r). Obviously, every edge from the left graph G_l and from the right graph G_r is traversed in the corresponding walk $f^{(1)}$ or $s^{(1)}$ the same number of times as in a gray walk $d^{(1)}$, i.e.,

$$\big(\forall e \in G_l \quad n_{d^{(1)}}(e) = n_{f^{(1)}}(e)\big) \quad \text{and} \quad \big(\forall e \in G_r \quad n_{d^{(1)}(e)} = n_{s^{(1)}}(e)\big).$$

Since the blue walk is completely in $s^{(2)}$, splitting is obvious, i.e.,

$$\big(\forall e \in G_l \quad n_{d^{(2)}}(e) = n_{f^{(2)}}(e) = 0\big) \quad \text{and} \quad \big(\forall e \in G_r \quad n_{d^{(2)}}(e) = n_{s^{(2)}}(e)\big).$$

Thus, the weight of the original double walk is equal to the product of weights of the first and second partitioned double walks up to the factor α_1^{-1}. Now we make them minimal by applying minimization mapping to them. At the same time, the weight of walks does not change.

Here is an example of double bipartite closed walk $d = \big(d^{(1)}, d^{(2)}\big)$ that illustrates the lemma.

$$d^{(1)} = (1, \tilde{1}, 2, \tilde{2}, 2, \tilde{3}, 2, \tilde{2}, 2, \tilde{1}, 3, \tilde{1}, 3, \tilde{1}, 1, \tilde{4}, 4, \tilde{4}, 5, \tilde{4}, 6,$$
$$\tilde{4}, 1, \tilde{5}, 1, \tilde{4}, 1, \tilde{1}, 1, \tilde{5}, 1, \tilde{1}, 1, \tilde{4}, 1, \tilde{1}, 3, \tilde{1}, 1);$$

$d^{(1)}$ is a gray walk;

$$d^{(2)} = (1, \tilde{6}, 7, \tilde{7}, 7, \tilde{6}, 7, \tilde{7}, 7, \tilde{6}, 1, \tilde{5}, 8, \tilde{5}, 1, \tilde{4}, 1, \tilde{4}, 9, \tilde{4}, 5, \tilde{4}, 1, \tilde{5}, 1);$$

$d^{(2)}$ is a blue walk, $r = 1$, $v = \tilde{1}$, $d \in {}^{(1)}\mathrm{Set}^{(g,\,l,\,f)}_{(=,\,c)}(19, 12; 9, 5; 7, 4)$;

$$G = \Big(\big\{1, \tilde{1}, 2, \tilde{2}, \tilde{3}, 3, \tilde{4}, 4, 5, 6, \tilde{5}, 8, \tilde{6}, 7, \tilde{7}, 9\big\},$$
$$\big\{(1, \tilde{1}), (\tilde{1}, 2), (2, \tilde{2}), (2, \tilde{3}), (\tilde{1}, 3), (1, \tilde{4}), (\tilde{4}, 4),$$
$$(\tilde{4}, 5), (\tilde{4}, 6), (1, \tilde{5}), (\tilde{5}, 8), (1, \tilde{6}), (\tilde{6}, 7), (7, \tilde{7}), (\tilde{4}, 9)\big\}\Big);$$

$$G_u = \Big(\big\{\tilde{1}, 2, \tilde{2}, \tilde{3}, 3\big\}, \big\{(\tilde{1}, 2), (2, \tilde{2}), (2, \tilde{3}), (\tilde{1}, 3)\big\}\Big);$$

$$G_l = \Big(\big\{1, \tilde{1}, 2, \tilde{2}, \tilde{3}, 3\big\}, \big\{(1, \tilde{1}), (\tilde{1}, 2), (2, \tilde{2}), (2, \tilde{3}); (\tilde{1}, 3)\big\}\Big);$$

$$G_r = \Big(\big\{1, \tilde{4}, 4, 5, 6, \tilde{5}, 8, \tilde{6}, 7, \tilde{7}, 9\big\},$$
$$\big\{(1, \tilde{4}), (\tilde{4}, 4), (\tilde{4}, 5), (\tilde{4}, 6), (1, \tilde{5}), (\tilde{5}, 8), (1, \tilde{6}), (\tilde{6}, 7), (7, \tilde{7}), (\tilde{4}, 9)\big\}\Big);$$

$l_g = 19$, $l_b = 12$, $f_g = 4$, $f_b = 0$, $r_g = 9$, $r_b = 5$, $u_g = 7$; $u_b = 0$, $v_g = 4$, $v_b = 0$, $d \to (\eta, \theta, c)$, $\eta = (\eta^{(1)}, \eta^{(2)})$; $\theta = (\theta^{(1)}, \theta^{(2)})$, $c \in \{0, 1\}^9$;

$$\eta^{(1)} = (1, \tilde{1}, 2, \tilde{2}, 2, \tilde{3}, 2, \tilde{2}, 2, \tilde{1}, 3, \tilde{1}, 3, \tilde{1}, 1, \tilde{1}, 1, \tilde{1}, 1, \tilde{1}, 3, \tilde{1}, 1);$$
$$\eta^{(2)} = (1);$$
$$\theta^{(1)} = (1, \tilde{4}, 4, \tilde{4}, 5, \tilde{4}, 6, \tilde{4}, 1, \tilde{5}, 1, \tilde{4}, 1, \tilde{5}, 1, \tilde{4}, 1);$$
$$\theta^{(2)} = (1, \tilde{6}, 7, \tilde{7}, 7, \tilde{6}, 7, \tilde{7}, 7, \tilde{6}, 1, \tilde{5}, 8, \tilde{5}, 1, \tilde{4}, 1, \tilde{4}, 9, \tilde{4}, 5, \tilde{4}, 1, \tilde{5}, 1);$$

$c = (1, 0, 0, 0, 1, 0, 1, 0, 1)$; $\eta \in {}^{(1)}\mathrm{Set}^{(1)}_{(1)}(11, 7)$, $\theta \in {}^{(1)}\mathrm{Set}_{(=,c)}(8, 12, 5, 5)$.

The bijectivity is proved by the following collection algorithm. We gradually renumber vertices of the first and second walks. The roots of these double walks are set in compliance number 1. Let us go along the first and second double walks. We start the construction from the root. If the next step of a double walk under construction ends at the root, then, if the next element of the code is 1, we continue going along the first subwalk f, otherwise continue going along the second subwalk s. If the final vertex of the current step along the subwalk does not have its own number in the large walk, then we set it in correspondence with the largest number of the already completed vertices of the large walk in the corresponding component plus 1 in the corresponding set. The result is a bipartite double walk from the required class. It is easy to see that splitting mapping and collection mapping are injective. It means that they are bijective since the area of definition and the area of values are finite. It remains to split the gray walk $f^{(1)}$.

Lemma 3.5 (The second cutting lemma)**.** *Let f_g be a natural number and u_g be a natural number or zero. Then the following equalities are true*:

$$ {}^{(1)}\mathrm{S}^{(1)}_{(1)}(f_g + u_g, f_g) = \sum_{v_g=0}^{u_g} {}^{(1)}\mathrm{S}^{(1,v)}_{(1)}(f_g + u_g, f_g, v_g), \tag{3.18}$$

$$ {}^{(1)}\mathrm{S}^{(1,v)}_{(1)}(f_g + u_g, f_g, v_g) = \binom{f_g + v_g - 1}{f_g - 1} \frac{\alpha_1 V_{2f_g}}{p^{f_g - 1}} {}^{(2)}\mathrm{S}^{(1)}(u_g, v_g), \tag{3.19}$$

$$ {}^{(2)}\mathrm{S}^{(1)}_{(1)}(f_g + u_g, f_g) = \sum_{v_g=0}^{u_g} {}^{(2)}\mathrm{S}^{(1,v)}_{(1)}(f_g + u_g, f_g, v_g), \tag{3.20}$$

$$ {}^{(2)}\mathrm{S}^{(1,v)}_{(1)}(f_g + u_g, f_g, v_g) = \binom{f_g + v_g - 1}{f_g - 1} \frac{\alpha_2 V_{2f_g}}{p^{f_g - 1}} {}^{(1)}\mathrm{S}^{(1)}(u_g, v_g). \tag{3.21}$$

This lemma is proved in the same way as the first cutting lemma. The first equality is obvious, and the second equality follows from the bijection

$$ {}^{(1)}\mathrm{Set}^{(1,v)}_{(1)}(f_g + u_g, f_g, v_g) \overset{bij}{\to} {}^{(2)}\mathrm{Set}^{(1)}(u_g, v_g) \times {}^{(1)}\mathrm{Set}^{(1,\varnothing)}_{1}(f_g) \times \mathrm{Code}^{(2)}(f_g + v_g, f_g), \tag{3.22}$$

where $\mathrm{Code}^{(2)}(f_g + v_g, f_g)$ is a set of sequences of zeros and ones of length $f_g + v_g$, which have exactly f_g ones and the last term is 1. The last term is 1 since the gray walk should return to the gray root r by the last step from the vertex v. It is obvious that ${}^{(1)}\mathrm{Set}_1^{(1,\varnothing)}(f_g)$ consists of a single walk with a weight $\frac{\alpha_1\alpha_2 V_{2f_g}}{p^{f_g-1}}$. Fig. 3.3 illustrates equality (3.19).

