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Generalized Resolvents of Symmetric Relations Generated on Semi-Axis by a Differential Expression and a Nonnegative Operator Function

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Generalized resolvents of a minimal symmetric relation generated on the semi-axis by a formally selfadjoint differential expression and a nonnegative operator function are described.

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1. Introduction

In [1], A.V. Straus described the generalized resolvents of the symmetric operator generated by a formally selfadjoint differential expression of even order in a scalar case. In [2] these results were used for the operator case. A differential expression with a nonnegative weight generates a linear relation. This relation is not an operator, in general. The generalized resolvents formulae for these relations are given in [3–5]. However, in these papers either the finite-dimensional case [3, 5] or the infinite-dimensional case [3, 4] under conditions that the kernel (the null space) of the maximal relation contained only solutions of the corresponding homogeneous equation was considered. In our paper a general situation is considered. We use projective and inductive limits of special spaces in the singular case to construct the spaces where a characteristic operator function acts. We consider the case of semi-axis instead of the general singular case only to simplify notations. The detailed bibliography is given in [1–5] and in the monograph [6].

2. Notations and Auxiliary Formulae

Let H be a separable Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$; $A(t)$ be an operator function strongly measurable on the interval $[a,\infty)$; the values of $A(t)$ are bounded operators in H such that for all $x \in H$ the scalar product $(A(t)x,x) \geq 0$ almost everywhere. Suppose the norm $\|A(t)\|$ is integrable on every compact interval $[a,\beta] \subset [a,\infty)$.

We denote by l the differential expression of order r ($r = 2n$ or $r = 2n+1$):

$$l[y] =$$

$$\begin{cases} \sum\limits_{k=1}^{n} (-1)^k \{(p_{n-k}(t)y^{(k)})^{(k)} - i[(q_{n-k}(t)y^{(k)})^{(k-1)} + (q_{n-k}(t)y^{(k-1)})^{(k)}]\} + p_n(t)y, \\ \sum\limits_{k=0}^{n} (-1)^k \{i[(q_{n-k}(t)y^{(k)})^{(k+1)} + (q_{n-k}(t)y^{(k+1)})^{(k)}] + (p_{n-k}(t)y^{(k)})^{(k)}\}. \end{cases}$$

Coefficients of l are bounded selfadjoint operators in H. The leading coefficients, $p_0(t)$ in the case of $r = 2n$ and $q_0(t)$ in the case of $r = 2n+1$, have the bounded inverse operator almost everywhere. The functions $p_{n-k}(t)$ are strongly differentiable k times and the functions $q_{n-k}(t)$ are strongly differentiable k times in the case $r = 2n$, and $k+1$ times in the case $r = 2n+1$. In general, we do not assume the coefficients of the expression l to be smooth as we have just said. According to [7] we treat l as a quasidifferential expression. Quasi-derivatives for the expression l are defined in [7]. Suppose the functions $p_j(t)$, $q_m(t)$ are strongly measurable, the function $q_0(t)$ in the case $r = 2n+1$ is strongly differentiable, and the norms of functions

$p_0^{-1}(t),\ p_0^{-1}(t)q_0(t),\ q_0(t)p_0^{-1}(t)q_0(t),\ p_1(t),\ \ldots,\ p_n(t),\ q_0(t),\ \ldots,q_{n-1}(t)$
(in the case $r = 2n$),
$q_0'(t),\ q_1(t),\ \ldots,q_n(t),\ p_0(t),\ \ldots,p_n(t)$
(in the case $r = 2n+1$)

are integrable on every compact interval $[a,\beta] \subset [a,\infty)$.

We define the scalar product

$$\langle y_1, y_2 \rangle = \int\limits_a^\infty (A(t)y_1(t), y_2(t))dt,$$

where $y_i(t)$ are H-valued functions continuous on $[a,\infty)$, and $\int\limits_a^\infty \left\|A^{1/2}(t)y_i(t)\right\|^2 dt < \infty$, $i = 1,2$. By identifying with zero the functions y such that $\langle y,y \rangle = 0$ and making the completion, we obtain the Hilbert space. We denote this space by $\mathrm{B} = L_2(H, A(t); a, \infty)$. Let $\tilde{y}$ be some element belonging to B, i.e., $\tilde{y}$ is a corresponding class of functions. If $y_1, y_2 \in \tilde{y}$, then y_1, y_2 are identified with respect to the norm generated by the scalar product $\langle \cdot, \cdot \rangle$. By $\tilde{y}$ we denote the class of functions containing y. Suppose $y \in \tilde{y}$. Without loss of generality, further we will often say that $y(t)$ belongs to B.

Let $(a_0, b_0) \subset [a, \infty)$ and $\mathrm{B}_0 = L_2(H, A(t); a_0, b_0)$. If $\tilde{y} \in \mathrm{B}_0$, then extending y by zero to the whole interval $[a, \infty)$ we can consider that $\tilde{y} \in \mathrm{B}$. If $\tilde{y} \in \mathrm{B}$, then restricting y to the interval (a_0, b_0) we can consider that $\tilde{y} \in \mathrm{B}_0$ (it is not excepted that $\tilde{y} \neq 0$ in B and $\tilde{y} = 0$ in B_0).

Let $G(t)$ be the set of elements $x \in H$ such that $A(t)x = 0$, and $H(t)$ be the orthogonal complement of $G(t)$ in H, $H = H(t) \oplus G(t)$, and $A_0(t)$ be the restriction of $A(t)$ to $H(t)$. Suppose $H_\tau(t)$, $-\infty < \tau < \infty$, is the Hilbert scale of spaces [8, Ch. 2] generated by the operator $A_0^{-1}(t)$. For the fixed t, operator $A_0^{1/2}(t)$ is a continuous one-to-one mapping of $H(t) = H_0(t)$ onto $H_{1/2}(t)$. We denote the adjoint operator of $A_0^{1/2}(t)$ by $\hat{A}_0^{1/2}(t)$. The operator $\hat{A}_0^{1/2}(t)$ is a continuous one-to-one mapping of $H_{-1/2}(t)$ onto $H(t)$ and $\hat{A}_0^{1/2}(t)$ is an extension of $A_0^{1/2}(t)$. Let $\tilde{A}_0(t) = A_0^{1/2}(t)\hat{A}_0^{1/2}(t)$. The operator $\tilde{A}_0(t)$ is a continuous one-to-one mapping of $H_{-1/2}(t)$ onto $H_{1/2}(t)$ and $\tilde{A}_0(t)$ is an extension of $A_0(t)$. We denote $\tilde{A}(t)$ (respectively $\tilde{A}^{1/2}(t)$) the operator defined on $H_{-1/2} \oplus G(t)$ such that $\tilde{A}(t)$ ($\tilde{A}^{1/2}(t)$) is equal to $\tilde{A}_0(t)$ (respectively $\hat{A}_0^{1/2}(t)$) on $H_{-1/2}(t)$ and $\tilde{A}(t)(\tilde{A}^{1/2}(t))$ is equal to zero on $G(t)$. The operator $\tilde{A}(t)$ ($\tilde{A}^{1/2}(t)$) is an extension of $A(t)$ ($A^{1/2}(t)$ respectively).

