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On the Characteristic Operators and Projections and on the Solutions of Weyl Type of Dissipative and Accumulative Operator Systems. II. Abstract Theory

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Special maximal semi-definite subspaces (maximal dissipative and accumulative relations) are considered. Particular cases of those arise in studying boundary problems for systems mentioned in the title. We provide a description of such subspaces and list their properties. A criterion is found that condition of semi-definiteness of sum of indefinite quadratic forms reduces to semi-definiteness of each of the summand forms, i.e it is separated. In the case when the forms depend on a parameter λ (e.g., a spectral parameter) within a domain $\Lambda \subset \mathbb{C}$, a condition is found under which separation of the semi-definiteness property at a single λ implies its separation for all λ.

Key words: maximal semi-definite subspace, maximal dissipative (accumulative) relation, idempotent.

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This work constitutes Part II of [32]. Notation, definitions, numeration of sections, statements, formulas etc., as well as the list of references, extend those of [32].

2. A Description and a Properties of Maximal Semi-definite Subspaces of a Special Form

Let $Q_j = Q_j^* \in B(\mathcal{H})$, $Q_j^{-1} \in B(\mathcal{H})$, $j = 1, 2$; $dim\mathcal{H}_\pm(Q_1) = dim\mathcal{H}_\pm(Q_2)$, with $\mathcal{H}_\pm(Q_j)$ being invariant subspaces for the operators Q_j, which correspond to positive and negative parts of their spectra. Then it is well know that there exists $\Gamma_j \in B(\mathcal{H})$ such that

$$\Gamma_j^{-1} \in B(\mathcal{H}), \quad \Gamma_j^* Q_j \Gamma_j = J, \tag{2.1}$$

where J is the canonical symmetry, that is $J = J^* = J^{-1}$ (for example [27], one can choose Γ_j so that $J = sgnQ_1$ or $J = sgnQ_2$). Represent J in the form

$$J = P_+ - P_-, \tag{2.2}$$

with $P_\pm$ being a pair of complementary orthogonal projections.

Introduce the notation

$$Q = diag(Q_1, -Q_2). \tag{2.3}$$

Let A_j, $j = 1, 2$, be linear operators in $\mathcal{H}$ (possibly unbounded and not densely defined) and suppose $\mathcal{D}_{A_1} = \mathcal{D}_{A_2} = \mathcal{D}$.

Consider the linear manifold

$$\mathcal{L} = \{A_1 f \oplus A_2 f | f \in \mathcal{D}\} \subset \mathcal{H}^2 \tag{2.4}$$

and the operators

$$S = P_+\Gamma_1^{-1}A_1 + P_-\Gamma_2^{-1}A_2, \qquad S_1 = P_+\Gamma_2^{-1}A_2 - P_-\Gamma_1^{-1}A_1. \tag{2.5}$$

Theorem 2.1. *$\mathcal{L}$ (2.4) is a maximal Q-nonnegative (Q-nonpositive) subspace in $\mathcal{H}^2$ if and only if the following conditions hold:*

1^o. $R(S) = \mathcal{H}$ ($R(S_1) = \mathcal{H}$).

2^o. There exists a compression K_+ (K_-) in $\mathcal{H}$ such that

$$S_1 f = K_+ S f \quad (S f = K_- S_1 f) \quad \forall f \in \mathcal{D}. \tag{2.6}$$

(Under 1^o K_+ (K_-) is unique).

Under (2.6), where linear operators $K_\pm$ are not necessary from $B(\mathcal{H})$, the operators A_j allow a parametrization as follows:

$$A_1 = \Gamma_1(P_+ - P_-K_+)S \qquad (A_1 = \Gamma_1(P_+K_- - P_-)S_1), \tag{2.7}$$

$$A_2 = \Gamma_2(P_- + P_+K_+)S \qquad (A_2 = \Gamma_2(P_+ + P_-K_-)S_1). \tag{2.8}$$

P r o o f. For certainty, we expound a proof for the case of Q-nonnegative $\mathcal{L}$.

Necessity. Suppose $\mathcal{L}$ (2.4) is a maximal Q-nonnegative subspace. Since

$$\mathbb{U}^* J_2 \mathbb{U} = \tilde{J}_2, \tag{2.9}$$

where

$$\mathbb{U} = \begin{pmatrix} P_+ & -P_- \\ P_- & P_+ \end{pmatrix} = \mathbb{U}^{*\,-1}, \tag{2.10}$$

$$J_2 = diag(J, -J), \;\; \tilde{J}_2 = diag(I, -I), \tag{2.11}$$

the subspace

$$\tilde{\mathcal{L}} = \mathbb{U}^* \Gamma^{-1} \mathcal{L} = \{Sf \oplus S_1 f |\, f \in \mathcal{D}\} \tag{2.12}$$

with

$$\Gamma = diag(\Gamma_1, \Gamma_2), \tag{2.13}$$

is maximal $\tilde{J}_2$-nonnegative. If so (see [24, p. 100], [25, Ch. I, § 8]), there exists a compression K_+ in $\mathcal{H}$ such that

$$\tilde{\mathcal{L}} = \{g \oplus K_+ g |\, g \in \mathcal{H}\}. \tag{2.14}$$

Compare (2.12), (2.14) to see that 1^o and 2^o hold.

Sufficiency. Suppose 1^o and 2^o hold. Multiply from the left both parts of the initial formulas in (2.5), (2.6) by P_+ and P_- respectively, and then sum up the resulting equalities to get the initial equality in (2.7). The initial equality from (2.8) can be deduced in a similar way.

With the notation

$$U_j[f] = (Q_j A_j f, A_j f), \quad f \in \mathcal{D}, \tag{2.15}$$

apply (2.1), (2.2), (2.7), (2.8) to deduce that

$$U_1[f] - U_2[f] = \|Sf\|^2 - \|K_+ Sf\| \geq 0, \tag{2.16}$$

since K_+ is a compression. Thus $\mathcal{L}$ (2.4) is Q-nonnegative. Prove its maximality. For that, as one can see from [23], [25, p. 38], in view of (2.1), (2.2), it suffices to verify that

$$\mathbb{P}_+ \mathcal{L} = \mathbb{P}_+ \mathcal{H}^2, \tag{2.17}$$

where

$$\mathbb{P}_+ = \Gamma \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix} \Gamma^{-1}. \tag{2.18}$$

Apply (2.18), (2.13), (2.7), (2.8), together with the fact that $R(S) = \mathcal{H}$, to deduce that

$$\begin{aligned} \mathbb{P}_+ \mathcal{L} &= \mathbb{P}_+ \{A_1 f \oplus A_2 f | f \in \mathcal{D}\} = \Gamma\{P_+ Sf \oplus P_- Sf | f \in \mathcal{D}\} \\ &= \Gamma\{P_+ g \oplus P_- g | g \in \mathcal{H}\} = \Gamma\{P_+ g \oplus P_- h | g, h \in \mathcal{H}\} = \mathbb{P}\mathcal{H}^2. \end{aligned}$$

Thus (2.17), along with Th. 2.1, is proved.