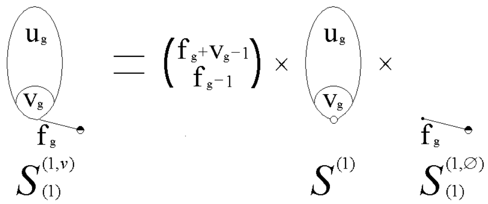

Fig. 3.3: Representation of ${}^{(1)}\mathrm{Set}_{(1)}^{(1,\,v)}$

Combining these two lemmas, changing the order of summation and taking out some expressions beyond sign of the sum, we get the formulas:

$$
\begin{aligned}
{}^{(1)}\mathrm{S}^{(g)}_{(=,c)}(l_g,l_b;r_g,r_b) &= \sum_{f_g=1}^{r_g}\binom{r_g-1}{f_g-1}\frac{V_{2f_g}}{p^{f_g-1}} \\
&\times \sum_{u_g=0}^{l_g-r_g} {}^{(1)}\mathrm{S}_{(=,c)}(l_g-u_g-f_g,l_b;r_g-f_g,r_b) \\
&\times \sum_{v_g=0}^{u_g}\binom{f_g+v_g-1}{f_g-1}{}^{(2)}\mathrm{S}^{(1)}(u_g,v_g),
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
{}^{(2)}\mathrm{S}^{(g)}_{(=,c)}(l_g,l_b;r_g,r_b) &= \sum_{f_g=1}^{r_g}\binom{r_g-1}{f_g-1}\frac{V_{2f_g}}{p^{f_g-1}} \\
&\times \sum_{u_g=0}^{l_g-r_g} {}^{(2)}\mathrm{S}_{(=,c)}(l_g-u_g-f_g,l_b;r_g-f_g,r_b) \\
&\times \sum_{v_g=0}^{u_g}\binom{f_g+v_g-1}{f_g-1}{}^{(1)}\mathrm{S}^{(1)}(u_g,v_g).
\end{aligned}
\tag{3.24}
$$

3.4. Conclusion of a recursive system of equations. In a similar way, (see also [12]) the next formulas are proved:

$$
\begin{aligned}
{}^{(1)}\mathrm{S}^{(1)}(l_g,r_g) &= \sum_{f_g=1}^{r_g}\binom{r_g-1}{f_g-1}\frac{V_{2f_g}}{p^{f_g-1}}\sum_{u_g=0}^{l_g-r_g}{}^{(1)}\mathrm{S}^{(1)}(l_g-u_g-f_g,r_g-f_g) \\
&\times \sum_{v_g=0}^{u_g}\binom{f_g+v_g-1}{f_g-1}{}^{(2)}\mathrm{S}^{(1)}(u_g,v_g),
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
{}^{(2)}\mathrm{S}^{(1)}(l_g,r_g) &= \sum_{f_g=1}^{r_g}\binom{r_g-1}{f_g-1}\frac{V_{2f_g}}{p^{f_g-1}}\sum_{u_g=0}^{l_g-r_g}{}^{(2)}\mathrm{S}^{(1)}(l_g-u_g-f_g,r_g-f_g) \\
&\times \sum_{v_g=0}^{u_g}\binom{f_g+v_g-1}{f_g-1}{}^{(1)}\mathrm{S}^{(1)}(u_g,v_g).
\end{aligned}
\tag{3.26}
$$

The formulas

$$
\begin{aligned}
{}^{(1)}\mathrm{S}^{(r)}_{(=,c)}(l_g, l_b; r_g, r_b) &= \sum_{f_g=1}^{r_g} \binom{r_g-1}{f_g-1} \sum_{f_b=1}^{r_b} \binom{r_b}{f_b} \frac{V_{2(f_g+f_b)}}{p^{f_g+f_b-1}} \\
&\times \sum_{u_g=0}^{l_g-r_g} {}^{(1)}\mathrm{S}_{(=,\not c)}(l_g-u_g-f_g, l_b-u_b-f_b; r_g-f_g, r_b-f_b) \\
&\times \sum_{v_g=0}^{u_g} \binom{f_g+v_g-1}{f_g-1} \sum_{v_b=0}^{u_b} \binom{f_b+v_b-1}{f_b-1} {}^{(2)}\mathrm{S}_{(=,\not c)}(u_g, v_g),
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
{}^{(2)}\mathrm{S}^{(r)}_{(=,c)}(l_g, l_b; r_g, r_b) &= \sum_{f_g=1}^{r_g} \binom{r_g-1}{f_g-1} \sum_{f_b=1}^{r_b} \binom{r_b}{f_b} \frac{V_{2(f_g+f_b)}}{p^{f_g+f_b-1}} \\
&\times \sum_{u_g=0}^{l_g-r_g} {}^{(2)}\mathrm{S}_{(=,\not c)}(l_g-u_g-f_g, l_b-u_b-f_b; r_g-f_g, r_b-f_b) \\
&\times \sum_{v_g=0}^{u_g} \binom{f_g+v_g-1}{f_g-1} \sum_{v_b=0}^{u_b} \binom{f_b+v_b-1}{f_b-1} {}^{(1)}\mathrm{S}_{(=,\not c)}(u_g, v_g)
\end{aligned}
\tag{3.28}
$$

follow from Lemmas 3.6 and 3.7 below.

Lemma 3.6. *Let l_g, l_b, r_g, r_b be natural numbers such that $l_g \geq r_g > 0$ and $l_b \geq r_b > 0$. Then the following equalities are true:*

$$
{}^{(1)}\mathrm{S}^{(r)}_{(=,c)}(l_g, l_b; r_g, r_b) = \sum_{u_g=0}^{l_g-r_g} \sum_{f_g=1}^{r_g} \sum_{u_b=0}^{l_b-r_b} \sum_{f_b=1}^{r_b} {}^{(1)}\mathrm{S}^{(r,l,f)}_{(=,c)}(l_g, l_b; r_g, r_b; u_g, u_b; f_g, f_b),
$$

$$
\begin{aligned}
&{}^{(1)}\mathrm{S}^{(r,l,f)}_{(=,c)}(l_g, l_b; r_g, r_b; u_g, u_b; f_g, f_b) \\
&\qquad = \alpha_1^{-1} \binom{r_g-1}{f_g-1} \binom{r_b}{f_b} {}^{(1)}\mathrm{S}^{(r,f)}_{(1,=,c)}(f_g+u_g, f_b+u_b; f_g, f_b) \\
&\qquad\quad \times {}^{(1)}\mathrm{S}_{(=,\not c)}(l_g-u_g-f_g, l_b-u_b-f_b; r_g-f_g, r_b-f_b),
\end{aligned}
$$

$$
{}^{(2)}\mathrm{S}^{(r)}_{(=,c)}(l_g, l_b; r_g, r_b) = \sum_{u_g=0}^{l_g-r_g} \sum_{f_g=1}^{r_g} \sum_{u_b=0}^{l_b-r_b} \sum_{f_b=1}^{r_b} {}^{(2)}\mathrm{S}^{(r,l,f)}_{(=,c)}(l_g, l_b; r_g, r_b; u_g, u_b; f_g, f_b),
$$

$$
\begin{aligned}
&{}^{(2)}\mathrm{S}^{(r,l,f)}_{(=,c)}(l_g, l_b; r_g, r_b; u_g, u_b; f_g, f_b) \\
&\qquad = \alpha_2^{-1} \binom{r_g-1}{f_g-1} \binom{r_b}{f_b} {}^{(2)}\mathrm{S}^{(r,f)}_{(1,=,c)}(f_g+u_g, f_b+u_b; f_g, f_b) \\
&\qquad\quad \times {}^{(2)}\mathrm{S}_{(=,\not c)}(l_g-u_g-f_g, l_b-u_b-f_b; r_g-f_g, r_b-f_b).
\end{aligned}
$$

The formula contains the factor $\binom{r_b}{f_b}$ because, unlike a gray walk, the first step of a blue walk does not have to be (r, v) (see Fig. 3.4).

