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On Trace Regularity of Solutions to a Wave Equation with Homogeneous Neumann Boundary Conditions

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We prove an additional regularity of time derivative of the trace of solution to the wave equation on the 3D half space with the homogeneous Neumann boundary conditions.

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1. Introduction

In this paper we consider the following equation on $\mathbb{R}^3_+$:

$$(\partial_t + U\partial_{x_1})^2\phi = \Delta\phi, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3_+ = \{x : \; x_3 > 0\}; \tag{1}$$

$$\left.\frac{\partial\phi}{\partial x_3}\right|_{x_3=0} = 0; \tag{2}$$

$$\phi(0) = \phi_0, \quad \phi_t(0) = \phi_1. \tag{3}$$

The equations of this type arise, for example, in aerodynamics of potential gas flows. We consider the following motivating model. Gas occupies the half-space $\mathbb{R}^3_+$ and moves along x_1-axis with the velocity $U \geq 0$, $U \neq 1$. Then the potential of the velocity of a perturbed gas flow ϕ satisfies (1)–(3).

The problem of the trace regularity of solutions to hyperbolic equations frequently arises in hybrid systems theory. In particular, for the purposes of [1, 2] we need to study the regularity of the function $(\partial_t + U\partial_{x_1})\phi(x_1, x_2, 0)$, where ϕ is a solution to (1)–(3). In the present paper we will prove several new results on the smoothness concerned with the equation (1)–(3).

The regularity of solutions to general hyperbolic equations and of their traces on the boundary were studied by I. Lasiecka and R. Triggiani (see [3, 4] and references therein). Their results [3, 4] give the following trace regularity for the problem (1)–(3).

Theorem 1. *Let $\phi(t)$ be a solution to (1)–(3) with the initial conditions (ϕ_0, ϕ_1), $\Sigma_T = \mathbb{R}^2 \times [0,T]$ and $\gamma[\cdot]$ be the Sobolev trace of a function defined on $\mathbb{R}^3_+$ onto the plane $\{x : x_3 = 0\}$. Then the mapping $(\phi_0, \phi_1) \longmapsto \gamma[\phi]$ is a continuous operator from $H^1(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)$ to $H^{3/4}(\Sigma_T)$ and from $L^2(\mathbb{R}^3_+) \times (H^1(\mathbb{R}^3_+))'$ to $H^{-1/4}(\Sigma_T)$ for every $0 < T < +\infty$.*

Set $U = 0$ and denote $\tilde{\phi} = \phi_t$. Formally differentiating (1), we obtain that $\tilde{\phi}$ satisfies (1)–(2) with the initial conditions $\tilde{\phi}(0) = \phi_1$, $\tilde{\phi}_t(0) = \Delta\phi_0$. If $(\phi_0, \phi_1) \in H^1(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)$, then $(\phi_1, \Delta\phi_0) \in L^2(\mathbb{R}^3_+) \times (H^1(\mathbb{R}^3_+))'$. Thus, Th. 1 can give us only $\partial_t\gamma[\phi] \in H^{-1/4}(\Sigma)$.

Our main result improves Th. 1 in two directions. First, we prove that $\partial_t\gamma[\phi] \in L^2(0,T;H^{-1/4}(\mathbb{R}^2))$ provided initial conditions $(\phi_0, \phi_1) \in H^1(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)$. Second, we can in some sense improve this result, finding an appropriate decomposition of $\partial_t\gamma[\phi]$ as a sum $\partial_t\gamma[\phi] = f_1 + f_2$. This new idea allows us to prove that $f_1 \in L^\infty(0,T;H^{-1/4-\epsilon}(\mathbb{R}^2))$ and $f_2 \in L^2(0,T;L^2(\mathbb{R}^2))$, provided initial conditions $(\phi_0, \phi_1) \in H^1(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)$. We also study, how the trace regularity can be improved when more smooth initial conditions are considered, and what happens if ϕ_0 lies in a homogeneous Sobolev space.

In the proof of Th. 3 we rely on some ideas borrowed from [3], and first study a trace regularity of nonomogeneous problem with zero initial conditions. These results are collected in Th. 4, which, we believe, is an interest on itself.

The structure of the paper is as follows. In Section 2 we introduce the definitions and notations we need and state our main results. In Section 3 we study the properties of solutions to wave equation (1)–(3) with smooth initial conditions and prove some results on the interpolation of functional spaces we use. In Section 4 we prove Th. 4. In Section 5 we use Th. 4 to prove Th. 3. In Section 6 we prove a "local" version of Th. 3.

2. Notations and Main Results

To describe the behaviour of a solution ϕ to (1)–(3) we use a homogeneous Sobolev space $\mathcal{H}^1(\mathbb{R}^3_+)$. We define the space $\mathcal{H}^1(\mathbb{R}^3)$ (see, e.g., [5]) as the closure of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $||u||_{\mathcal{H}^1(\mathbb{R}^3)} = ||\nabla u||_{\mathbb{R}^3}$. For $\mathcal{H}^1(\mathbb{R}^3_+)$ defined as the space of restrictions of functions from $\mathcal{H}^1(\mathbb{R}^3)$ onto $\mathbb{R}^3_+$ we use the equivalent norm $||\nabla\phi||_{\mathbb{R}^3_+}$.

If we consider the system (1)–(3) as a model for a perturbed gas flow, the norm for ϕ introduced above is natural. Indeed, in this case $\phi(t)$ represents the potential of velocity of a perturbed gas flow, and has no physical meaning itself, while $\nabla\phi(t)$ is the field of velocity of the perturbed flow and gives the complete information about the flow. For $(\phi_0,\phi_1)\in\mathcal{H}^1(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)$ we define a local energy by

$$\mathcal{E}_R(\phi_0,\phi_1)=\int\limits_{B_R^+}\left(|\nabla\phi_0(x)|^2+|\phi_1(x)|^2\right)dx,$$

where $B_R^+=\{x=(x_1,x_2,x_3):\ |x|<R,\ x_3>0\}$.

Now we find the appropriate spaces for initial data which are "more smooth" than $\mathcal{H}^1(\mathbb{R}^3_+)\times L^2(\mathbb{R}^3_+)$ and introduce a suitable notion of local energy for these spaces. First, we consider the wave equation in the whole space $\mathbb{R}^3$ with smooth initial conditions

$$\partial_t^2\phi=\Delta\phi,\quad x\in\mathbb{R}^3, \tag{4}$$

$$\phi(0)=\phi_0,\ \phi_t(0)=\phi_1. \tag{5}$$

Formally differentiating (4), we see that $\bar{\phi}(t)=\phi_t(t)$ satisfies (4) with the initial conditions $\bar{\phi}(0)=\phi_1,\ \bar{\phi}_t(0)=\Delta\phi_0$. Thus energy conservation law for $\bar{\phi}(t)$ gives us the following energy relation for ϕ:

$$||\nabla\phi_t(t)||^2_{0,\mathbb{R}^3}+||\Delta\phi(t)||^2_{0,\mathbb{R}^3}=||\nabla\phi_1||^2_{0,\mathbb{R}^3}+||\Delta\phi_0||^2_{0,\mathbb{R}^3}.$$

The classical conservation law for (4)–(5), together with the previous relation give us

$$||\nabla\phi(t)||^2_{0,\mathbb{R}^3}+||\Delta\phi(t)||^2_{0,\mathbb{R}^3}+||\phi_t(t)||^2_{1,\mathbb{R}^3}=||\nabla\phi_0||^2_{0,\mathbb{R}^3}+||\Delta\phi_0||^2_{0,\mathbb{R}^3}+||\phi_1||^2_{1,\mathbb{R}^3}. \tag{6}$$

Hence, if we define $\mathcal{W}^{1,1}(\mathbb{R}^3)$ as the closure of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $||\cdot||^2_{\mathcal{W}^{1,1}(\mathbb{R}^3)}=||\Delta\cdot||^2_{0,\mathbb{R}^3}+||\nabla\cdot||^2_{0,\mathbb{R}^3}$, we can easily verify that for initial data $(\phi_0,\phi_1)\in\mathcal{W}^{1,1}(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$ the problem (4)–(5) possesses precisely one solution $(\phi(t),\phi_t(t))\in C(0,T;\mathcal{W}^{1,1}(\mathbb{R}^3)\times H^1(\mathbb{R}^3))$ for any $T>0$ for which the energy relation (6) holds.

We consider initial data from the spaces that are "intermediate" between $\mathcal{H}^1(\mathbb{R}^3)$ and $\mathcal{W}^{1,1}(\mathbb{R}^3)$. These spaces can be defined via Fourier transform (see Prop. 2). The space $\mathcal{W}^{1,s}(\mathbb{R}^3),\ s\geq 0$ consists of all distributions $f\in S'(\mathbb{R}^3)$ such that its Fourier transform $\tilde{f}$ is a regular distribution and the integral

$$||f||^2_{\mathcal{W}^{1,s}(\mathbb{R}^3)}=\int\limits_{\mathbb{R}^3}d\xi(1+|\xi|)^{2s}|\xi|^2|\tilde{f}(\xi)|^2$$

is finite. It is easy to see that there is another description of $\mathcal{W}^{1,s}(\mathbb{R}^3)$:

$$\mathcal{W}^{1,s}(\mathbb{R}^3)=\{f\in\mathcal{H}^1(\mathbb{R}^3):\ \nabla f\in(H^s(\mathbb{R}^3))^3\} \tag{7}$$

with the equivalent norm $||f||_{\mathcal{W}^{1,s}(\mathbb{R}^3)}=||\nabla f||_{s,\mathbb{R}^3}$.

