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KdV Flow on Generalized Reflectionless Potentials

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The purpose of this article is to construct KdV flow on a space of generalized reflectionless potentials by applying Sato's Grassmannian approach. The point is that the base space contains not only rapidly decreasing potentials but also oscillating ones such as periodic ones, which makes it possible for us to discuss the shift invariant probability measures on it.

Key words: Korteweg de Vries equation, inverse spectral methods, reflectionless potentials.

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1. Introduction

The KdV equation is

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x},$$

and this describes the dynamics of shallow waters. As is well known, n-soliton solutions for the KdV equation are given by

$$u(t,x) = -2\frac{\partial^2}{\partial x^2}\log\det(I + A(t,x)),$$

where

$$A(t,x) = \left(\frac{\sqrt{m_i m_j}}{\eta_i + \eta_j} e^{-(\eta_i+\eta_j)x + 4\left(\eta_i^3+\eta_j^3\right)t}\right)_{1\le i,j\le n}$$

with $m_i, \eta_i > 0$. For each fixed $t \in \mathbf{R}$, $u(t,\cdot)$ is a reflectionless potential which appears in 1-D scattering theory. V.A. Marchenko [13, 14] considered the compact uniform closure of reflectionless potentials, which we call the space of generalized reflectionless potentials, and made an attempt to solve the KdV equation starting from an element of this closure. However, he had to impose the solvability condition on an integral equation, which made it impossible to solve the KdV

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equation in its full generality. On the other hand, M. Sato and Y. Sato established a unified approach for a large class of completely integrable systems. They constructed solutions based on dynamics (flows) on infinite dimensional Grassmann manifold, and it was rewritten from an analytic point of view by G. Segal and G. Wilson [16]. R.A. Johnson [9] mentioned the applicability of their approach to this space of generalized reflectionless potentials, which admits a certain class of oscillating functions. However, to apply this method we have to prove the transversality, which is equivalent to the solvability of the integral equation considered by Marchenko. The first purpose of this paper is to construct a KdV flow on this space by showing the transversality. In the case when the base space is a set of rapidly decreasing smooth functions, H.P. McKean [15] applied Sato's theory to construct the KdV flow on it.

For $\lambda_0 < 0$ let Ω_{λ_0} be the compact uniform closure of all reflectionless potentials whose associated Schrödinger operators have their spectrum in $[\lambda_0, \infty)$. Set

$$\Gamma = \left\{ \begin{array}{c} g;\ g(z) \text{ is holomorphic on } \mathbf{D},\ g(0) = 1,\ g(z) \neq 0 \text{ for } \forall z \in \mathbf{D}, \\ \text{takes real values on } \mathbf{R} \text{ and } g(-z) = g(z)^{-1} \text{ for } \forall z \in \mathbf{D} \end{array} \right\},$$

where $\mathbf{D}$ is the closed unit disc. We construct a homomorphism K between the group Γ and the group of all homeomorphisms on Ω_{λ_0}. This K induces the shift operation if we choose

$$g_x(z) = e^{-xz} \in \Gamma$$

and solutions for the KdV equation if we choose

$$g_{x,t}(z) = e^{-xz+4tz^3} \in \Gamma.$$

Any other higher order KdV equation can be solved in this way on Ω_{λ_0}. We also discuss the isospectral property under K.

The motivation of this paper is to construct a nice solution for the KdV equation starting from a certain random initial data. This problem was raised by V.E. Zakharov and the author was taught it by S.A. Molchanov. We would like to construct a solution as a typical random field $\{u(t,x)\}_{t,x\in\mathbf{R}}$ which is shift invariant with respect to t and x. In this respect, there are already solutions which are quasiperiodic in time and space, which is a special case of shift invariant random fields. However our aim is to give a very random solution. The construction of the KdV flow is a starting point in solving this problem. Since Ω_{λ_0} is compact and the KdV flow $\{K(g)\}_{g\in\Gamma}$ is commutative, the space of all probability measures on Ω_{λ_0} invariant with respect to $\{K(g)\}_{g\in\Gamma}$ is a non-empty compact convex set. Therefore we have many ergodic $K(g)-$invariant probability measures on Ω_{λ_0}. It is interesting to study the spectral property for the associated Schrödinger operators under these probability measures. The problem of V.E. Zakharov is just the problem on finding such a probability measure under which $K(g)q$ behaves as random as possible, especially the spectrum of the Schrödinger operators should have a dense point spectrum on $[\lambda_0, 0]$, whereas the spectrum in $(0, \infty)$ is always purely absolutely continuous for any potential from Ω_{λ_0}. Since a KdV-flow invariant probability measure is automatically shift invariant, hence we can define the Floquet exponent. We discuss the relationship between the KdV-flow and the Floquet exponent, although it is still unsatisfactory.

We try to give a self-contained explanation of this subject as far as possible, since it may be difficult to obtain a complete view only by citing necessary facts.

2. Spectral Theory of 1-D Schrödinger Operators and Dyson Formula

Let us consider a one-dimensional Schrödinger operator

$$L = -\frac{d^2}{dx^2} + q(x)$$

with potential q, which is a real valued function of $L^1_{loc}(\mathbf{R})$. In this section we introduce the Gelfand–Levitan inverse spectral theory and the Dyson formula which solves the inverse spectral problem by the Fredholm determinants of the integral operators associated with the spectral measures.

Suppose $q(x)$ satisfies

$$q(x) \geq -cx^2$$

for every sufficiently large $|x|$ with a constant c. Then it is known that L has a unique selfadjoint extension on $L^2(\mathbf{R})$. Under this condition, for $\lambda \in \mathbf{C}\backslash\mathbf{R}$ there exist unique solutions $f_{\pm}(x, \lambda)$ of

$$Lf = \lambda f, \quad f(0) = 1 \text{ and } f \in L^2(\mathbf{R}_{\pm}),$$

where $\mathbf{R}_+ = [0, \infty)$, $\mathbf{R}_- = (-\infty, 0]$. Set

$$m_{\pm}(\lambda) = m_{\pm}(\lambda, q) = \pm f'_{\pm}(0, \lambda).$$

These functions become the Herglotz ones which are holomorphic on the upper half-plane with positive imaginary parts. We call these functions as H-functions, see Appendix for the properties of H-functions. Let $g_\lambda(x, y)$ be the Green function for $L - \lambda$, that is

$$(L - \lambda)^{-1}(x, y) = g_\lambda(x, y).$$

It is well known that

$$g_\lambda(x, y) = g_\lambda(y, x) = -\frac{f_+(x, \lambda) f_-(y, \lambda)}{m_+(\lambda) + m_-(\lambda)} \quad \text{for } x \geq y \tag{1}$$

is valid. The Gelfand–Levitan inverse spectral theory says that the potential q on $\mathbf{R}_+$(resp.$\mathbf{R}_-$) can be recovered from m_+(resp.m_-) by solving the integral equation(2) of Fredholm type. Let

$$\phi_+(x,y) = \int_R \frac{\left(1-\cos\sqrt{\xi}x\right)\left(1-\cos\sqrt{\xi}y\right)}{\xi^2}\sigma_+(d\xi) - x\wedge y$$

with a measure σ_+ representing m_+. Define

$$F_+(x,y) = \frac{\partial^2}{\partial x\partial y}\phi_+(x,y)$$

and consider

$$K(x,y) + F_+(x,y) + \int_0^x F_+(y,t)K(x,t)dt = 0. \tag{2}$$

Then F_+ is continuous and the integral equation (2) is uniquely solvable in the space of continuous functions on $[0,x]$ for each fixed $x > 0$. Then the potential q is given by

$$q(x) = 2\frac{d}{dx}K(x,x) \quad \text{for } x > 0. \tag{3}$$

For details see V.A. Marchenko [12].

For later purpose we give another representation of q by a determinant. This kind of representation was remarked first by F. Dyson [3] in the scattering case (in which the potential q is decaying sufficiently fast at $\pm\infty$, see Th. 4 below) and by K. Iwasaki [8] in the case of boundary value problems on finite intervals. For the sake of completeness we give a proof of this formula here. Let F_+^x be the integral operator on $C([0,x])$ with kernel F_+.

Theorem 1. (***Dyson formula***)

$$q(x) = -2\frac{d^2}{dx^2}\log\det(I + F_+^x). \tag{4}$$

P r o f. Set

$$F_x(t,s) = xF_+(xt,xs)$$

and consider the integral operator F_x with kernel $F_x(t,s)$ on $L^2([0,1])$. Then it is easy to see that $\det(I + F_+^x) = \det(I + F_x)$ holds. Therefore

$$\begin{aligned}\frac{d}{dx}\log\det(I + F_+^x) &= \frac{d}{dx}\log\det(I + F_x)\\ &= tr\left\{(I + F_x)^{-1}\frac{\partial F_x}{\partial x}\right\}\\ &= \int_0^1 \frac{\partial F_x}{\partial x}(t,t)dt + \iint_{[0,1]^2}\Gamma_x(t,s)\frac{\partial F_x}{\partial x}(s,t)dtds,\end{aligned}$$

where $\Gamma_x = (I+F_x)^{-1} - I$ and $\Gamma_x(t,s)$ is the kernel for Γ_x. Observing

$$x\frac{\partial F_x}{\partial x}(t,s) = F_x(t,s) + t\frac{\partial F_x}{\partial t}(t,s) + s\frac{\partial F_x}{\partial s}(t,s),$$

we have

$$\begin{aligned}
x\frac{d}{dx}\log\det(I+F_x) &= tr\,(F_x + \Gamma_x F_x) \\
&\quad + \int_0^1 t\left(\frac{\partial F_x}{\partial t}(t,t) + \int_0^1 \frac{\partial F_x}{\partial t}(t,s)\Gamma_x(s,t)ds\right)dt \\
&\quad + \int_0^1 t\left(\frac{\partial F_x}{\partial s}(t,t) + \int_0^1 \Gamma_x(t,s)\frac{\partial F_x}{\partial s}(s,t)ds\right)dt \\
&= -\int_0^1 \Gamma_x(t,t)dt - \int_0^1 t\frac{\partial \Gamma_x}{\partial t}(t,t)dt - \int_0^1 t\frac{\partial \Gamma_x}{\partial s}(t,t)dt \\
&= -\int_0^1 \Gamma_x(t,t)dt - \int_0^1 t\frac{d}{dt}\left(\Gamma_x(t,t)\right)dt \\
&= -\Gamma_x(1,1),
\end{aligned}$$

where we have used the identity $F_x + \Gamma_x + F_x\Gamma_x = F_x + \Gamma_x + \Gamma_x F_x$. On the other hand, from (2) we see

$$\begin{aligned}
xK(x,xt) &= -xF_+(x,xt) - x\int_0^1 \Gamma_x(t,s)F_+(x,xs)ds \\
&= -F_x(t,1) - \int_0^1 \Gamma_x(t,s)F_x(s,1)ds \\
&= \Gamma_x(t,1),
\end{aligned}$$

which implies

$$\frac{d}{dx}\log\det(I+F_+^x) = -K(x,x).$$

Consequently, the proposition can be proved from (3).

The formula (4) may be called **Dyson formula**. ■

3. Inverse Scattering Problem and Reflectionless Potentials

If the potential satisfies

$$|q(x)|\,(1+|x|) \in L^1(\mathbf{R}),$$

one can define two linearly independent solutions $e^{\pm}(x,k)$ for $k \in \overline{\mathbf{C}_+}(=\{k\in\mathbf{C};\ \mathrm{Im}\,k \geq 0\})$ of $Le = k^2 e$ satisfying the following asymptotic behaviour

$$\begin{cases} e^{+}(x,k) \simeq e^{ikx} \text{ as } x \to +\infty, \\ e^{-}(x,-k) \simeq e^{-ikx} \text{ as } x \to -\infty\ . \end{cases}$$

Since the pairs $\{e^+(x,k), e^+(x,-k)\}, \{e^-(x,k), e^-(x,-k)\}$ form a fundamental system of solutions of $Le = k^2 e$ for nonzero real k, we can introduce $a(k), b(k)$ as

$$\begin{cases} e^+(x,k) = a(k)e^-(x,k) + b(k)e^-(x,-k), \\ e^-(x,-k) = a(k)e^+(x,-k) - b(-k)e^+(x,k). \end{cases}$$

The real valuedness of the potential V implies

$$\begin{cases} a(k) = \overline{a(-k)}, \\ b(k) = \overline{b(-k)}, \\ |a(k)|^2 = 1 + |b(k)|^2, \end{cases} \tag{5}$$

for nonzero real k. Set

$$\begin{cases} r^+(k) = -\dfrac{b(-k)}{a(k)}, \\ r^-(k) = \dfrac{b(k)}{a(k)}, \\ t(k) = \dfrac{1}{a(k)}. \end{cases}$$