In [3] it is proved that spaces $H_{-1/2}(t)$ are measurable with respect to parameter t [9, Ch. 1] whenever we take functions of the form $\tilde{A}_0^{-1}(t)A^{1/2}(t)h(t)$ instead of measurable functions, where $h(t)$ is a measurable H-valued function. The space B is a measurable sum of spaces $H_{-1/2}(t)$ and B consists of elements (i.e., classes of functions) with representatives of the form $\tilde{A}_0^{-1}(t)A^{1/2}(t)h(t)$, where $h(t) \in L_2(H; a, \infty)$, i.e., $\int_a^\infty \|h(t)\|^2 \, dt < \infty$. If y_1, y_2 are representatives of the class of functions $\tilde{y} \in \mathrm{B}$, then $\tilde{A}^{1/2}(t)y_1(t)$, $\tilde{A}^{1/2}(t)y_2(t)$ are the same functions in the space $L_2(H; a, \infty)$. We denote this function by $\tilde{A}^{1/2}(t)\tilde{y}$.

We define minimal and maximal relations generated by the expression l and the function $A(t)$ in the following way. Let D_0' be the set of finite on $(a; \infty)$ functions y satisfying the following conditions: a) the quasi-derivatives $y^{[0]}, \ldots, y^{[r]}$ of function y exist, they are absolutely continuous up to the order $r - 1$; b) $l[y](t) \in H_{1/2}(t)$ almost everywhere; c) the function $\tilde{A}_0^{-1}(t)l[y]$ belongs to B. To each class of functions identified in B with $y \in D_0'$ we assign the class of functions identified in B with $\tilde{A}_0^{-1}(t)l[y]$. This correspondence L_0' may not be an operator as it may happen that some function y is identified with zero in B and $\tilde{A}_0^{-1}(t)l[y]$

is not equal to zero. So, we get a linear relation L_0' in the space B. The closure of L_0' we denote by L_0. The relation L_0 is called as a minimal one. Let L_0^* be the relation adjoint of L_0. L_0^* is called the maximal relation.

Terminology concerning linear relations can be found in the monographs [6, 8]. Further the following notations are used: R as a range of values; $\{\cdot,\cdot\}$ as an ordered pair.

We consider the differential equation $l[y] = \lambda A(t)y$, where λ is a complex number. Let $W_j(t,\lambda)$ be the operator solution of this equation satisfying the initial conditions: $W_j^{[k-1]}(a,\lambda) = \delta_{jk}E$ (E is the identity operator, δ_{jk} is the Kronecker symbol, $j,k = 1,\ldots,r$). By $W(t,\lambda)$ we denote the one-row operator matrix $(W_1(t,\lambda),\ldots,W_r(t,\lambda))$. The operator $W(t,\lambda)$ maps continuously H^r into H for fixed t, λ. The adjoint operator $W^*(t,\lambda)$ maps continuously H into H^r. If $l[y]$ exists for the function y, then we denote $\hat{y} = (y, y^{[1]},\ldots,y^{[r-1]})$ (we treat $\hat{y}$ as a one-columned matrix). Let $z = (z_1,\ldots,z_m)$ be some system of functions such that $l[z_j]$ exists for each j. By $\hat{z}$ we denote the matrix $(\hat{z}_1,\ldots,\hat{z}_m)$. The analogous notations are used for the operator functions.

We consider the operator matrices of orders $2n$ and $2n+1$ for the expression l in cases $r = 2n$ and $r = 2n+1$ respectively:

$$J_{2n}(t) = \begin{pmatrix} & & & & & -E \\ & & & & \cdots & \\ & & & -E & & \\ & & E & & & \\ & \cdots & & & & \\ E & & & & & \end{pmatrix},$$

$$J_{2n+1}(t) = \begin{pmatrix} & & & & & & -E \\ & & & & & \cdots & \\ & & & & -E & & \\ & & & 2iq_0^{-1}(t) & & & \\ & & E & & & & \\ & \cdots & & & & & \\ E & & & & & & \end{pmatrix},$$

where all the elements, that are not indicated, are equal to zero. (In matrix $J_{2n+1}(t)$ the element $2iq_0^{-1}(t)$ stands on the intersection of the row $n+1$ and the column $n+1$.) Suppose the expression l is defined for the functions y, z, then, in these notations, Lagrange's formula has the following form:

$$\int_\alpha^\beta (l[y],z)dt - \int_\alpha^\beta (y,l[z])dt = (J_r(t)\hat{y}(t),\hat{z}(t))\Big|_\alpha^\beta, \quad a \le \alpha < \beta < \infty. \qquad (1)$$

It follows from "method of the variation of arbitrary constants" that general solution of the equation

$$l[y] - \lambda \tilde{A}(t)y = \tilde{A}(t)f(t)$$

is represented in the form:

$$y(t) = W(t,\lambda)\left(c + J_r^{-1}(a)\int_a^t W^*(s,\bar{\lambda})\tilde{A}(s)f(s)ds\right), \tag{2}$$

where $c \in H^r$. Consequently,

$$\hat{y}(t) = \hat{W}(t,\lambda)\left(c + J_r^{-1}(a)\int_a^t W^*(s,\bar{\lambda})\tilde{A}(s)f(s)ds\right). \tag{3}$$

3. Construction of a Space Containing the Range of the Characteristic Operator Function $M(\lambda)$

Let Q_0 be a set of elements $c \in H^r$ such that function $W(t,0)c$ is identified with zero in the space B, i.e., $\int_a^\infty \left\|A^{1/2}(s)W(s,0)c\right\|^2 ds = 0$. It follows from the equalities

$$W(t,\lambda)c = W(t,0)\left(c + \lambda J_r^{-1}(a)\int_a^t W^*(s,0)\tilde{A}(s)W(s,\lambda)cds\right), \tag{4}$$

$$W(t,0)c = W(t,\lambda)\left(c - \lambda J_r^{-1}(a)\int_a^t W^*(s,\bar{\lambda})\tilde{A}(s)W(s,0)cds\right) \tag{5}$$

that the function $W(t,\lambda)c$ is identified with zero in the space B if and only if $c \in Q_0$ (in the finite-dimensional case this fact was obtained in [7]). By Q we denote an orthogonal complement of Q_0 in H^r, $H^r = Q \oplus Q_0$.