Now we return to the wave equation in the half-space. Define the space $\mathcal{W}^{1,s}(\mathbb{R}^3_+)$, $s \in [0,1]$, as the space of restrictions of functions from $\mathcal{W}^{1,s}(\mathbb{R}^3)$ onto $\mathbb{R}^3_+$. Due to description (7) we see that $||\nabla f||_{s,\mathbb{R}^3_+}$ is an equivalent norm in $\mathcal{W}^{1,s}(\mathbb{R}^3_+)$. Similarly as in the case of $\mathcal{H}^1(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)$, we can define a local energy in the space $\mathcal{W}^{1,s}(\mathbb{R}^3_+) \times H^s(\mathbb{R}^3_+)$:

$$\mathcal{E}^s_R(\phi_0, \phi_1) = ||\nabla \phi_0||^2_{s,B^+_R} + ||\phi_1||^2_{s,B^+_R}, \quad s \in [0,1].$$

In order to obtain smooth solutions to (1)–(3) we need not only the smooth initial data but also consistency conditions imposed on these initial data. Therefore we need the spaces

$$\overline{\mathcal{W}}^{1,s}(\mathbb{R}^3_+) = \left\{ f \in \mathcal{W}^{1,s}(\mathbb{R}^3_+) : \left. \frac{\partial f}{\partial x_3} \right|_{x_3=0} = 0 \right\}, \quad s > 1/2,$$

with the same norm as in $\mathcal{W}^{1,s}(\mathbb{R}^3_+)$.

The following interpolation lemma is valid.

Lemma 1. $[\overline{\mathcal{W}}^{1,1}(\mathbb{R}^3_+), \mathcal{H}^1(\mathbb{R}^3_+)]_{[1-\theta]} = \mathcal{G}^\theta$, *where*

$$\begin{gathered} \mathcal{G}^\theta = \mathcal{W}^{1,\theta}(\mathbb{R}^3_+), \quad 0 \le \theta < 1/2, \\ \mathcal{G}^{1/2} = \{ f \in \mathcal{H}^1(\mathbb{R}^3_+) : \nabla f \in (L^2(\mathbb{R}_+; H^{1/2}(\mathbb{R}^2)))^2 \times H^{1/2}_{00}(\mathbb{R}_+; L^2(\mathbb{R}^2)) \}, \\ \mathcal{G}^\theta = \overline{\mathcal{W}}^{1,\theta}(\mathbb{R}^3_+), \quad 1/2 < \theta \le 1, \end{gathered} \tag{8}$$

where $H^{1/2}_{00}(\mathbb{R}_+; L^2(\mathbb{R}^2)) = \{ f \in H^{1/2}(\mathbb{R}_+; L^2(\mathbb{R}^2)) : x_3^{-1/2} f \in L^2(\mathbb{R}^3_+)\}$ *with the norm* $||f||^2_{H^{1/2}_{00}(\mathbb{R}_+;L^2(\mathbb{R}^2))} = ||f||^2_{H^{1/2}(\mathbb{R}_+;L^2(\mathbb{R}^2))} + ||x_3^{-1/2} f||^2_{L^2(\mathbb{R}^3_+)}$.

Now we can state the following existence theorem for the wave equation in the half-space.

Theorem 2. *Assume that initial data* $(\phi_0, \phi_1) \in \mathcal{G}^\theta \times H^\theta(\mathbb{R}^3_+)$. *Then:*

(i) For every $T > 0$ *there exists precisely one solution to (1)-(3)* $(\phi, \phi_t)(t) \in C(0,T; \mathcal{G}^\theta \times H^\theta(\mathbb{R}^3_+))$.

(ii) The norm of the solution does not increase. For $\theta \in [0,1]$, $\theta \ne 1/2$ *the following inequality is valid:*

$$||\nabla \phi(t)||^2_{\theta,\mathbb{R}^3_+} + ||\phi_t(t)||^2_{\theta,\mathbb{R}^3_+} \le ||\nabla \phi_0||^2_{\theta,\mathbb{R}^3_+} + ||\phi_1||^2_{\theta,\mathbb{R}^3_+}. \tag{9}$$

(iii) For every $R > 0$ *and* $\theta \in [0,1]$ *the local energy* $\mathcal{E}^\theta_R(\phi(t), \phi_t(t)) \to 0$, *when* $t \to +\infty$.

We will use weight spaces of X-valued functions. The space $L^2(\Omega; X, d\mu)$ consists of all functions mapping Ω into X such that

$$||f||^2_{L^2(\Omega;X,d\mu)} = \int_\Omega ||f(t)||^2_X d\mu < \infty$$

Here X is a normed space and $d\mu$ is a measure on Ω. In the case $X = \mathbb{R}$ we omit X in the notation.

Further we state our main results.

Theorem 3. *Assume that $\phi(t)$ is a solution to (1)–(3) with the initial conditions $\phi_0 \in \mathcal{G}^\theta(\mathbb{R}^3_+)$, $\phi_1 \in H^\theta(\mathbb{R}^3_+)$, $\theta \in [0,1]$, $\theta \neq 1/2$. Then:*

(i) $(\partial_t + U\partial_{x_1})\gamma[\phi] \in L^2(0,T; H^{-1/4+\theta}_{loc}(\mathbb{R}^2))$. The following estimate takes place

$$||(\partial_t + U\partial_{x_1})\gamma[\phi]||_{L^2(0,T;H^{-1/4+\theta}(B))} \leq C(T,U,B)\left(||\nabla\phi_0||_{\theta,\mathbb{R}^3_+} + ||\phi_1||_{\theta,\mathbb{R}^3_+}\right).$$

for any bounded set $B \subset \mathbb{R}^2$.

(ii) $(\partial_t + U\partial_{x_1})\gamma[\phi] = f_1 + f_2$, where $f_1 \in L^\infty(0,T; H^{-1/4+\theta-\epsilon}_{loc}(\mathbb{R}^2))$ for any $\epsilon > 0$, $f_2 \in L^2(0,T; H^\theta_{loc}(\mathbb{R}^2))$, and

$$||f_1||_{L^\infty(0,T;H^{-1/4+\theta-\epsilon}(B))} \leq C(T,U,\epsilon,B)\left(||\nabla\phi_0||_{\theta,\mathbb{R}^3_+} + ||\phi_1||_{\theta,\mathbb{R}^3_+}\right),$$
$$||f_2||_{L^2(0,T;H^\theta(B))} \leq C(T,U,B)\left(||\nabla\phi_0||_{\theta,\mathbb{R}^3_+} + ||\phi_1||_{\theta,\mathbb{R}^3_+}\right)$$

for any bounded set $B \subset \mathbb{R}^2$.

Remark 1. *Since the inclusion $\mathcal{G}^\theta \subset L^2(\mathbb{R}^3_+)$ does not take place, estimates for $(\partial_t + U\partial_{x_1})\gamma[\phi]$ have only local character. However, if $\phi_0 \in \mathcal{G}^\theta \bigcap L^2(\mathbb{R}^3_+) \subset H^{1+\theta}(\mathbb{R}^3_+)$ we have the estimate*

$$||(\partial_t + U\partial_{x_1})\gamma[\phi]||_{L^2(0,T;H^{-1/4+\theta}(\mathbb{R}^2))} \leq C(T,U)\left(||\phi_0||_{1+\theta,\mathbb{R}^3_+} + ||\phi_1||_{\theta,\mathbb{R}^3_+}\right)$$

in point (i) of Th. 3 and the estimates

$$||f_1||_{L^\infty(0,T;H^{-1/4+\theta-\epsilon}(\mathbb{R}^2))} \leq C(T,U,\epsilon)\left(||\phi_0||_{1+\theta,\mathbb{R}^3_+} + ||\phi_1||_{\theta,\mathbb{R}^3_+}\right),$$
$$||f_2||_{L^2(0,T;H^\theta(\mathbb{R}^2))} \leq C(T,U)\left(||\phi_0||_{1+\theta,\mathbb{R}^3_+} + ||\phi_1||_{\theta,\mathbb{R}^3_+}\right)$$

in point (ii) of Th. 3.

The theorem implies the following local energy estimate.

Corollary 1. *Assume that $\phi(t)$ is a solution to (1)–(3) with the initial conditions $\phi_0 \in \mathcal{G}^\theta(\mathbb{R}^3_+)$, $\phi_1 \in H^\theta(\mathbb{R}^3_+)$, $\theta \in [0,1]$, $\theta \neq 1/2$. Let Ω be a bounded smooth domain in $\mathbb{R}^2$, and r_Ω be the operator of restriction of a function defined on $\mathbb{R}^2$ to Ω. Then:*

(i) $r_\Omega(\partial_t + U\partial_{x_1})\gamma[\phi] \in L^2(0,T;H^{-1/4+\theta}(\Omega))$. The following estimate takes place

$$||r_\Omega(\partial_t + U\partial_{x_1})\gamma[\phi]||^2_{L^2(0,T;H^{-1/4+\theta}(\Omega))} \leq C\mathcal{E}^\theta_R(\phi_0,\phi_1).$$

The constants C, R depend only on T, Ω, and U.

(ii) $r_\Omega(\partial_t + U\partial_{x_1})\gamma[\phi] = f_1 + f_2$, where $f_1 \in L^\infty(0,T;H^{-1/4+\theta-\epsilon}(\Omega))$ for any $\epsilon > 0$, $f_2 \in L^2(0,T;H^\theta(\Omega))$, and

$$||f_1||^2_{L^\infty(0,T;H^{-1/4+\theta-\epsilon}(\Omega))} \leq C\mathcal{E}^\theta_R(\phi_0,\phi_1),$$
$$||f_2||^2_{L^2(0,T;H^\theta(\Omega))} \leq C\mathcal{E}^\theta_R(\phi_0,\phi_1).$$

The constants C, R depend on T, Ω, and U, the constant C in the first inequality depends also on ϵ.

