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Local Extremums of Trigonometric Polynomial

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Local extremes of the trigonometric polynomial

$$B_n(\theta) = \sum_{k=n}^{k=2n} \frac{\sin k\theta}{k}$$

are considered, and various inequalities between them are proved. In particular, the greatest and the least values of $B_n(\theta)$ are found.

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Let us consider a trigonometric polynomial

$$T_{mn}(\theta) = \frac{\sin m\theta}{m} + \ldots + \frac{\sin n\theta}{n}$$

with a derivative

$$T'_{mn}(\theta) = \cos m\theta + \ldots + \cos n\theta.$$

The local extremes $T_{mn}(\theta)$, obviously, are among the points

$$\alpha_p : \sin \frac{n-m+1}{2}\alpha_p = 0 \quad \text{and} \quad \beta_q : \cos \frac{n+m}{2}\beta_q = 0.$$

For $m = 1$, the sequences $\{\alpha_p\}$ and $\{\beta_q\}$ alternate, thus in points α_p they are local minima, and in points β_q they are local maxima. Moreover, local maxima decrease in q, and for $A_n(\theta) = T_{1,n}(\theta)$ we have ([1, p. 91])

$$A_n(\theta) \leq A_n(\frac{\pi}{n+1}), \quad 0 \leq \theta \leq \pi. \tag{1}$$

In the paper the similar results for local extreme of a polynomial $B_n(\theta) = T_{n,2n}(\theta)$ are obtained. This question arises, for example, when considering the series

$$\sum_{n=0}^{\infty} \epsilon_n T_{2^n, 2^{n+1}-1}, \quad \epsilon_n = \pm 1.$$

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In the proof of estimation (1) in ([1, p. 293]) the following method of averaging is used. Let a, b be two local extremes of $A_n(\theta)$. Consider

$$A_n(b) - A_n(a) = \int_a^b A_n'(\xi)d\xi = \left[c = \frac{a+b}{2}, d = \frac{b-a}{2}\right]$$

$$= \int_a^c [A_n'(\xi) + A_n'(\xi + d)]d\xi.$$

If the sum $[A_n'(\xi)+A_n'(\xi+d)]$ preserves a sign in the interval $[a, c]$, we shall obtain the sign of difference $A_n(b) - A_n(a)$. Let us put

$$B_n(\theta) = \frac{\sin n\theta}{n} + \ldots + \frac{\sin 2n\theta}{2n}, \quad B_n'(\theta) = \frac{\cos\dfrac{3n}{2}\theta \sin\dfrac{n+1}{2}\theta}{\sin\theta/2}. \tag{2}$$

As the $\cos 3n\theta/2$ frequency in (2) is approximately three times more than the $\sin(n+1)\theta/2$ one, it is convenient to consider the $\cos 3n\theta/2$ zeroes in the interval $(0, \pi)$ by groups of three, and we denote

$$a_q = \frac{\pi}{3n} + \frac{2\pi}{n}(q-1), \quad b_q = a_q + \frac{2\pi}{3n} = \frac{\pi}{n} + \frac{2\pi}{n}(q-1),$$
$$c_q = b_q + \frac{2\pi}{3n} = -\frac{\pi}{n} + \frac{2\pi}{n}q, \ \beta_q = \frac{2\pi}{n+1}q, \ q - series, \ q = 1, \ldots, \frac{n+1}{2}.$$

The last series is incomplete if n is odd. In this case $b_{[(n+1)/2]} = \beta_{[(n+1)/2]} = \pi$. It is easy to see that the relative arrangement of zeroes in a q-series is as follows:

$$a_q < b_q < c_q < \beta_q, \quad \left(1 \le q \le \frac{n+1}{6}\right), \quad seriesA,$$

$$a_q < b_q < \beta_q < c_q, \quad \left(\frac{n+1}{6} < q \le \frac{n+1}{2}\right), \quad seriesB.$$

