Journal of Mathematical Physics, Analysis, Geometry
2009, vol. 5, No. 4, pp. 386–395

Long-Time Asymptotic Behavior of an Integrable Model of the Stimulated Raman Scattering with Periodic Boundary Data

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Received July 6, 2009

The long-time asymptotic behavior of the initial-boundary value (IBV) problem in the quarter plane $(x > 0, t > 0)$ for nonlinear integrable equations of the stimulated Raman scattering is studied. Considered is the case of zero initial condition and single-phase boundary data $(pe^{i\omega t})$. By using the steepest descent method for oscillatory matrix Riemann–Hilbert problems it is shown that the solution of the IBV problem has different asymptotic behavior in different regions, namely:

- the selfsimilar vanishing (as $t \to \infty$) wave, when $x > \omega^2 t$;
- the modulated elliptic wave of finite amplitude, when $\omega_0^2 t < x < \omega^2 t$;
- the plane wave of finite amplitude, when $0 < x < \omega_0^2 t$.

The similar results are true for the same IBV problem with nonzero initial condition vanishing as $t \to \infty$.

Key words: nonlinear equations, Riemann–Hilbert problem, asymptotics.

Mathematics Subject Classification 2000: 37K15, 35Q15, 35B40.

1. Introduction

We consider the initial boundary value problem for integrable model of the stimulated Raman scattering (SRS equations):

$$2iq_t = \mu, \quad \mu_x = 2i\nu q, \quad \nu_x = i(\bar{q}\mu - q\bar{\mu}), \qquad x \in (0, \infty), \quad t \in (0, \infty), \tag{1}$$

with the vanishing (as $x \to \infty$) initial function and periodic boundary conditions:

$$q(x,0) = u(x), \qquad \mu(0,t) = pe^{\mathrm{i}\omega t}, \qquad p > 0, \qquad \nu(0,t) = l = const. \tag{2}$$

Since (1) implies $\frac{\partial}{\partial x}\Big(\nu^2(x,t) + |\mu(x,t)|^2\Big) = 0$, in what follows we assume that $\nu^2(x,t) + |\mu(x,t)|^2 \equiv 1$ and, particularly, $p^2 + l^2 = 1$. For definiteness we assume that $p = |\mu(0,t)| > 0$ and $\omega > 0$, while $l < 0$. The case $\omega < 0$ $l > 0$ is obtained by passing to the complex conjugated SRS equations.

The phenomenon of the stimulated Raman scattering is described by three coupled PDEs [1]. Initial boundary value problems for these equations in the domain $x \in (0,L), t \in (0,T)$ are well posed [2] for any $L > 0$ and $T > 0$. The SRS equations (1) are integrable reduction of them in a special case of the transient limit [1, 3]. In other words, the SRS equations admit the Lax pair, and the inverse scattering transform can be applied. We will use the version [4] of this transform when simultaneous spectral analysis of both the Lax equations is involved. The IBV problem for the SRS equations is a nice model of PDEs, which can be solved by using the method of simultaneous spectral analysis and the matrix Riemann–Hilbert problem without a restriction caused by the so-called global relation [4, 5] between spectral functions. Such a restriction takes place for the most of integrable equations because the method [4] involves more boundary values than in the corresponding well-posed IBV problem. Such an overdetermination of the boundary data implies the mentioned above global relation.

If $q(x,t)$ is real and $2q = v_x$, $\mu = \mathrm{i}\sin v$, $\nu = \cos v$, then the SRS equations are reduced to the sine-Gordon equation: $v_{xt} = \sin v$. The long-time asymptotic behavior of the rapidly decreasing (as $|x| \to \infty$) solution of this equation was studied in [6].