Remark 2. *Theorem 3 and Corollary 1 allow us to justify the stabilization results of [1, 2] for the initial data $(\phi(0),\phi_t(0)) \in \mathcal{G}^\theta \times H^\theta(\mathbb{R}^3_+)$, $\theta \in (0,1]$, $\theta \neq 1/2$.*

The theorem below deals with the trace regularity of the nonhomogeneous wave equation. Following [3], we choose a certain function f in (10) and obtain Th. 3 as a consequence of the following theorem.

Theorem 4. *Consider the problem*

$$(\partial_t + U\partial_{x_1})^2\phi = \Delta\phi + f(t,x), \quad x \in \mathbb{R}^3_+, \tag{10}$$
$$\left.\frac{\partial\phi}{\partial x_3}\right|_{x_3=0} = 0, \quad \phi(0) = \phi_t(0) = 0. \tag{11}$$

(i) Let $f \in L^2(0,T;H^\theta(\mathbb{R}^3_+))$. Then $(\partial_t + U\partial_{x_1})\gamma[\phi] \in L^2(0,T;H^{-1/4+\theta}(\mathbb{R}^2))$ and the following estimate takes place

$$||(\partial_t + U\partial_{x_1})\gamma[\phi]||_{L^2(0,T;H^{-1/4+\theta}(\mathbb{R}^2))} \leq C(T,U)||f||_{L^2(0,T;H^\theta(\mathbb{R}^3_+))}.$$

(ii) Let $f \in L^\infty(\mathbb{R}_+;H^\theta(\mathbb{R}^3_+))$. Then $(\partial_t + U\partial_{x_1})\gamma[\phi] = f_1 + f_2$, where $f_1 \in L^\infty(0,T;H^{-1/4+\theta-\epsilon}(\mathbb{R}^2))$, $\epsilon > 0$, $f_2 \in L^2(0,T;H^\theta(\mathbb{R}^2))$, and the following estimates are valid

$$||f_1||_{L^\infty(0,T;H^{-1/4+\theta-\epsilon}(\mathbb{R}^2))} \leq C(T,U,\epsilon)||f||_{L^\infty(\mathbb{R}_+;H^\theta(\mathbb{R}^3_+))},$$
$$||f_2||_{L^2(0,T;H^\theta(\mathbb{R}^2))} \leq C(T,U)||f||_{L^\infty(\mathbb{R}_+;H^\theta(\mathbb{R}^3_+))}.$$

3. Smooth Solutions to the Wave Equation and Interpolation Spaces

For the proof of Th. 3 we need to generalize spaces $\mathcal{W}^{1,s}$. Define space $\mathcal{W}^{\alpha,\beta}(\mathbb{R}^n)$ as the space of distributions $f \in S'(\mathbb{R}^n)$ such that their Fourier transforms $\tilde{f}$ are regular distributions and

$$||f||^2_{\mathcal{W}^{\alpha,\beta}(\mathbb{R}^n)} = \int\limits_{\mathbb{R}^n} |\xi|^{2\alpha}(1+|\xi|)^{2\beta}|\tilde{f}(\xi)|^2 d\xi < \infty.$$

The Sobolev trace theorem can be easily generalized for these spaces.

Lemma 2. *The Sobolev trace operator is continuous from $\mathcal{W}^{\alpha,\beta}(\mathbb{R}^n)$ to $\mathcal{W}^{\alpha-1/2,\beta}(\mathbb{R}^{n-1})$, if $\alpha > 1/2$.*

The following embedding takes place.

Lemma 3. *If $\theta < n/2$ then $\mathcal{W}^{\theta,0}(\mathbb{R}^n)$ is continuously embedded in $L^p(\mathbb{R}^n)$, $p = 2n/(n-2\theta)$.*

P r o o f. $f \in \mathcal{W}^{\theta,0}(\mathbb{R}^n)$ if and only if $g = F^{-1}|\xi|^\theta F f \in L^2(\mathbb{R}^n)$. Equivalently, $f = F^{-1}|\xi|^{-\theta} F g \in L^p(\mathbb{R}^n)$ if and only if $F^{-1}|\xi|^{-\theta}F$ is a continuous operator from $L^2(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Thus, the assertion of the lemma follows from Th. 1.11 [6].

This means that for $\theta < n/2$ functions form $\mathcal{W}^{\theta,0}(\mathbb{R}^n)$ are locally L^2-integrable.

First, we prove interpolation Lem. 1 which will be used in the proofs of Th. 2 on smooth solutions to the wave equation and in the main Th. 3.

P r o o f o f L e m m a 1. Let the following propositions be proved.

Proposition 1.

$$\mathcal{W}^{1,\theta}(\mathbb{R}^3_+) = \left\{ f \in \mathcal{H}^1(\mathbb{R}^3_+) : \ \nabla f \in (L^2(\mathbb{R}_+; H^\theta(\mathbb{R}^2)))^2 \times H^\theta(\mathbb{R}_+; L^2(\mathbb{R}^2)) \right\},$$

if $0 \le \theta \le 1$, and

$$\overline{\mathcal{W}}^{1,\theta}(\mathbb{R}^3_+) = \left\{ f \in \mathcal{H}^1(\mathbb{R}^3_+) : \ \nabla f \in (L^2(\mathbb{R}_+; H^\theta(\mathbb{R}^2)))^2 \times H^\theta_0(\mathbb{R}_+; L^2(\mathbb{R}^2)) \right\},$$

if $1/2 < \theta \le 1$.

Proposition 2.

$$[\mathcal{W}^{1,1}(\mathbb{R}^3), \mathcal{H}^1(\mathbb{R}^3)]_{[1-\theta]} = \mathcal{W}^{1,\theta}(\mathbb{R}^3), \quad \theta \in [0,1].$$

Proposition 3. *Let f be a continuous function on $\mathbb{R}^3_+$. Then we can define an operator of even extension as*

$$\tilde{f}(x_1, x_2, x_3) = \begin{cases} f(x_1, x_2, x_3), & x_3 \geq 0, \\ f(x_1, x_2, -x_3), & x_3 < 0. \end{cases}$$

The operator of even extension $\tilde{\cdot}$ can be extended to a continuous operator

$$\tilde{\cdot}: \ \mathcal{G}^\theta \to \mathcal{W}^{1,\theta}(\mathbb{R}^3), \quad \theta \in [0,1],$$

where $\mathcal{G}^\theta$ is defined by (8).

Proposition 4. *There exists an operator R such that*

$$R \in L(\mathcal{H}^1(\mathbb{R}^3), \mathcal{H}^1(\mathbb{R}^3_+)), \quad R \in L(\mathcal{W}^{1,1}(\mathbb{R}^3), \overline{\mathcal{W}}^{1,1}(\mathbb{R}^3_+)),$$
$$R\tilde{u} = u, \quad \forall u \in \mathcal{H}^1(\mathbb{R}^3_+), \quad \forall u \in \overline{\mathcal{W}}^{1,1}(\mathbb{R}^3_+),$$

where $\tilde{\cdot}$ is the operator of even extension defined in Prop. 3.

In this proof we assume that $\mathcal{G}^\theta$ is defined by the equalities (8).

First we prove that $[\overline{\mathcal{W}}^{1,1}(\mathbb{R}^3_+), \mathcal{H}^1(\mathbb{R}^3_+)]_{[1-\theta]} \subset \mathcal{G}^\theta$. Due to Prop. 1 we have that

$$\nabla: \ \mathcal{H}^1(\mathbb{R}^3_+) \to (L^2(\mathbb{R}_+; L^2(\mathbb{R}^2)))^2 \times L^2(\mathbb{R}_+; L^2(\mathbb{R}^2)),$$
$$\nabla: \ \overline{\mathcal{W}}^{1,1}(\mathbb{R}^3_+) \to (L^2(\mathbb{R}_+; H^1(\mathbb{R}^2)))^2 \times H^1_0(\mathbb{R}_+; L^2(\mathbb{R}^2))$$

is a continuous operator. Thus Ths. 11.6, 11.7 [7, Ch. 1] imply that

$$\nabla: [\overline{\mathcal{W}}^{1,1}(\mathbb{R}^3_+), \mathcal{H}^1(\mathbb{R}^3_+)]_{[1-\theta]} \to (L^2(\mathbb{R}_+; H^\theta(\mathbb{R}^2)))^2 \times H^\theta(\mathbb{R}_+; L^2(\mathbb{R}^2)),$$
$$0 \leq \theta < 1/2,$$
$$\nabla: [\overline{\mathcal{W}}^{1,1}(\mathbb{R}^3_+), \mathcal{H}^1(\mathbb{R}^3_+)]_{[1/2]} \to (L^2(\mathbb{R}_+; H^{1/2}(\mathbb{R}^2)))^2 \times H^{1/2}_{00}(\mathbb{R}_+; L^2(\mathbb{R}^2)),$$
$$\theta = 1/2,$$
$$\nabla: [\overline{\mathcal{W}}^{1,1}(\mathbb{R}^3_+), \mathcal{H}^1(\mathbb{R}^3_+)]_{[1-\theta]} \to (L^2(\mathbb{R}_+; H^\theta(\mathbb{R}^2)))^2 \times H^\theta_0(\mathbb{R}_+; L^2(\mathbb{R}^2)),$$
$$1/2 < \theta \leq 1,$$

is also a continuous operator. Since $u \in [\overline{\mathcal{W}}^{1,1}(\mathbb{R}^3_+), \mathcal{H}^1(\mathbb{R}^3_+)]_{[1-\theta]}$ implies $u \in \mathcal{H}^1(\mathbb{R}^3_+)$, using Prop. 1 we obtain the desired embedding.