$r^+(k)$(resp. $r^-(k)$) is called the **right reflection coefficient**(resp. the **left reflection coefficient**) and $t(k)$ is called the **transmission coefficient** (see V.A. Marchenko [12]). From (5) we see that

$$0 < |t(k)| \le 1 \text{ and } \left|r^+(k)\right| = \left|r^-(k)\right| \le 1$$

for every $k \in \mathbf{R} \setminus \{0\}$. It is known that the single $r^+(k)$(equivalently $r^-(k)$) determines the other $\{r^-(k), t(k)\}$. It is also known that $a(k)$ is holomorphic on $\mathbf{C}_+$ and has only finitely many simple poles $\{i\eta_j\}_{j=1}^n$on the pure imaginary axis. At $k = i\eta_j$, $e^+(x, i\eta_j)$ and $e^-(x, -i\eta_j)$ are linearly dependent and belong to $L^2(\mathbf{R})$. Therefore $\left\{-\eta_j^2\right\}_{j=1}^n$are eigenvalues of L. Set

$$\left(m_j^\pm\right)^{-2} = \int_R \left|e^\pm(x, i\eta_j)\right|^2 dx.$$

Then it is not difficult to see that

$$\left(m_j^-\right)^{-2} = -\left(m_j^+\right)^2 a'(i\eta_j)^2$$

holds. The triple $\left\{r^+(k), i\eta_j, m_j^+, 1 \le j \le n\right\}$ is called the **right scattering data**. The inverse scattering problem is to obtain the potential V from the right (or left) scattering data, and the basic part of the problem was solved by V.A. Marchenko. His procedure is as follows. Since $r^+(k) = O(|k|^{-1})$ as $|k| \to \infty$,

$$R^+(x) = \frac{1}{\pi}\int_R r^+(k)e^{2ikx}dk$$

is well-defined. It is known that $R^+(x)$ is locally absolutely continuous and

$$\int_0^\infty \left|R^+(x)\right| dx + \int_0^\infty (1+|x|) \left|R^{+\prime}(x)\right| dx < \infty. \tag{6}$$

Define

$$F^+(x) = R^+(x) + 2\sum_{j=1}^n e^{-2\eta_j x} m_j^+.$$

Then (5) makes it possible to consider an integral equation on $L^2([0,\infty))$ for each fixed $x \in \mathbf{R}$

$$K(t) + F^+(x+t) + \int_0^\infty F^+(x+t+s)K(s)ds = 0.$$

It is also known that this equation is uniquely solvable and we denote its solution by $K^+(x,t)$. Then the following theorem is valid.

Theorem 2. *It holds that*

$$q(x) = -\frac{\partial}{\partial x} K^+(x,0).$$

F. Dyson [3] discovered a compact expression of V. Let

$$F_x^+ f(t) = \int_0^\infty F^+(x+t+s)f(s)ds.$$

The property (6) implies the operator F_x^+ defines a trace class operator on $L^2([0,\infty))$ for each fixed $x \in \mathbf{R}$.

Theorem 3. (***Dyson formula***) *It holds that*

$$q(x) = -2\frac{d^2}{dx^2} \log\det(I + F_x^+). \tag{7}$$

A similar expression is possible by using the left scattering data $\{r^-(k), i\eta_j, m_j^-, 1 \le j \le n\}$. A potential q is called **reflectionless** if

$$r^+(k) \equiv 0 \text{ (and hence } r^-(k) \equiv 0).$$

Now it is easy to see from Th. 3 that q is reflectionless if and only if

$$V(x) = -2\frac{d^2}{dx^2} \log\det(I + A^+(x)) \tag{8}$$

with

$$A^{+}(x) = \left(\frac{\sqrt{m_i^+ m_j^+}}{\eta_i + \eta_j} e^{-(\eta_i+\eta_j)x} \right)_{1 \le i,j \le n}.$$

In this case the potential q is decaying exponentially fast and analytic on $\mathbf{R}$.

This reflectionless property can be interpreted by $m_\pm(\lambda)$ as follows. Since the definition of $m_\pm$ implies

$$m_+(\lambda) = \frac{e'_+(0,\sqrt{\lambda})}{e_+(0,\sqrt{\lambda})},\ m_-(\lambda) = -\frac{e'_-(0,-\sqrt{\lambda})}{e_-(0,-\sqrt{\lambda})} \quad \text{for} \quad \lambda \in \overline{\mathbf{C}_+},$$

and for $\xi > 0,\ e_\pm(0, \pm\sqrt{\xi + i0})$ exist finitely , we see

$$m_+(\xi + i0) + \overline{m_-(\xi + i0)} = \frac{2i\sqrt{\xi} b(\sqrt{\xi})}{e_+(0,\sqrt{\xi+i0})\overline{e_-(0,-\sqrt{\xi+i0})}}.$$

Hence we see that q is reflectionless if and only if

$$m_+(\xi + i0) = -\overline{m_-(\xi + i0)} \quad \text{for all } \xi > 0 \tag{9}$$

holds.

4. Generalized Reflectionless Potentials

In this section we give the closure of the class of all reflectionless potentials. To this end we characterize H-functions $m_\pm$ satisfying the property (9). We prepare a lemma.

Lemma 4. *An H-function m satisfies*

$$\operatorname{Re} m(\xi + i0) = 0 \quad a.e. \text{ on } \ (0, \infty) \tag{10}$$

if and only if there exists a measure ν on $(-\infty, 0]$ satisfying

$$\int_{-\infty}^{0} \frac{\nu(d\xi)}{1 + |\xi|} < \infty,$$

and $\gamma \le 0$ such that

$$m(\lambda) = -i\sqrt{\lambda}\gamma - i\sqrt{\lambda} \int_{-\infty}^{0} \frac{\nu(d\xi)}{\xi - \lambda}. \tag{11}$$

Moreover, setting

$$\rho(\lambda) = \gamma + \int_{-\infty}^{0} \frac{\nu(d\xi)}{\xi - \lambda},$$

for some real α we have

$$m(\lambda) = \alpha + \int_{-\infty}^{0} \left(\frac{1}{\xi - \lambda} - \frac{\xi}{1+\xi^2} \right) \sqrt{-\xi}\,\nu(d\xi)$$
$$- \frac{1}{\pi} \int_{0}^{\infty} \left(\frac{1}{\xi - \lambda} - \frac{\xi}{1+\xi^2} \right) \sqrt{\xi}\,\rho(\xi) d\xi. \tag{12}$$

P r o f. Suppose the characteristics of m are $\{\alpha, \beta, \sigma\}$. Set $\theta(\xi) = \arg\left(m(\xi + i0)\right)$. The identity (10) implies $\theta(\xi) = \frac{\pi}{2}$ a.e. on $(0, \infty)$. Therefore Appendix implies

$$m(\lambda) = \exp\left(c + \frac{1}{\pi} \int_{\mathbf{R}} \left(\frac{1}{\xi - \lambda} - \frac{\xi}{1+\xi^2} \right) \theta(\xi) d\xi \right)$$
$$= \frac{1}{-i\sqrt{\lambda}} \exp\left(c + \frac{1}{\pi} \int_{(-\infty,0]} \left(\frac{1}{\xi - \lambda} - \frac{\xi}{1+\xi^2} \right) \theta(\xi) d\xi \right).$$

Hence $-i\sqrt{\lambda} m(\lambda)$ is again an H-function which takes real values on $(0, \infty)$ and is analytic there. Therefore

$$-i\sqrt{\lambda} m(\lambda) = \alpha_1 + \beta_1 \lambda + \int_{-\infty}^{0} \left(\frac{1}{\xi - \lambda} - \frac{\xi}{1+\xi^2} \right) \sqrt{-\xi}\,\sigma(d\xi),$$

which implies

$$\int_{(-\infty,-1]} |\xi|^{-\frac{3}{2}} \sigma(d\xi) < \infty. \tag{13}$$

On the other hand, an H-function $-m^{-1}$ also satisfies (10), hence $m(\lambda)/\left(-i\sqrt{\lambda}\right)$ is an H-function such that

$$\frac{m(\lambda)}{-i\sqrt{\lambda}} = \alpha_2 + \beta_2 \lambda + \int_{-\infty}^{0-} \left(\frac{1}{\xi - \lambda} - \frac{\xi}{1+\xi^2} \right) \frac{\sigma(d\xi)}{\sqrt{-\xi}} + \frac{c}{-\lambda}$$

with some $c \geq 0$, which implies

$$\int_{[-1,0)} |\xi|^{-\frac{1}{2}} \sigma(d\xi) < \infty.$$

Moreover, we see $\beta_2 = 0$. Now, from (13) it follows

$$\frac{m(\lambda)}{-i\sqrt{\lambda}} = \gamma + \int_{-\infty}^{0+} \frac{\nu(d\xi)}{\xi - \lambda},$$

with

$$\begin{cases} \gamma = \alpha_2 + \int_{-\infty}^{0-} \left(\frac{-\xi}{1+\xi^2} \right) \frac{\sigma(d\xi)}{\sqrt{-\xi}} \\ \nu(d\xi) = I_{(-\infty,0)}(\xi) \frac{\sigma(d\xi)}{\sqrt{-\xi}} + c\delta_{\{0\}}(d\xi) \end{cases}.$$

Conversely assume m is given by (11). Then all we have to show is that m is an H-function. To see this, we note $m(\lambda) = -i\sqrt{\lambda}\rho(\lambda)$, and ρ is an H-function taking real values on $(0,\infty)$, from which the conclusion follows. ■

Proposition 5. *Herglotz functions $m_\pm$ satisfy the property (9) if and only if there exist measures $\nu_\pm$ on $(-\infty, 0]$ satisfying*

$$\int_{-\infty}^{0} \frac{\nu_+(d\xi) + \nu_-(d\xi)}{1+|\xi|} < \infty, \tag{14}$$

and $\alpha \in \mathbf{R}$, $\gamma \leq 0$ such that

$$\begin{aligned} m_\pm(\lambda) = \pm\alpha \pm \frac{1}{2}\int_{-\infty}^{0} \left(\frac{1}{\xi-\lambda} - \frac{\xi}{1+\xi^2}\right)\sqrt{-\xi}\,(\nu_+(d\xi) - \nu_-(d\xi)) \\ - \frac{i\sqrt{\lambda}\gamma}{2} - \frac{i\sqrt{\lambda}}{2}\int_{-\infty}^{0+} \frac{\nu_+(d\xi) + \nu_-(d\xi)}{\xi - \lambda}. \end{aligned} \tag{15}$$

P r o f. Let the characteristics of $m_\pm$ be $\{\alpha_\pm, \beta_\pm, \sigma_\pm\}$. First note $m_+(\lambda) + m_-(\lambda)$ is an H-function satisfying

$$\mathrm{Re}\,(m_+(\xi + i0) + m_-(\xi + i0)) = 0 \quad a.e. \text{ on } \quad (0,\infty),$$

since we have the condition (9). Hence we immediately see

$$\begin{cases} \sigma_+(d\xi) = \sigma_-(d\xi) = \dfrac{-\rho(\xi)}{2\pi}\sqrt{\xi}d\xi \quad \text{on } (0,\infty) \\ \text{with } \ \rho(\lambda) = \gamma + \displaystyle\int_{(-\infty,0)} \frac{\sigma_+(d\xi) + \sigma_-(d\xi)}{(\xi-\lambda)\sqrt{-\xi}} + \frac{c}{-\lambda}. \end{cases}$$

Introduce

$$\nu_\pm(d\xi) = I_{(-\infty,0)}(\xi)\frac{\sigma_\pm(d\xi)}{\sqrt{-\xi}} + \frac{1}{2}c\delta_{\{0\}}(d\xi).$$

Then $m_\pm$ can be represented as

$$\begin{aligned} m_\pm(\lambda) = \alpha_\pm + \beta_\pm\lambda + \int_{-\infty}^{0}\left(\frac{1}{\xi-\lambda} - \frac{\xi}{1+\xi^2}\right)\sqrt{-\xi}\nu_\pm(d\xi) \\ + \int_{0}^{\infty}\left(\frac{1}{\xi-\lambda} - \frac{\xi}{1+\xi^2}\right)\frac{-\rho(\xi)}{2\pi}\sqrt{\xi}d\xi. \end{aligned}$$

On the other hand, we know from Lem. 4 that

$$m_+(\lambda) + m_-(\lambda) = -i\sqrt{\lambda}\gamma - i\sqrt{\lambda}\int_{-\infty}^{0} \frac{\nu_+(d\xi) + \nu_-(d\xi)}{\xi - \lambda}, \tag{16}$$

hence $\beta_\pm = 0$, which concludes (15). ■

Now, for $\lambda_0 < 0$ we introduce Ω_{λ_0} a class of potentials q as all compact uniform limit on $\mathbf{R}$ of some reflectionless potentials whose associated Schrödinger operators have their spectrum in $[\lambda_0, \infty)$. An element of Ω_{λ_0} is called a **generalized reflectionless potential.** A potential of the form (8) is called the **classical reflectionless potential** and the set of all these potentials is denoted by $\Omega_{\lambda_0}^{cl}$. We try to parametrize the set by measures on $[-\sqrt{-\lambda_0}, \sqrt{-\lambda_0}]$ defined by

$$\Sigma_{\lambda_0} = \left\{ \begin{array}{c} \sigma; \text{a measure on } [-\sqrt{-\lambda_0}, \sqrt{-\lambda_0}] \text{ satisfying} \\ \displaystyle\int_{[-\sqrt{-\lambda_0}, \sqrt{-\lambda_0}]} \frac{\sigma(d\zeta)}{-\lambda_0 - \zeta^2} \leq 1 \end{array} \right\}.$$

V.A. Marchenko [14] showed the following result, which we prove again here.