The IBV problem in the finite domain $[0,L]\mathrm{x}[0,\mathrm{T}]$ was studied in [1], where the long-distance behavior of the system was established via the third Painleve transcendent. The problem in the finite domain was also considered in [7], where rigorous analysis of the Riemann–Hilbert problem was done. In the present paper, the IBV problem for the SRS equations is studied in the domain $(x > 0, t > 0)$ with zero initial function and simple periodic boundary data. The similar problem with nonzero initial function, vanishing at infinity, was studied in [8]. Using the steepest descent method of P. Deift and X. Zhou [9] for the oscillatory matrix RH problem, introduced in [8], there was obtained the asymptotics of the solution of the IBV problem in the form of a selfsimilar vanishing wave travelling in the region $x > \omega^2 t$. By using the ideas of [10] we obtained the explicit formula for the asymptotics of the solution of the IBV problem in the complementary region

$0 < x < \omega^2 t$. In the region $\omega_0^2 t < x < \omega^2 t$, where

$$\omega_0^2 = \frac{-8l^3\omega^2}{27 - 18l^2 - l^4 + \sqrt{(1-l^2)(9-l^2)^3}}, \qquad -1 < l < 0,$$

the solution takes the form of a modulated elliptic wave of finite amplitude while in the region $0 < x < \omega_0^2 t$ it takes the form of a plane wave. To make the asymptotic analysis more transparent, we consider the case when the initial function $u(x) \equiv 0$.

2. Riemann–Hilbert Problem

To formulate the Riemann–Hilbert problem, related to the IBV problem (1)–(2), we introduce the spectral functions corresponding to initial and boundary conditions. We consider the case $u(x) \equiv 0$. Therefore spectral functions are defined by boundary data only. The boundary values $\mu(0,t) = pe^{\mathrm{i}\omega t}$ and $\nu(0,t) = l$ $(p^2 + l^2 = 1)$ give the t-equation from the Lax pair:

$$\frac{\mathcal{E}(t,k)}{dt} = \frac{\mathrm{i}}{4k}\begin{pmatrix} l & \mathrm{i}p\mathrm{e}^{\mathrm{i}\omega t} \\ -\mathrm{i}p\mathrm{e}^{-\mathrm{i}\omega t} & -l \end{pmatrix}\mathcal{E}(t,k). \tag{3}$$

We choose the solution of (3) in the form

$$\mathcal{E}(t,k) = \frac{1}{2}\mathrm{e}^{\mathrm{i}\omega\sigma_3 t/2}\begin{pmatrix} \varkappa(k) + \dfrac{1}{\varkappa(k)} & \varkappa(k) - \dfrac{1}{\varkappa(k)} \\ \varkappa(k) - \dfrac{1}{\varkappa(k)} & \varkappa(k) + \dfrac{1}{\varkappa(k)} \end{pmatrix}\mathrm{e}^{-\mathrm{i}\Omega(k)\sigma_3 t}, \tag{4}$$

where

$$\varkappa(k) = \sqrt[4]{\frac{k-\bar{E}}{k-E}}, \qquad \Omega(k) = \frac{\omega}{2k}X(k), \qquad X(k) := \sqrt{(k-E)(k-\bar{E})},$$

and

$$E = \frac{l + \mathrm{i}p}{2\omega} = E_1 + \mathrm{i}E_2, \qquad \bar{E} = E_1 - \mathrm{i}E_2.$$

To fix the branches of the roots, we choose the cut in the complex k-plane along the curve $\gamma \cup \bar{\gamma}$, where $\operatorname{Im}\Omega(k) = 0$, and define $\varkappa(k)$ and $\Omega(k)$ in such a way that

$$\varkappa(k) = 1 + O(k^{-1}), \qquad \Omega(k) = \frac{\omega}{2} + O(k^{-1}) \qquad \text{as} \quad k \to \infty.$$

The set $\Sigma := \{k \in \mathbb{C} | \operatorname{Im}\Omega(k) = 0\}$ (Fig.) consists of the real line $\operatorname{Im} k = 0$ and the circle arc $\hat{\gamma} = \gamma \cup \bar{\gamma}$, which is defined by

$$\Big(k_1 - \frac{|E|^2}{2E_1}\Big)^2 + k_2^2 = \Big(\frac{|E|^2}{2E_1}\Big)^2, \qquad k_1^2 + k_2^2 \ge |E|^2.$$