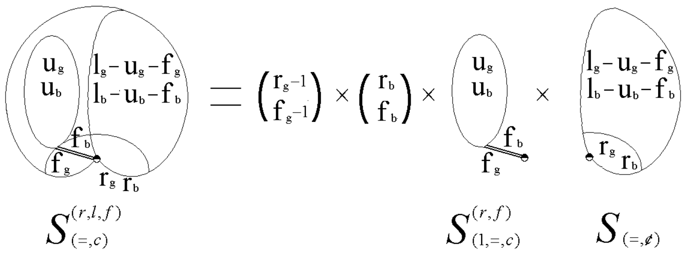

Fig. 3.4: Representation of ${}^{(1)}\mathrm{Set}^{(r,l,f)}_{(=,\,c)}$

Lemma 3.7. *Let f_g, f_b be natural numbers and u_g, u_b be natural numbers or zeros. Then the following equalities are true*:

$$
\begin{aligned}
&{}^{(1)}\mathrm{S}^{(r,f)}_{(1,=,c)}(f_g+u_g, f_b+u_b; f_g, f_b)\\
&\qquad = \sum_{v_g=0}^{u_g}\sum_{v_b=0}^{u_b} {}^{(1)}\mathrm{S}^{(v,r,f)}_{(1,=,c)}(f_g+u_g, f_b+u_b; f_g, f_b; v_g, v_b),\\
&{}^{(1)}\mathrm{S}^{(v,r,f)}_{(1,=,c)}(f_g+u_g, f_b+u_b; f_g, f_b; v_g, v_b)\\
&\qquad = \binom{f_g+v_g-1}{f_g-1}\binom{f_b+v_b-1}{f_b-1}\frac{\alpha_1 V_{2(f_g+f_b)}}{p^{f_g+f_b-1}}\,{}^{(2)}\mathrm{S}_{(=,\not c)}(u_g, u_b; v_g, v_b).\\
&{}^{(2)}\mathrm{S}^{(r,f)}_{(1,=,c)}(f_g+u_g, f_b+u_b; f_g, f_b)\\
&\qquad = \sum_{v_g=0}^{u_g}\sum_{v_b=0}^{u_b} {}^{(2)}\mathrm{S}^{(v,r,f)}_{(1,=,c)}(f_g+u_g, f_b+u_b; f_g, f_b; v_g, v_b),\\
&{}^{(2)}\mathrm{S}^{(v,r,f)}_{(1,=,c)}(f_g+u_g, f_b+u_b; f_g, f_b; v_g, v_b)\\
&\qquad = \binom{f_g+v_g-1}{f_g-1}\binom{f_b+v_b-1}{f_b-1}\frac{\alpha_2 V_{2(f_g+f_b)}}{p^{f_g+f_b-1}}\,{}^{(1)}\mathrm{S}_{(=,\not c)}(u_g, u_b; v_g, v_b).
\end{aligned}
$$

The second formula is illustrated in Fig. 3.5.

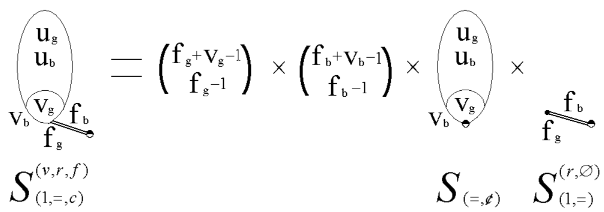

Fig. 3.5: Representation of ${}^{(1)}\mathrm{Set}^{(v,r,f)}_{(1,=,\,c)}$

A double bipartite walk from $\mathrm{S}_{(=,\not c)}(l_g, l_b; r_g, r_b)$ has a blue-gray edge or does not have it. In the first case, it is from $\mathrm{S}_{(=,c)}(l_g, l_b; r_g, r_b)$. And in the second, blue and gray walks do not have common vertices except the gray root r, therefore they are practically independent. This implies the following equations:

$$
{}^{(1)}\mathrm{S}_{(=,\not c)}(l_g, l_b; r_g, r_b) = {}^{(1)}\mathrm{S}_{(=,c)}(l_g, l_b; r_g, r_b) + \alpha_1^{-1}\,{}^{(1)}\mathrm{S}^{(1)}(l_g, r_g)\,{}^{(1)}\mathrm{S}^{(1)}(l_b, r_b), \tag{3.29}
$$

$$ {}^{(2)}\mathrm{S}_{(=,\not c)}(l_g, l_b; r_g, r_b) = {}^{(2)}\mathrm{S}_{(=,c)}(l_g, l_b; r_g, r_b) + \alpha_2^{-1(2)}\mathrm{S}^{(1)}(l_g, r_g)^{(1)}\mathrm{S}^{(2)}(l_b, r_b). \tag{3.30} $$

Going over all possible values of the parameters r_g and r_b, we deduce the equality

$$ \mathrm{S}^{(a)}_{(\neq,c)}(l_g, l_b) = \sum_{r_g=0}^{l_g} \sum_{r_b=0}^{l_b} \left({}^{(1)}\mathrm{S}_{(\neq,c)}(l_g, l_b; r_g, r_b) + {}^{(2)}\mathrm{S}_{(\neq,c)}(l_g, l_b; r_g, r_b) \right). \tag{3.31} $$

In any essential double bipartite walk from , the edge (r, v) is a blue-gray one or it is a pure gray one. Therefore, the following equalities hold:

$$ {}^{(1)}\mathrm{S}_{(\neq,c)}(l_g, l_b; r_g, r_b) = {}^{(1)}\mathrm{S}^{(g)}_{(\neq,c)}(l_g, l_b; r_g, r_b) + {}^{(1)}\mathrm{S}^{(r)}_{(\neq,c)}(l_g, l_b; r_g, r_b), \tag{3.32} $$

$$ {}^{(2)}\mathrm{S}_{(\neq,c)}(l_g, l_b; r_g, r_b) = {}^{(2)}\mathrm{S}^{(g)}_{(\neq,c)}(l_g, l_b; r_g, r_b) + {}^{(2)}\mathrm{S}^{(r)}_{(\neq,c)}(l_g, l_b; r_g, r_b). \tag{3.33} $$

If a blue root does not coincide with a gray one, then the blue root is either in the upper graph or it is in the lower one:

$$ {}^{(1)}\mathrm{S}^{(g)}_{(\neq,c)}(l_g, l_b; r_g, r_b) = {}^{(1)}\mathrm{S}^{(g,u)}_{(\neq,c)}(l_g, l_b; r_g, r_b) + {}^{(1)}\mathrm{S}^{(g,d)}_{(\neq,c)}(l_g, l_b; r_g, r_b), \tag{3.34} $$

$$ {}^{(2)}\mathrm{S}^{(g)}_{(\neq,c)}(l_g, l_b; r_g, r_b) = {}^{(2)}\mathrm{S}^{(g,u)}_{(\neq,c)}(l_g, l_b; r_g, r_b) + {}^{(2)}\mathrm{S}^{(g,d)}_{(\neq,c)}(l_g, l_b; r_g, r_b). \tag{3.35} $$

For a double walk from $\mathrm{S}^{(g,u)}_{(\neq,c)}(l_g, l_b; r_g, r_b)$, r_b is 0 since there is no edge (r, v) in the blue skeleton. In the second double walk s there is no blue component. We have the next lemma (see also Fig. 3.6).

Lemma 3.8. *Let l_g, l_b, r_g, r_b be natural numbers or zeros such that $l_g \geq r_g > 0$ and $l_b \geq r_b \geq 0$. Then the following equalities are true*:

$$ {}^{(1)}\mathrm{S}^{(g,u)}_{(\neq,c)}(l_g, l_b; r_g, r_b) = \delta_{r_b} \sum_{u_g=0}^{l_g-r_g} \sum_{f_g=1}^{r_g} {}^{(1)}\mathrm{S}^{(g,u,l,f)}_{(\neq,c)}(l_g, l_b; r_g; u_g, f_g), $$

$$ \begin{aligned} &{}^{(1)}\mathrm{S}^{(g,u,l,f)}_{(\neq,c)}(l_g, l_b; r_g; u_g, f_g) \\ &\quad = \alpha_1^{-1} \binom{r_g - 1}{f_g - 1} {}^{(1)}\mathrm{S}^{(g,f)}_{(1,\neq,c)}(f_g + u_g, l_b; f_g)^{(1)}\mathrm{S}^{(1)}(l_g - u_g - f_g, r_g - f_g), \end{aligned} $$

$$ {}^{(2)}\mathrm{S}^{(g,u)}_{(\neq,c)}(l_g, l_b; r_g, r_b) = \delta_{r_b} \sum_{u_g=0}^{l_g-r_g} \sum_{f_g=1}^{r_g} {}^{(2)}\mathrm{S}^{(g,u,l,f)}_{(\neq,c)}(l_g, l_b; r_g; u_g, f_g), $$

$$ \begin{aligned} &{}^{(2)}\mathrm{S}^{(g,u,l,f)}_{(\neq,c)}(l_g, l_b; r_g; u_g, f_g) \\ &\quad = \alpha_2^{-1} \binom{r_g - 1}{f_g - 1} {}^{(2)}\mathrm{S}^{(g,f)}_{(1,\neq,c)}(f_g + u_g, l_b; f_g)^{(2)}\mathrm{S}^{(1)}(l_g - u_g - f_g, r_g - f_g). \end{aligned} $$

If a blue root lies in the upper graph, then it either coincides with the vertex v or it does not coincide with it. In the first case, a double walk along the upper graph belongs to $\mathrm{S}_{(=,c)}(u_g, l_b; v_g, v_b)$, and in the second case, it belongs to $\mathrm{S}_{(\neq,c)}(u_g, l_b; v_g, v_b)$. And we have the next lemma.