Now we prove the embedding $\mathcal{G}^\theta \subset [\overline{\mathcal{W}}^{1,1}(\mathbb{R}^3_+), \mathcal{H}^1(\mathbb{R}^3_+)]_{[1-\theta]}$. Consider $u \in \mathcal{G}^\theta$. Using Props. 2 and 3, we have $\tilde{u} \in \mathcal{W}^{1,\theta}(\mathbb{R}^3) = [\mathcal{W}^{1,1}(\mathbb{R}^3), \mathcal{H}^1(\mathbb{R}^3)]_{[1-\theta]}$. Using Prop. 4 and interpolation, we obtain $R\tilde{u} = u \in [\overline{\mathcal{W}}^{1,1}(\mathbb{R}^3_+), \mathcal{H}^1(\mathbb{R}^3_+)]_{[1-\theta]}$. The proof of the lemma will be complete when we prove Props. 1–4.

P r o o f o f P r o p o s i t i o n 1. Using the definition of $\mathcal{W}^{1,\theta}(\mathbb{R}^3)$ via Fourier transform, we can easily verify that

$$\mathcal{W}^{1,\theta}(\mathbb{R}^3) = L^2(\mathbb{R};\mathcal{W}^{1,\theta}(\mathbb{R}^2)) \bigcap \mathcal{W}^{1,\theta}(\mathbb{R};L^2(\mathbb{R}^2)).$$

Thus, we obtain the first representation

$$\mathcal{W}^{1,\theta}(\mathbb{R}^3_+) = L^2(\mathbb{R}_+;\mathcal{W}^{1,\theta}(\mathbb{R}^2)) \bigcap \mathcal{W}^{1,\theta}(\mathbb{R}_+;L^2(\mathbb{R}^2)), \quad 0 \le \theta \le 1,$$
$$\overline{\mathcal{W}}^{1,\theta}(\mathbb{R}^3_+) = L^2(\mathbb{R}_+;\mathcal{W}^{1,\theta}(\mathbb{R}^2)) \bigcap \overline{\mathcal{W}}^{1,\theta}(\mathbb{R}_+;L^2(\mathbb{R}^2)), \quad 1/2 < \theta \le 1.$$

Using representation (7), we finish the proof.

P r o o f o f P r o p o s i t i o n 2. Following [7], in the space $\mathcal{H}^1(\mathbb{R}^3)$ we construct the operator $A = F^{-1}(1+|\xi|^2)^{1/2}F$ with $D(A) = \mathcal{W}^{1,1}(\mathbb{R}^3)$. It is easy to verify that we can define the operator $A^\theta = F^{-1}(1+|\xi|^2)^{\theta/2}F$, where F is a Fourier transform, with $D(A^\theta) = \mathcal{W}^{1,\theta}(\mathbb{R}^3)$. Obviously, A^θ is really A to power θ. Since A is a closed maximal accretive operator, $D(A^\theta) = [\mathcal{W}^{1,1}(\mathbb{R}^3), \mathcal{H}^1(\mathbb{R}^3)]_{[1-\theta]}$ [8].

P r o o f o f P r o p o s i t i o n 3. Since $C_0^\infty(\overline{\mathbb{R}}^3_+)$ is dense in $\mathcal{H}^1(\mathbb{R}^3_+)$ and $C_0(\mathbb{R}^3) \bigcap C^\infty(\mathbb{R}^3_+) \bigcap C^\infty(\mathbb{R}^3_-)$ is dense in $\mathcal{H}^1(\mathbb{R}^3)$, it is easy to prove that the even extension is a continuous operator from $\mathcal{H}^1(\mathbb{R}^3_+)$ in $\mathcal{H}^1(\mathbb{R}^3)$. Prop. 1 implies that the even extension of a function f is continuous if and only if the odd extension of $\partial_{x_3} f$ is continuous. Evidently, the odd extension of $\partial_{x_3} f$ is a continuous operator:

$$H^\theta(\mathbb{R}_+;L^2(\mathbb{R}^2)) \to H^\theta(\mathbb{R};L^2(\mathbb{R}^2)), \quad 0 \le \theta < 1/2,$$
$$H^\theta_{00}(\mathbb{R}_+;L^2(\mathbb{R}^2)) \to H^\theta(\mathbb{R};L^2(\mathbb{R}^2)), \quad \theta = 1/2,$$
$$H^\theta_0(\mathbb{R}_+;L^2(\mathbb{R}^2)) \to H^\theta(\mathbb{R};L^2(\mathbb{R}^2)), \quad 1/2 < \theta \le 1,$$

since the extension by zero is a continuous operator between the above mentioned spaces (see [7, Ch. 1], Th. 11.4). The proof is complete.

P r o o f o f P r o p o s i t i o n 4. Set $(Ru)(x) = 1/2(f(x) + f(-x))$. It is easy to verify that this operator satisfies all assertions of the proposition.

Now we can prove Th. 2.

P r o o f o f T h e o r e m 2. The existence and uniqueness of solution to (1)–(3) with the initial data $(\phi_0, \phi_1) \in \mathcal{H}^1(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)$ is a well-known fact, as well as the energy conservation law and the decay of local energy $\mathcal{E}^0_R(\phi(t), \phi_t(t))$ for this case. Using formal differentiation with respect to t, we can easily prove points (i), (ii), and (iii) of the theorem for the initial data $(\phi_0, \phi_1) \in \overline{\mathcal{W}}^{1,1}(\mathbb{R}^3_+) \times H^1(\mathbb{R}^3_+)$. Using interpolation Lem. 1, we complete the proof of points (i) and (ii). To prove (iii) for all $\theta \in [0, 1]$ we need the following criterium of the pointwise convergence of an operator sequence.

Proposition 5. *Let H_1 be a Banach space and H_2 be a pseudonormed space with a pseudonorm $p(\cdot)$. Assume that $\{A_n\}_{n=1}^{\infty}$ is a sequence of the linear operators $A_n : H_1 \to H_2$ such that $p(A_n h) \le C||h||_{H_1}$ $\forall h \in H_1$, and there exists a dense set $D \subset H_1$ such that $\forall h \in D$ $p(A_n h) \to 0$, when $n \to +\infty$. Then $\forall g \in H_1$ $p(A_n g) \to 0$, when $n \to +\infty$.*

When H_1 and H_2 are both Banach spaces, the result is well known. The proof of Prop. 5 is similar to the proof of this result.

Now we fix R and the set $H_1 = \mathcal{G}^\theta \times H^\theta(\mathbb{R}^3_+)$, $H_2 = H^{1+\theta}(B_R^+) \times H^\theta(B_R^+)$ with the pseudonorm $p(\varphi_0, \varphi_1) = (||\nabla\varphi_0||^2_{\theta,B_R^+} + ||\varphi_1||^2_{\theta,B_R^+})^{1/2}$, $A_n(\phi_0,\phi_1) = r_{B_R^+}(\phi(t_n), \phi_t(t_n))$, where $(\phi(t_n), \phi_t(t_n))$ is the solution to (1)–(3) with the initial conditions (ϕ_0, ϕ_1) at the moment t_n such that $t_n \to +\infty$ when $n \to +\infty$. The operator of restriction on B_R^+ $r_{B_R^+}$ is continuous from $\mathcal{G}^\theta$ to $H^{1+\theta}(B_R^+)$ (see [5] and interpolation Lem. 1). We chose $D = \overline{\mathcal{W}}^{1,1}(\mathbb{R}^3_+) \times H^1(\mathbb{R}^3_+)$. Using the interpolation inequality $||u||_{[X,Y]_{[\theta]}} \le ||u||_X^{1-\theta}||u||_Y^\theta$ (see, e.g., [7, Ch. 1]), we obtain that for $(\phi_0, \phi_1) \in D$

$$p(A_n(\phi_0,\phi_1)) = \mathcal{E}_R^\theta(\phi(t_n), \phi_t(t_n)) = ||\nabla\phi(t_n)||^2_{\theta,B_R^+} + ||\phi_t(t_n)||^2_{\theta,B_R^+}$$
$$\le ||\nabla\phi(t_n)||^{2\theta}_{1,B_R^+}||\nabla\phi(t_n)||^{2-2\theta}_{0,B_R^+} + ||\phi_t(t_n)||^{2\theta}_{1,B_R^+}||\phi_t(t_n)||^{2-2\theta}_{0,B_R^+} \to 0,$$

when $n \to +\infty$. It is easy to see that all assumptions of Prop. 5 are satisfied. Thus we prove (iii) for all $\theta \in [0,1]$. The proof of Th. 2 is complete.

4. Proof of Theorem 4

In this section we need some facts on the Laplace and Fourier transforms.

Denote by $\mathcal{D}'(\mathbb{R}, X)$ the space of distributions on $\mathbb{R}$ with the values in a Hilbert space X and by $S'(\mathbb{R}, X)$ the space of X-valued temperate distributions. We also set $\mathcal{D}'_+(X) = \{f \in \mathcal{D}'(\mathbb{R}, X) : \ \mathit{supp} f \subset \{x : \ x \ge 0\}\}$ and $\mathcal{D}'_+(a, X) = \{f \in \mathcal{D}'_+(X) : \ e^{-at}f(t) \in S'(\mathbb{R}, X)\}$. For the functions from $S(\mathbb{R}, X)$ (test functions for $S'(\mathbb{R}, X)$) we define the Fourier transform as

$$F[f(t)](\tau) = \int_{\mathbb{R}} dt e^{-it\tau} f(t) \tag{12}$$

and for functions from $S'(\mathbb{R}, X)$ by $(F[f], \phi) = (f, F[\phi])$, respectively. Further we define the Laplace transform for functions from $\mathcal{D}'_+(a, X)$ by

$$\mathcal{F}(s) = \mathcal{L}[f(t)](s) = F[e^{-\alpha t}f(t)](\beta), \quad s = \alpha + i\beta, \ \alpha > a. \tag{13}$$

The Laplace transform of a function from $\mathcal{D}'_+(a, X)$ is an analytic in the complex half-plane $\mathbb{C}_a = \{s \in \mathbb{C} : \ \mathit{Res} > a\}$ X-valued function.