Remark 2.1. *Condition 1º in Th. 2.1 in the case* $dim\mathcal{H} = \infty$ *could be replaced in general neither by*

$$\exists\alpha > 0 : \forall f \in D \quad \|Sf\| \geq \alpha\|f\| \quad (\|S_1 f\| \geq \alpha\|f\|), \tag{2.19}$$

nor by

$$\exists\beta > 0 : \forall f \in D \quad \|A_1 f\| + \|A_2 f\| \geq \beta\|f\|.^{\star} \tag{2.20}$$

P r o o f. Let $\mathcal{H} = l^2$. Set up

$$A_1 = \Gamma_1 P_+ U, A_2 = \Gamma_2 P_- U \qquad (A_1 = -\Gamma_1 P_- U, A_2 = \Gamma_2 P_+ U)$$

with U being the one-sided shift in l^2 [28]. Then $S = U$, $S_1 = 0$ ($S_1 = U$, $S = 0$), hence condition 2^o in Th. 2.1 holds with the compression $K_+ = 0$ ($K_- = 0$). Therefore, in view of (2.16) (an analog of (2.16) for equality $S = K_- S_1$), $\mathcal{L}$ (2.4) is Q-nonnegative (Q-nonpositive). On the other hand, $R(S) \neq \mathcal{H}$ ($R(S_1) \neq \mathcal{H}$), although (2.19), (2.20) hold. The Remark 2.1 is proved.

Theorem 2.1 implies

Corollary 2.1. *Let the linear manifold* $\mathcal{L}$ *and the operators* S, S_1 *be given by (2.4), (2.5), and suppose the following two conditions are satisfied:*

1) $\mathcal{L}$ *is* Q*-nonnegative (*Q*-nonpositive).*

2) $S^{-1} \in B(\mathcal{H})$ ($S_1^{-1} \in B(\mathcal{H})$).

Then $\mathcal{L}$ *is a maximal* Q*-nonnegative (respectively,* Q*-nonpositive) subspace.*

P r o o f is expounded here, e.g., for the Q-nonnegative case. Verify that 1), 2) imply the Conditions 1^o, 2^o of Th. 2.1. 2) implies 1^o together with (2.6) in which $K_+ = S_1 S^{-1}$. Then with this K_+ the representations (2.7), (2.8) are valid, hence also equality (2.16). On the other hand, 1) implies inequality (2.16), whence K_+ is a compression. The Corollary 2.1 is proved.

Remark 2.2. *The transformation*

$$\begin{pmatrix} iI & I \\ I & iI \end{pmatrix} \mathbb{U}^* \Gamma^{-1} \mathcal{L}$$

with $\mathbb{U}$, Γ *as in (2.10), (2.13), reduces the maximal* Q*-nonnegative (*Q*-nonpositive) subspace* $\mathcal{L}$ *(2.4) to a maximal accumulative (dissipative) relation in* $\mathcal{H}$*. Its Cayley transform* V*, relates to the compressions* $K_\pm$ *from Th. 2.1 as follows:* $V = \pm iK_\pm$.

P r o o f follows from the proof of Th. 2.1 and [22] (see also [2]).

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*(2.19)⇒(2.20). If (2.6) holds, where $B(\mathcal{H}) \ni K_\pm$ are not necessary compressions, then (2.20)⇒(2.19).

Remark 2.3. *(cf. [24, 25]). The formulae*

$$\begin{array}{c}\mathcal{L} = \{\Gamma_1(P_+ - P_-K_+)h \oplus \Gamma_2(P_- + P_+K_+)h | h \in \mathcal{H}\} \\ (\mathcal{L} = \{\Gamma_1(P_+K_- - P_-)h \oplus \Gamma_2(P_+ + P_-K_-)h | h \in \mathcal{H}\})\end{array} \qquad (2.21)$$

establish a one-to-one correspondence between compressions K_+ (K_-) in $\mathcal{H}$ and maximal Q-nonnegative (Q-nonpositive) subspaces $\mathcal{L}$ in $\mathcal{H}^2$. (In the case $\mathcal{L}$ being of the form (2.4), the compressions K_+ (K_-) in (2.7), (2.8) coincide to those in (2.21)). Besides that:

1) $\mathcal{L}$ (2.21) is maximal Q-neutral subspace if and only if K_+ (K_-) is an isometry in $\mathcal{H}$.*

2) $\mathcal{L}$ (2.21) is hypermaximal Q-neutral subspace if and only if K_+ (K_-) is a unitary in $\mathcal{H}$.

P r o o f is expounded here for certainty in the Q-nonnegative case. If $\mathcal{L}$ is of the form (2.21) with K_+ being a compression, then this $\mathcal{L}$ satisfies the assumptions of Th. 2.1 since with this $\mathcal{L}$ one has $S = I$, $S_1 = K_+S$. Thus by Th. 2.1 $\mathcal{L}$ is a maximal Q-nonnegative subspace.

Conversely, let $\mathcal{L}$ be a maximal Q-nonnegative subspace. Then one can use the idea of the proof of necessity in Th. 2.1 to deduce that $\mathcal{L} = \Gamma\mathbb{U}\tilde{\mathcal{L}}$ with Γ, $\mathbb{U}$, $\tilde{\mathcal{L}}$ as in (2.13), (2.10), (2.14), and additionally that in (2.14) K_+ is a compression, which implies (2.21).

A classification of $\mathcal{L}$ (2.21) in terms of the properties of compressions $K_\pm$ follows from (2.16) and [24, p. 100], [25, Ch. I, § 4, 8]. Since the correspondence (2.21) is obviously on-to-one, the statement of the remark is proved.

The following theorem allows one to characterize a maximal Q-definite subspace in terms of a linear equation, which provides an analog of the existing characterization for Hermitian [33] (see also [3]) and maximal dissipative or accumulative [22], (see also [2]) relations.

Theorem 2.2. *Suppose that the linear manifold $\mathcal{L}$ (e.g. $\mathcal{L}$ (2.4)) is a maximal Q-nonnegative (Q-nonpositive) subspace in $\mathcal{H}^2$. Then there exists a unique compression K_+ (K_-) in $\mathcal{H}$ such that*

$$f \oplus g \in \mathcal{L} \quad \Leftrightarrow \quad B_1 f - B_2 g = 0, \qquad (2.22)$$

where

$$\begin{array}{ll} B_1 = (K_+P_+ - P_-)\Gamma_1^* Q_1, & B_2 = (K_+P_- + P_+)\Gamma_2^* Q_2 \\ (\ B_1 = (P_+ - K_-P_-)\Gamma_1^* Q_1, & B_2 = (K_-P_+ + P_-)\Gamma_2^* Q_2\ ) \end{array} \qquad (2.23)$$

*In view of [25, p. 42] maximal Q-neutral subspace is maximal Q-nonnegative or maximal Q-nonpositive or both type.

and $\mathcal{L}$ admits representation (2.21) with these compressions $K_\pm$.

If in (2.23) $K_\pm$ are arbitrary compressions in $\mathcal{H}$, then

$$\hat{\mathcal{L}} = \{B_1^* f \oplus B_2^* f | f \in \mathcal{H}\} \subset \mathcal{H}^2 \tag{2.24}$$

is a maximal Q^{-1}-nonpositive (Q^{-1}-nonnegative) subspace in $\mathcal{H}^2$ and (as one can see from (2.23)),

$$\|B_1^* f\| + \|B_2^* f\| > 0, \qquad 0 \neq f \in \mathcal{H}. \tag{2.25}$$

If $\mathcal{L}$ is of the form (2.4) with $A_j \in B(\mathcal{H})$ and

$$\|A_1 f\| + \|A_2 f\| > 0, \qquad 0 \neq f \in \mathcal{H}, \tag{2.26}$$

then $S^{-1} \in B(\mathcal{H})$, ($S_1^{-1} \in B(\mathcal{H})$), where S, S_1 are as in (2.5), hence by (2.6) one has $K_+ = S_1 S^{-1}$ ($K_- = S S_1^{-1}$), i.e. B_j (2.23) admits an explicit expression in terms of A_j.