Let $[a,\beta_m]$, $m = 1,2,\ \ldots$, be a system of intervals such that $[a,\beta_m] \subset [a,\beta_{m+1})$ and $\beta_m \to \infty$ as $m \to \infty$. We denote $\mathrm{B}_m = L_2(H,A(t);a,\beta_m)$. Suppose $Q_0(m)$ is the set of elements $c \in Q$ such that the function $W(t,\lambda)c$ is identified with zero in the space B_m, i.e., $\int_a^{\beta m} \left\|A^{1/2}(s)W(s,\lambda)c\right\|^2 ds = 0$. It follows from (4), (5) that $Q_0(m)$ does not depend on λ. Let $Q(m)$ be the orthogonal complement of $Q_0(m)$

in Q, i.e., $Q = Q(m) \oplus Q_0(m)$. Obviously, $Q_0(1) \supset Q_0(2) \supset \ldots \supset Q_0(m) \supset \ldots$ and $Q(1) \subset Q(2) \subset \ldots \subset Q(m) \subset \ldots \subset Q$.

We define the quasiscalar product

$$(c,d)_-^{(i)} = \int\limits_a^{\beta i} (\tilde{A}(s)W(s,0)c, W(s,0)d)ds, \quad c,d \in Q,$$

in space Q. This quasiscalar product generates the semi-norm

$$\|c\|_-^{(i)} = \left(\int\limits_a^{\beta_i} \left\| A^{1/2}(s)W(s,0)c \right\|^2 ds \right)^{1/2} \leq \gamma \|c\|, \quad c \in Q, \ \gamma = \gamma(i) > 0. \tag{6}$$

Clearly, $\|\cdot\|_-^i \leq \|\cdot\|_-^{i+1}$.

Note that if $c \in Q(m)$, then $\|c\|_-^{(m)} > 0$ for $c \neq 0$. Therefore the semi-norm $\|\cdot\|_-^i$ is a norm on the set $Q(m)$ for $i \geq m$. By $Q_-^{(i)}(m)$ we denote the completion of $Q(m)$ with respect to this norm. It follows from (4), (5) that we obtain the same set $Q_-^{(i)}(m)$ with the equivalent norm whenever we replace $W(s,0)$ by $W(s,\lambda)$ in (6). The inclusion map $Q_-^{(k)}(m) \subset Q_-^{(i)}(m)$ is continuous for $k \geq i \geq m$. We denote $Q_-(m) = Q_-^{(m)}(m)$.

Let $\ker(a, \beta_m, \lambda)$ be a closure of the set of elements (i.e., of classes of functions) in the space B_m with the representatives of the form $W(t,\lambda)x$, where $x \in Q(m)$. (We denote these classes by $\tilde{W}(t,\lambda)x$.) It follows from (4–6) that the operator $c \to \tilde{W}(t,\lambda)c$ ($c \in Q_-(m)$) is the continuous one-to-one mapping of $Q_-(m)$ onto $\ker(a, \beta_m, \lambda)$. By $W_m(\lambda)$ we denote this operator. Here $\tilde{W}(t,\lambda)c$ is the class of functions such that the sequence $\{\tilde{W}(t,\lambda)c_k\}$ converges to $\tilde{W}(t,\lambda)c$ in the space B_m whenever $\{c_k\}$ converges to c in the space $Q_-(m)$.

By $Q(n,m)$ we denote the orthogonal complement of $Q(m)$ in $Q(n)$ for $n > m$, i.e., $Q(n) = Q(m) \oplus Q(n,m)$. Then

$$Q_-(n) = Q_-^{(n)}(m) \dot{+} Q_-^{(n)}(n,m), \tag{7}$$

where $Q_-^{(n)}(n,m)$ is the completion of $Q(n,m)$ with respect to the norm $\|\cdot\|_-^{(n)}$. Hence denoting

$$\ker(Q(m), a, \beta_n, \lambda) = W_n(\lambda)Q_-^{(n)}(m), \ \ker(Q(n,m), a, \beta_n, \lambda) = W_n(\lambda)Q_-^{(n)}(n,m),$$

we obtain

$$\ker(a, \beta_n, \lambda) = \ker(Q(m), a, \beta_n, \lambda) \dot{+} \ker(Q(n,m), a, \beta_n, \lambda). \tag{8}$$

We define the linear mappings $g_{mn} : \ker(a, \beta_n, \lambda) \to \ker(a, \beta_m, \lambda)$ $(n \geq m)$ in the following way. Let $g_{mn}z = W_m(\lambda) j_{mn} W_n^{-1}(\lambda) z$ for $z \in \ker(Q(m), a, \beta_n, \lambda)$ and $g_{mn} z = 0$ for $z \in \ker(Q(n,m), a, \beta_n, \lambda)$ (here j_{mn} is the inclusion map of $Q_-^{(n)}(m)$ into $Q_-(m)$). It follows from (7), (8) and the properties of the operators $W_k(\lambda)$ that mappings g_{mn} are continuous.

Moreover, we introduce the linear mappings $h_{mn} : Q_-(n) \to Q_-(m)$ $(n \geq m)$ in accordance with (7) in the following way. Since the inclusion map of $Q_-^{(n)}(m)$ into $Q_-(m)$ is continuous, we define $h_{mn}c = j_{mn}c$ whenever $c \in Q_-^{(n)}(m)$, and we define $h_{mn}c = 0$ whenever $c \in Q_-^{(n)}(n,m)$. Mappings h_{mn} are continuous.