Next we define a space $H_a(X)$. This space consists of the analytic in $\mathbb{C}_a$ X-valued functions $\mathcal{F}(s)$ that satisfy the following growth condition: for every $\epsilon > 0$ and $\sigma_0 > a$ there exist constants $C_\epsilon(\sigma_0) \geq 0$, $m = m(\sigma_0) \geq 0$ such that

$$||\mathcal{F}(s)||_X \leq C_\epsilon(\sigma_0)e^{\epsilon \cdot Res}(1 + |s|^m), \quad Res > \sigma_0. \tag{14}$$

The following fundamental theorem takes place (see, e.g., [9]).

Theorem 5. *A function $f(t) \in \mathcal{D}'_+(a, X)$ if and only if its Laplace transform $\mathcal{F}(s) \in H_a(X)$.*

Due to definition (13) $\mathcal{L}[f(t)](\alpha+i\beta) \in S'(\mathbb{R}, X)$ as a function of the variable β, provided α to be fixed, so the next representation takes place

$$\mathcal{L}^{-1}[\mathcal{F}(\alpha + i\beta)](t) = e^{\alpha t}F_\beta^{-1}[\mathcal{F}(\alpha + i\beta)](t). \tag{15}$$

Lemma 4. *Let B be a linear mapping that maps a function from $H_a(X)$ to a function from $H_a(Y)$, where X and Y are Hilbert spaces, and let there exist a constant $\gamma > a$ such that the operator $B_\gamma :\ \mathcal{F}(\gamma+i\beta) \mapsto (B\mathcal{F})(\gamma+i\beta)$ is a linear bounded operator from $L^2(\mathbb{R}; X)$ to $L^2(\mathbb{R}; Y)$. Then the operator $A = \mathcal{L}^{-1} \circ B \circ \mathcal{L}$ is a linear bounded operator from $L^2(\mathbb{R}_+; X, e^{-2\gamma t}dt)$ to $L^2(\mathbb{R}_+; Y, e^{-2\gamma t}dt)$.*

P r o o f. Since $B\mathcal{F} \in H_a(Y)$, we can use representation (15) for the inverse Laplace transform, thus

$$[Af](t) = e^{\gamma t}F_\beta^{-1}[B_\gamma F_t[e^{-\gamma t}f(t)](\beta)](t).$$

Using Plansherel's theorem (the Fourier transform is an isometry on $L^2(\mathbb{R}; X)$), one can easily verify the assertion of the lemma.

Corollary 2. *If A is a linear bounded operator from $L^2(\mathbb{R}_+; X, e^{-2\gamma t}dt)$ to $L^2(\mathbb{R}_+; Y, e^{-2\gamma t}dt)$ then $r_{(0,T)}A$ is a linear bounded operator from $L^2(\mathbb{R}_+; X)$ to $L^2(0, T; Y)$ and form $L^\infty(\mathbb{R}_+; X)$ to $L^2(0, T; Y)$, where $r_{(0,T)}$ is the operator of restriction of functions from $L^2(\mathbb{R}_+; Y, e^{-2\gamma t}dt)$ to $(0, T)$. The operator norm $||r_{(0,T)}A|| \leq C(T, \gamma)$.*

P r o o f o f T h e o r e m 4.

Remark 3. *We can apply the following change of variables to (1)–(3) (and also to (10)–(11)):*

$$s = t; \quad x_1 = y_1 + Ut, \quad x_2 = y_2, \quad x_3 = y_3.$$

Then equation (1) changes to $\partial_s^2\phi = \Delta_y\phi$, the operator $(\partial_t + U\partial_{x_1})\gamma[\phi]$ changes to $\partial_s\gamma[\phi]$, and initial conditions (3) remain unchanged. Thus, without loss of generality, we give the proofs of all the theorems above only for the case $U = 0$.

To prove part (i) of the theorem we consider the nonhomogeneous problem (10)–(11) with $f \in L^2(\mathbb{R}_+; H^\theta(\mathbb{R}^3_+))$ and $U = 0$. Similarly as in [3], we use the Fourier transform in x_1, x_2 and the Laplace transform in t. We denote $\hat{} = L_t F_{x_1 x_2}$. Applying the Fourier–Laplace transform to (10)–(11), we obtain

$$L_t F_{x_1 x_2}(\partial_t \gamma[\phi]) = s\hat{\phi}(s,\xi,x_3=0) = -\frac{s}{\sqrt{s^2+|\xi|^2}} \cdot \int_0^\infty e^{-y\sqrt{s^2+|\xi|^2}} \hat{f}(s,\xi,y)dy. \tag{16}$$

Thus

$$|s\hat{\phi}(s,\xi,x_3=0)|$$
$$\le \frac{|s|}{|\sqrt{s^2+|\xi|^2}|} \cdot \frac{(1+|\xi|^2)^{-\theta/2}}{\sqrt{2Re\sqrt{s^2+|\xi|^2}}} \times \left((1+|\xi|^2)^\theta \int_0^\infty dy |\hat{f}(s,\xi,y)|^2 \right)^{1/2}.$$

Now our aim is to find an appropriate Sobolev space $H^\sigma(\mathbb{R}^2)$ such that $\partial_t\gamma[\phi] \in L^2(\mathbb{R}_+; H^\sigma(\mathbb{R}^2), e^{-2\gamma t}dt)$, provided $f \in L^2(\mathbb{R}_+; H^\theta(\mathbb{R}^3_+), e^{-2\gamma t}dt)$. That is, we should find σ and such that

$$K(s,\xi) = \frac{|s|^2(1+|\xi|^2)^{\sigma-\theta}}{|s^2+|\xi|^2| Re\sqrt{s^2+|\xi|^2}} \le C_\alpha \tag{17}$$

for some fixed $\alpha > 0$. Here we denote $s = \alpha + i\beta$.

Similarly as in [3], we divide the first quarter of the half-plane $(\beta, |\xi|)$ into four domains:

$$\begin{aligned}
\mathcal{R}_0 &= \{\beta^2 + |\xi|^2 \le 1\},\\
\mathcal{R}_1 &= \{\beta/2 \le |\xi| \le 2\beta,\ \beta^2+|\xi|^2 \ge 1\},\\
\mathcal{R}_2 &= \{|\xi| \ge 2\beta,\ \beta^2+|\xi|^2 \ge 1\},\\
\mathcal{R}_3 &= \{|\xi| \le \beta/2,\ \beta^2+|\xi|^2 \ge 1\}.
\end{aligned}$$

and estimate $K(s,\xi)$ in each domain separately. We use the following inequalities for complex numbers:

$$\sqrt{2}|z|^{1/2} \ge \sqrt{2}Re\sqrt{z} = \sqrt{Rez + |z|} \ge |z|^{1/2}, \quad Rez \ge 0; \tag{18}$$

$$\sqrt{2}|z|^{1/2} \ge \sqrt{2}Re\sqrt{z} = \sqrt{|z| - |Rez|} = \frac{2\alpha\beta}{\sqrt{|z| - Rez}}, \quad Rez < 0. \tag{19}$$

Further we denote $z = s^2 + |\xi|^2 = \alpha^2 - \beta^2 + |\xi|^2 + 2i\alpha\beta$ and $D = |z| Re\sqrt{z}$.

Domain $\mathcal{R}_0$. It is easy to see that for every $\alpha > 0$ (17) takes place without any restriction on σ.

Domain $\mathcal{R}_2$. Here $Rez \geq \alpha^2 + 3\beta^2 \geq 0$, therefore due to (18)

$$Re\sqrt{z} \geq C|z|^{1/2} = C\left((\alpha^2 - \beta^2 + |\xi|^2)^2 + 4\alpha^2\beta^2\right)^{1/4} \geq C_\alpha|\xi|. \tag{20}$$

Thus

$$K(s,\xi) \leq C\frac{|s|^2(1+|\xi|^2)^{\sigma-\theta}}{|z|^{3/2}} \leq C\frac{(\alpha^2+1/2|\xi|^2)(1+|\xi|^2)^{\sigma-\theta}}{|\xi|^3} \leq C_\alpha$$

for $\sigma \leq 1/2 + \theta$.

Domain $\mathcal{R}_1$. In this domain we have $|z| \geq |\beta|$. If $Rez \geq 0$, then

$$Re\sqrt{z} \geq C_\alpha|z|^{1/2} \geq C_\alpha|\beta|^{1/2} \tag{21}$$

and $D \geq C_\alpha|z|^{3/2} \geq C_\alpha|\beta|^{3/2}$. If $Rez < 0$, then (19) implies

$$Re\sqrt{z} \geq \frac{\sqrt{2}\alpha\beta}{\sqrt{|z| - Rez}} \geq C_\alpha\frac{|\beta|}{\sqrt{|z|}} \geq C_\alpha|\beta|^{1/2} \tag{22}$$

and $D \geq C_\alpha|\beta|^{3/2}$. Thus, in $\mathcal{R}_1$

$$K(s,\xi) \leq C_\alpha\frac{(\alpha^2+\beta^2)(1+|\beta|^2/4)^{\sigma-\theta}}{|\beta|^{3/2}} \leq C_\alpha$$

for $\sigma \leq -1/4 + \theta$.