Conversely, suppose $\mathcal{L}$ is given by (2.22), with $B_j \in B(\mathcal{H})$, $j = 1, 2$, and $\hat{\mathcal{L}}$ (2.24) is a maximal Q^{-1}-nonpositive (Q^{-1}-nonnegative) subspace in $\mathcal{H}^2$. Then $\mathcal{L}$ is a maximal Q-nonnegative (Q-nonpositive) subspace in $\mathcal{H}^2$ (hence admits representation (2.21)). Furthermore, if (2.25) holds, then the compressions $K_\pm$ in (2.21) admit explicit expression in terms of B_j, specifically $K_+ = S_1^{*-1} S^*$ ($K_- = S^{*-1} S_1^*$) with S, S_1 being given by (2.5), where $A_1 = Q^{-1} B_1^*$, $A_2 = Q_2^{-1} B_2^*$ and $S_1^{-1} \in B(\mathcal{H})$ ($S^{-1} \in B(\mathcal{H})$).

P r o o f is expounded here for certainty in the Q-nonnegative case. Let $\mathcal{L}$ be a maximal Q-nonnegative subspace. Then by Remark 2.3 there exists a unique compression K_+, which makes valid (2.21), an equivalent of the initial equality in (2.12) with $\tilde{\mathcal{L}}$ (2.14). This implies by a virtue of [25, p. 73] that

$$\mathcal{L}^{[Q]} = Q^{-1} \hat{\mathcal{L}},$$

with $\hat{\mathcal{L}}$ being as in (2.24), (2.23); $\mathcal{L}^{[A]}$ stands here for A-orthogonal complement in $\mathcal{H}^2$. Therefore

$$f \oplus g \in \mathcal{L} \;\Leftrightarrow\; (Q_1 f, Q_1^{-1} B_1^* h) - (Q_2 g, Q_2^{-1} B_2^* h) = 0 \;\forall h \in \mathcal{H},$$

which implies (2.22), (2.23). Furthermore, $Q^{-1}\hat{\mathcal{L}}$ is of the form (2.21) with $K_- = K_+^*$, hence $\hat{\mathcal{L}}$ (2.24), (2.23) is a maximal Q^{-1}-nonpositive subspace by Remark 2.3.

If $\mathcal{L}$ (2.4) with $A_j \in B(\mathcal{H})$ being a maximal Q-nonnegative subspace, then $R(S) = \mathcal{H}$ by Th. 2.1. Besides that, $Ker S = \{0\}$ since if $Sf = 0$ for some nonzero $f \in \mathcal{H}$, then by condition (2.6) of Th. 2.1 $S_1 f = 0$ implies $A_1 f = A_2 f = 0$, which contradicts (2.26). Thus we have $S^{-1} \in B(\mathcal{H})$ by the Banach theorem.

Prove the converse. By our assumption, $Q^{-1}\hat{\mathcal{L}}$ is a maximal Q-nonpositive subspace. An application of Th. 2.1 provides the existence of a compression K_- such that

$$Q_1^{-1}B_1^* = \Gamma_1(P_+K_- - P_-)S_1, \qquad Q_2^{-1}B_2^* = \Gamma_2(P_+ + P_-K_-)S_1,$$

where S_1 is given by (2.5) with A_j being replaced by $Q_j^{-1}B_j^*$. Note that by a virtue of 1^o of Th. 2.1 one has $KerS_1^* = \{0\}$, which yields

$$B_1f - B_2g = 0 \Leftrightarrow (K_-^*P_+ - P_-)\Gamma_1^*Q_1f - (P_+ + K_-^*P_-)\Gamma_2Q_2g = 0.$$

Therefore $\mathcal{L} = (Q^{-1}\hat{\mathcal{L}})^{[Q]}$, hence [25, p. 73] $\mathcal{L}$ is a maximal Q-nonnegative subspace. An argument similar to that proving the direct statement demonstrates that for $\mathcal{L}$ in (2.21) operator $K_+ = K_-^*$, which allows to one deduce the rest of statements in a similar way. The theorem is proved.

Lemma 2.1. *(cf. [24, 25]). Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$; in (2.1) one has*

$$J = \begin{pmatrix} I_1 & 0 \\ 0 & -I_2 \end{pmatrix} \tag{2.27}$$

with I_j being the identity operators in $\mathcal{H}_j$, j=1,2. Then the formulae: $\mathcal{L} = A_1\mathcal{H}$ ($\mathcal{L} = A_2\mathcal{H}$), where

$$A_1 = \Gamma_1\begin{pmatrix} I_1 & 0 \\ K_{21} & 0 \end{pmatrix}, \qquad A_2 = \Gamma_2\begin{pmatrix} 0 & K_{12} \\ 0 & I_2 \end{pmatrix}, \tag{2.28}$$

establish a one to one correspondence between compressions $K_{21} \in B(\mathcal{H}_1, \mathcal{H}_2)$ ($K_{12} \in B(\mathcal{H}_2, \mathcal{H}_1)$) and maximal Q_1-nonnegative (Q_2-nonpositive) subspaces $\mathcal{L}$ in $\mathcal{H}$. Besides that:

$$f \in \mathcal{L} \quad \Leftrightarrow \quad \begin{pmatrix} 0 & 0 \\ K_{21} & I_2 \end{pmatrix}\Gamma_1^*Q_1f = 0 \quad \left(\begin{pmatrix} I_1 & K_{12} \\ 0 & 0 \end{pmatrix}\Gamma_2^*Q_2f = 0\right).$$

The Lemma 2.1 proves in the same way as (2.21), (2.22), (2.23) with using [24, 25].

Note that with $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$ and

$$Q_1 = Q_2 = \begin{pmatrix} 0 & iI_1 \\ -iI_1 & 0 \end{pmatrix},$$

the maximal Q_1-nonnegative (Q_1-nonpositive) subspace in $\mathcal{H}$ appears to be a maximal accumulative (dissipative) relation in $\mathcal{H}_1$, and, after a suitable change of notation, Lemma 2.1 provides a well known [22] (see also [2, 3]) description for them.

Lemma 2.2. *Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and the operator J in (2.1) is just (2.27). Then $\mathcal{L}$ (2.4) together with the operators A_1, A_2 as in (2.28) of Lemma 2.1 is a maximal Q-nonnegative subspace in $\mathcal{H}^2$.*

P r o o f. For $\mathcal{L}$ (2.4), (2.28) one has $S = I$, $S_1 = \begin{pmatrix} 0 & K_{12} \\ -K_{21} & 0 \end{pmatrix}$, so Lemma 2.2 is proved in view of Th. 2.1.

An analog for Lemma 2.2 is also valid for the Q-nonpositive case.

In addition to Th. 2.1, we have

Theorem 2.3. *Let $\mathcal{L}$ (2.4) be a maximal Q-nonnegative (Q-nonpositive) subspace in $\mathcal{H}^2$ (that is, the assumptions 1°, 2° of Th. 2.1 are satisfied), and suppose that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with the operator J in (2.1) being just (2.27). Then $(-1)^j(Q_jA_jf, A_jf) \le 0$ $((-1)^j(Q_jA_jf, A_jf) \ge 0)$ for $f \in D$, $j = 1, 2$, if and only if the compressions in (2.7),(2.8) are of the form*

$$K_+ = \begin{pmatrix} 0 & K_{12}^+ \\ K_{21}^+ & 0 \end{pmatrix}, \qquad \left(K_- = \begin{pmatrix} 0 & K_{12}^- \\ K_{21}^- & 0 \end{pmatrix}\right), \tag{2.29}$$

with $K_{ij}^\pm \in B(\mathcal{H}_j$, being obviously compressions.