Let us define the oriented contour Γ as follows: $\Gamma = \mathbb{R} \cup \gamma \cup \bar{\gamma}$. Denoting $\Omega_\pm(k)$, $\varkappa_\pm(k)$ the boundary values of $\Omega(k)$, $\varkappa(k)$ on the cut $\hat{\gamma}$ from the right (+) and left (-) sides of the cut (see Fig. 1), we have

$$\Omega_+(k) = -\Omega_-(k), \qquad \varkappa_-(k) = \mathrm{i}\varkappa_+(k).$$

The matrix-valued function $\mathcal{E}(t,k)$ in (4) is analytic in $\mathbb{C} \setminus \{\hat{\gamma} \cup \{0\}\}$ and it has an essential singularity at the point 0. The spectral data corresponding to the boundary values $\mu(0,t) = pe^{\mathrm{i}\omega t}$ and $\nu(0,t) = l$ $(p^2 + l^2 = 1)$ are defined as follows:

$$A(k) = \frac{1}{2}\left(\varkappa(k) + \frac{1}{\varkappa(k)}\right), \qquad B(k) = \frac{1}{2}\left(\varkappa(k) - \frac{1}{\varkappa(k)}\right) \qquad k \in \mathbb{C} \setminus \hat{\gamma}. \quad (5)$$

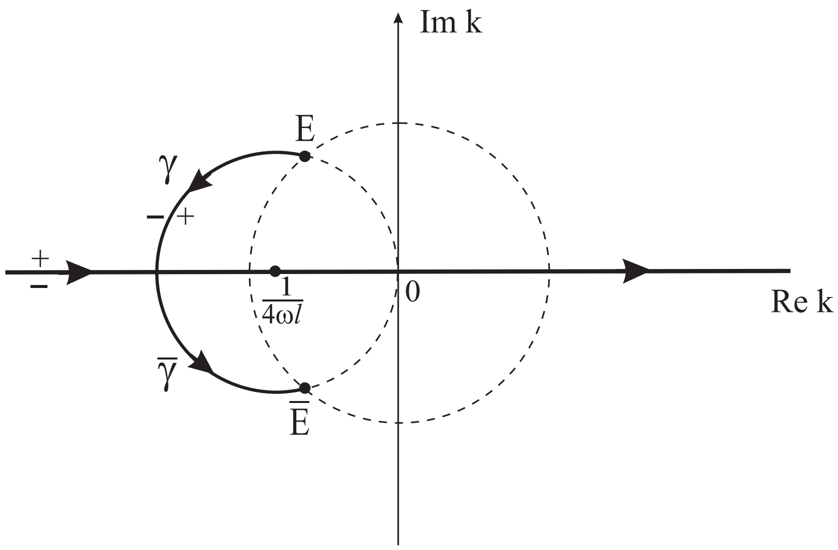

Fig. The set Σ and contour Γ.

Define the following functions:

$$\rho(k) := \frac{B(k)}{A(k)}, \qquad k \in \mathbb{C} \setminus \hat{\gamma}; \qquad f(k) := \frac{\mathrm{i}}{A_+(k)A_-(k)}, \qquad k \in \gamma, \qquad (6)$$

where the sign $\pm$ denotes boundary value of A(k) from the left (+) and from the right $(-)$ of the arc $\hat{\gamma} = \gamma \cup \bar{\gamma}$.

Consider the matrix Riemann–Hilbert problem on the contour Γ proposed in [8].

