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Uniqueness of Solution of the Inverse Problem of Scattering Theory for a Fourth Order Differential Bundle with Multiple Characteristics

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The system of four equations of Marchenko type allowing to restore the bundle by the scattering matrix is derived for a fourth order differential bundle in $L_2(0;+\infty)$ in the case of multiple $\pm i$ roots of the main characteristic polynomial. The uniqueness of solution of the inverse problem is proved when the bundle has a pure continuous spectrum.

Key words: bundle, spectrum, spectral expansion, scattering matrix, inverse problem.

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The author devotes this paper to the memory of academician of NAS of Azerbaijan, professor M.G. Gasymov

Let us consider a bundle L_λ in the space $L_2(0;+\infty)$ generated by a differential equation

$$l\left(x,\frac{d}{dx},\lambda\right)y=\left(\frac{d^2}{dx^2}+\lambda^2\right)^2 y+r(x)\,y'+(\lambda p(x)+q(x))\,y=0 \qquad (1)$$

with the boundary conditions

$$y(0,\lambda)=y'(0,\lambda)=0, \qquad (2)$$

where the complex-valued functions $p(x)$, $q(x)$, $r(x)$ satisfy the conditions

$$e^{\varepsilon x}\left|p(x)\right|\le c_1,\ e^{\varepsilon x}\left|r(x),\,r'(x)\right|\le c_2,\ e^{\varepsilon x}\left|q(x)\right|\le c_3,\ \varepsilon>0,$$

and λ is a spectral parameter.

In papers [1, 2] equation (1) is studied and the transformation operators transforming the solutions of the equation $\left(\frac{d^2}{dx^2}+\lambda^2\right)^2 y=0$ to the solutions of equation (1) are constructed. In particular, it is obtained in [2] that equation (1) has a fundamental system of solutions $F_j^{\pm}(x,\lambda)$, $j=\overline{0,1}$, satisfying the conditions

$$\lim_{x\to\infty}\left[F_j^{\pm}(x,\lambda)-x^j e^{\pm i\lambda x}\right]=0,\ \pm\mathrm{Im}\lambda>0, \tag{3}$$

and there exist the kernels $K_j^{\pm}(x,t)$ such that

$$F_j^{\pm}(x,\lambda)=x^j e^{\pm i\lambda x}+\int\limits_x^{\infty}K_j^{\pm}(x,t)\,e^{\pm i\lambda t}dt. \tag{4}$$

These kernels satisfy the equations

$$l\left(x,\frac{\partial}{\partial x},\pm i\frac{\partial}{\partial t}\right)K_j^{\pm}(x,t)=0,$$

and there holds

$$\lim_{x+t\to\infty}\frac{\partial^{\alpha+\beta}K_j^{\pm}(x,t)}{\partial x^{\alpha}\partial t^{\beta}}=0,\ \alpha+\beta\le 3,\ \int\limits_x^{\infty}\left|K_j^{\pm}(x,t)\right|^2dt<\infty. \tag{5}$$

Furthermore, for the kernels $K_j^{\pm}(x,t)$ the following conditions are fulfilled on the characteristics $t=x$:

$$\frac{d}{dx}K_j^{\pm}(x,t)=\pm\frac{1}{4i}\int\limits_x^{\infty}p_j(\xi)\,d\xi-\frac{1}{4}\int\limits_x^{\infty}r_j(\xi)\,d\xi,$$

$$\left.\left(\frac{\partial^2K_j^{\pm}}{\partial t^2}-\frac{\partial^2K_j^{\pm}}{\partial x^2}\right)\right|_{t=x}=\pm\frac{1}{4}p_j(x)-\frac{1}{4}r_j(x)+\frac{1}{2}\int\limits_x^{\infty}q_j(\xi)\,d\xi$$

$$\pm\frac{1}{4}\int\limits_x^{\infty}\left[ip(\xi)-r(\xi)\right]K_j^{\pm}(\xi,\xi)\,d\xi, \tag{6}$$

where $p_j(u)=u^jp(u)$, $r_j(u)=u^jr(u)$, $q_j(u)=u^j\left[q(u)-r'(u)\right]$.

In the given paper for a bundle L_λ we derive a system of four equations of Marchenko type allowing to restore the bundle by the scattering matrix. The uniqueness of the solution of the inverse problem is proved when the bundle has a pure continuous spectrum.