P r o o f is to be expounded here for certainty in the Q-nonnegative case. Necessity. Let $(-1)^j(QA_jf, A_jf) \le 0$ for $f \in D$, $j = 1, 2$. Then since $\mathcal{L}$ (2.4) is a maximal Q-nonnegative subspace, the linear manifolds $\{A_1f | \; f \in D\}$ and $\{A_2f | \; f \in D\}$ are, respectively, maximal Q_1-nonnegative and Q_2-nonpositive subspaces in $\mathcal{H}$. Thus by Th. 2.1 and Lemma 2.2 one has $\forall f \in D \, \exists h \in \mathcal{H}$: *

$$(P_+ - P_-K_+)Sf = \begin{pmatrix} I_1 & 0 \\ K_{21} & 0 \end{pmatrix} h, \tag{2.30}$$

$$(P_- + P_+K_+)Sf = \begin{pmatrix} 0 & K_{12} \\ 0 & I_2 \end{pmatrix} h, \tag{2.31}$$

where $Sf = g_1 \oplus g_2$, $h = h_1 \oplus h_2$; $g_j, h_j \in \mathcal{H}_j$, and the compression

$$K_+ = \begin{pmatrix} K_{11}^+ & K_{12}^+ \\ K_{21}^+ & K_{22}^+ \end{pmatrix}, \tag{2.32}$$

with $K_{ij}^+ \in B(\mathcal{H}_j, \mathcal{H}_i)$.

Multiply (2.30) from left by P_+ to get, in view of (2.27),

$$g_1 = h_1. \tag{2.33}$$

*And $\forall h \in \mathcal{H} \, \exists f \in D$:

In a similar way, multiply (2.30) from left by $\mathcal{P}_-$ to obtain in view (2.32)

$$-K_{21}^+ g_1 - K_{22}^+ g_2 = K_{21} h_1. \tag{2.34}$$

Since $R(S) = \mathcal{H}$ by Th. 2.1, the vectors $g_j \in \mathcal{H}_j$ in (2.33), (2.34) are arbitrary. Thus it follows from (2.33), (2.34) that $K_{21}^+ = -K_{21}$, $K_{22}^+ = 0$. Deduce similarly from (2.31) that $K_{12}^+ = K_{12}$, $K_{11}^+ = 0$, which proves the necessity.

Sufficiency. Since $\mathcal{L}$ (2.4) is a maximal Q-nonnegative subspace in $\mathcal{H}^2$, it follows from Th. 2.1 together with (2.7), (2.8), (2.27), (2.29), that

$$A_1 = \Gamma_1 \begin{pmatrix} I_1 & 0 \\ -K_{21}^+ & 0 \end{pmatrix} S, \qquad A_2 = \Gamma_2 \begin{pmatrix} 0 & K_{12}^+ \\ 0 & I_2 \end{pmatrix} S.$$

So by Lem. 2.1 sufficiency, along with theorem 2.3 is proved.

Consider examples (Th. 2.4–2.7) of Q-semi-definite subspaces which arise in investigation of boundary problems for the equation (0.1).

Let P be an orthogonal projection in $\mathcal{H}$ (in particular P can be an orthogonal projection onto $N^\perp$ (see [32])), and let $M_{\pm i}$ be a linear operators (not necessary bounded) in $\mathcal{H}$ with the property

$$M_{\pm i} = P M_{\pm i} P \tag{2.35}$$

(hence also $PD_{M_{\pm i}} \subseteq D_{M_{\pm i}}$, $(I-P)\mathcal{H} \subseteq D_{M_{\pm i}}$).

Let $G = G^* \in B(\mathcal{H}), G^{-1} \in B(\mathcal{H})$ (in particular G can be equal to $Q(c)$ (see [32])).

Represent $M_{\pm i}$ in the form

$$M_{\pm i} = \left(\mathcal{P}_{\pm i} - \frac{1}{2} I\right)(iG)^{-1}. \tag{2.36}$$

Consider linear manifolds in $\mathcal{H}^2$:

$$L_{\pm i} =$$

$$\left\{ \left[(\mathcal{P}_{\pm i} - I)(iG)^{-1}P + (I-P)\right] f \oplus \left[\mathcal{P}_{\pm i}(iG)^{-1}P + (I-P)\right] f | f \in D_{M_{\pm i}} \right\}^\star \tag{2.37}$$

and introduce the notation

$$G_2 = diag(G, -G).$$

*Which are subspaces if and only if the operators $M_{\pm i}$ are closed.

Lemma 2.3. *If $\overline{D}_{M_i} = \mathcal{H}$ and the operators $M_{\pm i}$ are related as follows*

$$M_{-i} = M_i^*,^\star \tag{2.38}$$

then the linear manifolds L_i and L_{-i} are G_2-orthogonal.

P r o o f reduces to a direct computation which uses that, in view of (2.38),

$$\mathcal{P}_{-i} = I - G^{-1}\mathcal{P}_i^* G. \tag{2.39}$$

Lemma 2.4. *The linear manifolds $L_{\pm i}$ are $\pm G_2$-nonnegative if and only if $\pm Im(M_{\pm i} f, f) \geq 0$ for all $f \in D_{M_{\pm i}}$.*

P r o o f reduces to a direct computation.

Theorem 2.4. *The linear manifolds $L_{\pm i}$ (2.37) are maximal $\pm G_2$-nonnegative subspaces in $\mathcal{H}^2$ if and only if $\pm M_{\pm i}$ are maximal dissipative operators in $\mathcal{H}$.*

P r o o f is expounded here for certainty in the case of L_i. Necessity. Suppose L_i is a maximal G_2-nonnegative subspace. Hence operator M_i is closed.

Prove that $\overline{D}_{M_i} = \mathcal{H}$. Clearly $\mathcal{H}$ can be represented in the form $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ so that there exists $\Gamma \in B(\mathcal{H})$ with $\Gamma^{-1} \in B(\mathcal{H})$, $\Gamma^* G \Gamma = J$ (2.27). For L_i (2.37) compute the operator S (2.5) with $\Gamma_1 = \Gamma_2 = \Gamma$. One has:

$$\Gamma S = M_i + \frac{i}{2}\Gamma\Gamma^* P + I - P. \tag{2.40}$$

Suppose there exists a nonzero $f_0 \in D_{M_i}^\perp$. Since $R(S) = \mathcal{H}$ by Th. 2.1, there exists $g_0 \in D_{M_i}$ such that $\Gamma S g_0 = f_0$. Then it follows from (2.40), (2.35) that

$$0 = (f_0, P g_0) = (M_i P g_0, P g_0) + \frac{i}{2}\|\Gamma^* P g_0\|^2,$$

whence

$$0 = Im(M_i P g_0, P g_0) + \frac{1}{2}\|\Gamma^* P g_0\|^2. \tag{2.41}$$

It follows from (2.41) that $P g_0 = 0$, since the first term in (2.41) is nonnegative by Lemma 2.4. On the other hand, (2.40), (2.35) imply that $0 = (f_0, (I-P)g_0) = \|(I-P)g_0\|^2$, hence $g_0 = 0 \Rightarrow \overline{D}_{M_i} = \mathcal{H}$. Thus M_i is closed dissipative operator (see [34]) by Lemma 2.4.

Prove that $Im(M_i^* f, f) \leq 0$ for $f \in D_{M_i^*}$. Since L_i is a maximal G_2-nonnegative subspace, it follows from Lemma 2.3 that L_{-i} (2.37), (2.38) is a G_2-nonpositive linear manifold in view of [25, p. 73]. Thus Lemma 2.4 together

*Alternatively, if $\overline{D}_{M_{-i}} = \mathcal{H}$ and $M_i = M_{-i}^*$

with (2.38) implies $Im(M_i^* f, f) \le 0$ for $f \in D_{M_i^*}$, which proves necessity in view of [34, p. 109].

Sufficiency. Suppose that M_i (2.35) is maximal dissipative. Hence the linear manifold L_i (2.37) is G_2-nonnegative by Lemma 2.4.

