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Sobolev's Type Optimal Topology in the Problem of Exact Observability for Hilbert Space Dynamical Systems Connected with Riesz Basis of Divided Differences

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This paper considers the problem of exact observability of a general class of linear distributed parameter systems in Hilbert spaces connected to Riesz basis properties of some families of exponential functions and the divided differences of those functions. Under some assumptions on asymptotic spectral analysis of the differential operator of the system, the conditions of exact observability are stated in the form of exact observable spaces being the direct sum of some specific Sobolev spaces. The main result consists of proving the optimality of these subspaces of observable states. The result was based on advanced non-harmonic analysis approach connected to the unusual fact that time-space Riesz basis does not consist only of exponential functions but also contains divided differences of these functions.

Key words: exact observability, partial differential equations, unbounded operators, Riesz basis of divided differences, optimality of observability subspaces

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1. Introduction and problem statement

Control theory research is commonly divided into some areas, such as controllability, stability, etc. One of the main areas is observability, which is aimed to describe the possibility of reconstructing the initial state of the system, and in consequence the whole trajectory of the system, basing on an incomplete information of the current state of the system. The problem of exact observability of distributed parameter systems in Hilbert spaces as opposed to approximate observability of such systems is still a subject of contemporary investigations. In recent years many different classes of systems and many different approaches where considered (see [6–9, 11, 13, 14, 16, 23, 25, 26] and references therein). In particular, it is worth noting that in [10] a setting, which is similar to that specified in this paper, was considered with a strong assumption of the operator of motion ($\mathcal{A}$) being a diagonal operator. Unfortunately, this interesting approach could not be applied in our case even after extending to block diagonal class of operators.

Lately in [20], a problem of exact observability of a specific class of systems, connected to Riesz basis of exponential functions and their divided differences, was considered. This paper extends those considerations providing not only a wider observability subspace of the systems in question, but also proving their optimality. Moreover, it appeared that this optimal subspace is not just a regular Sobolev space H^p, but it is a direct sum of such spaces. The authors are not acquainted with other works concerning sharpness of observability subspaces in this form.

More precisely, in this paper we consider a general class of dynamical systems with observation of the form

$$\begin{cases} \dot{Z} = \mathcal{A}Z, \\ Y = \mathcal{C}Z, \end{cases} \tag{1.1}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a C_0-semigroup $\mathcal{T}(t)$, $\mathcal{C} : \mathcal{H} \to \mathbb{C}$ is a linear (unbounded) observation operator, $\mathcal{H}$ is a Hilbert space. In addition, we assume that our observation is admissible [15, 20–22], that is,

Definition 1.1. The operator $\mathcal{C}$ is called an *admissible operator* for $\mathcal{T}(t)$ if, for some $T > 0$ (and hence for all $T > 0$), there exists a constant $\kappa \geq 0$ such that

$$\int_0^T |\mathcal{C}\mathcal{T}(t)z_0|^2 \, dt \leq \kappa^2 \|z_0\|^2 \quad \text{for all } z_0 \in D(\mathcal{A}).$$

For infinite-dimensional systems there are different concepts of observability notions [22]. Here we focus on the following

Definition 1.2 (see [15, 21, 22])**.** Let $\mathcal{K} : \mathcal{H} \to \mathcal{Y}$ be the output operator

$$z_0 \mapsto \mathcal{K}z_0 = \mathcal{C}\mathcal{T}(t)z_0,$$

where $\mathcal{Y}$ is a Hilbert space of time-dependent functions on the interval $(0, T)$. System (1.1) is said to be *approximately observable in time* T (or *observable in time* T) if $\ker \mathcal{K} = \{0\}$ and $\mathcal{Y} - \mathcal{H}$ *is exactly observable in time* T (or $\mathcal{Y} - \mathcal{H}$ *is continuously observable in time* T) if

$$\|\mathcal{K}z_0\|_{\mathcal{Y}}^2 \geq \kappa^2 \|z_0\|_{\mathcal{H}}^2, \quad \text{for all } z_0 \in \mathcal{H}, \tag{1.2}$$

for some constant $\kappa > 0$.

In general, approximate observability guarantees the possibility of reconstruction of the initial state of the system, and thus of the whole trajectory, using knowledge of the output. Exact observability means that we can find the initial and final states from the given output with infinitesimal precision, that is, for any convergent sequence of pairwise different initial states the resulting outputs are convergent and pairwise different as well. Of course, exact observability implies approximate observability, but not the other way around [22]. In this paper, we focus on analysis of exact observability notion. Namely, our goal is to consider the exact observability of system (1.1) with observation $\mathcal{C}$ under some assumptions we impose on the system in question:

(A1) The operator $\mathcal{A}$ has an orthonormal complete sequence of eigenelements $\{Y_k\}_{k\in\mathbb{Z}}$ with corresponding eigenvalues $\pm i\mu_k$, where $\mu_k \asymp k$. ($\alpha_k \asymp \beta_k$ iff $|\alpha_k| \leq C|\beta_k|$ and $|\beta_k| \leq C|\alpha_k|$ for some C and for sufficiently large k, see [5, p. 442].)

(A2) There exists an increasing sequence $\{k_n\}_{n\in\mathbb{Z}}$ of indices such that eigenvalues $i\mu_{k_n}$ and $i\mu_{k_n-1}$ are approaching each other with a certain speed, $|\mu_{k_n} - \mu_{k_n-1}| \asymp \frac{1}{|k_n|}$.

(A3) For some $T_0 > 0$, the system

$$\mathcal{E} = \left\{e^{i\mu_k t}\right\}_{k\in\mathbb{Z}\setminus\{k_n\}} \cup \left\{\frac{e^{i\mu_k t} - e^{i\mu_{k-1} t}}{\mu_k - \mu_{k-1}}\right\}_{k\in\{k_n\}} \tag{1.3}$$

is a Riesz basis for $L^2(0, T_0)$.