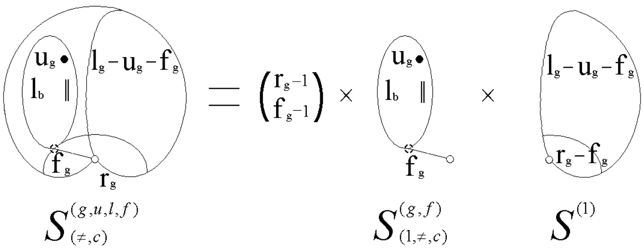

Fig. 3.6: Representation of ${}^{(1)}\mathrm{Set}^{(g,\,u,\,l,\,f)}_{(\neq,\,c)}$

Lemma 3.9. *Let f_g be a natural number, u_g be a natural number or zero, l_b be a natural number or zero. Then the following equalities are true*:

$$
{}^{(1)}\mathrm{S}^{(g,\,f)}_{(1,\,\neq,\,c)}(f_g+u_g,l_b;f_g)=\sum_{v_g=0}^{u_g}\sum_{v_b=0}^{l_b}{}^{(1)}\mathrm{S}^{(v,\,g,\,f)}_{(1,\,\neq,\,c)}(f_g+u_g,l_b;f_g;v_g),
$$

$$
\begin{aligned}
&{}^{(1)}\mathrm{S}^{(v,\,g,\,f)}_{(1,\,\neq,\,c)}(f_g+u_g,l_b;f_g;v_g)\\
&\quad=\alpha_1\binom{f_g+v_g-1}{f_g-1}\frac{V_{2f_g}}{p^{f_g-1}}\left({}^{(2)}\mathrm{S}_{(=,\,c)}(u_g,l_b;v_g,v_b)+{}^{(2)}\mathrm{S}_{(\neq,\,c)}(u_g,l_b;v_g,v_b)\right),
\end{aligned}
$$

$$
{}^{(2)}\mathrm{S}^{(g,\,f)}_{(1,\,\neq,\,c)}(f_g+u_g,l_b;f_g)=\sum_{v_g=0}^{u_g}\sum_{v_b=0}^{l_b}{}^{(2)}\mathrm{S}^{(v,\,g,\,f)}_{(1,\,\neq,\,c)}(f_g+u_g,l_b;f_g;v_g),
$$

$$
\begin{aligned}
&{}^{(2)}\mathrm{S}^{(v,\,g,\,f)}_{(1,\,\neq,\,c)}(f_g+u_g,l_b;f_g;v_g)\\
&\quad=\alpha_2\binom{f_g+v_g-1}{f_g-1}\frac{V_{2f_g}}{p^{f_g-1}}\left({}^{(1)}\mathrm{S}_{(=,\,c)}(u_g,l_b;v_g,v_b)+{}^{(1)}\mathrm{S}_{(\neq,\,c)}(u_g,l_b;v_g,v_b)\right).
\end{aligned}
$$

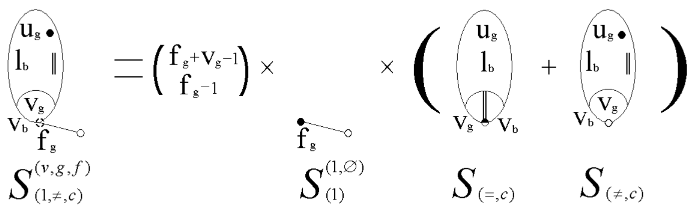

Fig. 3.7: Representation of ${}^{(1)}\mathrm{Set}^{(v,\,g,\,f)}_{(1,\,\neq,\,c)}$

The next equalities follow from Lemmas 3.8 and 3.9:

$$
\begin{aligned}
{}^{(1)}\mathrm{S}^{(g,\,u)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b)&=\delta_{r_b}\sum_{f_g=1}^{r_g}\binom{r_g-1}{f_g-1}\frac{V_{2f_g}}{p^{f_g-1}}\sum_{u_g=0}^{l_g-r_g}{}^{(1)}\mathrm{S}^{(1)}(l_g-u_g-f_g,r_g-f_g)\\
&\times\sum_{v_g=0}^{u_g}\binom{f_g+v_g-1}{f_g-1}\sum_{v_b=0}^{l_b}\left({}^{(2)}\mathrm{S}_{(=,\,c)}(u_g,l_b;v_g,v_b)+{}^{(2)}\mathrm{S}_{(\neq,\,c)}(u_g,l_b;v_g,v_b)\right),
\end{aligned}
\tag{3.36}
$$

$$
{}^{(2)}\mathrm{S}^{(g,\,u)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b)=\delta_{r_b}\sum_{f_g=1}^{r_g}\binom{r_g-1}{f_g-1}\frac{V_{2f_g}}{p^{f_g-1}}\sum_{u_g=0}^{l_g-r_g}{}^{(2)}\mathrm{S}^{(1)}(l_g-u_g-f_g,r_g-f_g)
$$

$$\times \sum_{v_g=0}^{u_g} \binom{f_g+v_g-1}{f_g-1} \sum_{v_b=0}^{l_b} \left({}^{(1)}\mathrm{S}_{(=,\,c)}(u_g,l_b;v_g,v_b) + {}^{(1)}\mathrm{S}_{(\neq,\,c)}(u_g,l_b;v_g,v_b) \right), \tag{3.37}$$

The following formulas:

$$ {}^{(1)}\mathrm{S}^{(g,d)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b) = \sum_{f_g=1}^{r_g} \binom{r_g-1}{f_g-1} \frac{V_{2f_g}}{p^{f_g-1}}$$
$$\times \sum_{u_g=0}^{l_g-r_g} {}^{(1)}\mathrm{S}_{(\neq,\,c)}(l_g-u_g-f_g,l_b;r_g-f_g,r_b) \sum_{v_g=0}^{u_g} \binom{f_g+v_g-1}{f_g-1} {}^{(2)}\mathrm{S}^{(1)}(u_g,v_g), \tag{3.38}$$

$$ {}^{(2)}\mathrm{S}^{(g,d)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b) = \sum_{f_g=1}^{r_g} \binom{r_g-1}{f_g-1} \frac{V_{2f_g}}{p^{f_g-1}}$$
$$\times \sum_{u_g=0}^{l_g-r_g} {}^{(2)}\mathrm{S}_{(\neq,\,c)}(l_g-u_g-f_g,l_b;r_g-f_g,r_b) \sum_{v_g=0}^{u_g} \binom{f_g+v_g-1}{f_g-1} {}^{(1)}\mathrm{S}^{(1)}(u_g,v_g) \tag{3.39}$$

are obtained from Lemmas 3.5 and 3.10 (see also Fig. 3.8).

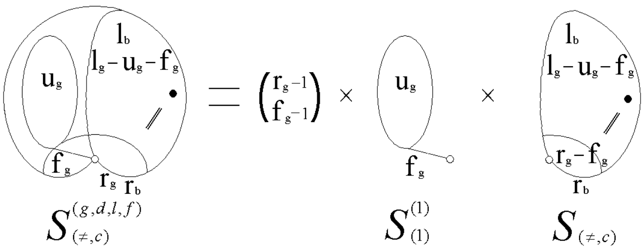

Fig. 3.8: Representation of ${}^{(1)}\mathrm{Set}^{(g,\,d,\,l,\,f)}_{(\neq,\,c)}$

Lemma 3.10. *Let l_g, l_b, r_g, r_b be natural numbers or zeros such that $l_g \geq r_g > 0$ and $l_b \geq r_b \geq 0$. Then the following equalities are true*:

$$ {}^{(1)}\mathrm{S}^{(g,d)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b) = \sum_{u_g=0}^{l_g-r_g} \sum_{f_g=1}^{r_g} {}^{(1)}\mathrm{S}^{(g,\,d,\,l,\,f)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b;u_g,f_g),$$

$$ {}^{(1)}\mathrm{S}^{(g,\,d,\,l,\,f)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b;u_g,f_g)$$
$$= \alpha_1^{-1} \binom{r_g-1}{f_g-1} {}^{(1)}\mathrm{S}^{(1)}_{(1)}(f_g+u_g,f_g) {}^{(1)}\mathrm{S}_{(\neq,\,c)}(l_g-u_g-f_g,l_b,r_g-f_g,r_b),$$

$$ {}^{(2)}\mathrm{S}^{(g,d)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b) = \sum_{u_g=0}^{l_g-r_g} \sum_{f_g=1}^{r_g} {}^{(2)}\mathrm{S}^{(g,\,d,\,l,\,f)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b;u_g,f_g),$$

$$ {}^{(2)}\mathrm{S}^{(g,\,d,\,l,\,f)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b;u_g,f_g) =$$

$$= \alpha_2^{-1}\binom{r_g-1}{f_g-1}{}^{(2)}\mathrm{S}^{(1)}_{(1)}(f_g+u_g,f_g){}^{(2)}\mathrm{S}_{(\neq,\,c)}(l_g-u_g-f_g,l_b,r_g-f_g,r_b).$$

The blue root which does not coincide with the gray one can be located either in the upper graph or in the lower (right) graph:

$$^{(1)}\mathrm{S}^{(r)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b) = {}^{(1)}\mathrm{S}^{(r,\,u)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b) + {}^{(1)}\mathrm{S}^{(r,\,d)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b), \tag{3.40}$$

$$^{(2)}\mathrm{S}^{(r)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b) = {}^{(2)}\mathrm{S}^{(r,\,u)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b) + {}^{(2)}\mathrm{S}^{(r,\,d)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b). \tag{3.41}$$

One more lemma for $\mathrm{S}^{(r,\,u)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b)$ looks like this (see also Fig. 3.9).