Find a 2x2 matrix-valued function $M(x,t,k)$ such that:

- $M(x,t,k)$ is sectionally analytic for $k \in \mathbb{C} \setminus \Gamma$;
- $M(x,t,k)$ is bounded in neighborhoods of the end points E and $\bar{E}$;
- $M(x,t,k) = I + O(k^{-1}), \quad k \to \infty$;
- $M(x,t,k) = \tilde{m}_0(x,t) + O(k), \qquad k \to 0$;
- $M_-(x,t,k) = M_+(x,t,k)J(x,t,k), \quad k \in \Gamma$, where

$$J(x,t,k) = \begin{cases} \begin{pmatrix} 1 & \rho(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ -\rho(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1-\rho^2(k) \end{pmatrix}, & k \in \mathbb{R}, \\ \begin{pmatrix} 1 & 0 \\ f(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 \end{pmatrix}, & k \in \gamma, \\ \begin{pmatrix} 1 & f(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix}, & k \in \bar{\gamma}, \end{cases} \tag{7}$$

with $\theta(k) = \theta(k,\xi) = 1/4k - k/4\xi^2$, $\xi^2 := t/4x$.

Theorem 2.1. *Let $\rho(k)$ and $f(k)$ be given as (6) and (5). Then the Riemann–Hilbert problem (7) has a unique solution $M(x,t,k)$. The functions $q(x,t)$, $\mu(x,t)$ and $\nu(x,t)$, defined by the equations*

$$q(x,t) := 2\mathrm{i} \lim_{k\to\infty} [kM(x,t,k)]_{12},$$

$$\begin{pmatrix} \nu(x,t) & \mathrm{i}\mu(x,t) \\ -\mathrm{i}\mu(x,t) & -\nu(x,t) \end{pmatrix} := -M(x,t,0)\sigma_3 M^{-1}(x,t,0),$$

are the solution of the IBV problem (1)-(2) with zero initial function $u(x)$.

This theorem is a corollary of the more general ($u(x) \neq 0$) theorem proved in [8].

3. Asymptotic Behavior of the Solution of the IBV Problem

In this section we study the long-time asymptotic behavior of the solution to the IBV problem (1)–(2). We show that there exist three different asymptotic formulas which describe the long-time behavior of the solution $q(x,t)$ of the IBV problem in the three different regions of the domain $x > 0, t > 0$.

The asymptotics of the solution in the region $x > \omega^2 t$ was obtained in [8]:

Theorem 3.1. *Let $q(x,t)$, $\mu(x,t)$ and $\nu(x,t)$ be the solution of the IBV problem (1)–(2). Then in the region $x > \omega^2 t$ the solution takes the form*

$$q(x,t) = 2\sqrt{\frac{\xi^3\eta(\xi)}{t}}\exp\left\{2\mathrm{i}\sqrt{xt} - \mathrm{i}\eta(\xi)\log\sqrt{xt} + i\varphi(\xi)\right\}$$
$$+2\sqrt{\frac{\xi^3\eta(-\xi)}{t}}\exp\left\{-2\mathrm{i}\sqrt{xt} + \mathrm{i}\eta(-\xi)\log\sqrt{xt} + i\varphi(-\xi)\right\} + o(t^{-1/2}), \quad t\to\infty,$$

where the functions $\eta(k)$ and $\varphi(k)$ are given by the equations:

$$\eta(k) = \frac{1}{2\pi}\log\left(1-\rho^2(k)\right), \qquad \xi^2 = \frac{t}{4x},$$
$$\varphi(k) = \frac{\pi}{4} - 3\eta(k)\log 2 - \arg\rho(k) - \arg\Gamma(-\mathrm{i}\eta(k)) + \frac{1}{\pi}\int_{-\xi}^{\xi}\log|s-k|d\log[1-\rho^2(s)].$$

Here $\Gamma(-\mathrm{i}\eta(k))$ is the Euler gamma-function, and $\rho(k) = \dfrac{\varkappa^2(k)-1}{\varkappa^2(k)+1}$.