(A4) $|C_k| \asymp \frac{1}{|k|}$, where $C_k := \mathcal{C}Y_k$.

A similar problem of exact observability has been solved in [20] by using less general assumptions on the system in question. It is worth noting that exact observability depends heavily on the choice of topology in the space as opposed to approximate observability [15]. There are two possible approaches to the study of exact observability of system (1.1). One can use a weaker topology on the right-hand side of inequality (1.2) (using, for example, the domain $D(\mathcal{A})$ or its power $D(\mathcal{A}^n)$ [22]), or a stronger topology on the left-hand side of this inequality. We continue to use the second approach in our research. Namely, using the results of [20], one can prove that the considered system is not exactly observable in default topologies setting (more precisely, neither is $L^2(0,T) - \mathcal{H}$ nor $H^1(0,T) - \mathcal{H}$ exactly observable in any time $T > 0$) and one can find a stronger topology for state observation for which the system becomes exactly observable (the system is $H^2(0,T) - \mathcal{H}$ exactly observable in any time $T \geq T_0$).

The above stated conditions (A1), (A2), (A4) imply (together with (A3)) details of the previous result. Tinkering only with conditions (A1), (A2), and (A4) would change the degree of the result (c.f. [17]), i.e., the system would not be $H^p - \mathcal{H}$ exact observable and would be $H^{p+1} - \mathcal{H}$ exact observable. Condition (A3) plays a crucial role in our further analysis. It turned out that the previously obtained degrees of observable/not observable topologies are not sharp. (Notice that this condition is stated in the form suitable in our analysis, based on complex analysis approach [2]; other possible formulations can be found in literature (cf. [3]).) Exploiting this Riesz basis of divided differences properties [1], we will find a subspace $\mathcal{O}$, $H^1 \subsetneq \mathcal{O} \subsetneq H^2$, giving the sharp $\mathcal{O} - \mathcal{H}$ exact observability result.

Example 1.3. Conditions (A1)–(A4) are satisfied in one of the models of vibrating Timoshenko beams. (For more details on Timoshenko beam theory, see [4].) Namely, following [20], we can provide a possible class of systems to which our results can be applied. Consider the following example of a vibrating Timoshenko beam system governed by partial differential equations of the form

$$\begin{cases} \ddot{w}(x,t) - w''(x,t) - \xi'(x,t) = 0, \\ \ddot{\xi}(x,t) - \gamma^2\xi''(x,t) + w'(x,t) + \xi(x,t) = 0 \end{cases} \tag{1.4}$$

for $x \in (0,1)$ and $t > 0$, where $\dot{w} = w_t$, $w' = w_x$, $\dot{\xi} = \xi_t$ and $\xi' = \xi_x$, and $\gamma^2 > 1$ depends on physical properties of the material of the beam (see [12, 18, 19]). Here, $w(x,t)$ denotes the deflection of the center line of the beam, $\xi(x,t)$ is the rotation angle of the cross section area and x, t stand respectively for the position along the beam and time. We assume the beam to be clamped at $x = 0$ and have a free end at $x = 1$. From that we have boundary conditions of the form

$$\begin{cases} w(0,t) = \xi(0,t) = 0, \\ w'(1,t) + \xi(1,t) = 0, \\ \xi'(1,t) = 0 \end{cases} \tag{1.5}$$

for $t > 0$. We observe the deflection of the center line of the beam at the free end, i.e.,

$$Y = \mathcal{C} \begin{pmatrix} w \\ \xi \\ \dot{w} \\ \dot{\xi} \end{pmatrix} = w(1,\cdot). \tag{1.6}$$

For such systems, one can use our results, stated below, presenting the optimal observability subspace after a suitable choice of Hilbert spaces and differential operators (cf. [20]).

2. $\mathcal{O} - \mathcal{H}$ exact observability

In this section, we present the first result of the paper, i.e., the precise exact observability analysis of system (1.1). We conduct the analysis for time $T = T_0$, for $T > T_0$ the result is classical. (The system exactly observable in time T_0 remains exactly observable for times $T \geq T_0$ [22].) At the beginning, we define the set of optimal observable states $\mathcal{O}$, which is necessary in the proof of our theorem. Due to the character of the Riesz basis (1.3) of $L^2(0,T_0)$, the following observable subspace is more natural in this setting than the previously considered $H^p(0,T_0)$ spaces.

Definition 2.1. Let us define the following decomposition of

$$L^2(0,T_0) = V_1 \oplus V_2,$$

where $V_1 = \overline{\operatorname{lin}\{e^{i\mu_k t}\}}_{k\in\mathbb{Z}\setminus\{k_n\}}$, $V_2 = \overline{\operatorname{lin}\left\{\frac{e^{i\mu_k t}-e^{i\mu_{k-1} t}}{\mu_k-\mu_{k-1}}\right\}}_{k\in\{k_n\}}$. The subspace $\mathcal{O}$ is called an *optimal observable state space* and is given by

$$\mathcal{O} = \Big\{ f \in L^2(0,T_0) \;\Big|\; f = f_1 + f_2,\ f_1 \in V_1 \cap H^1(0,T_0), \\ f_2 \in V_2 \cap H^2(0,T_0) \text{ and } \|f\|^2_{\mathcal{O}} < \infty \Big\},$$

with accompanying norm

$$\|f\|^2_{\mathcal{O}} = \|f\|^2_{L^2(0,T_0)} + 2\left\|\frac{d}{dt}f_1\right\|^2_{L^2(0,T_0)} + 2\left\|\frac{d^2}{dt^2}f_2\right\|^2_{L^2(0,T_0)}. \tag{2.1}$$

Now we proceed to the main result of this section. We prove that conditions (A1)–(A4) imply $\mathcal{O}-\mathcal{H}$ exact observability in time T_0 of system (1.1).