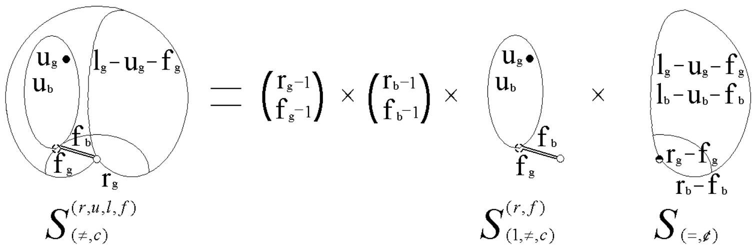

Fig. 3.9: Representation of $^{(1)}\mathrm{Set}^{(r,u,l,f)}_{(\neq,\,c)}$

Lemma 3.11. *Let l_g, l_b, r_g, r_b be natural numbers or zeros such that $l_g \geq r_g > 0$ and $l_b \geq r_b > 0$. Then the following equalities are true*:

$$^{(1)}\mathrm{S}^{(r,\,u)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b) = \sum_{u_g=0}^{l_g-r_g}\sum_{f_g=1}^{r_g}\sum_{u_b=0}^{l_b-r_b}\sum_{f_b=1}^{r_b} {}^{(1)}\mathrm{S}^{(r,u,l,f)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b;u_g,u_b;f_g,f_b),$$

$$\begin{aligned}&^{(1)}\mathrm{S}^{(r,u,l,f)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b;u_g,u_b;f_g,f_b)\\ &\quad= \binom{r_g-1}{f_g-1}\binom{r_b-1}{f_b-1}{}^{(1)}\mathrm{S}^{(r,\,f)}_{(1,\neq,\,c)}(f_g+u_g,f_b+u_b;f_g,f_b)\\ &\quad\times \alpha_1^{-1}{}^{(1)}\mathrm{S}_{(=,\not c)}(l_g-u_g-f_g,l_b-u_b-f_b;r_g-f_g,r_b-f_b),\end{aligned}$$

$$^{(2)}\mathrm{S}^{(r,\,u)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b) = \sum_{u_g=0}^{l_g-r_g}\sum_{f_g=1}^{r_g}\sum_{u_b=0}^{l_b-r_b}\sum_{f_b=1}^{r_b} {}^{(2)}\mathrm{S}^{(r,u,l,f)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b;u_g,u_b;f_g,f_b)$$

$$\begin{aligned}&^{(2)}\mathrm{S}^{(r,u,l,f)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b;u_g,u_b;f_g,f_b)\\ &\quad= \binom{r_g-1}{f_g-1}\binom{r_b-1}{f_b-1}{}^{(2)}\mathrm{S}^{(r,\,f)}_{(1,\neq,\,c)}(f_g+u_g,f_b+u_b;f_g,f_b)\\ &\quad\times \alpha_2^{-1}{}^{(2)}\mathrm{S}_{(=,\not c)}(l_g-u_g-f_g,l_b-u_b-f_b;r_g-f_g,r_b-f_b).\end{aligned}$$

The last formula contains the factor $\binom{r_b-1}{f_b-1}$ since the last step of the blue walk from the gray root of r should be $\overrightarrow{(r,v)}$. The second double walk has or it does not have blue-gray edges due to the fact that the edge (r,v) is gray-blue.

The following lemma for $\mathrm{S}^{(r,\,u)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b)$ looks like this (see also Fig. 3.10).

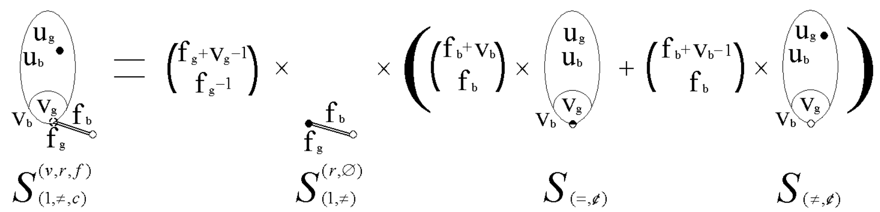

Fig. 3.10: Representation ${}^{(1)}\mathrm{Set}^{(v,\,r,\,f)}_{(1,\,\neq,\,c)}$

Lemma 3.12. *Let f_g, f_b be natural numbers and u_g, u_b be natural numbers or zeros. Then the following equalities are true*:

$$
{}^{(1)}\mathrm{S}^{(r,\,f)}_{(1,\,\neq,\,c)}(f_g+u_g, f_b+u_b; f_g, f_b) = \sum_{v_g=0}^{u_g}\sum_{v_b=0}^{u_b} {}^{(1)}\mathrm{S}^{(v,\,r,\,f)}_{(1,\,\neq,\,c)}(f_g+u_g, f_b+u_b; f_g, f_b; v_g, v_b),
$$

$$
{}^{(1)}\mathrm{S}^{(v,\,r,\,f)}_{(1,\,\neq,\,c)}(f_g+u_g, f_b+u_b; f_g, f_b; v_g, v_b) = \binom{f_g+v_g-1}{f_g-1}\frac{V_{2(f_g+f_b)}}{p^{f_g+f_b-1}}\,\alpha_1
$$
$$
\times\left(\binom{f_b+v_b}{f_b}{}^{(2)}\mathrm{S}_{(=,\,\not c)}(u_g,u_b;v_g,v_b) + \binom{f_b+v_b-1}{f_b}{}^{(2)}\mathrm{S}^{(s)}_{(\neq,\,\not c)}(u_g,u_b;v_g,v_b)\right),
$$

$$
{}^{(2)}\mathrm{S}^{(r,\,f)}_{(1,\,\neq,\,c)}(f_g+u_g, f_b+u_b; f_g, f_b) = \sum_{v_g=0}^{u_g}\sum_{v_b=0}^{u_b} {}^{(2)}\mathrm{S}^{(v,\,r,\,f)}_{(1,\,\neq,\,c)}(f_g+u_g, f_b+u_b; f_g, f_b; v_g, v_b),
$$

$$
{}^{(2)}\mathrm{S}^{(v,\,r,\,f)}_{(1,\,\neq,\,c)}(f_g+u_g, f_b+u_b; f_g, f_b; v_g, v_b) = \binom{f_g+v_g-1}{f_g-1}\frac{V_{2(f_g+f_b)}}{p^{f_g+f_b-1}}\,\alpha_2
$$
$$
\times\left(\binom{f_b+v_b}{f_b}{}^{(1)}\mathrm{S}_{(=,\,\not c)}(u_g,u_b;v_g,v_b) + \binom{f_b+v_b-1}{f_b}{}^{(1)}\mathrm{S}^{(s)}_{(\neq,\,\not c)}(u_g,u_b;v_g,v_b)\right).
$$

The blue root either matches the vertex v or it does not. In the first case, a double walk along the upper graph is from ${}^{(1)}\mathrm{S}_{(=,\,\not c)}(u_g,u_b;v_g,v_b)$, and in the second case, from ${}^{(1)}\mathrm{S}^{(s)}_{(\neq,\,\not c)}(u_g,u_b;v_g,v_b)$ (the blue walk along the upper graph should go along the vertex v since the blue root is in the upper graph, but the blue walk passes through the edge (r,v)). Different factors are in the expression in parentheses since in the second case the last step from the vertex v does not have to coincide with $\overrightarrow{(v,r)}$, but in the first case it is optional. So, we have the following formulas:

$$
{}^{(1)}\mathrm{S}^{(r,\,u)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b) = \sum_{f_g=1}^{r_g}\binom{r_g-1}{f_g-1}\sum_{f_b=1}^{r_b}\binom{r_b-1}{f_b-1}\frac{V_{2(f_g+f_b)}}{p^{f_g+f_b-1}}
$$
$$
\times\sum_{u_g=0}^{l_g-r_g}\sum_{u_b=0}^{l_b-r_b}{}^{(1)}\mathrm{S}_{(=,\,\not c)}(l_g-u_g-f_g, l_b-u_b-f_b; r_g-f_g, r_b-f_b)
$$
$$
\times\sum_{v_g=0}^{u_g}\binom{f_g+v_g-1}{f_g-1}\sum_{v_b=0}^{u_b}\left(\binom{f_b+v_b}{f_b}{}^{(2)}\mathrm{S}_{(=,\,\not c)}(u_g,u_b;v_g,v_b)\right.
$$

$$+\binom{f_b+v_b-1}{f_b}{}^{(2}\mathrm{S}^{(s)}_{(\neq,\not c)}(u_g,u_b;v_g,v_b)\Bigg), \tag{3.42}$$

$$\begin{aligned}{}^{(2)}\mathrm{S}^{(r,u)}_{(\neq,c)}(l_g,l_b;r_g,r_b) &= \sum_{f_g=1}^{r_g}\binom{r_g-1}{f_g-1}\sum_{f_b=1}^{r_b}\binom{r_b-1}{f_b-1}\frac{V_{2(f_g+f_b)}}{p^{f_g+f_b-1}}\\ &\times\sum_{u_g=0}^{l_g-r_g}\sum_{u_b=0}^{l_b-r_b}{}^{(2)}\mathrm{S}_{(=,\not c)}(l_g-u_g-f_g,l_b-u_b-f_b;r_g-f_g,r_b-f_b)\\ &\times\sum_{v_g=0}^{u_g}\binom{f_g+v_g-1}{f_g-1}\sum_{v_b=0}^{u_b}\Bigg(\binom{f_b+v_b}{f_b}{}^{(1)}\mathrm{S}_{(=,\not c)}(u_g,u_b;v_g,v_b)\\ &+\binom{f_b+v_b-1}{f_b}{}^{(1)}\mathrm{S}^{(s)}_{(\neq,\not c)}(u_g,u_b;v_g,v_b)\Bigg).\end{aligned} \tag{3.43}$$