By $\ker(a, \infty, \lambda)$ we denote a projective limit of the family $\{\ker(a, \beta_n, \lambda); n \in \mathrm{N}\}$ with respect to mappings g_{mn} and by Q_- we denote a projective limit of the family $\{Q_-(n)); n \in \mathrm{N}\}$ with respect to mappings h_{mn}, i.e.,

$$\ker(a, \infty, \lambda) = \lim(pr) g_{mn} \ker(a, \beta_n, \lambda), \quad Q_- = \lim(pr) h_{mn} Q_-(n).$$

It follows from the definition of projective limit [10, Ch. 2] that Q_- is a subspace of the product $\prod\limits_n Q_-(n)$ and Q_- consists of the elements $c = \{c_n\}$ such that $c_m = h_{mn} c_n$ for all $m \leq n$. Similarly, $\ker(a, \infty, \lambda)$ is a subspace of $\prod\limits_n \ker(a, \beta_n, \lambda)$ and the analogous statement is true in regard to $\ker(a, \infty, \lambda)$. By p_n, p'_n we denote the projections $\prod\limits_n Q_-(n)$ and $\prod\limits_n \ker(a, \beta_n, \lambda)$ onto $Q_-(n)$ and $\ker(a, \beta_n, \lambda)$ respectively.

The mappings g_{mn}, h_{mn} and the operators $W_n(\lambda) : Q_-(n) \to \ker(a, \beta_n, \lambda)$ satisfy the equality: $g_{mn} = W_m(\lambda) h_{mn} W_n^{-1}(\lambda)$. Consequently, the family of operators $\{W_n(\lambda)\}$ generates the isomorphism (i.e., the linear homeomorphism) $W(\lambda) : Q_- \to \ker(a, \infty, \lambda)$. If $c = \{c_n\} \in \prod\limits_n Q_-(n)$, then $W(\lambda)c = \{W_n(\lambda) c_n\}$ and $W(\lambda) Q_- = \ker(a, \infty, \lambda)$. Moreover,

$$p'_n(W(\lambda) Q_-) = W_n(\lambda) p_n(Q_-). \tag{9}$$

Lemma 1. *Let w_n be a representative of the class of functions $\tilde{w}_n = W_n(\lambda) d$ $(d \in Q_-(n))$ and let w_m be restriction of w_n to $[a, \beta_m]$ $(m \leq n)$. Then*

$$\tilde{w}_m = W_m(\lambda) h_{mn} d. \tag{10}$$

P r o o f. According to (7) we represent d in the form $d = d' + d''$, where $d' \in Q_-^{(n)}(m) \subset Q_-(m)$, $d'' \in Q_-^{(n)}(n,m)$. Suppose the sequences $\{d'_k\}$, $\{d''_k\}$ $(d'_k \in Q(m), d''_k \in Q(n,m))$ converge to d', d'' in the spaces $Q_-^{(n)}(m)$, $Q_-^{(n)}(n,m)$ respectively. Then the sequence $\{\tilde{W}(t,\lambda) d'_k\}$ converges to $W_n(\lambda) d'$ in the space B_n. Therefore $\{\tilde{W}(t,\lambda) d'_k\}$ converges to $W_m(\lambda) j_{mn} d'$ in B_m and the functions $W(t,\lambda) d''_k$ are identified with zero in B_m. Hence follows (10). Lemma 1 is proved.

Let $c' \in H^r$. Then the function $\tilde{A}^{1/2}(t)W(t,\lambda)c'$ belongs to $L_2(H; a, \beta_n)$ for all n and $\tilde{A}^{1/2}(t)W(t,\lambda)c'$ coincides with $\tilde{A}^{1/2}(t)W(t,\lambda)c''$ in this space, where $c'' = P_n P_0 c \in Q(n)$, P_0 is the orthogonal projection of H^r onto Q, P_n is the orthogonal projection of Q onto $Q(n)$. Suppose the sequence $\{d_k\}$, $d_k \in Q(n)$, converges to d in the space $Q_-(n)$; then classes of functions $\tilde{W}(t,\lambda)d_k \in \mathrm{B}_n$ with the representatives of $W(t,\lambda)d_k$ converge to the class of functions $\tilde{W}(t,\lambda)d$ in B_n. Therefore functions $\tilde{A}^{1/2}(t)W(t,\lambda)d_k$ converge to the function $z(t) = \tilde{A}^{1/2}(t)\tilde{W}(t,\lambda)d$ in the space $L_2(H; a, \beta_n)$. It follows from (10) that the restriction of $z(t)$ to the interval $[a, \beta_m]$, $m < n$, coincides with $\tilde{A}^{1/2}(t)\tilde{W}(t,\lambda)h_{mn}d$.

Suppose $c = \{c_n\} \in Q_-$; then $c_m = h_{mn}c_n$ $(m \leq n)$. It follows from (10) that the restriction of function $\tilde{A}^{1/2}(t)\tilde{W}(t,\lambda)c_n$ to the interval $[a, \beta_m]$ coincides with the function $\tilde{A}^{1/2}(t)\tilde{W}(t,\lambda)c_m$ in the space $L_2(H; a, \beta_m)$. Therefore by $\tilde{A}^{1/2}(t)\tilde{W}(t,\lambda)c$ we denote the function coinciding with $\tilde{A}^{1/2}(t)\tilde{W}(t,\lambda)c_n$ on any interval $[a, \beta_n]$. Correspondingly, by $\tilde{W}(t,\lambda)c$ we denote the $H_{-1/2}(t) \oplus G(t)$-valued function coinciding with $\tilde{W}(t,\lambda)c_n$ in the spaces B_n for all n. It follows from (9), (10) that $\tilde{W}(t,\lambda)c_n = \tilde{W}(t,\lambda)c_m$ in the space B_m, $m \leq n$.

4. Construction of a Domain of the Characteristic Operator Function $M(\lambda)$

The space $Q_-(n)$ can be treated as a negative one with respect to $Q(n)$. By $Q_+(n)$ we denote a corresponding space with the positive norm. It follows from (7) that $Q_+(n) = Q_+^{(n)}(m) \dot{+} Q_+^{(n)}(n,m)$, where $Q_+^{(n)}(m)$, $Q_+^{(n)}(n,m)$ are the corresponding positive spaces with respect to $Q_-^{(n)}(m)$, $Q(m)$ and $Q_-^{(n)}(n,m)$, $Q(n,m)$. The inclusion $Q_+(m) \subset Q_+^{(n)}(m)$ is dense and continuous. Consequently the inclusion map of $Q_+(m)$ into $Q_+(n)$ is continuous for $m \leq n$.