Theorem 2.2. *Assume that conditions (A1)–(A4) are satisfied. Then system (1.1) is $\mathcal{O}-\mathcal{H}$ exactly observable in time T_0.*

Remark 2.3. Notice that T_0 is the minimal possible time, where the exact observability phenomenon can hold (see Proposition 1 and Corollary 1 in [20]).

Proof. We start the proof with an estimation of the norm of the left-hand side of inequality (1.2), i.e., $\|\mathcal{K}z_0\|^2_{\mathcal{O}}$. In order to do that, we decompose an arbitrary state vector in the eigenspace (A1) and then rewrite it in the Riesz basis (1.3) from (A3). The arbitrary state $z_0 \in \mathcal{H}$ can be written in the normalized eigenvector space in the following form:

$$z_0 = \sum_{k\in\mathbb{Z}} \alpha_k Y_k, \tag{2.2}$$

where $\alpha_k = \langle z_0, Y_k\rangle \in \ell^2$. Further, after using the form of the operator $\mathcal{K}$, we obtain

$$\mathcal{K}z_0 = \mathcal{C}\mathcal{T}(t)z_0 = \sum_{k\in\mathbb{Z}} \alpha_k e^{i\mu_k t} C_k.$$

Then we decompose it in the Riesz basis (1.3),

$$\sum_{k\in\mathbb{Z}} \alpha_k e^{i\mu_k t} C_k = \sum_{k\in\mathbb{Z}\setminus\{k_n\}} \beta_k e^{i\mu_k t} + \sum_{k\in\{k_n\}} \gamma_k \frac{e^{i\mu_k t} - e^{i\mu_{k-1}t}}{\mu_k - \mu_{k-1}}, \tag{2.3}$$

where the coefficients are given by the formulas

$$\beta_k = C_k\alpha_k \quad \text{for } k \neq k_n - 1, k_n$$

and

$$\begin{aligned} \beta_{k_n-1} &= C_{k_n}\alpha_{k_n} + C_{k_n-1}\alpha_{k_n-1}, \\ \gamma_{k_n} &= (\mu_{k_n} - \mu_{k_n-1})\, C_{k_n-1}\alpha_{k_n-1} \end{aligned}$$

for remaining cases. Now we estimate from below the left-hand side of the observability inequality (1.2). For simplicity of further calculations, we present the norm in the space $\mathcal{O}$ from (2.1) in an equivalent way obtained using the parallelogram law. Hence, the norm of $\mathcal{K}z_0$ can be expressed as

$$\begin{aligned} \|f\|^2_{\mathcal{O}} &= \|f\|^2_{L^2(0,T_0)} + 2\left\|\frac{d}{dt}f_1\right\|^2_{L^2(0,T_0)} + 2\left\|\frac{d^2}{dt^2}f_2\right\|^2_{L^2(0,T_0)} \\ &= \|f\|^2_{L^2(0,T_0)} + \left\|\frac{d}{dt}f_1 - \frac{d^2}{dt^2}f_2\right\|^2_{L^2(0,T_0)} + \left\|\frac{d}{dt}f_1 + \frac{d^2}{dt^2}f_2\right\|^2_{L^2(0,T_0)}, \end{aligned} \tag{2.4}$$

where

$$f(t) = \mathcal{C}\mathcal{T}(t)z_0 = \sum_{k\in\mathbb{Z}\setminus\{k_n\}} \beta_k e^{i\mu_k t} + \sum_{k\in\{k_n\}} \gamma_k \frac{e^{i\mu_k t} - e^{i\mu_{k-1}t}}{\mu_k - \mu_{k-1}},$$

$$f_1(t) = \sum_{k\in\mathbb{Z}\setminus\{k_n\}} \beta_k e^{i\mu_k t},$$

$$f_2(t) = \sum_{k\in\{k_n\}} \gamma_k \frac{e^{i\mu_k t} - e^{i\mu_{k-1} t}}{\mu_k - \mu_{k-1}}.$$

Now we present necessary calculations for the second and third terms of (2.4),

$$\frac{d}{dt} f_1 = \sum_{k\in\mathbb{Z}\setminus\{k_n\}} i\mu_k \beta_k e^{i\mu_k t},$$

$$\begin{aligned}\frac{d^2}{dt^2} f_2 &= \sum_{k\in\{k_n\}} (-\mu_k)\,\gamma_k e^{i\mu_k t} + \sum_{k\in\{k_n\}} (-\mu_{k-1})\,\gamma_k \frac{\mu_k e^{i\mu_k t} - \mu_{k-1} e^{i\mu_{k-1} t}}{\mu_k - \mu_{k-1}}\\ &= \sum_{k\in\{k_n\}} (-\mu_k)\,\gamma_k e^{i\mu_k t} + \sum_{k\in\{k_n\}} (-\mu_{k-1})\,\gamma_k e^{i\mu_k t}\\ &\qquad + \sum_{k\in\{k_n\}} \left(-\mu_{k-1}^2\right)\gamma_k \frac{e^{i\mu_k t} - e^{i\mu_{k-1} t}}{\mu_k - \mu_{k-1}}.\end{aligned}$$

After rearranging the terms, we obtain

$$\begin{aligned}\frac{d^2}{dt^2} f_2 &= \sum_{k\in\{k_n\}} (-\mu_k - \mu_{k-1})\,\gamma_k e^{\mu_{k-1} t} + \sum_{k\in\{k_n\}} \left(-\mu_k^2\right)\gamma_k \frac{e^{\mu_k t} - e^{\mu_{k-1} t}}{\mu_k - \mu_{k-1}}\\ &= \sum_{k\in\{k_n-1\}} (-\mu_{k+1} - \mu_k)\,\gamma_{k+1} e^{\mu_k t} + \sum_{k\in\{k_n\}} \left(-\mu_k^2\right)\gamma_k \frac{e^{\mu_k t} - e^{\mu_{k-1} t}}{\mu_k - \mu_{k-1}}.\end{aligned}$$