Theorem 1. $B_n(a_q) > B_n(b_q), \quad q = 1, \ldots, (n+1)/2.$

P r o o f. In this case the averaging is not necessary, because in the interval $(a_q \le \theta \le b_q)$ the functions $\cos 3n\theta/2$ and $\sin(n+1)\theta/2$ preserve the sign and

$$\cos\frac{3n}{2}\theta = (-1)^q\delta_1, \quad \sin\frac{n+1}{2}\theta = (-1)^{q-1}\delta_2, \quad \delta_i > 0.$$

Let $\alpha_q, \alpha_{q+1}, \alpha_{q+2}, \alpha_{q+3}$ be the four consecutive zeroes of $\cos 3n\theta/2$ and

$$s_n(\theta) = B_n'(\theta) + B_n'\left(\theta + \frac{2\pi}{3n}\right) + B_n'\left(\theta + \frac{4\pi}{3n}\right), \ \alpha_q \le \theta \le \alpha_{q+1},$$

be the averaging of the derivative $B_n'(\theta)$ in the interval (α_q, α_{q+3}). Using obvious statements

Lemma 1.
$$\sin\left(\theta+\frac{2\pi}{3}\right)+\sin\theta=\sin\left(\theta+\frac{\pi}{3}\right)$$

and

Lemma 2.

$$\sin\alpha\sin(\beta+h)-\sin(\alpha+h)\sin\beta=\sinh\sin(\alpha-\beta),$$

it is easy to obtain an explicit expression for $s_n(\theta)$.

Lemma 3.

$$s_n(\theta)=\frac{\sin\dfrac{\pi}{3n}\cos\dfrac{3n}{2}\theta}{\sin\dfrac{\theta}{2}\sin\left(\dfrac{\theta}{2}+\dfrac{\pi}{3n}\right)\sin\left(\dfrac{\theta}{2}+\dfrac{2\pi}{3n}\right)}\left[-\sqrt{3}\cos\left(\frac{n\theta}{2}+\frac{\pi}{3}\right)\sin\frac{\theta}{2}\right.$$
$$\left.+2\sin\frac{\pi}{3n}\sin\frac{n\theta}{2}cos\left(\frac{\theta}{2}+\frac{\pi}{3n}\right)\right]=s_1+s_2. \quad (3)$$

The unobtrusive advantage of representation (3) in comparison with (2) is in a regular position of zeroes. The $\cos\left(\dfrac{\theta}{2}+\dfrac{\pi}{3n}\right)$ zeroes are in points a_q and the $\sin n\theta/2$ zeroes are in points $(c_q+a_{q+1})/2$. It follows, in particular, that in the intervals (a_q,b_q) and (b_q,c_q) these functions preserve the sign. At the same time, with the growth of q, the $\sin\dfrac{(n+1)\theta}{2}$ zeroes move from the interval (c_q,a_{q+1}) to the interval (b_q,c_q).

Theorem 2. $B_n(a_q)>B_n(a_{q+1}),\quad q=1,\dots,[(n-1)/2]$.

P r o o f. By Lemma 3, it is enough to check the inequality

$$s_n(\theta)=s_1+s_2\le 0,\quad a_q\le\theta\le b_q. \quad (4)$$

Let us prove that both terms in (4) are nonpositive. It follows from the relations

$$\cos\frac{3n\theta}{2}=(-1)^q\delta_1,\quad \sin\frac{n\theta}{2}=(-1)^{q-1}\delta_2,$$
$$\cos\left(\frac{n\theta}{2}+\frac{\pi}{3}\right)=(-1)^q\delta_3,\quad \delta_i>0,$$

that can easily be checked.

Theorem 3.

$$B_n(b_q) < B_n(b_{q+1}), \quad q = 1, \ldots, [(n-1)/2].$$

P r o o f. In view of Lem. 3, it is enough to check the inequality

$$s_n(\theta) = s_1 + s_2 \geq 0, \quad b_q \leq \theta \leq c_q. \tag{5}$$

A nonnegativity of both summands in (5) follows from the relations

$$\cos\frac{3n\theta}{2} = (-1)^{q-1}\delta_1, \quad \sin\frac{n\theta}{2} = (-1)^{q-1}\delta_2,$$

$$\cos\left(\frac{n\theta}{2} + \frac{\pi}{3}\right) = (-1)^q\delta_3, \quad \delta_i > 0,$$

since $\cos 3n\theta/2$ changes the sign, but $\sin(n\theta)/2$ and $\cos\left(\frac{n\theta}{2} + \frac{\pi}{3}\right)$ preserve it.