Suppose $h^+_{nm} : Q_+(m) \to Q_+(n)$, $n \geq m$, is the adjoint mapping of h_{mn}; then h^+_{nm} is the continuous inclusion map of $Q_+(m)$ into $Q_+(n)$. By Q_+ we denote inductive limit [10, Ch. 2] of the family $\{Q_+(n); n \in \mathbf{N}\}$ with respect to mappings h^+_{nm}, i.e., $Q_+ = \lim(ind) h^+_{nm} Q_+(n)$. It follows from [10, Ch. 4] that Q_+ is the adjoint space of Q_-. The space Q_+ can be treated as the union $Q_+ = \bigcup\limits_n Q_+(n)$ with the strongest topology such that all inclusion maps of $Q_+(n)$ into Q_+ are continuous [10, Ch. 2].

Let $\tilde{y} \in \mathrm{B}_m$ and $m \leq n$. Suppose y is a representative of the class of functions $\tilde{y}$, then we can treat $\tilde{y}$ as an element of the space B_n whenever we extend y by zero out of the interval $[a, \beta_m]$. If $m \leq n$, then the space B_m can be treated as a subspace B_n. The topology of B_m is induced by the topology of B_n. Let i_{nm} be the inclusion map of B_m into B_n. By $\tilde{B}$ we denote the inductive limit of the spaces B_n with respect to the mappings i_{nm}, i.e., $\tilde{B} = \lim(ind) i_{nm} \mathrm{B}_n$. The space $\tilde{B}$ can be treated as $\tilde{B} = \bigcup_{n} \mathrm{B}_n$ with the strongest topology such that all inclusion maps of B_n into $\tilde{B}$ are continuous.

Suppose $\{F_n\}$, $n \in \mathbf{N}$, is a family of locally convex spaces such that $F_m \subset F_n$ for $m \leq n$ and this inclusion map is continuous. According to [8, Ch. 1], an inductive limit $F = \lim(ind)F_n$ of the locally convex spaces F_n, $n \in \mathbf{N}$, is called a regular one if for every bounded set $S \subset F$ there is $n \in \mathbf{N}$ such that $S \subset F_n$ and S is a bounded set in F_n. It follows from [8, Ch. 1] that the inductive limits Q_+ and $\tilde{B}$ are regular. According to [10, Ch. 2], the inductive limit of bornological spaces is a bornological space. Since Q_+, $\tilde{B}$ are the inductive limits of the reflexive Banach spaces, we see that Q_+, $\tilde{B}$ are bornological.

Suppose F_n are bornological spaces such that their inductive limit F is regular. Let F_0 be a locally convex space. It follows from [10, Ch. 2] that a linear mapping $u : F \to F_0$ is continuous if and only if for every $n \in \mathbf{N}$ restriction of u to F_n maps every bounded set $S \subset F_n$ into the bounded set $u(S) \subset F_0$. According to [10, Ch. 2, Ex. 17], we can take a bounded sequence instead of the bounded set $S \subset F_n$. Further, these statements will be used for the proof of the continuity of corresponding operators. We take the space Q_- instead of F_0. Then the following conditions are equivalent: (i) the set $u(S)$ is bounded in Q_-; (ii) the sets $p_k u(S)$ are bounded in the spaces $p_k Q_- = Q_-(k)$ for every $k \in \mathbf{N}$; (iii) the sets $W_k(\lambda)p_k u(S)$ are bounded in the spaces B_k for every $k \in \mathbf{N}$; (iiii) the sets consisting of elements of the form $\tilde{W}(t,\lambda)c_k$ are bounded for every $k \in \mathbf{N}$, where $c_k \in p_k u(S) \subset Q_-(k)$.

Thus the following lemma is proved.

Lemma 2. *Suppose the spaces F_n are bornological and their inductive limit F is regular. The linear operator $u : F \to Q_-$ is continuous if and only if for every $n \in \mathbf{N}$ and every bounded set $S \subset F_n$ and every $k \in \mathbf{N}$ the sets consisting of elements of the form $\tilde{W}(t,\lambda)c_k$ are bounded in B_k, where $c_k \in p_k u(S) \subset Q_-(k)$. Any bounded sequence can be taken in place of bounded set S.*

Further, we shall take a family of space $\{Q_+(n)\}$ or $\{\mathrm{B}_n\}$ in place of $\{F_n\}$. Then $F = Q_+$ or $F = \tilde{\mathrm{B}}$ respectively. As it was mentioned above, the operator $W_n(\lambda)$ is a continuous one-to-one mapping of $Q_-(n)$ onto the closed subspace $\ker(a, \beta_n, \lambda)$ of the space B_n. Then the adjoint operator $W_n^*(\lambda)$ maps continuously B_n onto $Q_+(n)$. Therefore $W_n^*(\lambda)$ is the continuous operator of B_n into Q_+. The operator $W_n^*(\lambda)$ has the following form:

$$W_n^*(\lambda)\tilde{f} = \int_a^{\beta_n} W^*(s,\lambda)\tilde{A}(s)f(s)ds = \int_a^{\infty} W^*(s,\lambda)\tilde{A}(s)f(s)ds, \tag{11}$$

where $\tilde{f} \in \mathrm{B}$ and f vanishes outside $[a, \beta_n]$.

We note that the norms $\|W^*(s,\lambda)A^{1/2}(s)\|$, $\|A^{1/2}(s)f(s)\|$ belong to $L_2(a,\beta_n)$. Hence the integral in the right side of (11) exists.

Since $\tilde{B}$ consists of finite functions, in accordance with (11) we can define the operator $W_*(\lambda)$ mapping $\tilde{B}$ onto Q_+ by the formula

$$W_*(\lambda)\tilde{f} = \int_a^\infty W^*(s,\bar{\lambda})\tilde{A}(s)f(s)ds.$$

It follows from the reasoning given before Lemma 2 that the operator $W_*(\lambda)$: $\tilde{B} \to Q_+$ is continuous. Obviously, $W_*(\lambda)\tilde{f} = W_n^*(\bar{\lambda})\tilde{f}$ for $\tilde{f} \in \mathrm{B}_n$.

5. The Main Result

To prove the main theorem we need several lemmas.

Lemma 3. $\tilde{g} \in \mathrm{B}$ *belongs to the range* $R(L_0' - \bar{\lambda}E)$ *of the relation* $L_0' - \bar{\lambda}E$ *if and only if there is an interval* (a,β_n) *such that* g *is finite on* (a,β_n) *and*

$$\int_a^{\beta_n} W^*(s,\lambda)\tilde{A}(s)g(s)ds = 0. \tag{12}$$

P r o o f. Let g be finite and (12) is true. We denote

$$z(t) = W(t,\bar{\lambda})\left(J_r^{-1}(a)\int_a^t W^*(s,\lambda)\tilde{A}(s)g(s)ds\right).$$

From (2), (3), (12) we obtain that the ordered pair $\{\tilde{z},\tilde{g}\} \in L_0' - \bar{\lambda}E$.