Then,

$$\begin{aligned}\frac{d}{dt} f_1 \pm \frac{d^2}{dt^2} f_2 &= \sum_{k\in\mathbb{Z}\setminus\{k_n\}} \mu_k \beta_k e^{i\mu_k t} \mp \sum_{k\in\{k_n-1\}} (\mu_{k+1} + \mu_k)\,\gamma_{k+1} e^{i\mu_k t}\\ &\quad \mp \sum_{k\in\{k_n\}} \mu_k^2 \gamma_k \frac{e^{i\mu_k t} - e^{i\mu_{k-1} t}}{\mu_k - \mu_{k-1}}\\ &= \sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}} \mu_k \beta_k e^{i\mu_k t} + \sum_{k\in\{k_n-1\}} \left(\mu_k \beta_k \mp (\mu_{k+1} + \mu_k)\,\gamma_{k+1}\right) e^{i\mu_k t}\\ &\quad \mp \sum_{k\in\{k_n\}} \mu_k^2 \gamma_k \frac{e^{i\mu_k t} - e^{i\mu_{k-1} t}}{\mu_k - \mu_{k-1}}.\end{aligned} \tag{2.5}$$

Let us recall that if the family $\{\varphi_k\}$ is a Riesz basis, then there exist constants $m, M > 0$ such that for any sequence $(x_k) \in \ell^2$ one has

$$m \sum |x_k|^2 \le \left\| \sum x_k \varphi_k \right\|^2 \le M \sum |x_k|^2 \tag{2.6}$$

(see, e.g., [24]). Using (2.3), (2.5) and (2.6), we obtain an estimation from below for (2.4),

$$\|\mathcal{C}\mathcal{T}(t) z_0\|_{\mathcal{O}}^2 \ge m \left(\sum_{k\in\mathbb{Z}\setminus\{k_n\}} |\beta_k|^2 + \sum_{k\in\{k_n\}} |\gamma_k|^2 \right)$$

$$+ m \left(2 \sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}} |\mu_k\beta_k|^2 + \sum_{k\in\{k_n-1\}} |\mu_k\beta_k + (\mu_{k+1}+\mu_k)\,\gamma_{k+1}|^2 \right.$$
$$\left. + \sum_{k\in\{k_n-1\}} |\mu_k\beta_k - (\mu_{k+1}+\mu_k)\,\gamma_{k+1}|^2 + 2 \sum_{k\in\{k_n\}} \left|\mu_k^2\gamma_k\right|^2 \right),$$

in particular,

$$\|\mathcal{C}\mathcal{T}(t)z_0\|_{\mathcal{O}}^2 \geq m \left(2 \sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}} |\mu_k\beta_k|^2 + \sum_{k\in\{k_n-1\}} |\mu_k\beta_k + (\mu_{k+1}+\mu_k)\,\gamma_{k+1}|^2 \right.$$
$$\left. + \sum_{k\in\{k_n-1\}} |\mu_k\beta_k - (\mu_{k+1}+\mu_k)\,\gamma_{k+1}|^2 + 2 \sum_{k\in\{k_n\}} \left|\mu_k^2\gamma_k\right|^2 \right). \tag{2.7}$$

Now we estimate the right-hand side of inequality (1.2). Let us consider the norm of the arbitrary state $z_0 \in \mathcal{H}$. Due to the normalized expansion (2.2)), it is obvious that the norm of the state z_0 is given by

$$\|z_0\|_{\mathcal{H}}^2 = \sum_{k\in\mathbb{Z}} |\alpha_k|^2.$$

From (2.3), we can derive the formulas for the coefficients α_k:

$$\alpha_k = \frac{1}{C_k}\beta_k \quad \text{for } k \neq k_n - 1, k_n \tag{2.8a}$$

and

$$\alpha_{k_n-1} = \frac{1}{C_{k_n-1}} \left(\beta_{k_n-1} - \frac{1}{\mu_{k_n} - \mu_{k_n-1}}\gamma_{k_n} \right), \tag{2.8b}$$
$$\alpha_{k_n} = \frac{1}{C_{k_n}} \frac{1}{\mu_{k_n} - \mu_{k_n-1}}\gamma_{k_n} \tag{2.8c}$$

for remaining cases. Then the norm of the state z_0 can be presented as

$$\begin{aligned}
\|z_0\|^2 &= \sum_{k\in\mathbb{Z}} |\alpha_k|^2 \\
&= \sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}} \left|\frac{1}{C_k}\beta_k\right|^2 + \sum_{k\in\{k_n-1\}} \left|\frac{1}{C_k}\left(\beta_k - \frac{1}{\mu_{k+1}-\mu_k}\gamma_{k+1}\right)\right|^2 \\
&\quad + \sum_{k\in\{k_n\}} \left|\frac{1}{C_k}\frac{1}{\mu_k - \mu_{k-1}}\gamma_k\right|^2 \\
&\leq \sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}} \frac{1}{|C_k|^2}|\beta_k|^2 + \sum_{k\in\{k_n-1\}} 2\frac{1}{|C_k|^2}|\beta_k|^2 \\
&\quad + \sum_{k\in\{k_n\}} \left(2\frac{1}{|C_{k-1}|^2} + \frac{1}{|C_k|^2}\right)\frac{1}{|\mu_k - \mu_{k-1}|^2}|\gamma_k|^2,
\end{aligned} \tag{2.9}$$

where the sequences

$$\left(\frac{\frac{1}{|C_k|^2}}{|\mu_k|^2}\right)_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}}, \quad \left(\frac{\frac{1}{|C_k|^2}}{|\mu_k|^2}\right)_{k\in\{k_n-1\}}$$