A walk from $\mathrm{S}^{(s)}_{(\neq,\not c)}(u_g,u_b;v_g,v_b)$ either has gray edges with a gray root or it does not have. If the walk has gray edges with a gray root, then the edge (r,v) can be either blue-gray or pure gray. If the edge (r,v) is blue-gray, then it is just tree-like double walk with different roots and a blue-gray edge (r,v), i.e., it is from $\mathrm{S}^{(r)}_{(\neq,c)}(l_g,l_b;r_g,r_b)$. If the edge (r,v) is pure gray, then the blue root is in the lower graph since a blue walk passes through the gray root r. If there are no gray edges in the walk with a gray root, then there are no gray edges at all:

$$\begin{aligned}{}^{(1)}\mathrm{S}^{(s)}_{(\neq,\not c)}(l_g,l_b;r_g,r_b) &= {}^{(1)}\mathrm{S}^{(r)}_{(\neq,c)}(l_g,l_b;r_g,r_b)+\\ &+{}^{(1)}\mathrm{S}^{(s,g,d)}_{(\neq,\not c)}(l_g,l_b;r_g,r_b)+{}^{(1)}\mathrm{S}^{(s,n)}_{(\neq,\not c)}(l_g,l_b;r_g,r_b),\end{aligned} \tag{3.44}$$

$$\begin{aligned}{}^{(2)}\mathrm{S}^{(s)}_{(\neq,\not c)}(l_g,l_b;r_g,r_b) &= {}^{(2)}\mathrm{S}^{(r)}_{(\neq,c)}(l_g,l_b;r_g,r_b)+\\ &+{}^{(2)}\mathrm{S}^{(s,g,d)}_{(\neq,\not c)}(l_g,l_b;r_g,r_b)+{}^{(2)}\mathrm{S}^{(s,n)}_{(\neq,\not c)}(l_g,l_b;r_g,r_b).\end{aligned} \tag{3.45}$$

And we have the next lemma for $\mathrm{S}^{(s,g,d)}_{(\neq,\not c)}(l_g,l_b;r_g,r_b)$ (see also Fig. 3.11).

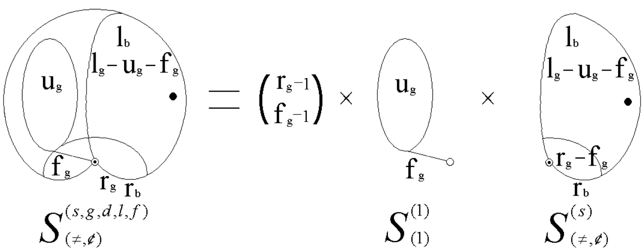

Fig. 3.11: Representation of ${}^{(1)}\mathrm{Set}^{(s,g,d,l,f)}_{(\neq,\not c)}$

Lemma 3.13. *Let l_g, l_b, r_g, r_b be natural numbers or zeros such that $l_g\ge r_g>0$ and $l_b\ge r_b>0$. Then the following equalities are true*:

$${}^{(1)}\mathrm{S}^{(s,g,d)}_{(\neq,\not c)}(l_g,l_b;r_g,r_b)=\sum_{u_g=0}^{l_g-r_g}\sum_{f_g=1}^{r_g}{}^{(1)}\mathrm{S}^{(s,g,d,l,f)}_{(\neq,\not c)}(l_g,l_b;r_g,r_b;u_g,f_g),$$

$$
\begin{aligned}
&{}^{(1)}\mathrm{S}^{(s,\,g,\,d\,,l,\,f)}_{(\neq,\,\not c)}(l_g,l_b;r_g,r_b;u_g,f_g)\\
&\qquad = \alpha_1^{-1}\binom{r_g-1}{f_g-1}{}^{(1)}\mathrm{S}^{(1)}_{(1)}(f_g+u_g,f_g)\,{}^{(1)}\mathrm{S}^{(s)}_{(\neq,\,\not c)}(l_g-u_g-f_g,l_b,r_g-f_g,r_b),\\
&{}^{(2)}\mathrm{S}^{(s,\,g,\,d)}_{(\neq,\,\not c)}(l_g,l_b;r_g,r_b) = \sum_{u_g=0}^{l_g-r_g}\sum_{f_g=1}^{r_g}{}^{(1)}\mathrm{S}^{(s,\,g,\,d\,,l,\,f)}_{(\neq,\,\not c)}(l_g,l_b;r_g,r_b;u_g,f_g),\\
&{}^{(2)}\mathrm{S}^{(s,\,g,\,d\,,l,\,f)}_{(\neq,\,\not c)}(l_g,l_b;r_g,r_b;u_g,f_g)\\
&\qquad = \alpha_2^{-1}\binom{r_g-1}{f_g-1}{}^{(2)}\mathrm{S}^{(1)}_{(1)}(f_g+u_g,f_g)\,{}^{(1)}\mathrm{S}^{(s)}_{(\neq,\,\not c)}(l_g-u_g-f_g,l_b,r_g-f_g,r_b).
\end{aligned}
$$

For proving we use Lemma 3.5:

$$
\begin{aligned}
{}^{(1)}\mathrm{S}^{(1)}_{(1)}(f_g+u_g,f_g) &= \sum_{v_g=0}^{u_g}\binom{f_g+v_g-1}{f_g-1}\frac{\alpha_1 V_{2f_g}}{p^{f_g-1}}\,{}^{(2)}\mathrm{S}^{(1)}(u_g,v_g),\\
{}^{(2)}\mathrm{S}^{(1)}_{(1)}(f_g+u_g,f_g) &= \sum_{v_g=0}^{u_g}\binom{f_g+v_g-1}{f_g-1}\frac{\alpha_2 V_{2f_g}}{p^{f_g-1}}\,{}^{(1)}\mathrm{S}^{(1)}(u_g,v_g).
\end{aligned}
$$

Therefore, the following equalities hold:

$$
\begin{aligned}
{}^{(1)}\mathrm{S}^{(s,\,g,\,d)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b) &= \sum_{f_g=1}^{r_g}\binom{r_g-1}{f_g-1}\frac{V_{2f_g}}{p^{f_g-1}}\\
&\quad\times\sum_{u_g=0}^{l_g-r_g}{}^{(1)}\mathrm{S}^{(s)}_{(\neq,\,\not c)}(l_g-u_g-f_g,l_b;r_g-f_g,r_b)\\
&\quad\times\sum_{v_g=0}^{u_g}\binom{f_g+v_g-1}{f_g-1}{}^{(2)}\mathrm{S}^{(1)}(u_g,v_g). \qquad (3.46)\\
{}^{(2)}\mathrm{S}^{(s,\,g,\,d)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b) &= \sum_{f_g=1}^{r_g}\binom{r_g-1}{f_g-1}\frac{V_{2f_g}}{p^{f_g-1}}\\
&\quad\times\sum_{u_g=0}^{l_g-r_g}{}^{(2)}\mathrm{S}^{(s)}_{(\neq,\,\not c)}(l_g-u_g-f_g,l_b;r_g-f_g,r_b)\\
&\quad\times\sum_{v_g=0}^{u_g}\binom{f_g+v_g-1}{f_g-1}{}^{(1)}\mathrm{S}^{(1)}(u_g,v_g). \qquad (3.47)
\end{aligned}
$$

If there are no gray edges, then $l_g = 0$ and $r_g = 0$. So, by the definition of ${}^{(1)}\mathrm{S}^{(s,n)}_{(\neq,\not c)}(l_g,l_b;r_g,r_b)$, the following formulas are correct:

$$
{}^{(1)}\mathrm{S}^{(s,n)}_{(\neq,\not c)}(l_g,l_b;r_g,r_b) = \delta_{l_g}\delta_{r_g}\,{}^{(1)}\mathrm{S}^{(1,s)}(l_b;r_b), \qquad (3.48)
$$

$$
{}^{(2)}\mathrm{S}^{(s,n)}_{(\neq,\not c)}(l_g,l_b;r_g,r_b) = \delta_{l_g}\delta_{r_g}\,{}^{(2)}\mathrm{S}^{(1,s)}(l_b;r_b). \qquad (3.49)
$$

The case $\mathrm{S}^{(1,\,s)}(l_b, r_b)$ also does not cause problems (see Figs. 3.12 and 3.13). We denote by b a vertex from which a blue walk first passes the gray root r. We cut the graph along the vertices of the edge (r, b). Here u_b means half of the length of the blue walk along the upper blue graph which was formed after removing the edge (r, b), and f_b means half of the length of the blue walk along the edge of (r, b). By v_b, we denote the number of steps leaving the vertex b other than $\overrightarrow{(b, r)}$. Obviously, the blue root lies in the upper graph. Again, two cases are possible: the blue root coincides with the vertex b and it does not coincide with the vertex b. In different cases different factors appear. And we have two more lemmas.