Prove that for this manifold the operator S given by (2.5) is such that $S^{-1} \in B(\mathcal{H})$ where $\mathcal{L}(2.4) = L_i$, $\Gamma_j = \Gamma$.

Prove that $0 \ne \sigma_p(S) \cup \sigma_c(S)$. If not, then there exists a sequence $\{f_n\}$ such that $f_n \in D(M_i)$, $\|f_n\| = 1$, and $\Gamma S f_n \to 0$, whence in view of (2.40) one has

$$Im(M_i P f_n, P f_n) + \frac{1}{2}\|\Gamma^* P f_n\|^2 \to 0. \tag{2.42}$$

Since the first term in (2.42) is nonnegative due to dissipativity of M_i, it follows from (2.42) that $Pf_n \to 0$. On the other hand, (2.40), (2.35) imply that $\|(I-P)f_n\|^2 = (\Gamma S f_n, (I-P)f_n) \to 0$, hence $f_n \to 0$. The contradiction we get proves that $0 \ne \sigma_p(S) \cup \sigma_c(S)$.

Prove that $0 \notin \sigma_r(S)$. If not, there exists a nonzero $f \in D_{M_i^*}$ such that $(\Gamma S)^* f = 0$, since $D_{(\Gamma S)^*} = D_{M_i^*}$ in view of (2.40). Then by a virtue of (2.40), (2.35) one has

$$PM_i^* Pf - \frac{i}{2} P\Gamma\Gamma^* f + (I-P)f = 0, \tag{2.43}$$

whence $(I-P)f = 0$. Thus by (2.43) one has

$$Im(M_i^* Pf, Pf) - \frac{1}{2}\|\Gamma^* Pf\| = 0. \tag{2.44}$$

It follows from maximal dissipativity of M_i that the first term in (2.44) is nonpositive [34, p. 109]. Thus by (2.44) $Pf = 0$, hence $f = 0$. It follows that $0 \notin \sigma_r(S)$, therefore $S^{-1} \in B(\mathcal{H})$, which completes the proof in view of Cor. 2.1.

For $P = I, M_{\pm i} \in B(\mathcal{H})$ Th. 2.4 is contained in [1].

Corollary 2.2. *If $\pm M_i$ are maximal dissipative operators in $\mathcal{H}$, then* $L_{\pm i} = \big\{[(\mathcal{P}_{\pm i} - I)G^{-1}f + (I-P)g] \oplus [\mathcal{P}_{\pm i}G^{-1}f + (I-P)g] | f \in D_{M_{\pm i}}, g \in \mathcal{H}\big\}$.

P r o o f follows from the fact that for linear manifolds in the right hand side the analog of Lemma 2.4 holds.*

Lemma 2.5. *Let $\overline{D}_{M_i} = \mathcal{H}$, the operators $M_{\pm i}$ be related by (2.38), and the operators $X_{\pm ij} \in B(\mathcal{H})$, $j = 1, 2$, be related by*

$$X_{-i1}^* Q_1 X_{i1} = G = X_{-i2}^* Q_2 X_{i2}. \tag{2.45}$$

*Note that for these manifolds the analog of Lemma 2.3 also holds.

Then the linear manifolds

$$\mathcal{L}_{\pm i} = diag(X_{\pm i1}, X_{\pm i2}) L_{\pm i} \tag{2.46}$$

are Q-orthogonal, with $L_{\pm i}$ being as in (2.37).

P r o o f follows from (2.45) and Lemma 2.3.

Lemma 2.6. *Suppose $\tilde{X}_{\pm ij}, \tilde{X}^{-1}_{\pm ij} \in B(\mathcal{H})$, $j = 1,2$, and the following three conditions are satisfied:*

1^o. $\tilde{L}_{\pm i}$ are a maximal $\pm G_2$-nonnegative subspaces in $\mathcal{H}^2$.

2^o. The subspaces

$$\tilde{\mathcal{L}}_{\pm i} = diag(\tilde{X}_{\pm i1}, \tilde{X}_{\pm i2}) \tilde{L}_{\pm i}$$

are $\pm Q$-nonnegative.

3^o.

$$\pm \tilde{X}^*_{\pm i1} Q_1 \tilde{X}_{\pm i1} \leq \pm G \leq \pm \tilde{X}^*_{\pm i2} Q_2 \tilde{X}_{\pm i2} \tag{2.47}$$

Then $\tilde{\mathcal{L}}_{\pm i}$ are a maximal $\pm Q$-nonnegative subspaces in $\mathcal{H}^2$.

P r o o f is presented here for certainty in the case of $\tilde{\mathcal{L}}_i$. Suppose that $\tilde{\mathcal{L}}_i$ is not maximal, that is $\mathcal{H}^2$ contains a Q-nonnegative subspace $T \supset \tilde{\mathcal{L}}_i$. Then the subspace $T_1 = diag(\tilde{X}^{-1}_{i1}, \tilde{X}^{-1}_{i2})T$ contains $\tilde{L}_i$. By a virtue of (2.47) for all $f_1 \oplus f_2 \in T$, one has

$$(G\tilde{X}^{-1}_{i1} f_1, \tilde{X}^{-1}_{i1} f_1) - (G\tilde{X}^{-1}_{i2} f_1, \tilde{X}^{-1}_{i2} f_2) \geq (Q_1 f_1, f_1) - (Q_2 f_2, f_2) \geq 0,$$

since T is Q-nonnegative. Thus T_1 is a Q-nonnegative subspace, which contradicts maximality of $\tilde{L}_i$. The lemma is proved.

Theorem 2.5. *Suppose L_i (L_{-i}) (2.37) is a maximal G_2-nonnegative (G_2-nonpositive) subspace in $\mathcal{H}^2$, and (2.38) holds. Let for $X_{\pm ij} \in B(\mathcal{H}), j = 1,2$, (2.45) holds.*

Then $\mathcal{L}_{-i}$ ($\mathcal{L}_i$) (2.46) is Q-nonpositive (Q-nonnegative) manifold in $\mathcal{H}^2$.

Additionally, if $X^{-1}_{ij} \in B(\mathcal{H})$, $X^{-1}_{-ij} \in B(\mathcal{H})$, $j = 1,2$, (2.47) for $\tilde{X}_{\pm ij} = X_{\pm ij}$ holds with + (-), and the spectrum of either of the operators Y_{i1}, Y_{i2} does not cover the unit circle, where

$$\Gamma_j Y_{\pm ij} = X_{\pm ij};\ \Gamma_j \in B(\mathcal{H}),\ \Gamma^{-1}_j \in B(\mathcal{H}),\ \Gamma^*_j Q_j \Gamma_j = G,\ j = 1,2, \tag{2.48}$$

(hence in view of (2.45) the spectrum of either of the operators Y_{-i1}, Y_{-i2} does not cover the unit circle).

Then $\mathcal{L}_{-i}$ ($\mathcal{L}_i$) (2.46) is a maximal Q-nonpositive (Q-nonnegative) subspace in $\mathcal{H}^2$.

P r o o f of Q-semidefiniteness for $\mathcal{L}_{-i}$ ($\mathcal{L}_i$) follows from [25, p. 73] in view of Lemma 2.5 and Th. 2.4.

The subsequent argument is expounded here for certainty in the case when condition (2.47) (with +) for $\tilde{X}_{+ij} = X_{ij}$ holds. In view of (2.48) we have

$$Y_{i1}^* G Y_{i1} \leq G \leq Y_{i2}^* G Y_{i2}.$$

Thus by (2.45) one has

$$Y_{-i1} G^{-1} Y_{-i1}^* \geq G^{-1} \geq Y_{-i2} G^{-1} Y_{-i2}^*,$$

whence in view of [24, p. 96], we deduce that

$$Y_{-i1}^* G Y_{-i1} \geq G \geq Y_{-i2}^* G Y_{-i2},$$

since the spectrum of either of the operators Y_{-i1}^*, Y_{-i2}^* does not cover the unit circle.