Domain $\mathcal{R}_3$. In this domain $Rez = \alpha^2 - \beta^2 + |\xi|^2 \leq \alpha^2 - 3/4\beta^2$. If $Rez \geq 0$, then

$$Re\sqrt{z} \geq C|z|^{1/2} \geq C_\alpha|\beta| \tag{23}$$

and $D \geq C|z|^{3/2} \geq C|\beta|^3$. If $Rez < 0$, (19) implies

$$2|Re\sqrt{z}|^2 \geq \frac{4\alpha^2\beta^2}{-Rez + \sqrt{(Rez)^2 + 4\alpha^2\beta^2}} \geq C_\alpha. \tag{24}$$

Since in $\mathcal{R}_3$ $D \geq C_\alpha|\beta|^2$ we have $K(s,\xi) \leq C_\alpha$.

Combining the inequalities for $\mathcal{R}_0 - \mathcal{R}_4$, we get $K(s,\xi) \leq C_\alpha$ for $\sigma = -1/4 + \theta$.

The proof of part (i) of Th. 4 is complete.

To prove (ii) we use a rather different technique. Here and further in this section we assume $f \in L^\infty(\mathbb{R}_+; H^\theta(\mathbb{R}^3_+))$. Denote $F_{x_1x_2} = \tilde{}$. Formally applying the inverse Laplace transform to (16), we get

$$\partial_t\tilde{\phi}(t,\xi,x_3=0)$$
$$= -\mathcal{L}^{-1}\left[\frac{s}{\sqrt{s^2+|\xi|^2}}\right](t,\xi) \overset{t}{*} \mathcal{L}^{-1}\left[\int_0^\infty e^{-y\sqrt{s^2+|\xi|^2}}\hat{f}(s,\xi,y)dy\right](t,\xi). \tag{25}$$

Define

$$[B\hat{f}](s,\xi) = \int\limits_0^\infty e^{-y\sqrt{s^2+|\xi|^2}} \hat{f}(s,\xi,y)dy. \qquad (26)$$

If $[B\hat{f}](s,\xi) \in H_a(Y)$, as soon as $\hat{f} \in H_a(X)$ for certain spaces X, Y, the equality (25) is meaningful due to Th. 5. In fact, we can prove even a stronger assertion on operator B.

Lemma 5. *The operator B has the following properties:*

(i) The function $[B\hat{f}](s,\xi) \in H_0(L^2(\mathbb{R}^2, (1+|\xi|)^{2\theta}d\xi))$ provided $\hat{f} \in H_0(L^2(\mathbb{R}^3_+, (1+|\xi|)^{2\theta}d\xi dy))$.

(ii) Denote $[B_\gamma \hat{f}](\beta,\xi) = [B\hat{f}](\gamma + i\beta, \xi)$. There exists $\gamma > 0$ such that $B_\gamma : L^2(\mathbb{R}; L^2(\mathbb{R}^3_+, (1+|\xi|)^{2\theta}d\xi dy)) \to L^2(\mathbb{R}; L^2(\mathbb{R}^2, (1+|\xi|)^{2\theta}d\xi))$ is a bounded linear operator.

P r o o f. First we prove that $[B\hat{f}](s,\xi)$ is a holomorphic function of the variable s in the complex half-plane $\{s : \ Res > 0\}$. Due to the Dunford theorem it is enough to prove that $[B\hat{f}](s,\xi)$ is weakly holomorphic. Let $g(\xi) \in L^2(\mathbb{R}^2, (1+|\xi|)^{-2\theta}d\xi)$. Then

$$(B\hat{f}, g) = \int\limits_{\mathbb{R}^2} d\xi g(\xi) \int\limits_0^\infty dy \cdot e^{-y\sqrt{s^2+|\xi|^2}} \hat{f}(s,\xi,y).$$

Consider the following expression

$$\frac{d}{ds} \int\limits_{|\xi|<Q} d\xi g(\xi) \int\limits_0^R dy \cdot e^{-y\sqrt{s^2+|\xi|^2}} \hat{f}(s,\xi,y)$$

$$= - \int\limits_{|\xi|<Q} d\xi g(\xi) \int\limits_0^R dy \cdot y e^{-y\sqrt{s^2+|\xi|^2}} \frac{s}{\sqrt{s^2+|\xi|^2}} \hat{f}(s,\xi,y)$$

$$+ \int\limits_{|\xi|<Q} d\xi g(\xi) \int\limits_0^R dy \cdot e^{-y\sqrt{s^2+|\xi|^2}} \frac{d}{ds} \hat{f}(s,\xi,y) = A_1(s,Q,R) + A_2(s,Q,R). \qquad (27)$$

Estimate the first term. Using the Schwartz inequality, we get

$$|A_1(s,Q,R)|$$
$$\le C\int\limits_{|\xi|<Q} d\xi g(\xi)\left|\frac{s}{\sqrt{s^2+|\xi|^2}}\right|\frac{(1+|\xi|)^{-\theta}}{(Re\sqrt{s^2+|\xi|^2})^{3/2}}\left((1+|\xi|)^{2\theta}\int\limits_0^R dy|\hat f(s,\xi,y)|^2\right)^{1/2}$$
$$\le C\left(\int\limits_{|\xi|<Q} d\xi(1+|\xi|)^{2\theta}\int\limits_0^R dy|\hat f(s,\xi,y)|^2\right)^{1/2}$$
$$\times\left(\int\limits_{|\xi|<Q} d\xi|g(\xi)|^2\cdot\frac{|s|^2(1+|\xi|)^{-2\theta}}{|s^2+|\xi|^2|(Re\sqrt{s^2+|\xi|^2})^3}\right)^{1/2}.$$

If we prove that

$$K_1(s,\xi)=\frac{|s|^2}{|s^2+|\xi|^2|(Re\sqrt{s^2+|\xi|^2})^3}\le C(s),\quad Res>a, \tag{28}$$

this will imply

$$|A_1(s,Q,R)|\le C(s)||g||_{L^2(\mathbb{R}^2,(1+|\xi|)^{-2\theta}d\xi)}||f(s)||_{L^2(\mathbb{R}^3_+,(1+|\xi|)^{2\theta}d\xi dy)},\quad Res>a. \tag{29}$$

We use the method of the proof of part (i) of the theorem: denote $s=\alpha+i\beta$, $z=s^2+|\xi|^2$, $D=|s^2+|\xi|^2|(Re\sqrt{s^2+|\xi|^2})^3$, divide the first quarter of the half-plane $(\beta,|\xi|)$ into four domains, and use inequalities (18), (19) .

For the domain $\mathcal{R}_0$ (28) is evident with $a=0$. In the domain $\mathcal{R}_2$ $D\ge C(\alpha)|z|^{5/2}\ge C(\alpha)|\xi|^5$, therefore $K_1(s,\xi)\le C(\alpha)$, $\alpha>0$. In the domain $\mathcal{R}_3$ (23), (24) imply $D\ge C(\alpha)|z|^2\ge C(\alpha)|\beta|^4$, hence, $K_1(s,\xi)\le C(\alpha)$, $\alpha>0$. In the domain $\mathcal{R}_1$ $D\ge C(\alpha)|z|^{5/2}\ge C(\alpha)|\beta|^{5/2}$ if $Rez\ge 0$ and, provided $Rez<0$, (22) yields $D\ge C(\alpha)|\beta|^{5/2}$, too. Thus, $K_1(s,\xi)\le C(\alpha)$, $\alpha>0$ for all $s,|\xi|:\ Res=\alpha>0$ and (28) together with (29) are proved.

Now we estimate the second term in (27). Using the Schwartz inequality, we obtain

$$|A_2(s,Q,R)|\le C\left(\int\limits_{|\xi|<Q} d\xi(1+|\xi|)^{2\theta}\int\limits_0^R dy\left|\frac{d}{ds}\hat f(s,\xi,y)\right|^2\right)^{1/2}$$
$$\times\left(\int\limits_{|\xi|<Q} d\xi|g(\xi)|^2\cdot\frac{(1+|\xi|)^{-2\theta}}{Re\sqrt{s^2+|\xi|^2}}\right)^{1/2}.$$

Estimates (20)–(22) imply

$$K_2(s,\xi) = \frac{1}{Re\sqrt{s^2+|\xi|^2}} \le C(\alpha), \quad \alpha = Res > 0, \tag{30}$$

and, consequently,

$$|A_2(s,Q,R)| \le C(s)||g||_{L^2(\mathbb{R}^2,(1+|\xi|)^{-2\theta}d\xi)} \left\| \frac{d}{ds} f(s) \right\|_{L^2(\mathbb{R}^3_+,(1+|\xi|)^{2\theta}d\xi dy)}. \tag{31}$$

Now, letting Q, R to tend to $+\infty$ in (27) and using (29), (31), we prove week (and, consequently, strong) analyticity of $[B\hat{f}](s,\xi)$.

It is easy to see that the constants $C(s)$ in (29), (31) grow as $|s|^m$, $|s| \to +\infty$ for some m, and therefore the growth condition (14) is fulfilled. Thus, part (i) of the lemma is proved.

The estimate (30) guarantees that part (ii) of the lemma is also true.

Corollary 3. *The operator $A = \mathcal{L}^{-1} \circ B \circ \mathcal{L}$ is bounded from $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3_+, (1+|\xi|)^{2\theta}d\xi dy))$ to $L^2(0,T; L^2(\mathbb{R}^2,(1+|\xi|)^{2\theta}d\xi))$.*

The assertion of the lemma follows directly from Lems. 4, 5 and Cor. 2.