Vice versa, let $\{\tilde{z},\tilde{g}\} \in L_0' - \bar{\lambda}E$. It follows from (2), (3) that there is a representative z of the class of functions $\tilde{z}$ such that the equality

$$\hat{z}(t) = \hat{W}(t,\bar{\lambda})\left(c + J_r^{-1}(a)\int_a^t W^*(s,\lambda)\tilde{A}(s)g(s)ds\right)$$

is true, where $c \in Q$. Since the function z is finite, we see that $c = 0$ and g is finite on some interval (a,β_n) and equality (12) is true. Lemma 3 is proved.

R e m a r k. In Lemma 3 we can replace the interval (a,β_n) by any interval such that the function z vanishes out of this interval, where $\{\tilde{z},\tilde{g}\} \in L_0' - \bar{\lambda}E$. Equality (12) and the equality $(\tilde{g},\tilde{W}(t,\lambda)c)_{\mathrm{B}_n} = 0$ are equivalent for all $c \in Q_-(n)$.

Lemma 4. *If the ordered pair* $\{\tilde{y},\tilde{f}\}\in L_0^*-\lambda E$, *then* $\tilde{y}$ *can be represented in the following form:*

$$\tilde{y}(t)=\tilde{W}(t,\lambda)c+\tilde{W}(t,\lambda)J_r^{-1}(a)\int_a^t W^*(s,\bar{\lambda})\tilde{A}(s)f(s)ds$$

$$=\tilde{W}(t,\lambda)\left(c+J_r^{-1}(a)\int_a^t W^*(s,\bar{\lambda})\tilde{A}(s)f(s)ds\right), \qquad (13)$$

where $c\in Q_-$.

P r o o f. We denote

$$u(t)=W(t,\lambda)\left(J_r^{-1}(a)\int_a^t W^*(s,\bar{\lambda})\tilde{A}(s)f(s)ds\right).$$

Let $\{\tilde{z},\tilde{g}\}\in L_0'-\bar{\lambda}E$ and $z(t)=0$ for $t\geq\beta_n$. From Lagrange's formula (1), we obtain

$$\int_a^{\beta_n}(\tilde{A}(s)g(s),u(s))ds-\int_a^{\beta_n}(\tilde{A}(s)z(s),f(s))ds=0.$$

The equality

$$\int_a^{\beta_n}(\tilde{A}(s)g(s),y(s))ds-\int_a^{\beta_n}(\tilde{A}(s)z(s),f(s))ds=0$$

is true for every ordered pair $\{\tilde{y},\tilde{f}\}\in L_0^*-\lambda E$. It follows from the last two equalities that $(\tilde{g},\tilde{y}-\tilde{u})_{\mathrm{B}_n}=0$. Since $\ker(a,\beta_n,\lambda)$ is closed and $g\in R(L_0'-\lambda E)$ is arbitrary, from Lemma 3 and remark we obtain the equality $\tilde{y}-\tilde{u}=\tilde{W}(t,\lambda)c_n$. Since the interval (a,β_n) is taken arbitrarily, we obtain (13) where $c=\{c_n\}\in Q_-$.

Note that Lemmas 3, 4 follow also from paper: V.M. Bruk, J. Math. Phys., Anal., Geom. **2** (2006), 1–10.

Theorem. *Every generalized resolvent* R_λ, $\mathrm{Im}\lambda\neq 0$, *of the relation* L_0 *is the integral operator*

$$R_\lambda\tilde{f}=\int_a^\infty K(t,s,\lambda)\tilde{A}(s)f(s)ds \quad (\tilde{f}\in\mathrm{B}).$$

The kernel $K(t,s,\lambda)$ *has the form*

$$K(t,s,\lambda)=\tilde{W}(t,\lambda)(M(\lambda)-(1/2)\mathrm{sgn}(s-t)J_r^{-1}(a))W^*(s,\bar{\lambda}),$$

where $M(\lambda) : Q_+ \to Q_-$ is the continuous operator such that $M(\bar{\lambda}) = M^*(\lambda)$ and

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(M(\lambda)x, x) \geq 0 \tag{14}$$

for every fixed λ, $\mathrm{Im}\lambda \neq 0$, and for every $x \in Q_+$. The operator function $M(\lambda)x$ is holomorphic for every $x \in Q_+$ in the semi-planes $\mathrm{Im}\lambda \neq 0$.

P r o o f. First, we prove the theorem for the functions finite on (a, ∞) . Suppose $\tilde{f} \in \mathrm{B}$ and f is a finite function. It follows from (13) that $\tilde{y} = R_\lambda \tilde{f}$ has the following form:

$$\tilde{y} = \tilde{y}(t, \tilde{f}, \lambda) = \tilde{W}(t, \lambda)\left(c(\tilde{f}, \lambda) + (1/2)J_r^{-1}(a)\int_a^t W^*(s, \bar{\lambda})\tilde{A}(s)f(s)ds\right.$$

$$\left. - (1/2)J_r^{-1}(a)\int_t^\infty W^*(s, \bar{\lambda})\tilde{A}(s)f(s)ds\right), \tag{15}$$

where $c(\tilde{f}, \lambda) \in Q_-$ and $c(\tilde{f}, \lambda)$ is uniquely determined by $\tilde{f}$ and λ, $\mathrm{Im}\lambda \neq 0$. Indeed, if it is not so, then $\tilde{W}(t, \lambda)c(\tilde{f}, \lambda) = R_\lambda 0 = 0$, and this equality is true whenever $c(\tilde{f}, \lambda) = 0$. Therefore, $c(\tilde{f}, \lambda) = C(\lambda)\tilde{f}$ where $C(\lambda) : \tilde{B} \to Q_-$ is the linear operator.

Now we show that the operator $C(\lambda)$ is continuous for every fixed λ. Let the sequence $\{\tilde{f}_k\}$ $(\tilde{f}_k \in \mathrm{B}_n)$ be bounded in B_n for a fixed number $n \in \mathbf{N}$. Then $\{\tilde{f}_k\}$ is bounded in B. Hence the sequence $\{R_\lambda \tilde{f}_k\}$ is bounded in B. Consequently, $\{R_\lambda \tilde{f}_k\}$ is bounded in B_m. It follows from (15) that the sequence $\{\tilde{W}(t, \lambda)p_m c(\tilde{f}_k, \lambda)\}$ is bounded in B_m. Since $n, m \in \mathbf{N}$ are arbitrary and according to Lemma 2, it follows that the operator $C(\lambda)$ is continuous.