Theorem 6. *Then two spaces Ω_{λ_0} and Σ_{λ_0} are homeomorphic and $m_\pm$ are given by*

$$m_\pm(-z^2) = -z - \int_{[-\sqrt{-\lambda_0}, \sqrt{-\lambda_0}]} \frac{\sigma(d\zeta)}{\pm\zeta - z}, \tag{17}$$

or the characteristic measures $\sigma_\pm$ of $m_\pm$ are

$$\sigma_\pm(d\xi) = \left\{ \begin{array}{l} \sqrt{-\xi}\sigma(d\zeta) \;\; on \; (-\infty, 0] \; with \; \zeta = \pm\sqrt{-\xi} \\ -\dfrac{1}{2\pi}\sqrt{\xi}\rho(\xi)d\xi \;\; on \; (0,\infty) \end{array} \right. ,$$

where

$$\rho(\lambda) = -2 + 2\int_{[-\sqrt{-\lambda_0}, \sqrt{-\lambda_0}]} \frac{\sigma(d\zeta)}{-\zeta^2 - \lambda}.$$

P r o f. Choose a $q \in \Omega_{\lambda_0}^{cl}$ whose associated Schrödinger operator has its spectrum in $[\lambda_0, \infty)$. Since q is rapidly decreasing, $\sigma_\pm$ have finitely many points in their supports in $[\lambda_0, 0)$. It is easy to see that

$$m_\pm(\lambda) = i\sqrt{\lambda} + O\left(\frac{1}{\sqrt{\lambda}}\right), \text{ as } \lambda \to \infty, \tag{18}$$

hence $\gamma = -2$. We introduce a measure σ on $[-\sqrt{-\lambda_0}, \sqrt{-\lambda_0}]$ by

$$\int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} f(\zeta)\sigma\,(d\zeta) = \frac{1}{2}\int_{\lambda_0}^{0} f(\sqrt{-\xi})\nu_+\,(d\xi) + \frac{1}{2}\int_{\lambda_0}^{0} f(-\sqrt{-\xi})\nu_-\,(d\xi)\,.$$

Since they satisfy the property (9), we can apply Prop. 5 and the formula (17) follows. On the other hand, the condition that the spectrum of the operator is contained in $[\lambda_0, \infty)$ implies

$$g_\lambda(0,0) = -\left(m_+(\lambda) + m_-(\lambda)\right)^{-1} > 0 \text{ for } \lambda < \lambda_0.$$

Hence (16) shows

$$\int_{[\lambda_0,0]} \frac{\nu_+(d\xi)+\nu_-(d\xi)}{2\,(\xi-\lambda)} < 1, \text{ for } \lambda < \lambda_0,$$

which implies the condition in the definition of Σ_{λ_0}. Now the expression (17) is easily deduced from (15). Then it is routine that for a compact uniform limit of some reflectionless potentials the associated $m_\pm$ also have the representation (17) by choosing a suitable σ from Σ_{λ_0}. In the next section we will prove that any element σ from Σ_{λ_0} with at most finitely many points as its support gives a classical reflectionless potential. This together with the theorem below completes the proof. ∎

For q, define a shift by

$$T_x q(\cdot) = q(\cdot + x) \quad \text{for } x \in \mathbf{R}.$$

Introducing another pair of linearly independent solutions $\{\varphi_\lambda, \psi_\lambda\}$ for

$$Lf = \lambda f \quad \text{satisfying} \quad \begin{cases} f(0)=1,\ f'(0)=0 \Longrightarrow \varphi_\lambda(x) \\ f(0)=0,\ f'(0)=1 \Longrightarrow \psi_\lambda(x). \end{cases} \tag{19}$$

Then the uniqueness of f_+ implies

$$f_+(y,\lambda;T_x q) = \frac{f_+(y+x,\lambda;q)}{f_+(x,\lambda;q)},$$

hence

$$\begin{aligned} m_+(\lambda;T_x q) &= f'_+(0,\lambda;T_x q) \\ &= \frac{f'_+(x,\lambda;q)}{f_+(x,\lambda;q)} \\ &= \frac{\varphi'_\lambda(x;q)+m_+(\lambda;q)\psi'_\lambda(x)}{\varphi_\lambda(x;q)+m_+(\lambda;q)\psi_\lambda(x)}. \end{aligned} \tag{20}$$

$m_-(\lambda;q)$ also has a similar expression, therefore it is easy to check that an identity

$$m_+(\xi+i0;T_x q) + \overline{m_-(\xi+i0;T_x q)} = 0 \text{ a.e. on } [0,\infty)$$

holds if so is the case $x=0$, which implies Ω_{λ_0} is a **shift-invariant space**. Moreover, an asymptotic expansion of the Green function shows that

$$\begin{cases} g_\lambda(x,x;q) = \dfrac{1}{-2i\sqrt{\lambda}} - \dfrac{q(x)}{4i\lambda\sqrt{\lambda}} + o\left(|\lambda|^{-\frac{3}{2}}\right), \\ \dfrac{\partial^2 g_\lambda(x,y;q)}{\partial x \partial y}\bigg|_{x=y} = \dfrac{i\sqrt{\lambda}}{2} - \dfrac{q(x)}{4i\sqrt{\lambda}} + o\left(|\lambda|^{-\frac{1}{2}}\right) \text{ as } \lambda \to \infty \end{cases}. \tag{21}$$

On the other hand, we have another expression for the Green function by using $\{\varphi_\lambda, \psi_\lambda\}$. Let

$$M(\lambda) = \begin{pmatrix} -\dfrac{1}{m_+(\lambda)+m_-(\lambda)} & -\dfrac{m_+(\lambda)}{m_+(\lambda)+m_-(\lambda)}+\dfrac{1}{2} \\ -\dfrac{m_+(\lambda)}{m_+(\lambda)+m_-(\lambda)}+\dfrac{1}{2} & \dfrac{m_+(\lambda)m_-(\lambda)}{m_+(\lambda)+m_-(\lambda)} \end{pmatrix},$$

which is a matrix valued Herglotz function. Set $\phi_\lambda(x) = (\varphi_\lambda(x), \psi_\lambda(x))^T$. Then (1) shows

$$g_\lambda(x,y) = (M(\lambda)\phi_\lambda(x), \phi_\lambda(y)). \tag{22}$$

Here the inner product on $\mathbf{C}^2$ is defined without taking the complex conjugate. For a potential $q \in \Omega_{\lambda_0}$, it is easy to see that for $\mathbf{z} \in \mathbf{C}^2$ an H-function $(M(\lambda)\mathbf{z}, \overline{\mathbf{z}})$ satisfies the identity (9) a.e. on $[0,\infty)$. Therefore, applying Lem. 4, we have

$$M(\lambda) = -\frac{i\sqrt{\lambda}}{2}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} - \frac{i\sqrt{\lambda}}{2}\int_{\lambda_0}^0 \frac{1}{\xi-\lambda}\Sigma(d\xi),$$

where $\Sigma(d\xi)$ is a real matrix valued non-negative definite measure on $[\lambda_0, 0]$. Here we have used the asymptotics (21). Now it follows from (22) that

$$\begin{aligned}\frac{g_\lambda(x,y)}{-\dfrac{i\sqrt{\lambda}}{2}} &= -\psi_\lambda(x)\psi_\lambda(y) + \int_{\lambda_0}^0 \frac{1}{\xi-\lambda}(\Sigma(d\xi)\phi_\lambda(x), \phi_\lambda(y)) \\ &= -\psi_\lambda(x)\psi_\lambda(y) + \int_{\lambda_0}^0 \frac{1}{\xi-\lambda}(\Sigma(d\xi)(\phi_\lambda(x)-\phi_\xi(x)), \phi_\lambda(y)-\phi_\xi(y)) \\ &\quad + \int_{\lambda_0}^0 \frac{1}{\xi-\lambda}(\Sigma(d\xi)\phi_\xi(x), \phi_\xi(y)).\end{aligned}$$

Since first two terms are holomorphic on $\mathbf{C}$ as a function of λ, the asymptotics of the Green function shows that the above right-hand side behaves like $O(\lambda^{-1})$ as $\lambda \to \infty$, which implies the sum of first two terms becomes zero. Hence we have

$$g_\lambda(x,y) = -\frac{i\sqrt{\lambda}}{2}\int_{\lambda_0}^0 \frac{1}{\xi-\lambda}(\Sigma(d\xi)\phi_\xi(x), \phi_\xi(y)). \tag{23}$$

This combined with (21) shows

$$\begin{cases} \displaystyle\int_{\lambda_0}^0 (\Sigma(d\xi)\phi_\xi(x), \phi_\xi(x)) = 1 \\ \displaystyle\int_{\lambda_0}^0 \xi\,(\Sigma(d\xi)\phi_\xi(x), \phi_\xi(x)) = \frac{V(x)}{2} \\ \displaystyle\int_{\lambda_0}^0 \left(\Sigma(d\xi)\phi'_\xi(x), \phi'_\xi(x)\right) = -\frac{V(x)}{2} \end{cases}. \tag{24}$$

This, in particular, implies

$$2\lambda_0 \le V(x) \le 0$$

for any $q \in \Omega_{\lambda_0}$. If we observe the identity

$$-\phi_\xi''(x) + q(x)\phi_\xi(x) = \xi\phi_\xi(x),$$

the repeating use of (24) shows that q is infinitely differentiable and their derivatives have bounds depending only on λ_0, which was proved by D.S. Lundina [11] through a different argument. We state the above argument as a theorem together with the refinements by V.A. Marchenko [14].

Theorem 7. *The followings hold:*

(*i*) *The shift acts on* Ω_{λ_0}.

(*ii*) *Any element of* Ω_{λ_0} *is infinitely differentiable and all its derivatives have bounds*

$$\left|q^{(n)}(x)\right| \le 2\left(\sqrt{-\lambda_0}\right)^{n+2}(n+1)!$$

for $n = 0, 1, 2, \ldots$. *In particular,* Ω_{λ_0} *becomes compact in the compact uniform metric.*

(*iii*) *Any element of* Ω_{λ_0} *is holomorphic on the strip* $\left\{\ |\mathrm{Im}\, z| < \sqrt{-\lambda_0}^{\,-1}\right\}$ *and satisfies*

$$|q(z)| \le -2\lambda_0\left(1 - \sqrt{-\lambda_0}\,|\mathrm{Im}\, z|\right)^{-2}.$$

Among potentials in Ω_{λ_0}, the potentials having finite band structure are of particular interest. We say a potential has a finite band structure if there exists a finite number of nonoverlapping intervals $[\lambda_i, \mu_i]$, $i = 1, 2, \ldots, n$, in $[\lambda_0, 0]$ on which

$$m_+(\xi + i0) = -\overline{m_-(\xi + i0)} \quad \text{for all } \xi \in [\lambda_i, \mu_i] \ (i = 1, 2, \ldots, n)$$

holds. To compute $\sigma_\pm$ in this case, first we consider

$$g(\lambda) = -\left(m_+(\lambda) + m_-(\lambda)\right)^{-1} (= g_\lambda(0,0)).$$

Taking log, we see

$$\begin{aligned} g(\lambda) &= \exp\left(\gamma + \frac{1}{\pi}\int_{\mathbf{R}}\left(\frac{1}{\xi-\lambda} - \frac{\xi}{1+\xi^2}\right)\arg g(\xi + i0)d\xi\right) \\ &= \frac{1}{-2i\sqrt{\lambda}}\exp\left(\frac{1}{\pi}\sum_{i=1}^{n}\int_{\lambda_i}^{\mu_i}\frac{\frac{\pi}{2}}{\xi-\lambda}d\xi + \frac{1}{\pi}\sum_{i=1}^{n}\int_{\mu_i}^{\lambda_{i+1}}\frac{\arg g(\xi+i0)}{\xi-\lambda}d\xi\right), \end{aligned}$$

where we set $\lambda_{n+1} = 0$. The constant factor is determined from the behaviour (18) of $g(\lambda)$. Let ξ_i be a unique zero of $g(\lambda)$ in $[\mu_i, \lambda_{i+1}]$ for $i = 1, 2, \ldots, n$. Then

$$\int_{\mu_i}^{\lambda_{i+1}} \frac{\arg g(\xi + i0)}{\xi - \lambda} d\xi = \pi \int_{\mu_i}^{\xi_i} \frac{1}{\xi - \lambda} d\xi = \pi \log \frac{\lambda - \xi_i}{\lambda - \mu_i},$$

hence

$$\begin{aligned} g(\lambda) &= \frac{1}{-2i\sqrt{\lambda}} \exp\left(\frac{1}{2} \sum_{i=1}^{n} \log \frac{\lambda - \mu_i}{\lambda - \lambda_i} + \sum_{i=1}^{n} \log \frac{\lambda - \xi_i}{\lambda - \mu_i} \right) \\ &= \frac{1}{-2i\sqrt{\lambda}} \sqrt{\prod_{i=1}^{n} \frac{(\lambda - \xi_i)^2}{(\lambda - \lambda_i)(\lambda - \mu_i)}}. \end{aligned}$$