Returning to (25), we note (see, e.g., [10]) that

$$\mathcal{L}^{-1}\left(-\frac{s}{\sqrt{s^2+|\xi|^2}}\right) = \theta(t)|\xi|J_1(|\xi|t) - \delta(t)J_0(0).$$

Thus $\partial_t\tilde{\phi}(t,\xi,0)$ can be represented as a sum $[A_1\tilde{f}](t,\xi) + [A_2\tilde{f}](t,\xi)$, where

$$[A_1\tilde{f}](t,\xi) = (\theta(t)|\xi|J_1(|\xi|t)) \overset{t}{*} [\mathcal{L}^{-1} \circ B \circ \mathcal{L}\tilde{f}](t,\xi)$$

$$[A_2\tilde{f}](t,\xi) = (-\delta(t)J_0(0)) \overset{t}{*} [\mathcal{L}^{-1} \circ B \circ \mathcal{L}\tilde{f}](t,\xi) = -J_0(0)[\mathcal{L}^{-1} \circ B \circ \mathcal{L}\tilde{f}](t,\xi).$$

It is easy to verify that

$$||F^{-1}_{x_1x_2} \circ A_2 \circ F_{x_1x_2} f||_{L^2(0,T;H^\theta(\mathbb{R}^2))} \le C||f||_{L^\infty(\mathbb{R}+;H^\theta(\mathbb{R}^3_+))}, \quad \theta \in [0,1]. \tag{32}$$

As for $[A_1\tilde{f}](t,\xi)$, denoting $g(t,\xi) = [\mathcal{L}^{-1} \circ B \circ \mathcal{L}\tilde{f}](t,\xi)$ and using the Schwartz inequality one can get

$$||[A_1\tilde{f}](t,\cdot)||^2_{L^2(\mathbb{R}^2,(1+|\xi|)^{-2\sigma}d\xi)}$$

$$= \int\limits_{\mathbb{R}^2} d\xi \frac{1}{(1+|\xi|)^{2\sigma}} \left| \int\limits_0^t |\xi| J_1(|\xi|(t-\tau)) g(\tau,\xi) \right|^2 \le \int\limits_{\mathbb{R}^2} d\xi \frac{|\xi|^{1-\alpha}}{(1+|\xi|)^{2(\sigma+\theta)}}$$

$$\times \int\limits_0^t d\tau \frac{1}{(t-\tau)^{2\alpha}} \int\limits_0^t ||\xi|^\alpha (t-\tau)^\alpha J_1(|\xi|(t-\tau))|^2 (1+|\xi|)^{2\theta} |g(\tau,\xi)|^2,$$

for all $t < T$. The asymptotical properties of the Bessel functions imply that $x^\alpha J_1(x)$ is bounded on $\mathbb{R}_+$ provided $0 \le \alpha \le 1/2$. The integral $\int_0^t (t-\tau)^{-2\alpha} d\tau$ converges for $\alpha < 1/2$, and for $\sigma - \theta = (1-\alpha)/2$ the function $|\xi|^{1-\alpha}/(1+|\xi|)^{2(\sigma-\theta)}$ is bounded for all $\xi \in \mathbb{R}^2$. Thus, using Cor. 2, we obtain

$$||A_1 \tilde{f}||_{L^\infty(0,T;L^2(\mathbb{R}^2,(1+|\xi|)^{-1+\alpha+2\theta} d\xi))} \le C(T,\alpha)||g||_{L^2(0,T;L^2(\mathbb{R}^2,(1+|\xi|)^{2\theta} d\xi))}$$
$$\le C(T,\alpha)||\tilde{f}||_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^3_+,(1+|\xi|)^{2\theta} d\xi dy))}, \quad \alpha < 1/2.$$

Hence, we have

$$||F^{-1}_{x_1x_2} \circ A_1 \circ F_{x_1x_2} f||_{L^\infty(0,T;H^{-1/4+\theta-\epsilon}(\mathbb{R}^2))} \le C(T,\epsilon)||f||_{L^\infty(\mathbb{R}_+;H^\theta(\mathbb{R}^3_+))}. \tag{33}$$

Combining (32) and (33), we obtain part (ii) of the theorem.

5. Proof of Theorem 3

Let us consider the problem (1)–(3). Due to Remark 3 we can prove this theorem only for the case $U = 0$, i. e., we consider the problem

$$\partial_{tt}\phi = \Delta\phi, \quad x \in \mathbb{R}^3_+, \tag{34}$$

$$\left.\frac{\partial \phi}{\partial x_3}\right|_{x_3=0} = 0, \quad \phi(0) = \phi_0, \quad \phi_t(0) = \phi_1. \tag{35}$$

We introduce the following operators concerning this problem.

An operator A on $L^2(\mathbb{R}^3_+)$ is defined as

$$A = -\Delta, \quad D(A) = \left\{ f \in H^2(\mathbb{R}^3_+) : \left.\frac{\partial f}{\partial x_3}\right|_{x_3=0} = 0 \right\}. \tag{36}$$

The operator $-A$ generates a s.c. cosine operator $C(t)$ on $L^2(\mathbb{R}^3_+)$ with the sine operator $S(t) = \int\limits_0^t d\tau C(\tau)$ [11]. For the problem (34)–(35) we can write them down explicitly:

$$C(t)\phi_0(x) = \frac{1}{(2\pi)^{3/2}} \int\limits_{\mathbb{R}^3} e^{i\xi x} \cos(|\xi|t) [F_x \hat{\phi}_0](\xi) d\xi, \tag{37}$$

$$S(t)\phi_0(x) = \frac{1}{(2\pi)^{3/2}} \int\limits_{\mathbb{R}^3} e^{i\xi x} \frac{\sin(|\xi|t)}{|\xi|} [F_x \hat{\phi}_0](\xi) d\xi, \tag{38}$$

where $\hat{\phi}_0$ is an even extension of ϕ_0 from $\mathbb{R}^3_+$ onto $\mathbb{R}^3$.

The solution to (34)–(35) can be represented as $\phi(t) = C(t)\phi_0 + S(t)\phi_1$. It is well known that

$$\frac{dC(t)x}{dt} = -AS(t)x, \quad x \in D(A^{1/2}). \tag{39}$$

The following lemma is also true.

Lemma 6. *Denote* $\mathcal{G}^\theta(\mathbb{R}^3_+) = [\overline{\mathcal{W}}^{1,1}(\mathbb{R}^3_+), \mathcal{H}^1(\mathbb{R}^3_+)]_{[1-\theta]}$. *Then*

$$C(t) : H^\theta(\mathbb{R}^3_+) \to C(0,T; H^\theta(\mathbb{R}^3_+)), \tag{40}$$

$$AS(t) : \mathcal{G}^\theta(\mathbb{R}^3_+) \to C(0,T; H^\theta(\mathbb{R}^3_+)) \tag{41}$$

are continuous operators.

P r o o f. The continuity of cosine operator is evident. For the sine operator and ϕ_0 smooth enough we have the following:

$$AS(t)\phi_0 = S(t)A\phi_0 = F_\xi^{-1}(\sin(|\xi|t)|\xi|[F_x\hat{\phi}_0](\xi)),$$

where $\hat{\phi}_0$ is an even extension of ϕ_0 from $\mathbb{R}^3_+$ onto $\mathbb{R}^3$. Assume that $\phi_0 \in \mathcal{H}^1(\mathbb{R}^3_+)$. Proposition 3 implies $||\hat{\phi}_0||_{\mathcal{H}^1(\mathbb{R}^3)} \le C||\phi_0||_{\mathcal{H}^1(\mathbb{R}^3_+)}$ and therefore $||AS(t)\phi_0||_{L^2(\mathbb{R}^3_+)} \le C||\phi_0||_{\mathcal{H}^1(\mathbb{R}^3_+)}$ for all $t > 0$. If $\phi_0 \in \overline{\mathcal{W}}^{1,1}(\mathbb{R}^3_+)$, then in the same way we obtain that $||AS(t)\phi_0||_{H^1(\mathbb{R}^3_+)} \le C||\phi_0||_{\overline{\mathcal{W}}^{1,1}(\mathbb{R}^3_+)}$ for all $t > 0$. Continuity with respect to t can be proved in the standard way. Finally, we use the interpolation Lem. 1 and prove the second assertion of the present lemma.

We will need the Neumann map N connected with the operator A. Since $Ker(A) = \{0\}$ (due to (36) it can be easily verified), N is defined as follows:

$$u = Nh \quad \Leftrightarrow \quad \left\{ -Au = 0 \text{ in } \mathbb{R}^3_+; \ \frac{\partial u}{\partial x_3}\bigg|_{x_3=0} = h \right\}.$$

The following fact also holds true [12]: $N^*A^*y = \gamma[y]$, where $\gamma[y]$ denotes the Sobolev trace of function y from $\mathbb{R}^3_+$ onto $\mathbb{R}^2$ and A^* is the $L^2(\mathbb{R}^3_+)$-adjoint of A.

Now we are ready to prove Th. 3. The idea of the proof was also borrowed from [3]. It was already mentioned that a solution to (34)–(35) can be represented as $\phi(t) = C(t)\phi_0 + S(t)\phi_1$. Then due to (39)

$$\partial_t\gamma[\phi](t) = N^*A^*AS(t)\phi_0 + N^*A^*C(t)\phi_1. \tag{42}$$

To analyze each term in (42), we consider the problem

$$\partial_{tt}\phi = \Delta\phi + f(x,t), \quad x \in \mathbb{R}^3_+, \tag{43}$$

$$\frac{\partial \phi}{\partial x_3}\bigg|_{x_3=0} = 0, \quad \phi(T) = \phi_t(T) = 0 \tag{44}$$

with a certain function f.