Now we prove that $c(\tilde{f}, \lambda)$ is uniquely determined by the element $W_*(\lambda)\tilde{f} \in Q_+$. We assume that $W_*(\lambda)\tilde{f} = 0$. Then the ordered pair $\{\tilde{z}, \tilde{f}\} \in L_0 - \lambda E$, where

$$\tilde{z}(t) = \tilde{W}(t, \lambda)\left((1/2)J_r^{-1}(a)\int_a^t W^*(s, \bar{\lambda})\tilde{A}(s)f(s)ds\right.$$

$$\left. - (1/2)J_r^{-1}(a)\int_t^\infty W^*(s, \bar{\lambda})\tilde{A}(s)f(s)ds\right).$$

Since $(L_0 - \lambda E)^{-1} \subset R_\lambda$, we obtain that $\tilde{W}(t, \lambda)c(\tilde{f}, \lambda)$ belongs to the range of the operator R_λ. Hence $c(\tilde{f}, \lambda) = 0$.

Thus $C(\lambda) = M(\lambda)W_*(\lambda)\tilde{f}$, where $M(\lambda) : Q_+ \to Q_-$ is an everywhere defined operator. We prove that $M(\lambda)$ is continuous for every fixed λ. Let the sequence $\{q_k\} = \{W_*(\lambda)\tilde{f}_k\}$ be bounded in $Q_+(n)$. By $\mathrm{B}_n^{(0)}$ we denote the orthogonal complement of $\ker W_n^*(\lambda)$ in the space B_n. The operator $W_n^*(\lambda)$ is a continuous one-to-one mapping of $B_n^{(0)}$ onto $Q_+(n)$. Consequently, there exists a bounded sequence $\{\tilde{g}_k\}$ ($\tilde{g}_k \in B_n^{(0)}$) in B_n such that $q_k = W_*(\lambda)\tilde{f}_k = W_n^*(\lambda)\tilde{g}_k$. Then the sequence $\{R_\lambda \tilde{g}_k\}$ is bounded in B. Consequently, $\{R_\lambda \tilde{g}_k\}$ is bounded in B_m for every $m \in \mathbf{N}$. Hence the sequence $\{\tilde{W}(t,\lambda)p_m M(\lambda)q_k\}$ is bounded in B_m. It follows from Lemma 2 that the operator $M(\lambda)$ is continuous.

Now we prove that the function $M(\lambda)x$ is holomorphic ($\mathrm{Im}\lambda \neq 0$) for every $x \in Q_+$. It follows from (15) and holomorphicity of R_λ that the function $\lambda \to \tilde{W}(t,\lambda)p_n C(\lambda)\tilde{f}$ is holomorphic in B_n for every $\tilde{f} \in B_j$, $n, j \in \mathbf{N}$. Substituting $p_n C(\lambda)\tilde{f}$ for c in equality (4), we get that function $\lambda \to \tilde{W}(t,0)p_n C(\lambda)\tilde{f}$ is holomorphic. Since the operator $x \to \tilde{W}(t,0)x$ is a continuous one-to-one mapping of $Q_-(n)$ onto $\ker(a,\beta_n)$ and $\ker(a,\beta_n)$ is closed in B_n, we obtain that $\lambda \to p_n C(\lambda)\tilde{f} = p_n M(\lambda)W_j^*(\bar{\lambda})\tilde{f}$ is the holomorphic function for every $\tilde{f} \in B_j$. Now holomorphicity of function $\lambda \to p_n M(\lambda)x$ follows from the lemma proved in [11].

Lemma 5. *Suppose bounded operators $S_3(\lambda) : B_1 \to B_3$, $S_1(\lambda) : B_1 \to B_2$, $S_2(\lambda) : B_2 \to B_3$ satisfy the equality $S_3(\lambda) = S_2(\lambda)S_1(\lambda)$ for every fixed λ belonging to some neighborhood of a point λ_0 and suppose the range of operator $S_1(\lambda_0)$ coincides with B_2, where B_1, B_2, B_3 are Banach spaces. If functions $S_1(\lambda)$, $S_3(\lambda)$ are strongly differentiable in the point λ_0, then in this point function $S_2(\lambda)$ is strongly differentiable.*

In this lemma it should be taken that $B_1 = B_j$, $B_2 = Q_+(j)$, $B_3 = Q_-(n)$, $S_1(\lambda) = W_j^*(\bar{\lambda})$, $S_2(\lambda) = p_n M(\lambda)$, $S_3(\lambda) = p_n C(\lambda)$.

So, the operator function $\lambda \to p_n M(\lambda)x$ is strongly differentiable for every $n \in \mathbf{N}$ and for every $x \in Q_+$. Now holomorphicity of the operator function $M(\lambda)x$ for every $x \in Q_+$ follows from the closeness of Q_- in the product of spaces $Q_-(n)$ [10, Ch. 2] and from the definition of topology of the product space.

It follows from the equality $R_\lambda^* = R_{\bar{\lambda}}$ that $M(\bar{\lambda}) = M^*(\lambda)$ and

$$\tilde{A}^{1/2}(s)K^*(t,s,\lambda)A^{1/2}(t) = \tilde{A}^{1/2}(s)K(s,t,\bar{\lambda})A^{1/2}(t). \tag{16}$$

Now we show inequality (14). First, we prove the following statement.