In the region $\omega_0^2 t < x < \omega^2 t$ the solution of the IBV problem takes the form of a modulated elliptic wave. In this region we use the new phase function instead of the function $\theta(k,\xi)$ (7)

$$g(k,\xi) = \left(\int_E^k + \int_{\bar{E}}^k\right)\left(1-\frac{\lambda_-(\xi)}{z}\right)\left(1-\frac{\lambda_+(\xi)}{z}\right)\sqrt{\frac{(z-d(\xi))(z-\bar{d}(\xi))}{(z-E)(z-\bar{E})}}\frac{dz}{8\xi^2},$$

where real $\lambda_-(\xi)$, $\lambda_+(\xi)$ and complex $d(\xi) = d_1(\xi) + \mathrm{i}d_2(\xi)$ parameters are determined by the equations

$$\lambda_- = -\lambda_+ + E_1\frac{1-\xi^4\lambda_-^{-2}\lambda_+^{-2}}{1-\xi^4|E|^2\lambda_-^{-3}\lambda_+^{-3}}\,, \qquad \lambda_+ = \frac{I_0(\lambda_-,d_1,d_2)}{I_{-1}(\lambda_-,d_1,d_2)}\,,$$

where

$$d_1 = E_1 - \lambda_- - \lambda_+, \qquad d_2 = \sqrt{\frac{\xi^4|E|^2}{\lambda_-^2\lambda_+^2} - (E_1-\lambda_--\lambda_+)^2}$$

and

$$I_m(\lambda_-,d_1,d_2) = \int_{d_1-\mathrm{i}d_2}^{d_1+\mathrm{i}d_2}\left(1-\frac{\lambda_-}{z}\right)\sqrt{\frac{(z-d_1)^2+d_2^2}{(z-E_1)^2+E_2^2}}\frac{dz}{z^m}, \quad m = 0,-1.$$

Let $w(k) = \sqrt{(k-E)(k-\bar{E})(k-d(\xi))(k-\bar{d}(\xi))}$ define the Riemann surface and $U(k)$ be the normalized Abelian integral

$$U(k) = \frac{1}{c}\int_E^k \frac{dz}{w(z)}, \quad c = 2\int_{\bar{d}}^d \frac{dz}{w(z)}, \qquad \tau = \frac{2}{c}\int_E^d \frac{dz}{w(z)}.$$

Introduce the following Riemann theta functions:

$$\Theta_{11}(t,\xi,k) = \frac{1}{2}\left[\tilde{\varkappa}(k) + \frac{1}{\tilde{\varkappa}(k)}\right]\frac{\theta_3[U(k)+U(E_0^-)-\tau/2-\alpha(\xi)t-\beta(\xi)]}{\theta_3[U(k)+U(E_0^-)-1/2-\tau/2]},$$
$$\Theta_{12}(t,\xi,k) = \frac{1}{2}\left[\tilde{\varkappa}(k) - \frac{1}{\tilde{\varkappa}(k)}\right]\frac{\theta_3[U(k)-U(E_0^-)+\tau/2+\alpha(\xi)t+\beta(\xi)]}{\theta_3[U(k)-U(E_0^-)+1/2+\tau/2]},$$
$$\Theta_{22}(t,\xi,k) = \frac{1}{2}\left[\tilde{\varkappa}(k) + \frac{1}{\tilde{\varkappa}(k)}\right]\frac{\theta_3[U(k)+U(E_0^-)+\tau/2+\alpha(\xi)t+\beta(\xi)]}{\theta_3[U(k)+U(E_0^-)+1/2+\tau/2]}.$$

Here

$$\theta_3(z) = \sum_{m\in\mathbb{Z}} e^{\pi i\tau m^2 + 2\pi i m z}, \qquad \operatorname{Im}\tau = \operatorname{Im}\tau(\xi) > 0,$$

is theta function. The branch of $\tilde{\varkappa}(k) = \left[\frac{(k-\bar{E})(k-\bar{d}(\xi))}{(k-E)(k-d(\xi))}\right]^{1/4}$ is fixed by its asymptotics

$$\tilde{\varkappa}(k) = 1 - \frac{d_2(\xi)+E_2}{2ik} + \mathrm{O}(k^{-2}), \quad k\to\infty.$$