and

$$\left(\frac{\frac{1}{|\mu_k-\mu_{k-1}|^2}\left(2\frac{1}{|C_{k-1}|^2}+\frac{1}{|C_k|^2}\right)}{\left|\mu_k^2\right|^2}\right)_{k\in\{k_n\}}$$

are bounded, so there exists a constant c given by

$$c^2 = \max\left\{\frac{1}{2}\sup_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}}\frac{\frac{1}{|C_k|^2}}{|\mu_k|^2}, \sup_{k\in\{k_n-1\}}\frac{\frac{1}{|C_k|^2}}{|\mu_k|^2},\right.$$
$$\left.\frac{1}{2}\sup_{k\in\{k_n\}}\frac{\frac{1}{|\mu_k-\mu_{k-1}|^2}\left(2\frac{1}{|C_{k-1}|^2}+\frac{1}{|C_k|^2}\right)}{\left|\mu_k^2\right|^2}\right\}.$$

Continuing to estimate the norm of the state z_0 of (2.9), we obtain

$$\begin{aligned}
\|z_0\|^2 \le c^2 &\left(2\sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}}|\mu_k|^2|\beta_k|^2\right.\\
&\left.+\,2\sum_{k\in\{k_n-1\}}|\mu_k|^2|\beta_k|^2+2\sum_{k\in\{k_n\}}\left|\mu_k^2\right|^2|\gamma_k|^2\right)\\
= c^2 &\left(2\sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}}|\mu_k|^2|\beta_k|^2\right.\\
&\left.+\frac{1}{2}\sum_{k\in\{k_n-1\}}|2\mu_k\beta_k|^2+2\sum_{k\in\{k_n\}}\left|\mu_k^2\right|^2|\gamma_k|^2\right)\\
= c^2 &\left(2\sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}}|\mu_k|^2|\beta_k|^2\right.\\
&\quad+\frac{1}{2}\sum_{k\in\{k_n-1\}}|\mu_k\beta_k+(\mu_{k+1}+\mu_k)\gamma_{k+1}+\mu_k\beta_k-(\mu_{k+1}+\mu_k)\gamma_{k+1}|^2\\
&\quad\left.+\,2\sum_{k\in\{k_n\}}\left|\mu_k^2\right|^2|\gamma_k|^2\right)\\
\le c^2 &\left(2\sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}}|\mu_k|^2|\beta_k|^2+\sum_{k\in\{k_n-1\}}|\mu_k\beta_k+(\mu_{k+1}+\mu_k)\gamma_{k+1}|^2\right.
\end{aligned}$$

$$+\sum_{k\in\{k_n-1\}}\left|\mu_k\beta_k-(\mu_{k+1}+\mu_k)\,\gamma_{k+1}\right|^2+2\sum_{k\in\{k_n\}}\left|\mu_k^2\right|^2|\gamma_k|^2\Bigg). \quad (2.10)$$

Let $\kappa^2=\frac{m}{c^2}$. Combining estimations (2.7) and (2.10), we obtain

$$\begin{aligned}\kappa^2\|z_0\|^2\le m\Bigg(&2\sum_{k\in\mathbb{Z}\setminus\{k_n-1,k_n\}}c_1^2|\mu_k|^2|\beta_k|^2\\&+\sum_{k\in\{k_n-1\}}c_2^2\left|\mu_k\beta_k+(\mu_{k+1}+\mu_k)\,\gamma_{k+1}\right|^2\\&+\sum_{k\in\{k_n-1\}}c_2^2\left|\mu_k\beta_k-(\mu_{k+1}+\mu_k)\,\gamma_{k+1}\right|^2+2\sum_{k\in\{k_n\}}c_3^2\left|\mu_k^2\right|^2|\gamma_k|^2\Bigg)\\ \le{}&\|\mathcal{C}\mathcal{T}(t)z_0\|^2_{\mathcal{O}},\end{aligned}$$

which means that system (1.1) is $\mathcal{O}-\mathcal{H}$ exactly observable for time $T=T_0$. $\square$

3. On the sharpness of the $\mathcal{O}-\mathcal{H}$ exact observability conditions

Let $T=T_0$. Comparing the previous results from [20], namely $H^2(0,T)-\mathcal{H}$ exact observability, and Theorem 2.2, that is, $\mathcal{O}-\mathcal{H}$ exact observability, we see that Theorem 2.2 is a significant improvement in our analysis. Now we are to prove that the estimation of the norm (2.1) cannot be improved any more. To this end, we will show that any further weakening of the left-hand side topology of (1.2) makes system (1.1) be not exactly observable anymore, in other words, it lacks of $\mathcal{N}-\mathcal{H}$ exact observability.

Firstly, we introduce an operator that allows us to describe the elements of space $\mathcal{O}$ in terms of convergence of series.

Definition 3.1. Let us recall that an arbitrary element $f\in L^2(0,T_0)$ can be decomposed in the Riesz basis (1.3) (see condition (A3)) as

$$f(t)=\sum_{k\in\mathbb{Z}\setminus\{k_n\}}\beta_k e^{i\mu_k t}+\sum_{k\in\{k_n\}}\gamma_k\frac{e^{i\mu_k t}-e^{i\mu_{k-1}t}}{\mu_k-\mu_{k-1}}.$$

Let $\mathcal{S}:L^2(0,T)\to\ell^2$ be a coefficient operator defined by

$$(\mathcal{S}(f))_k=\begin{cases}\beta_k & \text{if } k\in\mathbb{Z}\setminus\{k_n\},\\ \gamma_k & \text{if } k\in\{k_n\},\end{cases}$$

where $(\cdot)_k$ denotes the k-th coefficient of a given sequence. The set $\widetilde{\mathcal{O}}$ is called an *optimal coefficient observable state space* and is given by

$$\widetilde{\mathcal{O}}=\left\{f\in L^2(0,T_0)\;\middle|\;\|f\|^2_{\widetilde{\mathcal{O}}}<\infty\right\}$$

with accompanying norm

$$\begin{aligned}\|f\|_{\widetilde{\mathcal{O}}}^2 &= \|K_1(\mathcal{S}(f_1))\|_{\ell^2}^2 + \|K_2(\mathcal{S}(f_2))\|_{\ell^2}^2 \\ &= \sum_{k\in\mathbb{Z}\setminus\{k_n\}} |k|^2 |\beta_k|^2 + \sum_{k\in\{k_n\}} \left|k^2\right|^2 |\gamma_k|^2,\end{aligned}$$

where $K_p : \ell^2 \to \ell^2$, $K_p(\alpha_k) = (k^p \alpha_k)$.