Theorem 4. $B_n(c_q) > B_n(c_{q+1}), \quad q = 1, \ldots, [\frac{n-1}{2}]$.

P r o o f. As in the proof of Th. 2, it is enough to check the inequality

$$s_n(\theta) = s_1 + s_2 \leq 0, \quad a_q \leq \theta \leq c_{q+1}. \tag{6}$$

Similarly to the case above, it follows from the relations

$$\cos\frac{3n\theta}{2} = (-1)^{q-1}\delta_1, \quad \cos(\frac{n\theta}{2} + \frac{\pi}{3}) = (-1)^q\delta_2, \quad \delta_i > 0,$$

that $s_1 \leq 0$. At the same time, $\sin\frac{n\theta}{2}$ as well as s_2 in (6) changes the sign in the point $(c_q + a_{q+1})/2 = q2\pi/n$. Moreover, the numerical analysis shows that for small q the sum $s_1 + s_2$ near the point a_{q+1} takes positive values. Therefore, for the estimation of $\int_{c_q}^{a_{q+1}} s_2(\theta)d\theta$ one more averaging is necessary. For $h \in (0, \frac{\pi}{3n})$ we put $\theta_1 = 2\pi/q - h, \quad \theta_2 = 2\pi/q + h$. Then $\cos 3n\theta_1/2 = \cos 3n\theta_2/2$. Therefore, for $\sigma(h) = s_2(\theta_1) + s_2(\theta_2)$ we have

$$\sigma(h) = s_2(\theta_1) + s_2(\theta_2) = 2\sin^2\frac{\pi}{3n}\cos\frac{3n}{2}\theta_1\sin\frac{n}{2}\theta_1$$
$$\times\left[\frac{\cos\left(\frac{\theta_1}{2} + \frac{\pi}{3n}\right)}{\sin\frac{\theta_1}{2}\sin\left(\frac{\theta_1}{2} + \frac{\pi}{3n}\right)\sin\left(\frac{\theta_1}{2} + \frac{2\pi}{3n}\right)} + \frac{\cos\left(\frac{\theta_2}{2} + \frac{\pi}{3n}\right)}{\sin\frac{\theta_2}{2}\sin\left(\frac{\theta_2}{2} + \frac{\pi}{3n}\right)\sin\left(\frac{\theta_2}{2} + \frac{2\pi}{3n}\right)}\right] < 0,$$

since $\theta_1 < \theta_2$. Finally,

$$\int\limits_{c_q}^{a_{q+1}} s_2(\theta)d\theta = \int\limits_{c_q}^{\frac{c_q+a_{q+1}}{2}} s_2(\theta)d\theta + \int\limits_{\frac{c_q+a_{q+1}}{2}}^{a_{q+1}} s_2(\theta)d\theta = \int\limits_{0}^{\frac{2\pi}{n}q} \sigma(h)dh < 0,$$

and Theorem 4 is proved.

Theorem 5. $B_n(A_q) > B_n(c_q), \quad q = 1, \dots, [(n+1)/6]$.

P r o o f. In this case, for $a_q \le \theta \le b_q$ we put

$$s_n(\theta) = B_n'(\theta) + B_n'(\theta + 2\pi/3n)$$

$$= \frac{\cos\frac{3n}{2}\theta}{\sin\frac{\theta}{2}\sin(\frac{\theta}{2}+\frac{\pi}{3n})}\left[2\sin\frac{n+1}{2}\theta\sin\frac{\pi}{6n}\cos\left(\frac{\theta}{2}+\frac{\pi}{6n}\right)\right.$$
$$\left.-2\sin\left(\frac{\pi}{6}+\frac{\pi}{6n}\right)\sin\frac{\theta}{2}\cos\left((n+1)\frac{\theta}{2}+\frac{\pi}{6}+\frac{\pi}{6n}\right)\right]. \tag{7}$$