$-\xi_i$ is a pole of $m_+(\lambda) + m_-(\lambda)$, hence $\sigma_+(\{\xi_i\}) + \sigma_-(\{\xi_i\}) > 0$. However if $\sigma_\pm(\{\xi_i\}) > 0$, then the Schrödinger operator L has two non-trivial solutions $f_\pm$ satisfying

$$Lf_\pm = \xi_i f_\pm, \; f_\pm(0) = 0 \text{ and } f_\pm \in L^2(\mathbf{R}_\pm),$$

which means $f_\pm$ are linearly dependent, hence ξ_i is an eigenvalue of L on $L^2(\mathbf{R})$. This contradicts the fact that L has no spectrum outside the set

$$S = \bigcup\nolimits_{1 \le i \le n} [\lambda_i, \mu_i] \cup [0, \infty).$$

Therefore $\sigma_+(\{\xi_i\})\, \sigma_-(\{\xi_i\}) = 0$. Let

$$\varepsilon_i^+ = \begin{cases} 1 & \text{if } \; \sigma_+(\{\xi_i\}) > 0 \\ 0 & \text{otherwise} \end{cases}, \; \varepsilon_i^- = 1 - \varepsilon_i^+.$$

Noting $\sigma_+ = \sigma_-$ on S, we have

$$\sigma_\pm(d\xi) = \Delta(\xi) I_S(\xi) d\xi + \sum_{i=1}^{n} \varepsilon_i^\pm \sigma_i \delta_{(\{\xi_i\})}(d\xi), \tag{25}$$

with

$$\Delta(\xi) = \begin{cases} \dfrac{1}{\pi}\sqrt{-\xi} \dfrac{\sqrt{|(\xi - \lambda_i)(\xi - \mu_i)|}}{|\xi - \xi_i|} \sqrt{\displaystyle\prod_{j:j \ne i} \frac{(\xi - \lambda_j)(\xi - \mu_j)}{(\xi - \xi_j)^2}} & \text{if } \xi \in [\lambda_i, \mu_i] \\ \dfrac{1}{\pi}\sqrt{\xi} \sqrt{\displaystyle\prod_{i=1}^{n} \frac{(\xi - \lambda_i)(\xi - \mu_i)}{(\xi - \xi_i)^2}} \; \text{if } \xi \ge 0, \end{cases}$$

and

$$\sigma_i = 2\sqrt{-\xi_i (\lambda_i - \xi_i)(\mu_i - \xi_i)} \sqrt{\prod_{j:j \ne i} \frac{(\xi_i - \lambda_j)(\xi_i - \mu_j)}{(\xi_i - \xi_j)^2}}.$$

A measure σ on $[-\sqrt{-\lambda_0}, \sqrt{-\lambda_0}]$ is defined by

$$\sigma(d\zeta) = \begin{cases} \dfrac{1}{\sqrt{-\xi}}\sigma_+(d\xi), & if \quad \zeta = \sqrt{-\xi} > 0 \\ \dfrac{1}{\sqrt{-\xi}}\sigma_-(d\xi), & if \quad \zeta = -\sqrt{-\xi} < 0 \end{cases} .$$

In this case we can determine q by using the Theta function on the compact Riemann surface for a hyperelliptic curve

$$w^2 = \lambda \prod_{i=1}^{n} (\lambda - \lambda_i)(\lambda - \mu_i). \tag{26}$$

We choose a homology basis $\{\alpha_i, \beta_j\}_{1 \le i,j \le n}$ on the surface and a basis of differential forms of the first kind $\{\omega_i\}_{1 \le i \le n}$ satisfying

$$\frac{1}{2\pi i} \oint_{\alpha_i} \omega_j = \delta_{ij}.$$

Set

$$B_{ij} = \oint_{\beta_i} \omega_j.$$

This matrix is called a period matrix for the surface, and it is known that B is symmetric and its real part is negative definite. If all the points $\{-\lambda_i, -\mu_j\}$ lie on the real line, the matrix B becomes real. Hence in this case B is a real symmetric negative definite matrix. Define the Theta function

$$\theta(z_1, z_2, \ldots, z_n) = \sum_{\mathbf{m} \in \mathbf{Z}^n} \exp\left\{\frac{1}{2}(B\mathbf{m}, \mathbf{m}) + (\mathbf{z}, \mathbf{m})\right\}.$$

Then it is known (see A.R. Its–V.B. Matveev [7]) that

Proposition 8. *There exist $c \in \mathbf{R}$, $\mathbf{a}, \mathbf{b} \in \mathbf{R^n}$ such that*

$$q(x) = c - 2\frac{d^2}{dx^2} \log \theta(x\mathbf{a} + \mathbf{b}). \tag{27}$$

5. Characterization of Classical Reflectionless Potentials

In this section we characterize classical reflectionless potentials in terms of σ and give a concrete description of the measure

$$m(d\eta) = \sum_{i=1}^{n} m_i^+ \delta_{\{\eta_i\}}(d\eta)$$

associated with the scattering data.

For σ from Σ_{λ_0}, first we compute F_+ of Sect. 2. Set

$$\phi_+(x,y) = -\frac{1}{2\pi}\int_0^\infty \frac{(1-\cos\sqrt{\xi}x)(1-\cos\sqrt{\xi}y)}{\xi^2}\sqrt{\xi}\rho(\xi)\,d\xi + \int_{[\lambda_0,0]} \frac{(1-\cos\sqrt{\xi}x)(1-\cos\sqrt{\xi}y)}{\xi^2}\sigma_+(d\xi) - x\wedge y.$$

Define

$$F_+(x,y) = \frac{\partial^2 \phi_+(x,y)}{\partial x \partial y}.$$

Here we remark the positive-definiteness of F_+, which will be useful when applying Sato theory.

Lemma 9. *F_+ is positive definite.*

Proof. A routine calculation shows

$$F_+(x,y) = \frac{-1}{2\pi}\int_0^\infty \frac{\sin\sqrt{\xi}x\sin\sqrt{\xi}y}{\sqrt{\xi}}(\rho(\xi)+2)\,d\xi + \int_{\lambda_0}^0 \frac{\sinh\sqrt{-\xi}x\sinh\sqrt{-\xi}y}{-\xi}\sigma_+(d\xi),$$

which implies the positive-definiteness of F_+. ∎

Further calculation shows

$$F_+(x,y) = \frac{1}{4}\int_{[\lambda_0,0]} \frac{e^{\sqrt{-\xi}(x+y)} - e^{\sqrt{-\xi}|x-y|}}{-\xi}\sigma_+(d\xi) + \frac{1}{4}\int_{[\lambda_0,0]} \frac{e^{-\sqrt{-\xi}|x-y|} - e^{-\sqrt{-\xi}(x+y)}}{-\xi}\sigma_-(d\xi).$$

Replacing $\{\sigma_+,\sigma_-\}$ with σ, we have

$$F_+(x,y) = \int_{[-\sqrt{-\lambda_0},\sqrt{-\lambda_0}]} \frac{e^{\zeta|x+y|} - e^{\zeta|x-y|}}{2\zeta}\sigma(d\zeta). \tag{28}$$

Now we compute the Fredholm determinant of the integral operator F_+ in $L^2([0,a],dx)$ with kernel $F_+(x,y)$. For later purpose we decompose F_+ into two parts:

$$F_+ = -V + B,$$

with

$$V(x,y) = \begin{cases} \displaystyle\int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{\sinh \zeta(x-y)}{\zeta} \sigma(d\zeta) & \text{for } x \geq y \\ 0 & \text{for } x < y \end{cases},$$

$$B(x,y) = \int_{-\sqrt{-\lambda_0}}^{-\sqrt{\lambda_0}} \frac{\sinh \zeta x}{\zeta} e^{\zeta y} \sigma(d\zeta).$$

To compute the inverse $(I-V)^{-1}$ we set

$$m(\lambda) = \int_0^\infty e^{\sqrt{-\lambda}x} dx \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{\sinh \zeta x}{\zeta} \sigma(d\zeta) = \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{\sigma(d\zeta)}{-\zeta^2 - \lambda}. \tag{29}$$

We note that m is a function of Herglotz type. Further, in this case m takes negative values on $(0,\infty)$. Therefore

$$\widetilde{m}(\lambda) \equiv \frac{m(\lambda)}{1-m(\lambda)} = -1 + \frac{1}{1-m(\lambda)}$$

is an H-function as well and takes negative values on $(0,\infty)$. Here we use the condition

$$\int_{[-\sqrt{-\lambda_0},\sqrt{-\lambda_0}]} \frac{\sigma(d\zeta)}{-\lambda_0 - \zeta^2} \leq 1,$$

which implies $\widetilde{m}(\lambda)$ has no singularity on $(-\infty,\lambda_0)$, hence there exists a unique measure $\widetilde{\sigma}$ on $\mathbf{R}$ such that

$$\widetilde{m}(\lambda) = \int_{[0,\sqrt{-\lambda_0}]} \frac{\widetilde{\sigma}(d\xi)}{-\xi^2 - \lambda}. \tag{30}$$

Defining

$$\widetilde{V}(x,y) = \begin{cases} \displaystyle\int_{[0,\sqrt{-\lambda_0}]} \frac{\sinh \xi(x-y)}{\xi} \widetilde{\sigma}(d\xi) & \text{for } x \geq y \\ 0 & \text{for } x < y \end{cases},$$

we see

$$(I-V)^{-1} = I + \widetilde{V}.$$

Now we employ the method used by Ikeda–Kusuoka–Manabe [6]

$$\begin{aligned} I + F_+ &= I - V + B \\ &= (I-V)(I + (I-V)^{-1}B) \\ &= (I-V)(I + (I+\widetilde{V})B), \end{aligned}$$

which leads us to

$$\left(\widetilde{V}B\right)(x,y) = \int_0^x \widetilde{V}(x,z)B(z,y)dz$$
$$= \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \int_0^{\sqrt{-\lambda_0}} \frac{1}{\xi^2-\zeta^2}\left(\frac{\sinh\xi x}{\xi} - \frac{\sinh\zeta x}{\zeta}\right) e^{\zeta y}\widetilde{\sigma}\,(d\xi)\,\sigma(d\zeta).$$

Now set

$$g(x,\zeta) \equiv \frac{\sinh\zeta x}{\zeta} + \int_0^{\sqrt{-\lambda_0}} \frac{1}{\xi^2-\zeta^2}\left(\frac{\sinh\xi x}{\xi} - \frac{\sinh\zeta x}{\zeta}\right)\widetilde{\sigma}(d\xi)$$
$$= \left(1+\widetilde{m}(-\zeta^2)\right)\frac{\sinh\zeta x}{\zeta} + \int_{[0,\infty)} \frac{1}{\xi^2-\zeta^2}\frac{\sinh\xi x}{\xi}\widetilde{\sigma}(d\xi).$$

We assume σ has its support only on a finite set $\{\zeta_i\}_{1\le i\le n}$ of $[-\sqrt{-\lambda_0}, \sqrt{-\lambda_0}]$ and

$$\zeta_i^2 \neq \zeta_j^2 \text{ if } i,j \in \{1,2,\ldots,n\}\,,\; i\neq j. \tag{31}$$

We try to compute the determinant keeping its generalization to σ with infinite support in mind. (30) shows that $\widetilde{\sigma}$ has a finite support $\{\eta_i\}_{1\le i\le n}$ in $(0,\sqrt{-\lambda_0}]$. (30) implies also

$$\zeta \;\text{ or }\; -\zeta \in \operatorname{supp}\sigma \implies \widetilde{m}(-\zeta^2)+1=0. \tag{32}$$

Therefore, for ζ or $-\zeta$ $\in$suppσ

$$g(x,\zeta) = \int_{[0,\infty)} \frac{1}{\xi^2-\zeta^2}\frac{\sinh\xi x}{\xi}\widetilde{\sigma}(d\xi),$$

and we have

$$\widetilde{B}(x,y) \equiv B(x,y) + \left(\widetilde{V}B\right)(x,y)$$
$$= \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} g(x,\zeta)e^{\zeta y}\sigma\,(d\zeta)$$
$$= \int_{[0,\sqrt{-\lambda_0}]} \frac{\sinh\xi x}{\xi}\widetilde{\sigma}(d\xi) \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{1}{\xi^2-\zeta^2}e^{\zeta y}\sigma(d\zeta). \tag{33}$$

To compute the determinant further we remark here a duality relation for determinants. Let $(X,\mathcal{F},\mu)$, $(\Sigma,\mathcal{M},\sigma)$ be the measure spaces and

$$K(x,\xi), L(x,\xi) \in L^2(X\times\Sigma, \mu\times\sigma).$$

Define

$$\begin{cases} F(x,y) = \int_\Sigma K(x,\xi)L(y,\xi)\sigma(d\xi) \\ \widehat{F}(\xi,\eta) = \int_X K(x,\xi)L(x,\eta)\mu(dx) \end{cases}.$$