Hence by (2.48) the condition (2.47) (with −) for $\tilde{X}_{-ij} = X_{-ij}$ holds. Finally, maximality of L_i implies maximality for L_{-i} in view of (2.38), Th. 2.4, and [34, p. 109]. Thus $\mathcal{L}_{-i}$ is a maximal Q-nonpositive subspace by Lemma 2.6. The theorem is proved.

The next theorem allows one to use Remark 1.1 for producing c.o. of a boundary problem for the equation (0.1) with a non-separated boundary condition, whose special case is the periodic boundary condition.

Theorem 2.6. *Suppose:*

1°.

$$\Gamma, \Gamma^{-1} \in B(\mathcal{H}), \quad Q_2 = \Gamma^* Q_1 \Gamma. \tag{2.49}$$

2°.

$$\mathbf{U} \in B(\mathcal{H}), \mathbf{U}^* Q_1 \mathbf{U} - Q_1 \leq 0 \ (\geq 0). \tag{2.50}$$

3°. The spectrum of $\mathbf{U}$ *does not cover the unit circle.*

Then $\mathcal{L}$ *(2.4) with*

$$A_1 = I, \qquad A_2 = \Gamma^{-1}\mathbf{U} \tag{2.51}$$

is a maximal Q*-nonnegative (*Q*-nonpositive) subspace in* $\mathcal{H}^2$.

P r o o f is expounded here for certainty in the Q-nonnegative case. It follows from (2.49), (2.50) that $\mathcal{L}$ (2.4), (2.51) is Q-nonnegative.

Since by (2.1), (2.49)

$$\Gamma_2^* \Gamma^* Q_1 \Gamma \Gamma_2 = J, \tag{2.52}$$

one can set up in (2.1) $\Gamma_1 = \Gamma\Gamma_2 \stackrel{def}{=} \Gamma_3$. Once this is done, the operator S for $\mathcal{L}$ (2.4), (2.51) acquires the form

$$S = P_+ \Gamma_3^{-1} + P_- \Gamma_3^{-1} \mathbf{U}. \tag{2.53}$$

Prove that $S^{-1} \in B(\mathcal{H})$. Start with demonstrating that $0 \notin \sigma_p(S) \cup \sigma_c(S)$. If not, there exists a sequence $\{f_n\}$ such that

$$f_n \in \mathcal{H}, \quad \|f_n\| = 1, \quad Sf_n \to 0. \tag{2.54}$$

It follows from (2.53), (2.54) that

$$P_-\Gamma_3^{-1} f_n - \Gamma_3^{-1} f_n \to 0, \qquad P_+\Gamma_3^{-1}\mathbf{U} f_n - \Gamma_3^{-1}\mathbf{U} f_n \to 0, \tag{2.55}$$

whence

$$\begin{aligned} &\Big\{ \big[(J\Gamma_3^{-1}\mathbf{U}f_n, \Gamma_3^{-1}\mathbf{U}f_n) - (J\Gamma_3^{-1}f_n, \Gamma_3^{-1}f_n)\big] \\ &- \big[(JP_+\Gamma_3^{-1}\mathbf{U}f_n, P_+\Gamma_3^{-1}\mathbf{U}f_n) - (JP_-\Gamma_3^{-1}f_n, P_-\Gamma_3^{-1}f_n)\big] \Big\} \to 0. \end{aligned} \tag{2.56}$$

On the other hand, by a virtue of (2.52), the first bracket in (2.56) is just $(\mathbf{U}^* Q_1 \mathbf{U} f_n, f_n) - (Q_1 f_n, f_n)$, hence nonpositive in view of (2.50). By (2.2), the second bracket in (2.56) equals

$$\|P_+\Gamma_3^{-1}\mathbf{U}f_n\|^2 + \|P_-\Gamma_3^{-1}f_n\|^2.$$

Thus we deduce from (2.56) that

$$P_+\Gamma_3^{-1}\mathbf{U}f_n \to 0, \qquad P_-\Gamma_3^{-1}f_n \to 0,$$

whence $f_n \to 0$ by (2.55). The contradiction we get proves that $0 \neq \sigma_p(S) \cup \sigma_c(S)$.

Prove that $0 \notin \sigma_r(S)$. If not, then for some nonzero $f \in \mathcal{H}$ one has

$$\mathbf{U}^*\Gamma_3^{*-1}P_- f = -\Gamma_3^{*-1}P_+ f. \tag{2.57}$$

On the other hand, since the spectrum of $\mathbf{U}$ does not cover the unit circle, it follows from [24, p. 96] that

$$(Q_1^{-1}\mathbf{U}^*\Gamma_3^{*-1}P_- f, \mathbf{U}^*\Gamma_3^{*-1}P_- f) + \big[-(Q_1^{-1}\Gamma_3^{*-1}P_- f, \Gamma_3^{*-1}P_- f)\big] \le 0. \tag{2.58}$$

Now by (2.57), (2.52), (2.2), the first term in (2.58) equals $\|P_+ f\|$, while the second term by (2.52), (2.2) equals $\|P_- f\|^2$, whence $f = 0$. Hence $0 \in \sigma_r(S)$, which finishes the proof in view of Cor. 2.1.

Remark 2.4. *The proof show that condition 3º in the Th. 2.6 is unnecessary, when $Q_j \gg 0$ ($Q_j \ll 0$), $j = 1, 2$, and when $Q_j \ll 0$ ($Q_j \gg 0$), $\mathbf{U}^{-1} \in B(\mathcal{H})$. If Q_j are indefinite or if $Q_j \ll 0$ ($Q_j \gg 0$) it is impossible in general to get rid of 3º.*

In fact, if $Q_1 = Q_2 = W$, $\mathbf{U} = T$, where T, indefinite W see [24, p.67], then (2.50), (≥ 0) holds and hence the linear manifold (2.4), (2.51) is Q-nonnegative, but for it $Ker S^* \neq \{0\}$. Hence (2.4), (2.51) isn't maximal by Th. 2.1. If $\mathcal{H} = l^2$, $Q_1 = Q_2 = -I(I)$, $\mathbf{U}$ is the one-side shift in l^2 [28], then (2.50) (with $= 0$) holds and for (2.4), (2.51) $Ker S^*(S_1^*) \neq \{0\}$. Hence (2.4), (2.51) isn't maximal by Th. 2.1.

Lemma 2.7. *Let A_j, $j = 1, 2$, be linear operators in $\mathcal{H}$, $D_{A_j} = D$, $(-1)^j(Q_j A_j f, A_j f) \leq 0$ (hence $\mathcal{L}$ (2.4) is a Q-nonnegative manifold in $\mathcal{H}^2$), and suppose $\mathcal{L}$ (2.4) is a maximal Q-nonnegative subspace in $\mathcal{H}^2$ (hence $\mathcal{L}_j = \{A_j f | f \in \mathcal{D}\}$ are maximal $(-1)^j Q_j$-nonpositive subspaces in $\mathcal{H}$). Then*

$$\mathcal{L}^{[Q]} = \mathcal{L}_1^{[Q_1]} \oplus \mathcal{L}_2^{[Q_2]}, \tag{2.59}$$

where $[A]$ stands for the A-orthogonal complement in the associated Hilbert subspace.