Lemma 3.14. *Let l_b, r_b be natural numbers such that $l_b \geq r_b > 0$. Then the following equalities are true*:

$$
\begin{aligned}
&{}^{(1)}\mathrm{S}^{(1,\,s)}(l_b, r_b)) = \sum_{u_b=0}^{l_b-r_b} \sum_{f_b=1}^{r_b} {}^{(1)}\mathrm{S}^{(1,\,s,\,l,\,f)}(l_b; r_b; u_b; f_b),\\
&{}^{(1)}\mathrm{S}^{(1,\,s,\,l,\,f)}(l_b; r_b; u_b; f_b\\
&\qquad = \binom{r_b-1}{f_b-1} \alpha_1^{-1}\, {}^{(1)}\mathrm{S}^{(1,\,s,\,f)}_{(1)}(f_b + u_b, f_b)\, {}^{(1)}\mathrm{S}^{(1)}(l_b - u_b - f_b, r_b - f_b),\\
&{}^{(2)}\mathrm{S}^{(1,\,s)}(l_b, r_b)) = \sum_{u_b=0}^{l_b-r_b} \sum_{f_b=1}^{r_b} {}^{(1)}\mathrm{S}^{(1,\,s,\,l,\,f)}(l_b; r_b; u_b; f_b),\\
&{}^{(2)}\mathrm{S}^{(1,\,s,\,l,\,f)}(l_b; r_b; u_b; f_b)\\
&\qquad = \binom{r_b-1}{f_b-1} \alpha_2^{-1}\, {}^{(2)}\mathrm{S}^{(1,\,s,\,f)}_{(1)}(f_b + u_b, f_b)\, {}^{(2)}\mathrm{S}^{(1)}(l_b - u_b - f_b, r_b - f_b).
\end{aligned}
$$

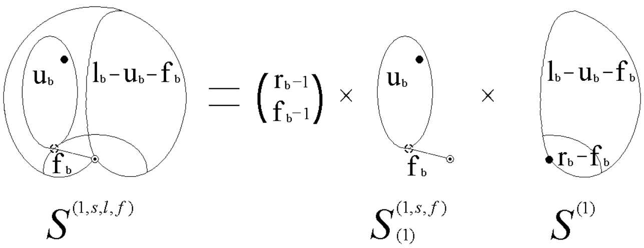

Fig. 3.12: Representation of ${}^{(1)}\mathrm{Set}^{(1,\,s,\,l,\,f)}$

Lemma 3.15. *Let f_b be a natural number, u_b be a natural number or zero. Then the following equalities are true*:

$$
\begin{aligned}
&{}^{(1)}\mathrm{S}^{(1,\,s,\,f)}_{(1)}(f_b + u_b, f_b) = \sum_{v_b=0}^{u_b} {}^{(1)}\mathrm{S}^{(1,\,s,\,v,\,f)}_{(1)}(f_b + u_b, f_b, v_b),\\
&{}^{(1)}\mathrm{S}^{(1,\,s,\,v,\,f)}_{(1)}(f_b + u_b, f_b, v_b)\\
&\qquad = \frac{\alpha_1 V_{2f_b}}{p^{f_b-1}} \left(\binom{f_b + v_b - 1}{f_b} {}^{(1)}\mathrm{S}^{(1,\,s)}(u_b, v_b) + \binom{f_b + v_b}{f_b} {}^{(1)}\mathrm{S}^{(1)}(u_b, v_b) \right),
\end{aligned}
$$

$$
{}^{(2)}\mathrm{S}^{(1,\,s,\,f)}_{(1)}(f_b+u_b,f_b)=\sum_{v_b=0}^{u_b}{}^{(2)}\mathrm{S}^{(1,\,s,\,v,\,f)}_{(1)}(f_b+u_b,f_b,v_b),
$$

$$
\begin{aligned}
&{}^{(2)}\mathrm{S}^{(1,\,s,\,v,\,f)}_{(1)}(f_b+u_b,f_b,v_b)\\
&\quad=\frac{\alpha_2 V_{2f_b}}{p^{f_b-1}}\left(\binom{f_b+v_b-1}{f_b}{}^{(2)}\mathrm{S}^{(1,\,s)}(u_b,v_b)+\binom{f_b+v_b}{f_b}{}^{(2)}\mathrm{S}^{(1)}(u_b,v_b)\right).
\end{aligned}
$$

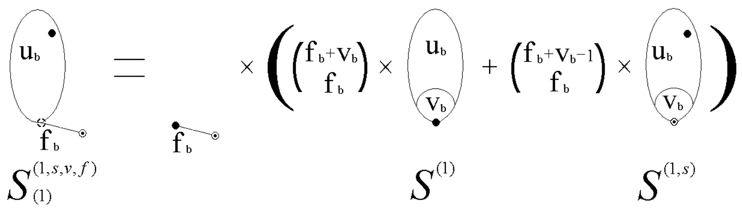

Fig. 3.13: Representation of $\mathrm{Set}^{(1,\,s,\,v,\,f)}_{(1)}$

These two lemmas imply the following equalities:

$$
\begin{aligned}
{}^{(1)}\mathrm{S}^{(1,\,s)}(l_b,r_b)&=\sum_{f_b=1}^{r_b}\binom{r_b-1}{f_b-1}\frac{V_{2f_b}}{p^{f_b-1}}\sum_{u_b=0}^{l_b-r_b}{}^{(1)}\mathrm{S}^{(1)}(l_b-u_b-f_b,r_b-f_b)\\
&\times\sum_{v_b=0}^{u_b}\left(\binom{f_b+v_b}{f_b}{}^{(2)}\mathrm{S}^{(1)}(u_b,v_b)+\binom{f_b+v_b-1}{f_b}{}^{(2)}\mathrm{S}^{(1,\,s)}(u_b,v_b)\right),
\end{aligned}
\tag{3.50}
$$

$$
\begin{aligned}
{}^{(2)}\mathrm{S}^{(1,\,s)}(l_b,r_b)&=\sum_{f_b=1}^{r_b}\binom{r_b-1}{f_b-1}\frac{V_{2f_b}}{p^{f_b-1}}\sum_{u_b=0}^{l_b-r_b}{}^{(1)}\mathrm{S}^{(2)}(l_b-u_b-f_b,r_b-f_b)\\
&\times\sum_{v_b=0}^{u_b}\left(\binom{f_b+v_b}{f_b}{}^{(1)}\mathrm{S}^{(1)}(u_b,v_b)+\binom{f_b+v_b-1}{f_b}{}^{(1)}\mathrm{S}^{(1,\,s)}(u_b,v_b)\right).
\end{aligned}
\tag{3.51}
$$

The lemmas for ${}^{(1)}\mathrm{S}^{(r,\,d)}_{(\neq,\,c)}(l_b,r_b)$ are formulated as follows.