Lemma 6. *Suppose $\tilde{u}$, $\tilde{u}_0$, $\tilde{v}$, $\tilde{v}_0 \in B_n$ satisfy the equalities*

$$\begin{aligned} \tilde{u}(t) &= \tilde{W}(t,\lambda)(c + J_r^{-1}(a)\int_a^t W^*(s,\bar{\lambda})\tilde{A}(s)u_0(s)ds), \\ \tilde{v}(t) &= \tilde{W}(t,\lambda)(d + J_r^{-1}(a)\int_a^t W^*(s,\bar{\lambda})\tilde{A}(s)v_0(s)ds), \end{aligned} \tag{17}$$

where $d \in Q_-(n)$, $c = -J_r^{-1}(a) \int\limits_a^{\beta_n} W^*(s, \bar\lambda)\tilde A(s)u_0(s)ds$. *Then*

$$\int\limits_a^{\beta_n} (\tilde A(t)u_0(t), v(t))dt - \int\limits_a^{\beta_n} (\tilde A(t)u(t), v_0(t))dt$$

$$= -(J_r(a)c, d) - (\lambda - \bar\lambda) \int\limits_a^{\beta_n} (\tilde A(t)u(t), v(t))dt. \qquad (18)$$

P r o o f. Since $J_r(a)c \in Q_+(n)$, we see that the right-hand side (18) exists. Let $d_k \in Q(n)$ and the sequence $\{d_k\}$ converges to d as $k \to \infty$ in the space $Q_-(n)$. If we replace d by d_k in (17), then we obtain the function denoted by $\tilde v_k(t)$. The sequence $\{\tilde v_k\}$ converges to $\tilde v$ in the space B_n. We apply Lagrange's formula (1) to the functions u, v_k. From the equalities $\hat u(\beta_n) = 0$, $\tilde A(t)u_0(t) = l[u] - \lambda\tilde A(t)u$, $\tilde A(t)v_0(t) = l[v_k] - \lambda\tilde A(t)v_k$, we obtain the equality of the form (18), where v is replaced by v_k. By calculating to the limit as $k \to \infty$, we obtain (18). The lemma is proved.

In order to prove inequality (14), we take the arbitrary element $x \in Q_+$. Then there is $n \in \mathbf{N}$ such that $x \in Q_+(n)$. Consequently, there exists $\tilde f \in \mathrm{B}_n$ such that

$$\int\limits_a^{\beta_n} W^*(s, \bar\lambda)\tilde A(s)f(s)ds = W_*(\lambda)\tilde f = x.$$

Let $\tilde z(t) = \tilde W(t, \lambda)(M(\lambda)x + (1/2)J_r^{-1}(a)x)$. Suppose $\tilde y = R_\lambda \tilde f$ has the form of (15), where $c(\tilde f, \lambda) = M(\lambda)x$. Having made some elementary transformations we can apply Lemma 6 to the functions $\tilde u = \tilde y - \tilde z$, $\tilde u_0 = \tilde f$, $\tilde v = \tilde y + \tilde z$, $\tilde v_0 = \tilde f$. Then we have

$$\int\limits_a^{\beta_n} (\tilde A(t)f, z + y)dt - \int\limits_a^{\beta_n} (\tilde A(t)(y - z), f)dt$$
$$= 2(x, M(\lambda)x) - (\lambda - \bar\lambda) \int\limits_a^{\beta_n} (\tilde A(t)(y - z), y + z)dt.$$

Consequently,

$$(\mathrm{Im}\lambda)^{-1}\mathrm{Im}(M(\lambda)x, x)$$
$$= (z, z)_{\mathrm{B}_n} + \{(\lambda - \bar\lambda)^{-1}[(R_\lambda\tilde f, \tilde f)_{\mathrm{B}_n} - (\tilde f, R_\lambda\tilde f)_{\mathrm{B}_n}] - (R_\lambda\tilde f, R_\lambda\tilde f)_{\mathrm{B}_n}\}. \qquad (19)$$

The operator function R_λ is a generalized resolvent of the minimal relation generated in the space B_n by the expression l and the function $A(t)$ (the proof is similar to the proof of the corresponding statement for the operator from [1]).

Consequently, the addend in figurate brackets in the right-hand side (19) is nonnegative. Now (14) follows from (19).

Now we assume that $\tilde f \in \mathrm{B}$ is not finite, in general. By V_1 we denote the operator $x \to \tilde W(t,\lambda)(M(\lambda)x + (1/2)J_r^{-1}(a)x)$. The operator V_1 maps continuously $Q_+(n)$ into B for every $n \in \mathbf{N}$. Indeed, for any bounded sequence $\{x_k\}$ in $Q_+(n)$ there exists a bounded sequence $\{\tilde g_k\}$ ($\tilde g_k \in \mathrm{B}_n^{(0)}$) in B_n such that $x_k = W_n^*(\lambda)\tilde g_k$. Then the sequence $\{R_\lambda \tilde g_k\}$ is bounded in B. The functions g_k vanish out of the interval $[a,\beta_n]$. Consequently, the equality

$$R_\lambda \tilde g_k = \tilde W(t,\lambda)(M(\lambda)x_k + (1/2)J_r^{-1}(a)x_k)$$

is true out of the interval $[a,\beta_n]$. Therefore the sequence $\{\tilde W(t,\lambda)(M(\lambda)x_k + (1/2)J_r^{-1}(a)x_k)\}$ is bounded in the space B. This implies that the operator V_1 is continuous. Hence we obtain the inequality

$$\int_a^\infty (\tilde A(t)\tilde W(t,\lambda)(M(\lambda)x + (1/2)J_r^{-1}(a)x, \tilde W(t,\lambda)(M(\lambda)x + (1/2)J_r^{-1}(a)x)dt$$

$$\le k(n,\lambda)\,\|x\|^2_{Q_+(n)}\,, \quad k(n,\lambda) > 0. \tag{20}$$

Now suppose $\tilde f \in \mathrm{B}$ and f is a nonfinite function, in general. We take the sequence $\{\tilde f_n\}$ converging to $\tilde f$ in the space B, where $\tilde f_n \in \mathrm{B}$ and functions f_n are finite. Using (20) and (16), we obtain that for every finite function g ($\tilde g \in \mathrm{B}$) there exists the limit

$$\lim_{n\to\infty}\int_a^\infty (\tilde A^{1/2}(t)\int_a^\infty K(t,s,\lambda)\tilde A(s)f_n(s)ds, \tilde A^{1/2}(t)g(t))dt$$
$$= \lim_{n\to\infty}\int_a^\infty (\tilde A^{1/2}(s)f_n(s), \tilde A^{1/2}(s)\int_a^\infty K(s,t,\bar\lambda)\tilde A(t)g(t)dt)$$
$$= \int_a^\infty (\tilde A^{1/2}(s)f(s), \tilde A^{1/2}(s)\int_a^\infty K(s,t,\bar\lambda)\tilde A(t)g(t)dt).$$

Hence the sequence $\left\{\int_a^\infty K(t,s,\lambda)\tilde A(s)f_n(s)ds\right\}$ converges to $R_\lambda \tilde f$ as $n\to\infty$ at least weakly in the space B. The theorem is proved.

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