Lemma 10. *F and $\hat{F}$ define trace class operators on $L^2(X,\mu)$ and $L^2(\Sigma,\sigma)$ respectively and it holds that*

$$\det(I+F) = \det(I+\widehat{F}).$$

P r o f. This is an infinite dimensional version of the identity

$$\det(I+AB) = \det(I+BA)$$

for any $n\times m$ matrix A and $m\times n$ matrix B. We omit the proof. ∎

Setting

$$K(x,\xi^2) = \frac{\sinh \xi x}{\xi},\ L(y,\xi^2) = \int_{\mathbf{R}} \frac{1}{\xi^2-\zeta^2} e^{\zeta y}\sigma(d\zeta),$$

we have

$$\widetilde{B}(x,y) = \int_{[0,\infty)} K(x,\xi^2)L(y,\xi^2)\widetilde{\sigma}(d\xi).$$

Hence the corresponding $\widehat{B}$ becomes

$$\widehat{B}(\xi^2,\eta^2) = \int_0^a K(x,\xi^2)L(x,\eta^2)dx,$$

we see from Lem. 10

$$\det(I+\widetilde{B}) = \det(I+\widehat{B}). \tag{34}$$

Now we compute $\widehat{B}(\xi^2,\eta^2)$.

$$\begin{aligned} \widehat{B}(\xi^2,\eta^2) &= \int_{\mathbf{R}} \frac{1}{\eta^2-\zeta^2}\sigma(d\zeta)\int_0^a \frac{\sinh \xi x}{\xi} e^{\zeta x}dx \\ &= \int_{\mathbf{R}} \frac{1}{\eta^2-\zeta^2}\sigma(d\zeta)\frac{1}{2\xi}\left(\frac{e^{(\zeta+\xi)a}-1}{\zeta+\xi} - \frac{e^{(\zeta-\xi)a}-1}{\zeta-\xi}\right) \\ &= \int_{\mathbf{R}} \frac{e^{\zeta a}}{2\xi\left(\eta^2-\zeta^2\right)}\sigma(d\zeta)\left(\frac{e^{\xi a}}{\zeta+\xi} - \frac{e^{-\xi a}}{\zeta-\xi}\right) + \int_{\mathbf{R}} \frac{\sigma(d\zeta)}{\left(\eta^2-\zeta^2\right)\left(\zeta^2-\xi^2\right)}. \end{aligned} \tag{35}$$

However

$$\int_{\mathbf{R}} \frac{\sigma(d\zeta)}{(\eta^2-\zeta^2)(\zeta^2-\xi^2)} = \begin{cases} \dfrac{m(-\eta^2)-m(-\xi^2)}{\eta^2-\xi^2} = 0 & \text{if } \eta^2 \neq \xi^2 \\ -\tilde{\sigma}(\{\eta\})^{-1} & \text{if } \eta^2 = \xi^2 \end{cases}, \tag{36}$$

hence setting $\sigma_k = \sigma(\{\zeta_k\})$, $\tilde{\sigma}_j = \tilde{\sigma}(\{\eta_j\})$ and

$$\begin{cases} a_{ik} = \dfrac{1}{\eta_i^2 - \zeta_k^2}, \\ b_{jk} = \dfrac{1}{\eta_j + \zeta_k} + \dfrac{e^{-2\eta_j a}}{\eta_j - \zeta_k}, \end{cases} \tag{37}$$

we have

$$\begin{aligned} \det(I+\widehat{B}) &= \det\left(\left(\sum_k a_{ik} e^{(\eta_j+\zeta_j)a} \frac{\widetilde{\sigma}_j}{2\eta_j} b_{jk}\right)\right) \\ &= \exp\left(a\sum_{j=1}^n (\eta_j+\zeta_j)\right)\left(\prod_j \frac{\widetilde{\sigma}_j\sigma_j}{2\eta_j}\right)\det\left((a_{ij})\right)\det((b_{ij})). \end{aligned} \tag{38}$$

To compute $\det((b_{ij}))$ we set

$$\begin{cases} C_{ij}^+ = \dfrac{1}{\eta_i+\zeta_j}, \quad C_{ij}^- = \dfrac{1}{\eta_i - \zeta_j} \\ \Lambda_{ij}(a) = e^{-2\eta_i a}\delta_{ij} \end{cases}. \tag{39}$$

Then

$$b_{ij} = \left(C^+ + \Lambda(a)C^-\right)_{ij}.$$

Introduce

$$H(\lambda) = 1 - \prod_{i=1}^n \frac{\lambda-\eta_i}{\lambda+\zeta_i} \text{ and } \widetilde{H}(\lambda) = \frac{H(\lambda)}{1-H(\lambda)} = -1 + \prod_{i=1}^n \frac{\lambda+\zeta_i}{\lambda-\eta_i}.$$

Then we can show that there exist real numbers $\{\mu_i\}_{1\le i\le n}, \{\nu_i\}_{1\le i\le n}$ such that

$$H(\lambda) = \sum_{i=1}^n \frac{\mu_i}{\lambda+\zeta_i} \text{ and } \widetilde{H}(\lambda) = \sum_{i=1}^n \frac{\nu_i}{\lambda-\eta_i}.$$

Lemma 11. *We have the identities:*

(*i*) $\displaystyle\sum_{i=1}^n \frac{\mu_i}{\eta_j+\zeta_i} = 1, \ \sum_{i=1}^n \frac{\nu_i}{\eta_i+\zeta_j} = 1$ *for* $\forall j = 1,2,\ldots,n$;

(*ii*) $\displaystyle\mu_i = \left(\sum_{j=1}^n \frac{\nu_j}{(\eta_j+\zeta_i)^2}\right)^{-1}, \ \nu_i = (\eta_i+\zeta_i)\prod_{j:j\neq i} \frac{\eta_i+\zeta_j}{\eta_i-\eta_j}.$

P r o f. We omit the proof, since the computation is elementary. ∎

Lemma 12. $\left(C^{+}\right)^{-1} = \left(\dfrac{\mu_i \nu_j}{\eta_j + \zeta_i}\right)$ *and* $C^{-}\left(C^{+}\right)^{-1} = \left(\dfrac{\nu_j\left(1 - H(-\eta_i)\right)}{\eta_i + \eta_j}\right)$.

P r o f. Set

$$P_{ij} = \frac{\mu_i}{\eta_j + \zeta_i},\ Q_{ij} = \frac{\nu_i}{\eta_i + \zeta_j} \text{ and } S = \left(\nu_i \delta_{ij}\right).$$

Then Lem. 11 shows

$$\begin{aligned}
(PQ)_{ij} &= \sum_{k=1}^{n} \frac{\mu_i}{\eta_k + \zeta_i} \frac{\nu_k}{\eta_k + \zeta_j} \\
&= \begin{cases} \dfrac{\mu_i}{\zeta_i - \zeta_j} \displaystyle\sum_{k=1}^{n} \left(\frac{\nu_k}{\eta_k - \zeta_i} - \frac{\nu_k}{\eta_k - \zeta_j}\right) = 0 & \text{if } i \neq j \\ \mu_i \displaystyle\sum_{k=1}^{n} \frac{\nu_k}{\left(\eta_k + \zeta_i\right)^2} = 1 & \text{if } i = j \end{cases} \\
&= \delta_{ij}.
\end{aligned}$$

Therefore

$$\left(C^{+}\right)^{-1} = Q^{-1} S = PS,$$

which implies

$$\begin{aligned}
C^{-}\left(C^{+}\right)^{-1}_{ij} &= \sum_{k=1}^{n} \frac{1}{\eta_i - \zeta_k} \frac{\mu_k}{\eta_j + \zeta_k} \nu_j \\
&= \frac{\nu_j}{\eta_i + \eta_j} \sum_{k=1}^{n} \left(\frac{\mu_k}{\eta_i - \zeta_k} + \frac{\mu_k}{\eta_j + \zeta_k}\right) \\
&= \frac{\nu_j\left(1 - H(-\eta_i)\right)}{\eta_i + \eta_j}.
\end{aligned}$$

∎

Lemma 13. $\nu_i\left(1 - H(-\eta_i)\right) = \dfrac{\left(1 - H(-\eta_i)\right)^2}{2\eta_i} \widetilde{\sigma}_i > 0$ *for* $i = 1, 2, \ldots, n$.

P r o f. Observe $-1 - \widetilde{m}(-\lambda)$ has zeroes $\left\{\zeta_i^2\right\}_{1 \leq i \leq n}$ and poles $\left\{\eta_i^2\right\}_{1 \leq i \leq n}$, hence

$$-\widetilde{m}(-\lambda) = \sum_{i=1}^{n} \frac{\widetilde{\sigma}_i}{\eta_i^2 - \lambda} = 1 - \prod_{i=1}^{n} \frac{\lambda - \zeta_i^2}{\lambda - \eta_i^2}.$$

On the other hand, similarly as we obtained ν_i in Lemma 11, we have

$$\begin{aligned}\widetilde{\sigma}_i &= \left(\eta_i^2 - \zeta_i^2\right) \prod_{j:j\neq i}^{n} \frac{\zeta_j^2 - \eta_i^2}{\eta_j^2 - \eta_i^2} \\ &= \left(\eta_i - \zeta_i\right)\left(\eta_i + \zeta_i\right) \prod_{j:j\neq i}^{n} \frac{\left(\eta_i - \zeta_j\right)\left(\eta_i + \zeta_j\right)}{\left(\eta_i - \eta_j\right)\left(\eta_i + \eta_j\right)} \\ &= \nu_i \left(\eta_i - \zeta_i\right) \prod_{j:j\neq i}^{n} \frac{\eta_i - \zeta_j}{\eta_j + \eta_i} \\ &= \frac{2\nu_i \eta_i}{1 - H(-\eta_i)},\end{aligned}$$

hence

$$\nu_i \left(1 - H(-\eta_i)\right) = \frac{\left(1 - H(-\eta_i)\right)^2}{2\eta_i} \widetilde{\sigma}_i > 0.$$

■

Setting

$$m_i = \nu_i \left(1 - H(-\eta_i)\right) > 0,$$

we see

Proposition 14. *$V \in \Omega_{\lambda_0}^{cl}$ if and only if the associated σ has a finite support. Moreover, the Fredholm determinant is given by*

$$\det(I + F_+) = const. \times \det\left(\delta_{ij} + \frac{\sqrt{m_i m_j}}{\eta_i + \eta_j} e^{-a(\eta_i + \eta_j)}\right),$$

with

$$const. = \exp\left(a \sum_j \left(\eta_j + \zeta_j\right)\right) \left(\prod_j \left(\frac{\widetilde{\sigma}_j \sigma_j}{2\eta_j}\right)\right) \det\left(C^+\right) \det\left((a_{ij})\right),$$

and the measure $m(d\eta) = \sum_{i=1}^{n} m_i \delta_{\{\eta_i\}}(d\eta)$ can be represented as

$$m(d\eta) = \frac{\left(1 - H(-\eta)\right)^2}{2\eta} \widetilde{\sigma}(d\eta).$$

P r o f. Summing up the above argument, we have

$$\begin{aligned}\det(I+F_+) &= \exp\left(a\sum_j(\eta_j+\zeta_j)\right)\prod_j\left(\frac{\widetilde{\sigma}_j\sigma_j}{2\eta_j}\right)\det\left((a_{ij})\right)\\ &\quad\times\det\left(C^+ + \Lambda(a)C^-\right)\\ &= const.\det\left(I+\Lambda(a)C^-\left(C^+\right)^{-1}\right)\\ &= const.\det\left(\delta_{ij}+e^{-2a\eta_i}\frac{\nu_j\left(1-H(-\eta_i)\right)}{\eta_i+\eta_j}\right)\\ &= const.\det\left(\delta_{ij}+e^{-2a\eta_i}\frac{\nu_i\left(1-H(-\eta_i)\right)}{\eta_i+\eta_j}\right)\\ &= const.\det\left(\delta_{ij}+\frac{\sqrt{m_im_j}}{\eta_i+\eta_j}e^{-a(\eta_i+\eta_j)}\right).\end{aligned}$$

■

Remark 15. *If we replace σ with $\widehat{\sigma}$ constructed by reflection from σ, that is, $\widehat{\sigma}(d\xi)=\sigma(-d\xi)$. Then $\widetilde{\sigma}$ remains unchanged and $C^+(resp. C^-)$ turns to $C^-(resp. C^+)$, hence*

$$\det(I+\widehat{F}_+)(a)=\det(I+F_+)(-a).$$

Remark 16. *The condition (31) can be removed if we approximate σ by a sequence of measures satisfying (31).*

6. Construction of KdV-Flow

In this section we construct the KdV-flow on Ω_{λ_0} by applying the theory of M. Sato–Y. Sato. They gave a very transparent view for a class of integrable systems including the KdV equation and later it was developed by Date–Jimbo–Kashiwara–Miwa [1]. However, their original argument is quite algebraic. So we imply here a more analytic version by S. Segal–G. Wilson [15] and give a complete proof by calculating the τ-functions for classical reflectionless potentials.