One can observe that the norm $\|\cdot\|_{\mathcal{O}}$ is equivalent to the norm $\|\cdot\|_{\widetilde{\mathcal{O}}}$, which means that $\mathcal{O} = \widetilde{\mathcal{O}}$. Thus, $\mathcal{O} - \mathcal{H}$ exact observability is equivalent to $\widetilde{\mathcal{O}} - \mathcal{H}$ exact observability and the lack of $\widetilde{\mathcal{N}} - \mathcal{H}$ exact observability would give us sharpness of the obtained result.

Theorem 3.2. *The result of Theorem 2.2 is sharp, i.e., there is no weaker norm of the left-hand side of inequality (1.2) for which system (1.1) is exactly observable.*

Proof. For every pair $\varepsilon = (\varepsilon_1, \varepsilon_2)$, $\varepsilon_1,\ \varepsilon_2 \geqslant 0$, $\varepsilon_1^2 + \varepsilon_2^2 > 0$, let us define a subspace $\widetilde{\mathcal{N}}_\varepsilon \subset L^2(0, T_0)$ with accompanying norm

$$\begin{aligned}\|f\|_{\widetilde{\mathcal{N}}_\varepsilon}^2 &= \|K_{1-\varepsilon_1}(\mathcal{S}(f_1))\|_{\ell^2}^2 + \|K_{2-\varepsilon_2}(\mathcal{S}(f_2))\|_{\ell^2}^2 \\ &= \sum_{k\in\mathbb{Z}\setminus\{k_n\}} \left|k^{1-\varepsilon_1}\right|^2 |\beta_k|^2 + \sum_{k\in\{k_n\}} \left|k^{2-\varepsilon_2}\right|^2 |\gamma_k|^2.\end{aligned} \tag{3.1}$$

For proving that system (1.1) is not $\widetilde{\mathcal{N}}_\varepsilon - \mathcal{H}$ exactly observable for any choice of parameters $\varepsilon_1,\ \varepsilon_2$, we consider 3 cases.

Case 1. In the first considered case we assume that $\varepsilon_1 > 0$ and $\varepsilon_2 = 0$. Then the norm of an arbitrary element $f \in \widetilde{\mathcal{N}}_\varepsilon$ is given by

$$\|f\|_{\widetilde{\mathcal{N}}_\varepsilon}^2 = \sum_{k\neq k_n} \left|k^{1-\varepsilon_1}\right|^2 |\beta_k|^2 + \sum_{k=k_n} \left|k^2\right|^2 |\gamma_k|^2.$$

The second term of the above sum is not changed by passing from $\widetilde{\mathcal{O}}$ to $\widetilde{\mathcal{N}}_\varepsilon$. Therefore, we will look for a counterexample for which $\gamma_k = 0$. Consider the sequence $\{z_N\}_{N\in\mathbb{N}}$, where

$$z_N = \sum_{\substack{k=1\\ k\notin\{k_n\}}}^{N} \alpha_k Y_k.$$

Using (2.8), we obtain

$$z_N = \sum_{\substack{k=1\\ k\notin\{k_n\}}}^{N} \frac{1}{C_k} \beta_k Y_k.$$

Hence, the norm of z_N can be expressed as

$$\|z_N\|_{\mathcal{H}}^2 = \sum_{\substack{k=1\\ k\notin\{k_n\}}}^{N} \frac{1}{|C_k|^2} |\beta_k|^2.$$

Letting $\beta_k = k^{-\frac{3}{2}}$ and using condition (A4), i.e., $|C_k| \asymp \frac{1}{|k|}$, we estimate the norm

$$\|z_N\|^2_{\mathcal{H}} \asymp \sum_{k=1}^{N} |k|^2 \left|k^{-\frac{3}{2}}\right|^2 = \sum_{k=1}^{N} |k|^{-1} \to \infty \quad \text{as } N \to \infty.$$

Now we proceed with the estimation of the norm of $\mathcal{K}z_N$,

$$\begin{aligned}\|\mathcal{K}z_N\|^2_{\widetilde{\mathcal{N}}_\varepsilon} &= \sum_{\substack{k=1 \\ k\notin\{k_n\}}}^{N} \left|k^{1-\varepsilon_1}\right|^2 |\beta_k|^2 \\ &\leq \sum_{k=1}^{N} |k|^{-1-2\varepsilon_1} \to \zeta\left(1+2\varepsilon_1\right) \quad \text{as } N \to \infty,\end{aligned}$$

where $\zeta(\cdot)$ is the Riemann zeta function. Then

$$\frac{\|\mathcal{K}z_N\|^2_{\widetilde{\mathcal{N}}_\varepsilon}}{\|z_N\|^2_{\mathcal{H}}} \to 0 \quad \text{as } N \to \infty.$$

Thus, inequality (1.2) cannot hold, hence system (1.1) is not $\widetilde{\mathcal{N}}_\varepsilon - \mathcal{H}$ exactly observable in time $T = T_0$, which finishes the proof for Case 1.