Lemma 3.16. *Let l_g, l_b, r_g, r_b be natural numbers or zeros such that $l_g\geq r_g>0$ and $l_b\geq r_b>0$. Then the following equalities are true*:

$$
{}^{(1)}\mathrm{S}^{(r,\,d)}_{(\neq,\,c)}(l_g,l_b,r_g,r_b)=\sum_{u_g=0}^{l_g-r_g}\sum_{f_g=1}^{r_g}\sum_{u_b=0}^{l_b-r_b}\sum_{f_b=1}^{r_b}{}^{(1)}\mathrm{S}^{(r,\,d,\,l,\,f)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b;u_g,u_b;f_g,f_b),
$$

$$
{}^{(2)}\mathrm{S}^{(r,\,d)}_{(\neq,\,c)}(l_g,l_b,r_g,r_b)=\sum_{u_g=0}^{l_g-r_g}\sum_{f_g=1}^{r_g}\sum_{u_b=0}^{l_b-r_b}\sum_{f_b=1}^{r_b}{}^{(2)}\mathrm{S}^{(r,\,d,\,l,\,f)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b;u_g,u_b;f_g,f_b),
$$

$$
\begin{aligned}
&{}^{(2)}\mathrm{S}^{(r,\,d,\,l,\,f)}_{(\neq,\,c)}(l_g,l_b;r_g,r_b;u_g,u_b;f_g,f_b)\\
&\qquad=\binom{r_g-1}{f_g-1}\binom{r_b-1}{f_b}{}^{(2)}\mathrm{S}_{(1,\,=,\,c)}(f_g+u_g,f_b+u_b;f_g,f_b)\\
&\qquad\times\alpha_2^{-1}{}^{(2)}\mathrm{S}^{(s)}_{(\neq,\,\not{c})}(l_g-u_g-f_g,l_b-u_b-f_b;r_g-f_g,r_b-f_b).
\end{aligned}
$$

Lemma 3.17. *Let l_g, l_b, r_g, r_b, f_g, f_b be natural numbers and u_g, v_g be natural numbers or zeros such that $l_g \geq r_g \geq f_g > 0$ and $l_b \geq r_b \geq f_b > 0$. Then the following equalities are true*:

$$ {}^{(1)}\mathrm{S}^{(r,\,d,\,l,\,f)}_{(\neq,\,c)}(l_g, l_b; r_g, r_b; u_g, u_b; f_g, f_b) = \binom{r_g-1}{f_g-1}\binom{r_b-1}{f_b}\,{}^{(1)}\mathrm{S}_{(1,\,=,\,c)}(f_g+u_g, f_b+u_b; f_g, f_b) \times \alpha_1^{-1}\,{}^{(1)}\mathrm{S}^{(s)}_{(\neq,\,\not{c})}(l_g-u_g-f_g, l_b-u_b-f_b; r_g-f_g, r_b-f_b), $$

$$ {}^{(1)}\mathrm{S}_{(1,\,=,\,c)}(f_g+u_g, f_b+u_b; f_g, f_b) = \sum_{v_g=0}^{u_g}\sum_{v_b=0}^{u_b} {}^{(1)}\mathrm{S}^{(v)}_{(1,\,=,\,c)}(f_g+u_g, f_b+u_b; f_g, f_b; v_g, v_b), $$

$$ {}^{(1)}\mathrm{S}^{(v)}_{(1,\,=,\,c)}(f_g+u_g, f_b+u_b; f_g, f_b; v_g, v_b) = \binom{f_g+v_g-1}{f_g-1}\binom{f_b+v_b-1}{f_b-1} \times \alpha_1 \frac{V_{2(f_g+f_b)}}{p^{f_g+f_b-1}}\,{}^{(2)}\mathrm{S}_{(=,\,\not{c})}(u_g, u_b; v_g, v_b), $$

$$ {}^{(2)}\mathrm{S}_{(1,\,=,\,c)}(f_g+u_g, f_b+u_b; f_g, f_b) = \sum_{v_g=0}^{u_g}\sum_{v_b=0}^{u_b} {}^{(2)}\mathrm{S}^{(v)}_{(1,\,=,\,c)}(f_g+u_g, f_b+u_b; f_g, f_b; v_g, v_b), $$

$$ {}^{(2)}\mathrm{S}^{(v)}_{(1,\,=,\,c)}(f_g+u_g, f_b+u_b; f_g, f_b; v_g, v_b) = \binom{f_g+v_g-1}{f_g-1}\binom{f_b+v_b-1}{f_b-1} \times \alpha_2 \frac{V_{2(f_g+f_b)}}{p^{f_g+f_b-1}}\,{}^{(1)}\mathrm{S}_{(=,\,\not{c})}(u_g, u_b; v_g, v_b). $$

Lemmas 3.16 and 3.17 imply the following equalities:

$$ {}^{(1)}\mathrm{S}^{(r,\,d)}_{(\neq,\,c)}(l_g, l_b; r_g, r_b) = \sum_{f_g=1}^{r_g}\binom{r_g-1}{f_g-1}\sum_{f_b=1}^{r_b}\binom{r_b-1}{f_b}\frac{V_{2(f_g+f_b)}}{p^{f_g+f_b-1}} \times \sum_{u_g=0}^{l_g-r_g}\sum_{u_b=0}^{l_b-r_b} {}^{(1)}\mathrm{S}^{(s)}_{(\neq,\,\not{c})}(l_g-u_g-f_g, l_b-u_b-f_b; r_g-f_g, r_b-f_b) \times \sum_{v_g=0}^{u_g}\binom{f_g+v_g-1}{f_g-1}\sum_{v_b=0}^{u_b}\binom{f_b+v_b-1}{f_b-1}\,{}^{(2)}\mathrm{S}_{(=,\,\not{c})}(u_g, u_b; v_g, v_b), \tag{3.52} $$

$$ {}^{(2)}\mathrm{S}^{(r,\,d)}_{(\neq,\,c)}(l_g, l_b; r_g, r_b) = \sum_{f_g=1}^{r_g}\binom{r_g-1}{f_g-1}\sum_{f_b=1}^{r_b}\binom{r_b-1}{f_b}\frac{V_{2(f_g+f_b)}}{p^{f_g+f_b-1}} \times \sum_{u_g=0}^{l_g-r_g}\sum_{u_b=0}^{l_b-r_b} {}^{(2)}\mathrm{S}^{(s)}_{(\neq,\,\not{c})}(l_g-u_g-f_g, l_b-u_b-f_b; r_g-f_g, r_b-f_b) \times \sum_{v_g=0}^{u_g}\binom{f_g+v_g-1}{f_g-1}\sum_{v_b=0}^{u_b}\binom{f_b+v_b-1}{f_b-1}\,{}^{(1)}\mathrm{S}_{(=,\,\not{c})}(u_g, u_b; v_g, v_b). \tag{3.53} $$

The system (3.9)–(3.12), (3.23)–(3.52) is recursive since by each essential step we decrease the total length of the double (single) walk. For unambiguous solvability, it is necessary to impose on the system the following initial conditions:

$$^{(1)}\mathrm{S}^{(1)}(0,x) = \delta_x\alpha_1, \qquad ^{(2)}\mathrm{S}^{(1)}(0,x) = \delta_x\alpha_2 \tag{3.54}$$

$$^{(1)}\mathrm{S}^{(1,\,s)}(0,x) = 0, \qquad ^{(2)}\mathrm{S}^{(1,\,s)}(0,x) = 0. \tag{3.55}$$

Let x, y, z, w be natural numbers or zeros such that at least one of the next inequalities $z \geq x > 0$ $w \geq y > 0$ is violated. Then the following equalities hold:

$$^{(1)}\mathrm{S}^{(g)}_{(=,\,c)}(z,w;x,y) = 0, \qquad ^{(2)}\mathrm{S}^{(g)}_{(=,\,c)}(z,w;x,y) = 0, \tag{3.56}$$

$$^{(1)}\mathrm{S}^{(r)}_{(=,\,c)}(z,w;x,y) = 0, \qquad ^{(2)}\mathrm{S}^{(r)}_{(=,\,c)}(z,w;x,y) = 0, \tag{3.57}$$

$$^{(1)}\mathrm{S}^{(r,\,u)}_{(\neq,\,c)}(z,w;x,y) = 0, \qquad ^{(2)}\mathrm{S}^{(r,\,u)}_{(\neq,\,c)}(z,w;x,y) = 0, \tag{3.58}$$

$$^{(1)}\mathrm{S}^{(r,d)}_{(\neq,\,c)}(z,w;x,y) = 0, \qquad ^{(2)}\mathrm{S}^{(r,d)}_{(\neq,\,c)}(z,w;x,y) = 0, \tag{3.59}$$

$$^{(1)}\mathrm{S}^{(s,\,g,\,d)}_{(\neq,\,\not c)}(z,w;x,y) = 0, \qquad ^{(2)}\mathrm{S}^{(s,\,g,\,d)}_{(\neq,\,\not c)}(z,w;x,y) = 0. \tag{3.60}$$

Let x, y, z, w be natural numbers or zeros such that at least one of the next inequalities $z \geq x > 0$ $w \geq y \geq 0$ is violated. Then the following equalities hold:

$$^{(1)}\mathrm{S}^{(g,\,u)}_{(\neq,\,c)}(z,w;x,y) = 0, \qquad ^{(2)}\mathrm{S}^{(g,\,u)}_{(\neq,\,c)}(z,w;x,y) = 0, \tag{3.61}$$

$$^{(1)}\mathrm{S}^{(g,d)}_{(\neq,\,c)}(z,w;x,y) = 0, \qquad ^{(2)}\mathrm{S}^{(g,d)}_{(\neq,\,c)}(z,w;x,y) = 0. \tag{3.62}$$

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Асимптотика кореляторів розбавлених дводольних випадкових графів

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Досліджено асимптотику кореляційних функцій дводольних розбавлених зважених випадкових $N \times N$ матриць. Показано, що основний член кореляційної функції k-го та m-го моментів інтегральної щільності станів дорівнює $N^{-1}n_{k,m}$. Одержано замкнуту систему рекурентних співвідношень для коефіцієнтів $\{n_{k,m}\}_{k,m=1}^{\infty}$.

Ключові слова: дводольний розбавлений випадковий граф, корелятор моментів, асимптотика, основний член, система рекурентних рівнянь