Let S^1 be the unit circle in $\mathbf{C}$, and $H=L^2(S^1)$. Introduce two orthogonal subspaces $H_\pm$ of H

$$\begin{cases} H_+=\left\{f\in H;\ f(z)=\sum_{n\geq 0}f_nz^n \text{ with } \sum_{n\geq 0}|f_n|^2<\infty\right\}\\ H_-=\left\{f\in H;\ f(z)=\sum_{n\leq -1}f_nz^n \text{ with } \sum_{n\leq -1}|f_n|^2<\infty\right\}.\end{cases}$$

Then it is easy to see that

$$H=H_+\oplus H_- \text{ (orthogonal sum).} \tag{40}$$

Let $P_{H_\pm}$ be the orthogonal projections to $H_\pm$, respectively. Let W be a closed subspace of H satisfying:

(i) $P_{H_+} : W \longrightarrow H_+$ is a Fredholm operator (i.e., has finite dimensional kernel and cokernel) with index 0, that is $\dim Ker = \dim CoKer$.

(ii) $P_{H_-} : W \longrightarrow H_-$ is a trace class operator;

(iii) $f \in W \longrightarrow z^2 f \in W$;

(iv) $H_- \cap W = \{0\}$(**transversality**).

We denote by $Gr^{(2)}(H)$ the set of all closed subspaces W satisfying the conditions (i), (ii), (iii) and (iv). The properties (i) and (iv) assure the unique existence of a bounded operator A from H_+ to H_- such that

$$W = \{f + Af;\ f \in H_+\}.$$

This is because (iv) implies $\dim Ker = 0$, and hence (i) implies $P_{H_+}(W) = H_+$. Conversely, if such an operator A exists, then (iv) holds. Introduce

$$\Gamma = \left\{ \begin{array}{c} g;\ g(z) \text{ is holomorphic on } \mathbf{D}, g(0) = 1\ ,\ g(z) \neq 0 \text{ for } \forall z \in \mathbf{D}, \\ \text{takes real values on } \mathbf{R} \text{ and } g(-z) = g(z)^{-1} \text{ for } \forall z \in \mathbf{D} \end{array} \right\},$$

where $\mathbf{D} = \{z \in \mathbf{C}; |z| \leq 1\}$. Apparently Γ is a commutative group acting on $Gr^{(2)}(H)$ by multiplication but for the condition (iv). For $g \in \Gamma$, we represent it as

$$g^{-1} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

corresponding to the decomposition (40). Now, for $g \in \Gamma$ and $W \in Gr^{(2)}(H)$ we define

$$\tau_W(g) = \det(I + a^{-1}bA). \tag{41}$$

Although the following lemma was proved in [16], we give a proof for the sake of completeness.

Lemma 17. *$\tau_W(g) \neq 0$ if and only if $g^{-1}W$ is transverse to H_-.*

P r o f. Suppose for $f \in H_+$, $f + a^{-1}bAf = 0$. Set

$$f_1 = g^{-1}Af - bAf \in H_-.$$

Then

$$gf_1 = Af - a^{-1}bAf = Af + f \in W,$$

which completes the proof. ∎

Throughout this section we assume $-1 < \lambda_0 < 0$, which is not essential. For $\sigma \in \Sigma_{\lambda_0}$, define $m_+(-z^2)$ by (17). Set

$$W_\sigma = \left\{ A(z^2) + m_+(-z^2)B(z^2);\ A, B \in H_+ \right\}. \tag{42}$$

Here it should be noted that the condition

$$\int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{\sigma(d\zeta)}{1-\zeta^2} \le \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{\sigma(d\zeta)}{-\lambda_0-\zeta^2} \le 1$$

assures $\left|m_+(-z^2)\right| \le 3$ if $|z| = 1$, hence W_σ becomes a closed subspace of H. This space was considered by R. Johnson [9] as an application of Sato theory. We compute the operator $a^{-1}bA$ for this space and identify its Fredholm determinant with a Fredholm determinant of an F_+, which makes it possible to show the transversality of gW_σ.

Lemma 18. *For $f \in H_+$, the equation*

$$f(z^2) = B(z^2) - \int_{-\sqrt{\lambda_0}}^{\sqrt{\lambda_0}} \frac{B(z^2) - B(\zeta^2)}{z^2 - \zeta^2} \sigma(d\zeta) \tag{43}$$

is uniquely solvable in H_+, and the solution is given by

$$B(z^2) = f(z^2) + \int_0^{\sqrt{\lambda_0}} \frac{f(z^2) - f(\zeta^2)}{z^2 - \zeta^2} \widetilde{\sigma}(d\zeta). \tag{44}$$

P r o f. Suppose the support σ is finite. Note first

$$B(z^2) - \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{B(z^2) - B(\zeta^2)}{z^2 - \zeta^2} \sigma(d\zeta) = B(z^2)(1 - m(-z^2)) + \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{B(\zeta^2)\sigma(d\zeta)}{z^2 - \zeta^2}$$
$$f(z^2) + \int_0^{\sqrt{-\lambda_0}} \frac{f(z^2) - f(\zeta^2)}{z^2 - \zeta^2} \widetilde{\sigma}(d\zeta) = f(z^2)(1 + \widetilde{m}(-z^2)) - \int_0^{\sqrt{-\lambda_0}} \frac{f(\zeta^2)}{z^2 - \zeta^2} \widetilde{\sigma}(d\zeta),$$

where m was introduced in (29). Then, substituting (44) into (43), we see

$$B(z^2) - \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{B(z^2) - B(\zeta^2)}{z^2 - \zeta^2} \sigma(d\zeta)$$

$$= f(z^2)(1 - m(-z^2))(1 + \widetilde{m}(-z^2)) - (1 - m(-z^2)) \int_0^{\sqrt{-\lambda_0}} \frac{f(\zeta^2)}{z^2 - \zeta^2} \widetilde{\sigma}(d\zeta)$$
$$+ \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{f(\zeta^2)(1 + \widetilde{m}(-\zeta^2))\sigma(d\zeta)}{z^2 - \zeta^2} - \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{\sigma(d\zeta)}{z^2 - \zeta^2} \int_0^{\sqrt{-\lambda_0}} \frac{f(\eta^2)\widetilde{\sigma}(d\eta)}{\zeta^2 - \eta^2}$$

$$\begin{aligned}
&= f(z^2) - (1 - m(-z^2)) \int_0^{\sqrt{-\lambda_0}} \frac{f(\zeta^2)}{z^2 - \zeta^2} \widetilde{\sigma}(d\zeta) \\
&- \int_0^{\sqrt{-\lambda_0}} f(\eta^2) \widetilde{\sigma}(d\eta) \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{\sigma(d\zeta)}{(\zeta^2 - \eta^2)(z^2 - \zeta^2)} \\
&= f(z^2).
\end{aligned}$$

In this calculation we have used the fact

$$1 + \widetilde{m}(-\zeta^2) = 0 \text{ if } \zeta \in \operatorname{supp}\sigma \text{ and } 1 - m(-\zeta^2) = 0 \text{ if } \zeta \in \operatorname{supp}\widetilde{\sigma}.$$

Now the rest of the proof is easy if we approximate a general $\sigma \in \Sigma_{\lambda_0}$ by a sequence of σ's with finite supports. ■

For $f \in H_+$, we define

$$\begin{cases} Pf(z) = \dfrac{f(z) - f(-z)}{2z}, \\ Kf(z^2) = f(z^2) + \displaystyle\int_0^{\sqrt{-\lambda_0}} \frac{f(z^2) - f(\zeta^2)}{z^2 - \zeta^2} \widetilde{\sigma}(d\zeta). \end{cases}$$

Lemma 19. *For the space of (42) we have*

$$\left(a^{-1} bA\right) f(z) = \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{1 - g(z) g(\zeta)^{-1}}{\zeta - z} KPf(\zeta^2) \sigma(d\zeta).$$

P r o f. Let $f \in H_+$. Then the definition of the operator A implies

$$f(z) + Af(z) = A(z^2) + m_+(-z^2) B(z^2),$$

with $A, B \in H_+$. However

$$m_+(-z^2) B(z^2) = -zB(z^2) - \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{B(z^2) - B(\zeta^2)}{\zeta - z} \sigma(d\zeta) - \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{B(\zeta^2) \sigma(d\zeta)}{\zeta - z},$$

and first two terms are contained in H_+ and the last term is contained in H_- since $\lambda_0 > -1$. Hence we have

$$\begin{aligned}
f(z) &= A(z^2) - zB(z^2) - \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{B(z^2) - B(\zeta^2)}{\zeta - z} \sigma(d\zeta) \\
Af(z) &= -\int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{B(\zeta^2)}{\zeta - z} \sigma(d\zeta),
\end{aligned}$$

which implies

$$\frac{f(z)-f(-z)}{2z} = -B(z^2) + \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{B(z^2)-B(\zeta^2)}{z^2-\zeta^2}\sigma(d\zeta).$$

Then Lemma 18 shows

$$Af(z) = -\int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{KPf(\zeta^2)}{\zeta - z}\sigma(d\zeta).$$

On the other hand, for $f \in H$

$$P_{H_+}f(z) = \frac{1}{2\pi i}\oint_{|z'|=1} \frac{f(z')}{z'-z}dz',$$

hence

$$\begin{aligned}
a^{-1}bAf(z) &= \frac{g(z)}{2\pi i}\oint_{|z'|=1} \frac{g(z')^{-1}}{z'-z}dz' \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} \frac{KPf(\zeta^2)}{\zeta - z'}\sigma(d\zeta) \\
&= g(z)\int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} KPf(\zeta^2)\sigma(d\zeta)\frac{1}{2\pi i}\oint_{|z'|=1} \frac{g(z')^{-1}}{(z'-z)(\zeta-z')}dz' \\
&= \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} KPf(\zeta^2)\frac{1-g(z)g(\zeta)^{-1}}{\zeta - z}\sigma(d\zeta),
\end{aligned}$$

which concludes the lemma. ■

Now we compute the Fredholm determinant when the support of σ consists of a finite set $\{\zeta_i\}_{1\le i\le n}$. First note

$$\det(I + a^{-1}bA) = \det(I + B),$$

with B defined by

$$Bf(z^2) \equiv \int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}} Kf(\zeta^2)\left(\frac{1-g(z)g(\zeta)^{-1}}{\zeta - z} - \frac{1-g(-z)g(\zeta)^{-1}}{\zeta+z}\right)\frac{\sigma(d\zeta)}{2z},$$

for $f \in H_+$. Since σ has a finite support, the following calculation is possible:

$$\begin{aligned}
Kf(\zeta^2) &= f(\zeta^2) + \int_0^{\sqrt{-\lambda_0}} \frac{f(\zeta^2)-f(\eta^2)}{\zeta^2-\eta^2}\widetilde{\sigma}(d\eta) \\
&= \left(1+\widetilde{h}(-\zeta^2)\right)f(\zeta^2) - \int_0^{\sqrt{-\lambda_0}} \frac{f(\eta^2)}{\zeta^2-\eta^2}\widetilde{\sigma}(d\eta) \\
&= \int_0^{\sqrt{-\lambda_0}} \frac{f(\eta^2)}{\eta^2-\zeta^2}\widetilde{\sigma}(d\eta),
\end{aligned}$$

for $1+\widetilde{m}(-\zeta^2)=0$ if $\zeta\in\operatorname{supp}\sigma$ as we saw in (32). Hence

$$
\begin{aligned}
&Bf(z^2)\\
&=\int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}}\left(\frac{g(-z)g(\zeta)^{-1}}{2z\,(\zeta+z)}-\frac{g(z)g(\zeta)^{-1}}{2z\,(\zeta-z)}+\frac{1}{\zeta^2-z^2}\right)\sigma(d\zeta)\int_0^{\sqrt{-\lambda_0}}\frac{f(\eta^2)\widetilde{\sigma}(d\eta)}{\eta^2-\zeta^2}\\
&=\int_0^{\sqrt{-\lambda_0}}f(\eta^2)\widetilde{\sigma}(d\eta)\int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}}\left(\frac{g(-z)g(\zeta)^{-1}}{2z\,(\zeta+z)}-\frac{g(z)g(\zeta)^{-1}}{2z\,(\zeta-z)}+\frac{1}{\zeta^2-z^2}\right)\frac{\sigma(d\zeta)}{\eta^2-\zeta^2}\\
&=\int_0^{\sqrt{-\lambda_0}}f(\eta^2)\widetilde{\sigma}(d\eta)\int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}}\left(\frac{g(-z)g(\zeta)^{-1}}{2z\,(\zeta+z)}-\frac{g(z)g(\zeta)^{-1}}{2z\,(\zeta-z)}\right)\frac{\sigma(d\zeta)}{\eta^2-\zeta^2}\\
&\quad+\int_0^{\sqrt{-\lambda_0}}f(\eta^2)\widetilde{\sigma}(d\eta)\int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}}\frac{\sigma(d\zeta)}{(\eta^2-\zeta^2)\,(\zeta^2-z^2)}.
\end{aligned}
$$