The eigenfunctions $\varphi_i^{\pm}(x,\lambda)$, $i=\overline{1,2}$, of the continuous spectrum $-\infty<\lambda<\infty$ of the bundle L_λ are determined in the following way:

$$\varphi_i^{\pm}(x,\lambda)=F_0^{\pm}(x,\lambda)\,S_{i1}^{\pm}(\lambda)+F_1^{\pm}(x,\lambda)\,S_{i2}^{\pm}(\lambda)-F_{i-1}^{\pm}(x,\lambda)\,,\; i=\overline{1,2}. \qquad (7)$$

Here $S_{ik}^{\pm}(\lambda)$, $i,k=\overline{1,2}$, are found from the conditions $\varphi_i^{\pm}(0,\lambda)=\varphi_i^{\pm\prime}(0,\lambda)=0$, $i=\overline{1,2}$, and for them as $|\lambda|\to\infty$ there hold

$$\begin{gathered}
S_{11}^{+}(\lambda)=1-2K_1^{+}(0,0)+O\left(\frac{1}{\lambda}\right),\\
S_{12}^{+}(\lambda)=-2i\lambda+2\left(K_0^{+}(0,0)-\overline{K_0^{+}(0,0)}\right)+O\left(\frac{1}{\lambda}\right),\\
S_{21}^{+}(\lambda)=O\left(\frac{1}{\lambda}\right),\; S_{22}^{+}(\lambda)=1+O\left(\frac{1}{\lambda}\right),\\
S_{11}^{-}(\lambda)=\frac{1}{1-K^{-}(0,0)}+O\left(\frac{1}{\lambda}\right),\; S_{21}^{-}(\lambda)=O\left(\frac{1}{\lambda}\right),\\
S_{12}^{-}(\lambda)=\frac{i\lambda}{1-K^{-}(0,0)}-\frac{K^{+}(0,0)}{1-K^{-}(0,0)}+O\left(\frac{1}{\lambda}\right),\\
S_{22}^{-}(\lambda)=\frac{2i\lambda}{1-K^{-}(0,0)}-\frac{K^{+}(0,0)-K^{-}(0,0)}{1-K^{-}(0,0)}+O\left(\frac{1}{\lambda}\right).
\end{gathered} \qquad (8)$$

Thus, assuming

$$1-2K_1^{+}(0,0)=a_{11}^{+},\; -2i\lambda+2\left(K_0^{+}(0,0)-\overline{K_0^{+}(0,0)}\right)=a_{12}^{+},\; a_{21}^{+}=0,\; a_{22}^{+}=1 \qquad (9)$$

and denoting $S_{ik}^{-}(\lambda)=a_{ik}^{-}+O\left(\frac{1}{\lambda}\right)$, we find that the functions

$$F_{kj}^{\pm}(t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left\{S_{kj}^{\pm}(\lambda)-a_{kj}^{\pm}(\lambda)\right\}e^{i\lambda t}d\lambda \qquad (10)$$

belong to the space $L_2(-\infty;+\infty)$.

Using paper [3, Th. 4], we can write a spectral expansion of the bundle L_λ. There holds the following theorem:

Theorem 1. *Let the bundle have neither eigenvalues nor spectral properties and let a smooth up to sixth order function $f(x)$ finite in the vicinity of zero and infinity be given. Then it holds a uniformly converging for all $x\in[0;\infty)$ spectral expansion*

$$f(x)=\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\lambda^3\left[E_1(\lambda)\,\varphi_1(x,\lambda)+E_2(\lambda)\,\varphi_2(x,\lambda)\right]d\lambda, \qquad (11)$$

where $E_1(\lambda) = \int\limits_0^\infty \varphi_1^*(\xi,\lambda) f(\xi)\, d\xi$, $E_2(\lambda) = \int\limits_0^\infty \varphi_2^*(\xi,\lambda) f(\xi)\, d\xi$ *and* $\varphi_1^*(x,\lambda)$, $\varphi_2^*(x,\lambda)$ *are the solutions of the adjoint equation* $l^*\left(x, \frac{d}{dx}, \lambda\right) z = 0$ *satisfying the conditions* $\varphi_i^*(0,\lambda) = \varphi_i^{*\prime}(0,\lambda) = 0$, $i = \overline{1,2}$.