P r o o f. Since $\mathcal{L}$ is a maximal Q-nonnegative subspace, one deduces by [25, p. 73] that $\mathcal{L}^{[Q]}$ is a maximal Q-nonpositive subspace:

$$\mathcal{L}^{[Q]} = (\mathcal{L}_1 \oplus \mathcal{L}_2)^{[Q]} \supseteq \mathcal{L}_1^{[Q_1]} \oplus \mathcal{L}_2^{[Q_2]}, \tag{2.60}$$

with $\mathcal{L}_j^{[Q_j]}$ being maximal $(-1)^j Q_j$-nonnegative subspaces by [25, p. 73]. Thus by an analogue of Lemma 2.2 for the Q-nonpositive case, the subspace in the right hand side of the inclusion (2.60) is maximal Q-nonpositive. Hence the equality in (2.59), together with the Lemma, is proved.

The case $P = I$ in Th. 2.4 is supplemented by

Theorem 2.7. *Let $\mathcal{P}$ be linear operator in $\mathcal{H}$. Set*

$$A_1 = \mathcal{P} - I, \qquad A_2 = \mathcal{P}. \tag{2.61}$$

1^o. Suppose $\mathcal{L}$ (2.4), (2.61) is a maximal G_2-nonnegative subspace in $\mathcal{H}^2$ *, hence, in particular,

$$(GA_1 f, A_1 f) - (GA_2 f, A_2 f) \geq 0, \;\; f \in D_{\mathcal{P}}. \tag{2.62}$$

Let inequality (2.62) is separated, i.e., is equivalent to the pair of inequalities being simultaneously satisfied:

$$(-1)^j (GA_j f, A_j f) \leq 0, \qquad j = 1, 2; \; f \in D_{\mathcal{P}}. \tag{2.63}$$

*By a virtue of Th. 2.4, this is equivalent to maximal dissipativity of M_i (2.36), (2.35), ($\mathcal{P}_i = \mathcal{P}$, $P = I$), hence $\overline{D}_{\mathcal{P}} = \mathcal{H}$.

Then

$$D_{\mathcal{P}^2} = D_{\mathcal{P}}, \quad \mathcal{P}^2 = \mathcal{P}, \tag{2.64}$$

that is, $\mathcal{P}$ is an idempotent.

2°. Conversely, let $\mathcal{L}$ (2.4), (2.61) be G_2-nonnegative, that is, (2.62) holds, and let $\mathcal{P}$ be an idempotent, i.e., (2.64) holds.

Then (2.62) is separated, that is, (2.63) holds.

P r o o f. 1^o. Lemmas 2.7, 2.3 imply

$$\mathcal{L}_1^{[G]} \oplus \mathcal{L}_2^{[G]} = \mathcal{L}^{[G_2]} \supseteq \left\{-G^{-1}\mathcal{P}^*Gg \oplus (I - G^{-1}\mathcal{P}^*G)g | g \in D_{\mathcal{P}^*G}\right\}.$$

It follows that

$$\mathcal{L}_1^{[G]} \supseteq \left\{G^{-1}\mathcal{P}^*Gg | g \in D_{\mathcal{P}^*G}\right\},$$

hence one has

$$((\mathcal{P} - I)f, \mathcal{P}^*h) = 0, \qquad \forall f \in D_{\mathcal{P}},\ h \in D_{\mathcal{P}^*}. \tag{2.65}$$

On the other hand, since the operator

$$M = (\mathcal{P} - \frac{1}{2}I)(iG)^{-1} \tag{2.66}$$

is maximal dissipative by Th. 2.4, $\mathcal{P}$ is densely defined, closed*, hence [30, p. 335] $\mathcal{P}^*$ is densely defined, and $\mathcal{P}^{**} = \mathcal{P}$. Thus (2.65) means that $(\mathcal{P} - I)f \in D_{\mathcal{P}^{**}} = D_{\mathcal{P}}$ and

$$\mathcal{P}(\mathcal{P} - I)f = 0, \qquad \forall f \in D_{\mathcal{P}},$$

which proves (2.64).

2^o. Set up subsequently in (2.62), (2.61) $f = \mathcal{P}h$, $h \in D_{\mathcal{P}}$, and $f = (\mathcal{P} - I)h$, we obtain (2.63) in view of (2.64). The theorem is proved.

Replace G with $-G$ to see that an analogue for Th. 2.7 is valid for G_2-nonpositive $\mathcal{L}$ (2.4), (2.61).

For $\mathcal{P} \in B(\mathcal{H})$ Th. 2.7 is contained in [1].

Remark 2.5. *There exists a maximal G_2-nonnegative subspace of the form $\mathcal{L}$ (2.4), (2.61), with $\mathcal{P}$ being an unbounded idempotent, defined densely in $\mathcal{H}$.*

In fact, represent $M(\lambda)$ (1.104), (1.103), (1.102) in the form (1.20) and set $\mathcal{P} = \mathcal{P}(i)$. As the operator $M(\lambda)$ (1.104) is maximal dissipative if $Im\lambda > 0$, it follows from Th. 2.4 that $\mathcal{P}$ is the desired idempotent.

Theorem2.7 implies

*Closeness of $\mathcal{P}$ also follows from the fact that $\mathcal{L}$ (2.4), (2.61) is subspace (see the footnote to (2.37)).

Corollary 2.3. *Let for linear operators $\mathcal{A}_1$, $\mathcal{A}_2$ in $\mathcal{H}$ the following conditions hold: 1) $D_{\mathcal{A}_1} = D_{\mathcal{A}_2} = \mathcal{D}$, 2) $(\mathcal{A}_2 + \mathcal{A}_1)^{-1} \in B(\mathcal{H})$ $((\mathcal{A}_2 - \mathcal{A}_1)^{-1} \in B(\mathcal{H}))$ and hence one can define an operator*

$$\mathcal{P} = \mathcal{A}_2(\mathcal{A}_2 + \mathcal{A}_1)^{-1} \ \ (\mathcal{P} = \mathcal{A}_2(\mathcal{A}_2 - \mathcal{A}_1)^{-1}), \tag{2.67}$$

3) $(-1)^j(G\mathcal{A}_jf, \mathcal{A}_jf) \le 0, f \in \mathcal{D}, j = 1,2$, and hence by Lemma 2.4 an operator M (2.66), (2.67) is dissipative (see [25]), 4) an operator M (2.66), (2.67) is maximal dissipative.

Then (2.64) holds for $\mathcal{P}$ (2.67).

For $\mathcal{A}_1$, $\mathcal{A}_2 \in B(\mathcal{H})$ Cor. 2.2 is contained in [1].

Next consider $\mathcal{L}$ (2.4) with the operators $A_j = A_j(\lambda)$ depending analytically on λ.

Suppose one has operator functions $A_j = A_j(\lambda)$, $j = 1,2$, in $\mathcal{H}$ (possibly unbounded and not densely defined), with λ varying in a domain $\Lambda \subseteq C$, and assume that $\mathcal{D}_{A_j} = D$ does not depend on j and λ.

Lemma 2.8. *Suppose that the vector functions $A_j(\lambda)f$, $j = 1,2$, depend analytically on $\lambda \in \Lambda$ for all $f \in D$. With $S = S(\lambda)$, $S = S_1(\lambda)$ being the vector functions associated to $A_j = A_j(\lambda)$ by (2.5), assume that for $\lambda \in \Lambda$:*

1^o. $R(S(\lambda)) = \mathcal{H}$, $(R(S_1(\lambda)) = \mathcal{H})$.

2^o. There exists $K(\lambda) \in B(\mathcal{H})$ such that $S_1(\lambda) = K(\lambda)S(\lambda)$ $(S(\lambda) = K(\lambda)S_1(\lambda))$, with $\|K(\lambda)\|$ being locally bounded.

Then $K(\lambda)$ depends analytically on $\lambda \in \Lambda$.