Note here

$$\det(I+B)=\det(I+QB)$$

if we denote the restriction of $f\in H_+$ to $L^2(\widetilde{\sigma})$ by Q. Then (36) shows

$$
\begin{aligned}
&(I+QB)\,f(\xi^2)\\
&=\int_0^{\sqrt{-\lambda_0}}f(\eta^2)\widetilde{\sigma}(d\eta)\int_{-\sqrt{-\lambda_0}}^{\sqrt{-\lambda_0}}\left(\frac{g(-\xi)g(\zeta)^{-1}}{2\xi\,(\zeta+\xi)}-\frac{g(\xi)g(\zeta)^{-1}}{2\xi\,(\zeta-\xi)}\right)\frac{\sigma(d\zeta)}{\eta^2-\zeta^2},
\end{aligned}
$$

which is the same as (35) if we replace $e^{-\xi a}$ with g. Now the computation is quite analogous to that of Sect. 5, and we have

Lemma 20. *Suppose the support of σ is finite $\{\zeta_i\}_{1\le i\le n}$. Then*

$$\tau_{W_\sigma}(g)=const.\times\det\left(\delta_{ij}+\frac{\sqrt{m_im_j}}{\eta_i+\eta_j}g(-\eta_i)^{-1}g(\eta_j)\right),$$

with

$$const.=\prod_{j=1}^n\left(\frac{g(-\eta_j)g(\zeta_j)^{-1}\sigma_j\widetilde{\sigma}_j}{2\eta_j}\right)\det\left(C^+\right)\det\left((a_{ij})\right).$$

Now we define the KdV flow. For $g\in\Gamma$ and $x\in\mathbf{R}$, introduce $g^x\in\Gamma$ by

$$g^x(z)=e^{-xz}g(z).$$

Lemma 21. *For $\sigma\in\Sigma_{\lambda_0}$ define W_σ by (42). Then for any $g\in\Gamma$, there exists a $\sigma_g\in\Sigma_{\lambda_0}$ such that*

$$\tau_{W_\sigma}(g^x)=\det(I+F^x_{+,\sigma_g}).\tag{45}$$

P r o f. First suppose the support of σ is finite. Then Lemma 20 implies

$$\tau_{W_\sigma}(g^x) = \exp\left(a\sum_j(\eta_j+\zeta_j)\right)\prod_{j=1}^n (g(\eta_j)g(\zeta_j))^{-1}\prod_{j=1}^n\left(\frac{\sigma_j\widetilde{\sigma}_j}{2\eta_j}\right)\det(C^+)\det(a_{ij})$$
$$\times\det(\delta_{ij}+\frac{\sqrt{m_im_j}}{\eta_i+\eta_j}e^{-x(\eta_i+\eta_j)}g(\eta_i)g(\eta_j)),$$

with some constants $\widetilde{c}_1(g), \widetilde{c}_2(g)$. Hence, regarding $m_i g(\eta_i)^2$ as a new weight, we can construct a classical reflectionless potential q_g with the scattering data $\left\{r^+(k)=0, i\eta_j, m_j g(\eta_j)^2\right\}$. Let $\sigma_g \in \Sigma_{\lambda_0}$ be its spectral measure. Then from Prop. 14 the identity (45) follows. For a general $\sigma \in \Sigma_{\lambda_0}$, approximate it by a sequence of measures $\{\sigma_n\}$ from Σ_{λ_0} with finite supports. Then it is easy to see that $\tau_{W_{\sigma_n}}(g^x)$ and $\det(I+F^x_{+,\sigma_g^n})$ converge as $n\to\infty$ to $\tau_{W_\sigma}(g^x)$ and $\det(I+F^x_{+,\sigma_g})$, respectively, if we choose a subsequence of $\left\{\sigma_g^n\right\}$ converging to a $\sigma_g \in \Sigma_{\lambda_0}$ if necessary. Hence we have (45) for a general σ. ∎

This lemma combined with Lem. 9 has the following conclusion.

Theorem 22. *For $\sigma \in \Sigma_{\lambda_0}$ and $g \in \Gamma, \tau_{W_\sigma}(g) > 0$, hence $g^{-1}W_\sigma$ is transverse to H_-.*

For $g \in \Gamma$ and $q \in \Omega_{\lambda_0}$, we define

$$(K(g)q)(x) = -2\frac{d^2}{dx^2}\log\tau_{W_\sigma}(g^x) \tag{46}$$

as an element of Ω_{λ_0}. Then we can prove the following

Theorem 23. *$K(g)$ is a homeomorphism on Ω_{λ_0} satisfying*

$$K(g_1g_2) = K(g_1)K(g_2) \text{ and } K(1) = id$$

and $K(g_x)q(\cdot) = q(\cdot + x)$, $K(g_{x,t})q$ satisfies the KdV equation, where $g_x(z) = e^{-xz}$ and $g_{x,t}(z) = e^{-xz+4tz^3}$.

P r o f. Since everything is valid if $K(g)$ is restricted to the classical reflectionless potentials, all we have to do is to approximate $\sigma \in \Sigma_{\lambda_0}$ by a sequence of measures $\{\sigma_n\}$ from Σ_{λ_0} with finite supports. The multiplicativity of K comes from the cocycle property of τ_W

$$\tau_{g_1^{-1}W}(g_2) = \frac{\tau_W(g_1g_2)}{\tau_W(g_1)}.$$

∎

We call this K as **KdV-flow** on Ω_{λ_0}. It is well known also that $K(g)q$ satisfies the higher order KdV equations if we choose suitable one-parameter groups on Γ.

7. Characterization of W_σ

We have introduced a space W_σ. It is natural to ask when W coincides with a W_σ. We combine the two properties (i), (iv) of W and summarize them again. A closed subspace W of H is in $Gr^{(2)}(H)$ if and only if:

(i) $P_{H_+} : W \longrightarrow H_+$ is one-to-one and onto.

(ii) $P_{H_-} : W \longrightarrow H_-$ is a trace class operator.

(iii) $f \in W \Longrightarrow z^2 f \in W$.

We set new conditions. For $f \in H$, define $\overline{f}(z) = \overline{f(\overline{z})}$.

(iv) $f \in W \Longrightarrow \overline{f} \in W$.

Let

$$g_x(z) = e^{-xz}.$$

(v) For any $x \in \mathbf{R}$, $g_x^{-1}W$ satisfies (i).

Suppose the conditions (i)$\sim$(v). Then, for any $x \in \mathbf{R}$, there exists a unique $f_W(x, \cdot) \in W$ such that

$$f_W(x,z) = e^{-xz}\left(1 + \frac{a_1(x)}{z} + \frac{a_2(x)}{z^2} + \cdots\right). \tag{47}$$

Since

$$\overline{f_W}(x,z) = e^{-xz}\left(1 + \frac{\overline{a_1(x)}}{z} + \frac{\overline{a_2(x)}}{z^2} + \cdots\right),$$

and the property (iv) implies $\overline{f_W}(x, \cdot) \in W$, the uniqueness shows $f_W(x,z) = \overline{f_W}(x,z)$, hence $a_i(x) \in \mathbf{R}$.

Lemma 23 (Segal–Wilson). *f_W satisfies*

$$-\frac{d^2}{dx^2} f_W(x,z) - 2a_1'(x) f_W(x,z) = -z^2 f_W(x,z),$$

and $\{a_i(x)\}_{1 \le i < \infty}$ satisfy

$$2a_i'(x) = a_{i-1}''(x) + 2a_1'(x) a_{i-1}(x) \text{ for } i = 2, 3, \dots . \tag{48}$$

P r o f. Differentiating both sides of (47), we have

$$\begin{aligned}
&\frac{d^2}{dx^2} f_W(x,z) \\
&= z^2 f_W(x,z) - 2z e^{-xz}\left(\frac{a_1'(x)}{z} + \frac{a_2'(x)}{z^2} + \cdots\right) + e^{-xz}\left(\frac{a_1''(x)}{z} + \frac{a_2''(x)}{z^2} + \cdots\right) \\
&= z^2 f_W(x,z) - 2a_1'(x) f_W(x,z) + e^{-xz} \sum_{i=1}^{\infty} \frac{a_i''(x) - 2a_{i+1}'(x) + 2a_1'(x) a_i(x)}{z^i}.
\end{aligned}$$

The linearity of the space W and (iii) imply

$$\frac{d^2}{dx^2} f_W(x,z) - z^2 f_W(x,z) + 2a_1'(x) f_W(x,z) \in W.$$

Therefore from (vi) it follows that

$$\begin{cases} \dfrac{d^2}{dx^2} f_W(x,z) - z^2 f_W(x,z) + 2a_1'(x) f_W(x,z) = 0, \\ 2a_i'(x) = a_{i-1}''(x) + 2a_1'(x) a_{i-1}(x) \text{ for } i = 2,3,\ldots. \end{cases}$$

■

Suppose $V(x)$ is a real valued infinitely differentiable function on $\mathbf{R}$. Consider a Schrödinger operator

$$L = -\frac{d^2}{dx^2} + q$$

on $L^2(\mathbf{R})$. If the boundaries $\pm\infty$ are of the limit circle type, then we have to impose suitable boundary conditions at $\pm\infty$. For $\lambda \in \mathbf{C}$ such that $\mathrm{Im}\,(\lambda) \neq 0$ let $f_+(x,\lambda)$ be the solution for

$$Lf_+ = \lambda f_+ \ , \ f_+(0,\lambda) = 1$$

and

$$\begin{cases} f_+ \in L^2(\mathbf{R}_+) \text{ if } +\infty \text{ is a limit point }, \\ f_+ \text{ satisfies the boundary condition at } +\infty \text{ if } +\infty \text{ is a limit circle.} \end{cases}$$

Set

$$m_+(\lambda) = f_+'(0,\lambda),$$

which is called the Weyl function.

Lemma 25. *There exist real valued smooth functions $\{b_i(x)\}_{1\leq i<\infty}$ such that for each fixed $n \geq 0$*

$$f_+(x,-z^2) = e^{-xz}\left(1 + \frac{b_1(x)}{z} + \frac{b_2(x)}{z^2} + \cdots + \frac{b_n(x)}{z^n} + O\left(\frac{1}{z^{n+1}}\right)\right) \tag{49}$$

holds as $|z| \to \infty$ in a region $-\dfrac{\pi}{2} + \varepsilon < \arg z < -\varepsilon$ for any $\varepsilon > 0$. Moreover, they satisfy

$$\begin{cases} 2b_i'(x) = b_{i-1}''(x) + 2b_1'(x) b_{i-1}(x) \textit{ for } i = 2,3,\ldots, \\ b_i(0) = 0 \textit{ for } i = 1,2,3,\ldots. \end{cases} \tag{50}$$

P r o f. Let φ, ψ be the solutions for $Lf = -z^2 f$ satisfying $\varphi(0) = \psi'(0) = 1$, $\varphi'(0) = \psi(0) = 0$. Then

$$f_+(x, -z^2) = \varphi(x, z) + m_+(-z^2)\psi(x, z).$$

F. Gesztesy–B. Simon [5] proved for each $a > 0$ there exists a unique function $A(\alpha)$ on $[0, a]$ such that

$$m_+(-z^2) = -z - \int_0^a A(\alpha)e^{-2\alpha z}d\alpha + O\left(e^{-2(a-\varepsilon)z}\right)$$

holds as $|z| \to \infty$ in a region $-\dfrac{\pi}{2} + \varepsilon < \arg z < -\varepsilon$ for any $\varepsilon > 0$. The function A is determined from m on $[0, a]$ and smooth if so is m. On the other hand, φ, ψ can be represented as

$$\begin{cases} \varphi(x, z) = \cosh zx + \displaystyle\int_0^x K_1(x, y)\cosh zy dy, \\ \psi(x, z) = \dfrac{\sinh zx}{z} + \displaystyle\int_0^x K_2(x, y)\dfrac{\sinh zy}{z}dy, \end{cases}$$

with smooth functions K_1, K_2. With these identities we see that $f_+(x, -z^2)$ has an asymptotic expansion (49) together with its derivatives $f_+^{(k)}$ of any order. Substituting this expansion to the equation $Lf_+ = -z^2 f_+$, we have

$$\begin{aligned} 0 &= e^{xz}\left(-f_+''(x, -z^2) + q(x)f_+(x, -z^2) + z^2 f_+(x, -z^2)\right) \\ &= 2b_1'(x) + q(x) + \frac{2b_2'(x) - b_1''(x) + b_1(x)q(x)}{z} + \frac{2b_3'(x) - b_2''(x) + b_2(x)q(x)}{z^2} \\ &\quad + \cdots + \frac{2b_{n+1}'(x) - b_n''(x) + b_n(x)q(x)}{z^n} + O\left(\frac{1}{z^{n+1}}\right), \end{aligned}$$

hence

$$\begin{cases} 2b_1'(x) + q(x) = 0 \\ 2b_{n+1}'(x) - b_n''(x) + b_n(x)q(x) = 0 \end{cases}$$

holds for any $n \geq 1$, which implies (50). ∎

We introduce one more condition:
(vi) $1 \in W$.
Now we can state

Theorem 26. *Suppose a subspace W of H satisfies the conditions (i) $\sim$ (vi). Then there exists a unique $\sigma \in \Sigma_{-1}$ such that $W = W_\sigma$.*

P r o f. If a subspace W of H satisfies the conditions (i)$\sim$(vi), then we can define f_W in (50). The condition (vi) shows $f_W(0,z)=1$, and hence

$$a_i(0)=0 \text{ for any } i=1,2,\ldots.$$

Then, setting $q(x)=-2a_1'(x)$, from Lems. 24 and 25 we see $f_W(x,z)=f_+(x,-z^2)$ identically. On the other hand, for this q we introduce $f_-(x,-z^2)$ satisfying the boundary condition at $-\infty$, and we can show $f_W(x,-z)=f_-(x,-z^2)$ identically.