The point E_0^- is the preimage of the complex number

$$E_0 = \frac{Ed(\xi)-\bar{E}\bar{d}(\xi)}{E-\bar{E}+d(\xi)-\bar{d}(\xi)}$$

on the second sheet of the Riemann surface of the function $w(k)$. Parameters $\alpha = \alpha(\xi)$ and $\beta = \beta(\xi)$ are periods of the normalized Abelian integrals of the second kind $g(k)$ and

$$\zeta(k) = \frac{1}{2}\left(\int_E^k + \int_{\bar{E}}^k\right)\frac{z^2 - e_1 z + e_0}{w(z)}dz,$$

i.e.,

$$\alpha(\xi) = \frac{1}{\pi}\int_E^{d(\xi)} dg(k), \quad \int_{\bar{d}(\xi)}^{d(\xi)} dg(k) = 0, \quad \beta(\xi) = \frac{1}{\pi}\int_E^{d(\xi)} d\zeta(k), \quad \int_{\bar{d}(\xi)}^{d(\xi)} d\zeta(k) = 0.$$

The definition of the Abelian integral $\zeta(k)$ implies

$$e_1 = \frac{E + \bar{E} + d(\xi) + \bar{d(\xi)}}{2}, \qquad e_0 = -\left(\int_d^{\bar{d}} (z^2 - e_1 z)\frac{dz}{w(z)}\right)\left(\int_d^{\bar{d}} \frac{dz}{w(z)}\right)^{-1}.$$

The large k expansion of $\zeta(k)$ is of the form

$$\zeta(k) = k + \zeta_\infty(\xi) + O(k^{-1}), \quad k \to \infty,$$

where

$$\zeta_\infty = \frac{1}{2}\left(\int_E^\infty + \int_{\bar{E}}^\infty\right)\left[\frac{z^2 - e_1 z + e_0}{w(z)} - 1\right] dz + E_2$$

is a real function of ξ.

Theorem 3.2. *If $\omega_0^2 t < x < \omega^2 t$, then for $t \to \infty$ the solution of the IBV problem (1)–(2) takes the form of a modulated elliptic wave:*

$$\begin{aligned} q(x,t) =& 2\mathrm{i}\frac{\Theta_{12}(t,\xi,\infty)}{\Theta_{11}(t,\xi,\infty)}\exp[2\mathrm{i}tg_\infty(\xi) - 2\mathrm{i}\phi(\xi)] + \mathrm{O}(t^{-1/2}), \\ \nu(x,t) =& -1 + 2\frac{\Theta_{11}(t,\xi,0)\Theta_{22}(t,\xi,0)}{\Theta_{11}(t,\xi,\infty)\Theta_{22}(t,\xi,\infty)} + \mathrm{O}(t^{-1/2}), \\ \mu(x,t) =& 2\mathrm{i}\frac{\Theta_{11}(t,\xi,0)\Theta_{12}(t,\xi,0)}{\Theta_{11}^2(t,\xi,\infty)}\exp[2\mathrm{i}tg_\infty(\xi) - 2\mathrm{i}\phi(\xi)] + \mathrm{O}(t^{-1/2}), \end{aligned}$$

where

$$g_\infty(\xi) = \left(\int_E^\infty + \int_{\bar{E}}^\infty\right)$$

$$\times\left[\left(1 - \frac{\lambda_-(\xi)}{z}\right)\left(1 - \frac{\lambda_+(\xi)}{z}\right)\sqrt{\frac{(z - d(\xi))(z - \bar{d}(\xi))}{(z-E)(z-\bar{E})}} - 1\right]\frac{dz}{8\xi^2} - \frac{l}{8\omega\xi^2},$$