P r o o f is expounded here for certainty in the case $S_1(\lambda) = K(\lambda)S(\lambda)$. First prove that the operator-valued function $K(\lambda)$ is strongly continuous at any $\lambda_0 \in \Lambda$.

Denote by Δy an increment of the operator function $y = y(\lambda)$ at λ_0. For all $f \in \mathcal{H}$ one has

$$(\Delta S_1)f = (\Delta(KS))f = (\Delta K)S(\lambda_0 + \Delta\lambda)f + K(\lambda_0)(\Delta S)f,$$

whence

$$(\Delta K)S(\lambda_0 + \Delta\lambda)f \to 0 \tag{2.68}$$

as $\Delta\lambda \to 0$ by continuity of $S(\lambda)f$ and $S_1(\lambda)f$. On the other hand,

$$\|(\Delta K)(\Delta S)f\| \le \|\Delta K\|\|(\Delta S)f\| \to 0 \tag{2.69}$$

as $\Delta\lambda \to 0$ by local boundedness of $\|K(\lambda)\|$. It follows from (2.68), (2.69) that $(\Delta K)S(\lambda_0)f \to 0$ as $\Delta\lambda \to 0$, hence $K(\lambda)$ is strongly continuous at λ_0 since $R(S(\lambda_0)) = \mathcal{H}$.

Now prove that $K(\lambda)$ is analytic at λ_0. Since for all $f \in \mathcal{H}$

$$\frac{\Delta(KS)}{\Delta\lambda} f = \frac{\Delta K}{\Delta\lambda} S(\lambda_0) f + K(\lambda_0 + \Delta\lambda) \frac{\Delta S}{\Delta\lambda} f,$$

one can take into account that as $\Delta\lambda \to 0$ one has $K(\lambda_0 + \Delta\lambda) f \xrightarrow{S} K(\lambda_0)$,

$$\frac{\Delta(KS)}{\Delta\lambda} f = \frac{\Delta S_1}{\Delta\lambda} f \to \frac{d}{d\lambda}(S_1 f), \qquad \frac{\Delta S}{\Delta\lambda} f \to \frac{d}{d\lambda}(S f).$$

This allows one to deduce that there exists $\lim\limits_{\Delta\lambda \to 0} \frac{\Delta K}{\Delta\lambda} g$ for all $g \in \mathcal{H}$ since $R(S(\lambda_0)) = \mathcal{H}$. Thus for all $g, h \in \mathcal{H}$ the scalar function $(K(\lambda) g, h)$ is analytic in the domain Λ, hence [30, p. 195] $K(\lambda)$ is analytic in Λ. The Lemma is proved.

Theorem 2.8. *Suppose that the vector-functions $A_j f = A_j(\lambda) f$, $j = 1, 2$, are analytic in $\lambda \in \Lambda$, for all $f \in D$, and assume $\mathcal{L} = \mathcal{L}(\lambda)$ (2.4) for $\lambda \in \Lambda$ is a maximal Q-nonnegative (Q-nonpositive) subspace, hence, in particular,*

$$U_1(\lambda, f) - U_2(\lambda, f) \geq 0 \ (\leq 0), \qquad \lambda \in \Lambda, \tag{2.70}$$

with $U_j(\lambda, f) = (Q_j A_j(\lambda) f, A_j(\lambda) f)$, $f \in D$.

Then: 1°. If for some $\lambda = \lambda_0 \in \Lambda$, for all $f \in D$ one has an equality in (2.70), then this equality also holds for all $\lambda \in \Lambda$.

If, in addition, for some $\lambda = \mu_0 \in \Lambda$ and all $f \in D$ the inequality (2.70) is separated, i.e., it is equivalent to the following two inequalities being valid simultaneously:

$$U_1(\lambda, f) \geq 0 \ (\leq 0), \qquad U_2(\lambda, f) \leq 0 \ (\geq 0), \tag{2.71}$$

then (2.70) is separated for all $\lambda \in \Lambda$.

2°. Suppose that $A_j(\lambda) \in B(\mathcal{H})$ for $\lambda \in \Lambda$ and (2.26) holds. Then if at some $\lambda = \lambda_0 \in \Lambda$ for all nonzero $f \in \mathcal{H}$ one has a strict inequality in (2.70), then the strict inequality also holds for all $\lambda \in \Lambda$ and all nonzero $f \in \mathcal{H}$.

P r o o f is expounded here for certainty in the Q-nonnegative case.

1°. By Th. 2.1, $A_j = A_j(\lambda)$ admits representations (2.7), (2.8), with $K_+ = K_+(\lambda)$ being a compression in $\mathcal{H}$ which depends analytically on $\lambda \in \Lambda$ by Lemma 2.8. If we have an equality in (2.70) at $\lambda = \lambda_0$, then it follows from Remark 2.3 (alternatively, by (2.16)) that $K_+(\lambda_0)$ is an isometry. Hence one can use e.g., [35, p. 210] to deduce that $K_+(\lambda) = K_+(\lambda_0)$, for all $\lambda \in \Lambda$, which implies equality in (2.70) for all $\lambda \in \Lambda$ by Remark 2.3 (alternatively, by (2.16)).

Suppose that at $\lambda = \mu_0$ (2.70) is separated. Assume that the operators Γ_j in (2.7), (2.8) are chosen so that (2.1), (2.27) hold. Then by Th. 2.3, $K_+(\mu_0)$ is of the form (2.29), hence by the above argument, $K_+(\lambda) = K_+(\mu_0)$ is of the same form. Thus by Th. 2.3 the inequality (2.70) is separated for all $\lambda \in \Lambda$, which proves 1^o.

2^0. Suppose that for $\lambda = \lambda_0$, for all nonzero $f \in \mathcal{H}$ one has strict inequality in (2.70), but there exist $\lambda = \gamma_0 \in \Lambda$ and a nonzero $f = f_0 \in \mathcal{H}$ which make (2.70) an equality. Thus $\|K_+(\gamma_0)S(\gamma_0)f_0\| = \|S(\gamma_0)f_0\|$ by (2.16), where $S^{-1}(\lambda) \in B(\mathcal{H})$ for all $\lambda \in \Lambda$ in view of Th. 2.2. Hence it follows from [35, p. 210] that for all $\lambda \in \Lambda$

$$K_+(\lambda)S(\gamma_0)f_0 = S(\gamma_0)f_0,$$

whence

$$K_+(\lambda_0)S(\lambda_0)g_0 = S(\lambda_0)g_0. \tag{2.72}$$

with $g_0 = S^{-1}(\lambda_0)S(\gamma_0)f_0 \neq 0$. Now (2.72) implies that (2.70) becomes equality with $\lambda = \lambda_0$, $f = g_0$ in view of (2.16) . The contradiction we get demonstrates that 2^o and the theorem are proved.

Remark 2.6. *Suppose we are under assumptions of Th. 2.8 which precede its $n^o 1^o$, and suppose that for all $\lambda \in \Lambda$ (2.70) (≥ 0 or ≤ 0) is a strict inequality with some $f = f(\lambda) \in D$. Then the assumption that (2.70) is separated for some $\lambda = \mu_0 \in \Lambda$ does not imply its separation for all $\lambda \in \Lambda$.*

In fact, let

$$Q_1 = Q_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \left(Q_1 = Q_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right),$$
$$A_1 = \begin{pmatrix} \lambda - i & \lambda + i \\ 0 & 1 \end{pmatrix}, \; A_2 = \begin{pmatrix} -\lambda - i & i - \lambda \\ 1 & 0 \end{pmatrix}. \tag{2.73}$$

Then with $Im\lambda > 0$, $\mathcal{L}$ (2.4), (2.73) is a maximal Q-positive (Q-negative) subspace such that the associated inequality (2.70) separates only at $\lambda = i$.

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