As $\cos 3n\theta/2 = (-1)^q\delta, \quad \delta > 0, \quad a_q \le \theta \le b_q$, it is enough to check the positivity of the square brackets, multiplied by $(-1)^{q-1}$, that is equivalent to the inequality

$$\sin\frac{n+1}{2}\theta\left[\sin\frac{\pi}{6n}\cos\left(\frac{\theta}{2}+\frac{\pi}{6n}\right)+2\sin^2\left(\frac{\pi}{6}+\frac{\pi}{6n}\right)\sin\frac{\theta}{2}\right]$$
$$\ge \sin\left(\frac{\pi}{3}+\frac{\pi}{3n}\right)\sin\frac{\theta}{2}\cos(n+1)\frac{\theta}{2}, \tag{8}$$

with odd q, and we have the opposite inequality for even q. For definiteness we consider the case with an odd q. In (8) let us omit a positive term $2\sin^2\left(\frac{\pi}{6}+\frac{\pi}{6n}\right)$ $\times\sin\frac{\theta}{2}$. Then we are restricted with the interval $a_q \le \theta \le \frac{\pi}{n+1} + \frac{2\pi}{n}(q-1)$, because $\cos(n+1)\theta/2 \le 0$ holds on the interval $\frac{\pi}{n+1} + \frac{2\pi}{n}(q-1) \le \pi$, and (8) is obvious. After division by positive $\sin(n+1)\theta/2$, in the case of odd q we obtain the inequality

$$\frac{\sin\frac{\pi}{6n}\cos\left(\frac{\theta}{2}+\frac{\pi}{6n}\right)}{\sin\frac{\theta}{2}} \ge \sin\left(\frac{\pi}{3}+\frac{\pi}{3n}\right)\cot\frac{(n+1)\theta}{2},$$

or

$$\sin\frac{\pi}{3n}\cot\frac{\theta}{2} - \sin\left(\frac{\pi}{3}+\frac{\pi}{3n}\right)\cot\frac{(n+1)\theta}{2} \ge 2\sin^2\frac{\pi}{6n}. \tag{9}$$

Lemma 4. $\dfrac{\cot \frac{(n+1)\theta}{2}}{\cot \frac{\theta}{2}}$ *decreases monotonously when*

$$a_q \le \theta \le \frac{\pi}{n+1} + \frac{2\pi}{n}(q-1).$$

P r o o f. A nonnegativity of the logarithmic derivative of fraction follows immediately from the well-known inequality $|\sin(n\theta)| \le n|\sin\theta|$ and from the positivity $\sin(n+1)\theta$ when $a_q \le \theta \le \dfrac{\pi}{n+1} + \dfrac{2\pi}{n}(q-1)$.
By Lemma 4, $\cot\dfrac{(n+1)\theta}{2} \le \dfrac{\cot(n+1)\pi/6n}{\cot\pi/6n}\cot\theta/2$ and inequality (9) follows from the inequality

$$\cot\frac{\pi}{2(n+1)}[1+\cos\frac{\pi}{3n} - \sin\left(\frac{\pi}{3}+\frac{\pi}{3n}\right)\cot(n+1)\frac{\pi}{6n}] \ge \sin\frac{\pi}{3n}.$$

The last inequality, except, perhaps, some initial values n, is given by the following

Lemma 5. $\sin\left(\dfrac{\pi}{3}+\dfrac{\pi}{3n}\right)\cot(n+1)\dfrac{\pi}{6n} \le \dfrac{3}{2}$.