Step 1. *Term* $N^*A^*C(t)\phi_1$. Let in (43)–(44) $f(t,x) = C(t)\phi_1$. Due to Lemma 6 $f \in L^\infty(\mathbb{R}_+; H^\theta(\mathbb{R}^3_+))$ provided $\phi_1 \in H^\theta(\mathbb{R}^3_+)$. Then for this problem

$$\partial_t\gamma[\phi](t) = N^*A^*\int_t^T C(\tau - t)C(\tau)\phi_1.$$

Using the formula $C(t_1)\cdot C(t_2) = 1/2\,(C\,(t_1+t_2) + C\,(t_1-t_2))$, we obtain that

$$\partial_t\gamma[\phi](t) = \frac{1}{2}N^*A^*C(t)\phi_1(T-t) + \frac{1}{4}N^*A^*(S(2T-t) - S(t))\phi_1. \tag{45}$$

Theorem 4 (i) and (40) imply

$$||\partial_t\gamma[\phi](t)||_{L^2(0,T;H^{-1/4+\theta}(\mathbb{R}^2))} \le C_T||C(t)\phi_1||_{L^\infty(\mathbb{R}_+;H^\theta(\mathbb{R}^3_+))} \le C_T||\phi_1||_{H^\theta(\mathbb{R}^3_+)}. \tag{46}$$

Lemma 2 yields to the following estimate of the second term in (45):

$$||N^*A^*(S(2T-t) - S(t))\phi_1||_{C(0,T;\mathcal{W}^{1/2,\theta}(\mathbb{R}^2))} \le C||\phi_1||_{H^\theta(\mathbb{R}^3_+)}, \tag{47}$$

where C does not depend on T. Estimates (46) and (47) together with Lem. 3 give us that $N^*A^*C(t)\phi_1 \in L^2(0,T-1;H^{-1/4-\theta}_{loc}(\mathbb{R}^2))$ and

$$||N^*A^*C(t)\phi_1||_{L^2(0,T-1;H^{-1/4-\theta}(B))} \le C(T,B)||\phi_1||_{H^\theta(\mathbb{R}^3_+)} \tag{48}$$

for any bounded set $B \subset \mathbb{R}^2$. Form the other hand, using Th. 4 (ii) and (47), we get

$$N^*A^*C(t)\phi_1 = A_1 + A_2, \tag{49}$$

where $A_1 \in L^\infty(0,T;H^{-1/4+\theta-\epsilon}_{loc}(\mathbb{R}^2))$, $A_2 \in L^2(0,T;H^\theta_{loc}(\mathbb{R}^2))$, and

$$||A_1||_{L^\infty(0,T;H^{-1/4+\theta-\epsilon}(B))} \le C(T,B,\epsilon)||\phi_1||_{H^\theta(\mathbb{R}^3_+)}, \tag{50}$$

$$||A_2||_{L^2(0,T;H^\theta(B))} \le C(T,B)||\phi_1||_{H^\theta(\mathbb{R}^3_+)} \tag{51}$$

for any bounded set $B \subset \mathbb{R}^2$.

Step 2. *Term* $N^*A^*AS(t)\phi_0$. Let in (43)–(44) $f(t,x) = AS(t)\phi_0$. Due to Lemma 6 $f \in L^\infty(\mathbb{R}_+; H^\theta(\mathbb{R}^3_+))$ provided $\phi_0 \in \mathcal{G}^\theta(\mathbb{R}^3_+)$. Then for this problem

$$\partial_t\gamma[\phi](t) = N^*A^*\int_t^T C(\tau - t)AS(\tau)\phi_0.$$

Using the formula $S(t_1)\cdot C(t_2) = 1/2\,(S\,(t_1+t_2) - S\,(t_1-t_2))$, we obtain

$$\partial_t\gamma[\phi](t) = \frac{1}{2}N^*A^*AS(t)\phi_0(T-t) + \frac{1}{4}N^*A^*(C(2T-t) - C(t))\phi_0.$$

Theorem 4 yields

$$||\partial_t \gamma[\phi](t)||_{L^2(0,T;H^{-1/4+\theta}(\mathbb{R}^2))} \le C_T ||AS(t)\phi_0||_{L^\infty(\mathbb{R}_+;H^\theta(\mathbb{R}^3_+))} \le C_T ||\phi_0||_{\mathcal{G}^\theta(\mathbb{R}^3_+)}. \tag{52}$$

Similarly as in Step 1,

$$||N^* A^* (C(2T-t) - C(t))\phi_0||_{C(0,T;\mathcal{W}^{1/2,\theta}(\mathbb{R}^2))} \le C||\phi_0||_{\mathcal{G}^\theta(\mathbb{R}^3_+)}, \tag{53}$$

where C does not depend on T, and, finally, (52), (53), and Lem. 3 imply $N^*A^*AS(t)\phi_0 \in L^2(0,T-1; H^{-1/4+\theta}_{loc}(\mathbb{R}^2))$ and

$$||N^*A^*AS(t)\phi_0||_{L^2(0,T-1;H^{-1/4+\theta}(B))} \le C(T,B)||\phi_0||_{\mathcal{G}^\theta(\mathbb{R}^3_+)} \tag{54}$$

for any bounded set $B \subset \mathbb{R}^2$. From the other hand, Th. 4 and (53) give us

$$N^*A^*AS(t)\phi_0 = B_1 + B_2, \tag{55}$$

where $B_1 \in L^\infty(0,T; H^{-1/4+\theta-\epsilon}_{loc}(\mathbb{R}^2))$, $B_2 \in L^2(0,T; H^\theta_{loc}(\mathbb{R}^2))$, and

$$||B_1||_{L^\infty(0,T;H^{-1/4+\theta-\epsilon}(B))} \le C(T,B,\epsilon)||\phi_0||_{\mathcal{G}^\theta(\mathbb{R}^3_+)}, \tag{56}$$

$$||B_2||_{L^2(0,T;H^\theta(B))} \le C(T,B)||\phi_0||_{\mathcal{G}^\theta(\mathbb{R}^3_+)} \tag{57}$$

for any bounded set $B \subset \mathbb{R}^2$.

Since T was chosen arbitrary, (48) and (54) give us assertion (i) of the theorem, and (49)–(51), (55)–(57) give us assertion (ii) of the same theorem.

Remark 4. *The norm in $\mathcal{G}^{1/2}$ is not equivalent to $||\nabla \cdot ||_{1/2,\mathbb{R}^3_+}$ (see [7, Ch. 1], Th. 11.7), therefore the case $\theta = 1/2$ is excluded from Th. 3 and Cor. 1.*

6. Proof of Corollary 1

The representation of a solution to (34)–(35) given by Th. 3.3 [13] implies, that the solution $(\phi,\phi_t)(t_0)$ in a half-ball of radius R depends only on the initial data values in the ball of radius $R_1(R,t_0)$. Moreover, $\nabla\phi(t_0)$ depends only on $\nabla\phi_0$ in this ball. Chose new initial data. For ϕ_0 the Poincaré inequality holds

$$||\phi_0||^2_{1,B^+_{R_1}} \le C\left(||\nabla\phi_0||^2_{0,B^+_{R_1}} + \left|\int\limits_{B^+_{R_1}} \phi_0(x)dx\right|^2\right).$$

We set $\tilde{\phi}_0 = \phi_0 - C_1$, where $C_1 = 1/mes(B^+_{R_1})\int_{B^+_{R_1}} \phi_0(x)dx$. Thus, for $\tilde{\phi}_0$ we have $||\tilde{\phi}_0||^2_{1,B^+_{R_1}} \le C||\nabla\tilde{\phi}_0||^2_{0,B^+_{R_1}}$ and, consequently, $||\tilde{\phi}_0||^2_{1+\theta,B^+_{R_1}} \le C||\nabla\tilde{\phi}_0||^2_{\theta,B^+_{R_1}}$,

$\theta \in [0,1]$. There exists an operator of finitary extension L_θ from $H^\theta(B^+_{R_1})$ to $H^\theta(\mathbb{R}^3_+)$, $\theta \in [0,2]$ such that $suppLu \subset B^+_{2R_1}$. Since for $\theta > 3/2$ we need boundary conditions (2) to be satisfied by the initial datum ϕ_0, we apply the operator R defined in Prop. 4, to $L_\theta\tilde{\phi}_0$. We keep the same notation for the obtained operator. Hence, we have

$$||\nabla L\tilde{\phi}_0||_{\theta,B^+_{2R_1}} \leq C||L\tilde{\phi}_0||_{\theta+1,\mathbb{R}^3_+} \leq C||\tilde{\phi}_0||_{\theta+1,B^+_{R_1}}$$
$$\leq C||\nabla\tilde{\phi}_0||_{\theta,B^+_{R_1}} = C||\nabla\phi_0||_{\theta,B^+_{R_1}},$$

for $\theta \in [0,1]$. We define new initial data as follows:

$$\phi(0) = L_{1+\theta}\tilde{\phi}_0, \quad \phi_t(0) = L_\theta\phi_1, \quad \theta \in [0,1],\ \theta \neq 1/2.$$

Denote by $(\bar{\phi},\bar{\phi}_t)(t)$ the solution to (34)–(35) with this new initial data. Obviously, $(\bar{\phi},\bar{\phi}_t)(t_0) = (\phi,\phi_t)(t_0)$ in B^+_R. Thus, Th. 3 implies Cor. 1.

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