and the phase shift $\phi(\xi)$ is defined by

$$\phi(\xi) = \int_{\gamma_d \cup \bar{\gamma}_d} (k - k_0(\xi))\log\left[\frac{h(k)}{\delta^2(k,\xi)}\right]\frac{dk}{2\pi w_+(k)}, \quad h(k) = \begin{cases} [A_-(k)A_+(k)]^{-1}, \ k \in \gamma_d \\ A_-(k)A_+(k), \ k \in \bar{\gamma}_d; \end{cases}$$

$$\delta(k) = \exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{\lambda_-(\xi)}^{\lambda_+(\xi)} \frac{\log(1-\rho^2(s))ds}{s-k}\right\}, \quad k \in \mathbb{C}\setminus[\lambda_-(\xi), \lambda_+(\xi)], \ |E| \le \xi \le \frac{1}{2\omega_0},$$

where $A(k)$, $\rho(k)$ *are spectral functions (5), (6), and* $k_0(\xi) = e_1(\xi) + \zeta_\infty(\xi)$, $\lambda_\pm(\xi)$ *are stationary points of the phase functions* $g(k,\xi)$, *and* γ_d ($\bar\gamma_d$) *is an arc connecting* E *and* $d(\xi)$ ($\bar E$ *and* $\bar d(\xi)$), *where* $\operatorname{Im} g(k) = 0$.

In the region $0 < x < \omega_0^2 t$ the phase g-function takes the form:

$$\hat g(k,\xi) := \left(\frac{\omega}{4k} + \frac{1}{4\xi^2}\right)\sqrt{(k-E)(k-\bar E)}.$$

This function allows us to prove the following

Theorem 3.3. *The solution of the IBV problem (1)-(2) for* $t \to \infty$ *takes the form of a plane wave:*

$$\begin{aligned} q(x,t) &= -\frac{p}{2\omega}\exp\left[\mathrm{i}\omega t - \mathrm{i}\frac{l}{\omega}x - 2\mathrm{i}\hat\phi(\xi)\right] + O(t^{-1/2}),\\ \mu(x,t) &= p\exp\left[\mathrm{i}\omega t - \mathrm{i}\frac{l}{\omega}x - 2\mathrm{i}\hat\phi(\xi)\right] + O(t^{-1/2}),\\ \nu(x,t) &= l + O(t^{-1/2}), \end{aligned}$$

where

$$\hat\phi(\xi) = \frac{1}{2\pi}\left(\int_{-\infty}^{\lambda_-(\xi)} + \int_{\lambda_+(\xi)}^{\infty}\right)\log A^2(k)\frac{dk}{X(k)},$$

$\lambda_\pm(\xi)$ *are the stationary points of the function* $\hat g(k,\xi)$ ($\frac{1}{2\omega_0} \le \xi \le \infty$), *and* $A(k)$ *is defined by (5). Here the contour* $\gamma_g \cup \bar\gamma_g$ *connects* E *and* $\bar E$ *along the arc, where* $\operatorname{Im}\hat g(k) = 0$.

R e m a r k 3.1. *If* $x = 0$ ($\xi = \infty$), *then* $\lambda_-(\infty) = -\infty$ *and* $\lambda_+(\infty) = +\infty$. *Hence*

$$\hat\phi(\infty) = 0.$$

Therefore the plane wave $\mu(0,t)$ *and* $\nu(0,t)$ *match with the boundary conditions.*

Since $g_\infty(\xi_0) = \hat g_\infty(\xi_0) = \omega/2 - l/8\omega\xi_0^2$, $\phi(\xi_0) = \hat\phi(\xi_0)$, $\operatorname{Im} d(\xi_0) = 0$,*where* $\xi_0 = \dfrac{1}{2\omega_0}$, *and theta-function* $\theta_3(*)|_{\xi=\xi_0} \equiv 1$, *we have that elliptic waves match with the plane waves at* $\xi = \xi_0$.

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