The proof of Lem. 5 follows from the inequality

$$\sin\left(\frac{\pi}{3}+2\alpha\right)\cot\left(\frac{\pi}{6}+\alpha\right) \le 3/2, \quad 0 \le \alpha \le \frac{\pi}{2},$$

with $\alpha = \pi/6n$. The case of even q is considered in a similar way, and Th. 5 is completely proved.
The relation between $B_n(b_q)$ and $B_n(c_q)$, $q = 1, \ldots, [(n+1)/6]$, is a little more complicated. Using the averaging over zero $\beta_q = \dfrac{2\pi}{n+1}q$, similar to that applied in Th. 4, we receive

Proposition 1. *If* $\beta_q = 2\pi q/(n+1) > (b_q + c_q)/2$, *that holds when* $0 \le q \le (n-1)/3$, *then* $B_n(b_q) < B_n(c_q)$, *otherwise* $B_n(c_q) < B_n(b_q)$.

From Theorems 3 and 4 it follows that $B_n(b_q)$ monotonously increases in q, and $B_n(c_q)$ monotonously decreases. The following theorem is valid.

Theorem 6. $B_n(b_0) < B_n(c_{[\frac{n-1}{2}]})$.

P r o o f. Really,

$$B_n(b_0) = B_n\left(\frac{\pi}{n}\right) = \sum_{k=n}^{2n} \frac{\sin\frac{k\pi}{n}}{k} = [k = l+n] = -\sum_{l=0}^{n} \frac{\sin\frac{l\pi}{n}}{l+n}$$
$$\simeq -\int_0^1 \frac{\sin(\pi t)}{l+t}dt = -0.433785.$$

On the other hand, for example, when n is even

$$B_n\left(c_{|\frac{n-1}{2}|}\right) = B_n\left(\pi - \frac{\pi}{3n}\right) = \sum_{k=n}^{2n} \frac{\sin(k\pi - \frac{k\pi}{3n})}{k}$$

$$= \sum_{k=n}^{2n} (-1)^{k+1} \frac{\sin\frac{k\pi}{3n}}{k} = (-1)^{n+1} \sum_{l=0}^{n} (-1)^l \frac{\sin(\frac{\pi}{3} + \frac{l\pi}{3n})}{l+n}$$

$$= (-1)^{n+1} \sum_{l=0}^{\frac{n}{2}} \left[\frac{\sin(\frac{\pi}{3} + \frac{l\pi}{3n})}{l+n} - \frac{\sin(\frac{\pi}{3} + \frac{(l+1)\pi}{3n})}{l+1+n} \right].$$

As

$$\left| \left[\frac{\sin(\frac{\pi}{3} + \frac{l\pi}{3n})}{l+n} - \frac{\sin(\frac{\pi}{3} + \frac{(l+1)\pi}{3n})}{l+1+n} \right] \right| \leq \frac{2\sin\frac{\pi}{3n}}{1+n} + \frac{1}{(l+n)(l+n+1)},$$

then

$$\left| B_n\left(c_{[\frac{n-1}{2}]}\right) \right| \leq 2\ln 2 \sin\frac{\pi}{3n} + \frac{1}{2n},$$

$$B_n(b_0) < B_n\left(c_{[\frac{n-1}{2}]}\right), \quad n > 6.$$

For initial values n the inequality is checked by direct calculation. Theorem 6 is proved.

As above, let $\beta_q = 2\pi q/(n+1)$, $q = 1, \ldots, [(n-1)/2]$, be the zeroes of $\sin(n+1)\theta/2$ in the q-series. The following assertion is valid.

Proposition 2.

$$\begin{gathered} B_n(\beta_q) \leq B_n(c_q), \quad B_n(\beta_q) \leq B_n(a_{q+1}), \\ B_n(c_q) \leq B_n(a_{q+1}), \quad 1 \leq q \leq [(n+1)/6]\,, \\ B_n(c_q) \leq B_n(\beta_q), \quad B_n(c_q) \leq B_n(a_{q+1}), \\ B_n(\beta_q) \leq B_n(a_{q+1}), \quad [(n+1)/6] < q \leq [(n-1)/2]\,. \end{gathered} \tag{10}$$

The first and the second inequalities in (10) can be easily checked without averaging as in Th. 1. The last inequalities can be proved similarly to Th. 5, but the proofs are more cumbersome because of the difference of denominators in β_q and c_q, a_{q+1}.

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