Therefore we have

$$m_+(-z^2)=f_W'(0,z),\ m_-(-z^2)=-f_W'(0,-z). \tag{51}$$

Now we consider the analytic continuation of f_W. $f_W(x,z)$ is holomorphic on $\{|z|>1\}$ and $f_W(x,z)=\overline{f_W(x,\overline{z})}$. Therefore $m_\pm$ are holomorphic on $\mathbf{C}\backslash[-1,\infty)$

and take real values on $(-\infty,-1)$. Therefore the representing measure $\sigma_\pm$ has no mass on $(-\infty,-1)$. This combined with (51) shows that $f_W(x,z)$ is analytically continuable to $\mathbf{C}\backslash([-1,1]\cup[-i,i])$ keeping the property $f_W(x,z)=\overline{f_W(x,\overline{z})}$. Then it is immediate that

$$m_+(\xi+i0)=f_W'(0,i\sqrt{\xi}+0)=\overline{f_W'(0,-i\sqrt{\xi}+0)}=-\overline{m_-(\xi+i0)} \text{ for a.e. } \xi>0.$$

Then Proposition 6 shows that

$$m_\pm(-z^2)=-z-\int_{-1}^{1}\frac{\sigma(d\zeta)}{\pm\zeta-z}$$

with a measure σ on $[-1,1]$. We have to prove

$$\int_{-1}^{1}\frac{\sigma(d\zeta)}{1-\zeta^2}\le 1. \tag{52}$$

Suppose (52) is not valid. Then there exists $\lambda_0<-1$ such that

$$\int_{-1}^{1}\frac{\sigma(d\zeta)}{-\lambda_0-\zeta^2}=1. \tag{53}$$

Set

$$f(z)=-\int_{-1}^{1}\frac{B(\zeta^2)\sigma(d\zeta)}{\zeta-z} \text{ with } B(z^2)=\frac{1}{z^2-\lambda_0}.$$

Then $f\in H_-$ since $m_+(-z^2)\in W$. On the other hand, (53) implies

$$B(z^2)=\int_{-1}^{1}\frac{B(z^2)-B(\zeta^2)}{z^2-\zeta^2}\sigma(d\zeta).$$

Let

$$A(z^2) = -\int_{-1}^{1} \frac{B(z^2) - B(\zeta^2)}{z^2 - \zeta^2} \zeta \sigma(d\zeta).$$

Then

$$f(z) = A(z^2) + m_+(-z^2)B(z^2)$$

holds. However the Taylor expansions for A, B around the origin converge uniformly on $\{|z| \le 1\}$, and we know $m_+(-z^2) \in W$. Hence the property (iii) implies $f \in W$, hence $W \cap H_- \ne \{0\}$, which contradicts (i). Therefore (52) holds, thus we have $q \in \Omega_{-1}$. It is easy to see that $W_\sigma \subset W$ since $1, m_+(-z^2) \in W$, which shows $W_\sigma = W$ in view of the property (i). The uniqueness of σ is trivial since σ is determined from f_W, which completes the proof. ■

Remark 27. *It is interesting to remove the condition (vi). Without (vi) one can proceed in parallel with the above argument up to a certain point. In this case we have to consider a meromorphic function on* $1 < |z| \le \infty$

$$\widetilde{f_W}(x,z) = \frac{f_W(x,z)}{f_W(0,z)}$$

instead of $f_W(x,z)$ *itself. Since* $f_W(0,\infty) = 1$, *there exists* $R \ge 1$ *such that* $f_W(0,z) \ne 0$ *for all* z *such that* $|z| > R$. $\widetilde{f_W}$ *satisfies* $L\widetilde{f_W} = -z^2 \widetilde{f_W}$ *and has an expansion*

$$\widetilde{f_W}(x,z) = e^{-xz}\left(1 + \frac{\widetilde{a_1}(x)}{z} + \frac{\widetilde{a_2}(x)}{z^2} + \cdots\right)$$

on $|z| > R$. *Since* $\{\widetilde{a_i}(x)\}_{1 \le i < \infty}$ *satisfy the same equations (50), we see* $\widetilde{a_i}(x) = b_i(x)$ *for all* $i = 1, 2, \ldots$. *Hence*

$$\widetilde{f_W}(x,z) = f_+(x, -z^2) \tag{54}$$

identically, which implies

$$\frac{f_W'(0,z)}{f_W(0,z)} = m_+(-z^2). \tag{55}$$

Then we can show as above that the left-hand side of (55) is analytically continuable to $\mathbf{C} \backslash ([-R, R] \cup [-iR, iR])$, *which shows*

$$m_+(\xi + i0) = -\overline{m_-(\xi + i0)} \text{ for a.e. } \xi > 0.$$

Hence h_+ *has an expression*

$$m_+(-z^2) = -z - \int_{-R}^{R} \frac{\sigma(d\zeta)}{\zeta - z}.$$

Let $\lambda_0 \le -R^2$ be any number such that

$$\int_{-R}^{R} \frac{\sigma(d\zeta)}{-\lambda_0 - \zeta^2} \le 1,$$

then $2\lambda_0 \le q(x) \le 0$, which implies the boundaries $\pm\infty$ are of limit point type. It is not clear if we can avoid the possibility that there exists a zero of $f_W(0,z)$ in $1 < |z| < R$.

8. Isospectral Property of KdV-Flow

We introduce a subgroup Γ_0 of Γ by

$$\Gamma_0 = \{g \in \Gamma;\ \log g \text{ is a polynomial of odd degree with real coefficients}\}.$$

Let

$$L^q = -\frac{d^2}{dx^2} + q.$$

Theorem 28. *For $q \in \Omega_{\lambda_0}$ and $g \in \Gamma_0$, there exists a unitary operator $U(g,q)$ on $L^2(\mathbf{R}, dx)$ satisfying*

$$L^{K(g)q} = U(g,q)^{-1} L^q U(g,q).$$

P r o f. We prove this theorem only for $g_x(z) = e^{-xz}$ and $g_{x,t}(z) = e^{-xz+4tz^3}$. For a general $g \in \Gamma_0$ the proof is analogous. For $g_x(z) = e^{-xz}$, the proof is trivial, since $K(g_x)q(\cdot) = q(\cdot + x)$ and we have only to set $U(g_x, q) = T_x$ independently of V. For $g_{x,t}(z) = e^{-xz+4tz^3}$, we employ the Lax representation. Set

$$u(t,x) = (K(g_{t,0})q)(x) = (K(g_{t,x})q)(0),$$

and define an antisymmetric operator

$$A^q = 4D^3 - 6qD - 3Dq$$

with $D = \dfrac{d}{dx}$. Then it is easy to see

$$[L^q, A^q] = L^q A^q - A^q L^q = 6qq_x - q_{xxx},$$

where the left-hand side is a multiplication operator. Since $u(t,x)$ satisfies the KdV equation, we see

$$\frac{d}{dt} L^{u(t)} = [L^{u(t)}, A^{u(t)}],$$

where $u(t)(\cdot) = u(t, \cdot)$. This formula is called the Lax representation of the KdV equation. Define a one-parameter family of unitary operators by solving an equation

$$\frac{dU(t)}{dt} = U(t)A^{u(t)},\ U(0) = I.$$

Then it is obvious that $L^{u(t)} = U(t)^{-1}L^q U(t)$, which concludes the proof. ■

Remark 29. *One can expect Theorem 28 holds for any $g \in \Gamma$, however to show this we have to construct a pseudodifferential operator for which the commutator with L^q reduces to a multiplication operator.*

9. Floquet Exponent

Let $\Omega = \Omega_{\lambda_0}$ be the space of generalized reflectionless potentials and $\{K(g)\}_{g\in\Gamma}$ be the KdV flow. We consider the set of all probability measures $\mathcal{P}_K(\Omega)$ on Ω which are invariant under $\{K(g)\}_{g\in\Gamma}$ and denote by $\mathcal{P}_{shift}(\Omega)$ the set of all shift invariant probability measures on Ω. Since the set of all probability measures $\mathcal{P}(\Omega)$ on Ω is compact and the flow $\{K(g)\}_{g\in\Gamma}$ is commutative, we easily see that $\mathcal{P}_K(\Omega)$ is a non-empty compact convex set. For $\mu \in \mathcal{P}_{shift}(\Omega)$, set

$$w(\lambda) = w_\mu(\lambda) = \int_\Omega m_+(\lambda, q)\mu(dq), \tag{56}$$

which is called **Floquet exponent** for $\mu \in \mathcal{P}_{shift}(\Omega)$. This was first introduced by Johson–Moser. Then it is well known that

$$w(\lambda) = \int_\Omega m_-(\lambda, q)\mu(dq) = -\frac{1}{2}\int_\Omega g_\lambda(0, 0, q)^{-1}\mu(dq), \tag{57}$$

and

$$w'(\lambda) = \int_\Omega g_\lambda(0, 0, q)\mu(dq) \tag{58}$$

hold (see S. Kotani [10]). Then the commutativity and the isospectral property under K imply easily the following

Theorem 30. *If $\mu \in \mathcal{P}_{shift}(\Omega)$, then for all $g \in \Gamma$ it holds $K(g)\mu \in \mathcal{P}_{shift}(\Omega)$ and we have $w_{K(g)\mu} = w_\mu$.*

Let $\mathcal{W}_K$ be the set of all functions w satisfying
(i) $w, -iw, w'$ are Herglotz functions,
(ii) $w(\lambda) < 0$ on $(-\infty, -\lambda_0]$,
(iii) $\operatorname{Re} w(\xi + i0) = 0$ on $[0, \infty)$,
(iv) $w(\lambda)/\sqrt{-\lambda} \to -1$ as $\lambda \to -\infty$.
Then from the compactness of Ω we have

Theorem 31. *For any $\mu \in \mathcal{P}_K(\Omega)$, $w_\mu \in \mathcal{W}_K$ is valid. Conversely, for any $w \in \mathcal{W}_K$ there exists a $\mu \in \mathcal{P}_K(\Omega)$ such that $w = w_\mu$.*

It is natural to ask

$$\text{To what extent does } w \text{ determine } \mu \in \mathcal{P}_K(\Omega) \text{ ?} \tag{59}$$

If w comes from a finite bands spectrum, then the characterization of all potentials with finite bands spectrum (see [7]) shows that w determines uniquely μ.

Appendix. In this Appendix we collect several fundamental facts of H-functions (Herglotz functions). For the proofs see P.L. Duren [2]. As we defined in Sect. 2, a holomorphic function defined on $\mathbf{C}_+$ is called a Herglotz function if it maps $\mathbf{C}_+$ into $\overline{\mathbf{C}_+}$. It is well known that an H-function m has an expression

$$m(\lambda) = \alpha + \beta\lambda + \int_{\mathbf{R}} \left(\frac{1}{\xi - \lambda} - \frac{\xi}{1+\xi^2} \right) \sigma(d\xi)$$

with $\alpha, \beta \in \mathbf{R}$, $\beta \geq 0$ and a nonnegative measure σ on $\mathbf{R}$ satisfying

$$\int_{\mathbf{R}} (1+\xi^2)^{-1} \sigma(d\xi) < \infty.$$

The triple $\{\alpha, \beta, \sigma\}$ is called the characteristics for m. The measure σ can be recovered from m by

$$\sigma([a,b)) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_a^b \operatorname{Im} m(\xi + i\varepsilon) d\xi$$

if $\sigma(\{a\}) = \sigma(\{b\}) = 0$. It is also known that for a.e. $\xi \in \mathbf{R}$

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \operatorname{Im} m(\xi + i\varepsilon) = \sigma'(\xi)$$

holds, where $\sigma'(\xi)$ denotes the density of absolutely continuous part of σ. m itself has a finite limit

$$\lim_{\varepsilon \downarrow 0} m(\xi + i\varepsilon) = m(\xi + i0) \neq 0 \quad \text{a.e.} \xi \in \mathbf{R}$$

unless $m \equiv 0$. It is of some use to note that if m is an H-function, so is $-m^{-1}$ and $\log m$. Especially $\log m$ is useful when we have to factorize m in an appropriate way. Since

$$\operatorname{Im} \log m(\lambda) = \arg m(\lambda) \in [0, \pi],$$

applying the above representation for m, we easily see that

$$\log m(\lambda) = \gamma + \frac{1}{\pi} \int_{\mathbf{R}} \left(\frac{1}{\xi - \lambda} - \frac{\xi}{1+\xi^2} \right) \arg m(\xi + i0) d\xi$$

with $\gamma \in \mathbf{R}$.

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