Case 2. Here we assume that $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$. Considerations in this case are similar to those of the previous case. The norm of an arbitrary element $f \in \widetilde{\mathcal{N}}_\varepsilon$ is given by

$$\|f\|^2_{\widetilde{\mathcal{N}}_\varepsilon} = \sum_{k\neq k_n} |k|^2 |\beta_k|^2 + \sum_{k=k_n} \left|k^{2-\varepsilon_2}\right|^2 |\gamma_k|^2.$$

Now we will look for a counterexample for which $\beta_k = 0$. Consider the sequence $\{z_N\}_{N\in\mathbb{N}}$, where

$$z_N = \sum_{\substack{k=1 \\ k\in\{k_n\}}}^{N} \alpha_k Y_k.$$

From (2.8), we obtain

$$z_N = \sum_{\substack{k=1 \\ k\in\{k_n\}}}^{N} \frac{1}{C_k} \frac{1}{\mu_k - \mu_{k-1}} \gamma_k Y_k.$$

Thus, the norm of z_N can be expressed as

$$\|z_N\|^2_{\mathcal{H}} = \sum_{\substack{k=1 \\ k\in\{k_n\}}}^{N} \frac{1}{|C_k|^2} \frac{1}{|\mu_k - \mu_{k-1}|^2} |\gamma_k|^2.$$

Letting $\gamma_k = k^{-\frac{5}{2}}$ and taking into account conditions (A2) and (A4), i.e., $|\mu_{k_n} - \mu_{k_n-1}| \asymp \frac{1}{|k_n|}$ and $|C_k| \asymp \frac{1}{|k|}$, we estimate the norm

$$\|z_N\|^2_{\mathcal{H}} \asymp \sum_{\substack{k=1\\k\in\{k_n\}}}^{N} |k|^2 \frac{1}{\frac{1}{|k|^2}} \left|k^{-\frac{5}{2}}\right|^2 = \sum_{\substack{k=1\\k\in\{k_n\}}}^{N} |k|^{-1} \to \infty \quad \text{as } N \to \infty.$$

Now we estimate the left-hand side of inequality (1.2),

$$\|\mathcal{K}z_N\|^2_{\widetilde{\mathcal{N}}_\varepsilon} = \sum_{\substack{k=1\\k\in\{k_n\}}}^{N} \left|k^{2-\varepsilon_2}\right|^2 |\gamma_k|^2 \le \sum_{\substack{k=1\\k\in\{k_n\}}}^{N} |k|^{-1-2\varepsilon_2} \to \zeta\,(1+2\varepsilon_2) \quad \text{as } N \to \infty.$$

Then

$$\frac{\|\mathcal{K}z_N\|^2_{\widetilde{\mathcal{N}}_\varepsilon}}{\|z_N\|^2_{\mathcal{H}}} \to 0 \quad \text{as } N \to \infty.$$

Thus, inequality (1.2) cannot hold, hence system (1.1) is not $\widetilde{\mathcal{N}}_\varepsilon - \mathcal{H}$ exactly observable in time $T = T_0$, which finishes the proof for Case 2.

Case 3. In the last considered case we assumed that $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ and the norm of an arbitrary element $f \in \widetilde{\mathcal{N}}_\varepsilon$ is given by (3.1). Let us consider the sequence $\{z_N\}_{N\in\mathbb{N}}$, where

$$z_N = \sum_{k=1}^{N} \alpha_k Y_k.$$

From (2.8), we obtain

$$z_N = \sum_{\substack{k=1\\k\notin\{k_n-1,k_n\}}}^{N} \frac{1}{C_k}\beta_k Y_k + \sum_{\substack{k=1\\k\in\{k_n-1\}}}^{N} \frac{1}{C_k}\left(\beta_k - \frac{1}{\mu_{k+1}-\mu_k}\gamma_{k+1}\right) Y_k + \sum_{\substack{k=1\\k\in\{k_n\}}}^{N} \frac{1}{C_k}\frac{1}{\mu_k - \mu_{k-1}}\gamma_k Y_k.$$

Thus, the norm of z_N is given by

$$\|z_N\|^2_{\mathcal{H}} = \sum_{\substack{k=1\\k\notin\{k_n-1,k_n\}}}^{N} \left|\frac{1}{C_k}\beta_k\right|^2 + \sum_{\substack{k=1\\k\in\{k_n-1\}}}^{N} \left|\frac{1}{C_k}\left(\beta_k - \frac{1}{\mu_{k+1}-\mu_k}\gamma_{k+1}\right)\right|^2 + \sum_{\substack{k=1\\k\in\{k_n\}}}^{N} \left|\frac{1}{C_k}\frac{1}{\mu_k - \mu_{k-1}}\gamma_k\right|^2.$$

Let us assume that $\beta_k = k^{-\frac{3}{2}}$, $\gamma_k = (\mu_k - \mu_{k-1})\,(k-1)^{-\frac{3}{2}}$ and taking into account conditions (A2) and (A4), we estimate the norm of z_N,

$$\|z_N\|^2_{\mathcal{H}} \asymp \sum_{k=1}^{N} |k|^2 \left|k^{-\frac{3}{2}}\right|^2 + \sum_{k=1}^{N} |k|^2 \frac{1}{\frac{1}{|k|^2}} \left|k^{-\frac{5}{2}}\right|^2 = \sum_{k=1}^{N} |k|^{-1} \to \infty \quad \text{as } N \to \infty.$$

Now we show the boundedness of the norm of $\mathcal{K}z_N$,

$$\begin{aligned}\|\mathcal{K}z_N\|^2_{\widetilde{\mathcal{N}}_\varepsilon} &= \sum_{\substack{k=1\\k\notin\{k_n\}}}^{N} \left|k^{1-\varepsilon_1}\right|^2 |\beta_k|^2 + \sum_{\substack{k=1\\k\in\{k_n\}}} \left|k^{2-\varepsilon_2}\right|^2 |\gamma_k|^2 \\ &\leq \sum_{k=1}^{N} |k|^{-1-2\varepsilon_1} + \sum_{k=1}^{N} |k|^{-1-2\varepsilon_2} \\ &\to \zeta(1+2\varepsilon_1) + \zeta(1+2\varepsilon_2) \quad \text{as } N\to\infty.\end{aligned}$$