From (7) we take the function $\varphi_i^+(x,\lambda)$ and multiply it by $\frac{1}{2\pi} e^{i\lambda t}$, then we integrate it with respect to λ on $(-\infty;\infty)$ under assumption that $t > x$. Thus, in the sense of convergence in a class of distributions we get

$$\int\limits_{-\infty}^{\infty} e^{i\lambda x} e^{i\lambda t} S_{11}^+(\lambda) d\lambda + \int\limits_{-\infty}^{\infty} S_{11}^+(\lambda) \left(\int\limits_x^\infty K_0^+(x,\xi)\, e^{i\lambda\xi} d\xi \right) e^{i\lambda t} d\lambda$$
$$+ \int\limits_{-\infty}^{\infty} x e^{i\lambda x} e^{i\lambda t} S_{11}^+(\lambda) d\lambda + \int\limits_{-\infty}^{\infty} S_{12}^+(\lambda) \left(\int\limits_x^\infty K_1^+(x,\xi)\, e^{i\lambda\xi} d\xi \right) e^{i\lambda t} d\lambda$$
$$- \int\limits_{-\infty}^{\infty} e^{-i\lambda x} e^{i\lambda t} d\lambda - \int\limits_{-\infty}^{\infty} \left(\int\limits_x^\infty \overline{K_0^+(x,\xi)} e^{-i\lambda\xi} d\xi \right) e^{i\lambda t} d\lambda = 0,$$
$$\int\limits_{-\infty}^{\infty} e^{i\lambda x} e^{i\lambda t} S_{12}^+(\lambda) d\lambda + \int\limits_{-\infty}^{\infty} S_{11}^+(\lambda) \left(\int\limits_x^\infty K_0^+(x,\xi)\, e^{i\lambda\xi} d\xi \right) e^{i\lambda t} d\lambda$$
$$+ \int\limits_{-\infty}^{\infty} x e^{i\lambda x} e^{i\lambda t} S_{22}^+(\lambda) d\lambda + \int\limits_{-\infty}^{\infty} S_{22}^+(\lambda) \left(\int\limits_x^\infty K_1^+(x,\xi)\, e^{i\lambda\xi} d\xi \right) e^{i\lambda t} d\lambda$$
$$- \int\limits_{-\infty}^{\infty} e^{-i\lambda x} e^{i\lambda t} d\lambda - \int\limits_{-\infty}^{\infty} \left(\int\limits_x^\infty \overline{K_1^+(x,\xi)} e^{-i\lambda\xi} d\xi \right) e^{i\lambda t} d\lambda = 0. \tag{12}$$

Analogously, by multiplying the function $\varphi_i^-(x,\lambda)$ by $\frac{1}{2\pi} e^{-i\lambda t}$, we get the similar relations from (7). In sequel, we consider the asymptotic formulas (8) and arrive at the following theorem.

Theorem 2. *If the bundle L_λ has neither eigenvalues nor spectral properties, then for each $x \geq 0$ the kernels $K_i^\pm(x,t)$, $(x < t < \infty)$, of the transformation operator satisfy the following system of the main equations of Marchenko type [4]:*

$$F_{11}^\pm(x+t) + \int\limits_x^\infty K_0^\pm(x,\xi) F_{11}^\pm(t+\xi)\, d\xi + x F_{12}^\pm(x+t)$$

$$+\int_x^\infty K_1^\pm(x,\xi)F_{12}^\pm(t+\xi)\,d\xi-\overline{K_0^\pm(x,t)}=0,$$

$$F_{21}^\pm(x+t)+\int_x^\infty K_0^\pm(x,\xi)F_{21}^\pm(t+\xi)\,d\xi+xF_{22}^\pm(x+t)$$

$$+\int_x^\infty K_1^\pm(x,\xi)F_{22}^\pm(t+\xi)\,d\xi-\overline{K_1^\pm(x,t)}=0. \tag{13}$$

Let us study the properties of the transition function $F_{kj}^\pm(x)$. In the main equations (13) we assume $t=x$ and $t=2x$, and find

$$F_{11}^\pm(2x)+xF_{12}^\pm(2x)+\int_x^\infty K_0^\pm(x,\xi)F_{11}^\pm(x+\xi)\,d\xi$$

$$+\int_x^\infty K_1^\pm(x,\xi)F_{12}^\pm(x+\xi)\,d\xi=\overline{K_0^\pm(x,x)},$$

$$F_{11}^\pm(3x)+xF_{12}^\pm(3x)+\int_x^\infty K_0^\pm(x,\xi)F_{11}^\pm(2x+\xi)\,d\xi$$

$$+\int_x^\infty K_1^\pm(x,\xi)F_{12}^\pm(2x+\xi)\,d\xi=\overline{K_0^\pm(x,2x)}.$$