Then

$$\frac{\|\mathcal{K}z_N\|^2_{\widetilde{\mathcal{N}}_\varepsilon}}{\|z_N\|^2_{\mathcal{H}}} \to 0 \quad \text{as } N\to\infty.$$

It means that inequality (1.2) cannot hold, hence system (1.1) is not $\widetilde{\mathcal{N}}_\varepsilon - \mathcal{H}$ exactly observable in time $T = T_0$, which finishes the proof for Case 3. □

Remark 3.3. After considering a suitable operator form of system (1.4)–(1.6), one can apply Theorems 2.2 and 3.2 and deduce that this system is $\mathcal{O}-\mathcal{H}$ exactly observable in time $T_0 = 2\left(1+\frac{1}{\gamma}\right)$ and is not $\mathcal{N}-\mathcal{H}$ exactly observable for any subspace $\mathcal{N}$ with a weaker topology than $\mathcal{O}$ (see [12] for the minimal time T_0 derivation).

Remark 3.4. One should notice that even for concrete examples it is hard to provide a precise form of the subspaces V_1 and V_2. It is an open problem that requires further investigations. For example, the subspace V_1 consists of all the possible infinite sums of the form $f_1(t) = \sum_{k=-\infty}^{\infty} \beta_k e^{i\mu_k t}$. Thus, the assumption of $f_1 \in V_1 \cap H^1$ is hard to examine because it requires considering infinite sums of this form, as all finite sums $\left(\sum_{k=-N}^{N} \beta_k e^{i\mu_k t}\right)$ belong to $V_1 \cap H^\infty$.

One may expect to be able to describe the subspaces V_1 and V_2 exploiting some special properties of families spanning those subspaces, that is, $\left\{e^{i\mu_k t}\right\}$ and $\left\{\frac{e^{i\mu_k t}-e^{i\mu_{k-1} t}}{\mu_k-\mu_{k-1}}\right\}$. For example, if one assumes that $\mu_k = 2k\pi$, it is obvious that $V_1 \subset L^2(0,4)$ consists of 2-periodic functions. In this case, the function

$$f(t) = \begin{cases} 16x^2 & \text{for } 0 \leqslant x \leqslant \dfrac{1}{4}, \\ -16\left(x-\dfrac{1}{2}\right)^2 + 2 & \text{for } \dfrac{1}{4} \leqslant x \leqslant \dfrac{3}{4}, \\ 16(x-1)^2 & \text{for } \dfrac{3}{4} \leqslant x \leqslant \dfrac{5}{4}, \\ -16\left(x-\dfrac{3}{2}\right)^2 + 2 & \text{for } \dfrac{5}{4} \leqslant x \leqslant \dfrac{7}{4}, \\ 16(x-2)^2 & \text{for } \dfrac{7}{4} \leqslant x \leqslant 2 \end{cases}$$

belongs to $V_1 \cap H^1$, but not to $V_1 \cap H^2$, which means that it is an example presenting the possible usage of sharpness of the $\mathcal{O} - \mathcal{H}$ exact observability result as oppose to the previously found $H^2 - \mathcal{H}$ result [20]. The family spanning subspace V_2 is much harder to be analyzed even in the simplest cases, it remains an open problem for future investigations.

4. Conclusions

In this paper, we presented the conditions of exact observability of a general class of distributed parameter systems in Hilbert spaces in terms of some specific assumptions on generating operators. We extended some previous results about analysis of suitable topology for state observation (of left-hand side of inequality (1.2)). In [20], it was proven that system (1.1) is neither $L^2(0,T) - \mathcal{H}$ (default topologies setting), nor $H^1(0,T) - \mathcal{H}$ exactly observable in any time $T > 0$. We found a stronger topology for the state observation for which the system becomes $H^2(0,T) - \mathcal{H}$ exactly observable in time $T \geq T_0$. A careful analysis of the obtained results turned out to be not sharp. We considered a question that naturally arose, what would be the optimal topology of the observable space such that the system could be $\mathcal{Y} - \mathcal{H}$ exactly observable and not be exactly observable for any weaker topology? To this end, we found a subspace $\mathcal{O}$ which is proved to give a sharp $\mathcal{O} - \mathcal{H}$ exact observability. Our considerations required the usage of advanced non-harmonic analysis approach connected to the crucial fact that the time-space Riesz basis does not consist only of exponential functions but also contains divided differences of these functions.

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Оптимальна топологія типу Соболєва у задачі точної спостережуваності для динамічних систем у гільбертовому просторі, пов'язаних із базисом Рісса з розділених різниць

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У цій статті розглядається проблема точної спостережуваності загального класу лінійних систем з розподіленими параметрами, пов'язаних з властивостями базису Рісса деяких сімейств експоненціальних функцій та розділених різниць цих функцій, в гільбертових просторах. За деяких припущень щодо асимптотичного спектрального аналізу диференціального оператора системи, сформульовано умови точної спостережуваності у вигляді того, що простори точної спостережуваності є прямою сумою деяких специфічних просторів Соболєва. Основний результат полягає у доведенні оптимальності цих підпросторів спостережуваних станів. Результат ґрунтувався на підході розширеного негармонічного аналізу, пов'язаному з незвичним фактом, що часово-просторовий базис Рісса складається не лише з експоненціальних функцій, а й містить розділені різниці цих функцій.

Ключові слова: точна спостережуваність, диференціальні рівняння з частинними похідними, необмежені оператори, базис Рісса розділених різниць, оптимальність підпросторів спостережуваності