Hence

$$F_{11}^\pm(\eta)+\frac{\eta}{2}F_{12}^\pm(\eta)+\int_{\frac{\eta}{2}}^\infty K_0^\pm\left(\frac{\eta}{2},\xi\right)F_{11}^\pm\left(\frac{\eta}{2}+\xi\right)d\xi$$

$$+\int_{\frac{\eta}{2}}^\infty K_1^\pm\left(\frac{\eta}{2},\xi\right)F_{12}^\pm\left(\frac{\eta}{2}+\xi\right)d\xi=\overline{K_0^\pm\left(\frac{\eta}{2},\frac{\eta}{2}\right)},$$

$$F_{11}^\pm(\eta)+\frac{\eta}{3}F_{12}^\pm(\eta)+\int_{\frac{\eta}{3}}^\infty K_0^\pm\left(\frac{\eta}{3},\xi\right)F_{11}^\pm\left(\frac{2\eta}{3}+\xi\right)d\xi$$

$$+\int_{\frac{\eta}{3}}^\infty K_1^\pm\left(\frac{\eta}{3},\xi\right)F_{12}^\pm\left(\frac{2\eta}{3}+\xi\right)d\xi=\overline{K_0^\pm\left(\frac{\eta}{3},\frac{2\eta}{3}\right)}.$$

From formula (52) of paper [2] for the estimation of the kernels of transformation operators

$$\left|K_j^{\pm}(x,t)\right| \le \frac{1}{4}\sigma_j\left(\frac{x+t}{2}\right)e^{\tau(x)},$$

where

$$\sigma_j\left(\frac{x+t}{2}\right) = \int\limits_{\frac{x+t}{2}}^{\infty} s^{j+1}\left\{|p(s)| + |r(s)| + s\left[\left|r'(s)\right| + |q(s)|\right]\right\}ds,\ j=\overline{0,1},$$

$$\tau(x) = \int\limits_{x}^{\infty} s^2\left\{|p(s)| + |r(s)| + s\left(\left|r'(s)\right| + |q(s)|\right)\right\}ds,$$

and for their derivatives, for example, for the first derivatives, where the estimations

$$\left\{\left|\frac{\partial K_j^{\pm}(x,t)}{\partial x}\right|, \left|\frac{\partial K_j^{\pm}(x,t)}{\partial t}\right|\right\} \le \int\limits_{\frac{x+t}{2}}^{\infty} \xi\left\{\left|p_j'(\xi)\right| + \left|r'(\xi)\right| + |q_j(\xi)|\right\}d\xi$$

$$+\frac{1}{2}\int\limits_{\frac{x+t}{2}}^{\infty}\left\{|p_j(\xi)| + |r_j(\xi)|\right\}d\xi + \frac{1}{4}\sigma_j\left(\frac{x+t}{2}\right)\sigma_0(x)\,e^{\tau(x)},$$

hold, it follows that

$$\left|F_{11}^{\pm}(\eta) + \frac{\eta}{2}F_{12}^{\pm}(\eta)\right| \le \left|K_0^{\pm}\left(\frac{\eta}{2},\frac{\eta}{2}\right)\right| + \left(\int\limits_{\frac{\eta}{2}}^{\infty}\left|K_0^{\pm}\left(\frac{\eta}{2},\xi\right)\right|^2 d\xi\int\limits_{\frac{\eta}{2}}^{\infty}\left|F_{11}^{\pm}\left(\frac{\eta}{2}+\xi\right)\right|^2 d\xi\right)^{\frac{1}{2}}$$

$$+\left(\int\limits_{\frac{\eta}{2}}^{\infty}\left|K_1^{\pm}\left(\frac{\eta}{2},\xi\right)\right|^2 d\xi\int\limits_{\frac{\eta}{2}}^{\infty}\left|F_{12}^{\pm}\left(\frac{\eta}{2}+\xi\right)\right|^2 d\xi\right)^{\frac{1}{2}} \le C_1 e^{-\delta\eta},\ \delta>0 \tag{14}$$

and also

$$\left|F_{11}^{\pm}(\eta) + \frac{\eta}{3}F_{12}^{\pm}(\eta)\right| \le Ce^{-\delta\eta},$$

where C_1, C and δ are constant numbers, $\delta < \frac{\varepsilon}{2}$.

In the similar way we show

$$\left|F_{21}^{\pm}(\eta) + \frac{\eta}{2}F_{22}^{\pm}(\eta)\right| \le C_2 e^{-\delta\eta},\ \eta>0. \tag{15}$$

Using these estimations, from (13) we find that $F_{kj}^{\pm}(x)$ has the same number of the derivatives as $K_i^{\pm}(x,t)$. Furthermore, the derivatives of the function exponentially decrease.

For each fixed $x = x_0 \in [0;\infty)$ the main system of equations (13) gives four equations for determining $K_0^{\pm}(x,t)$ and $K_1^{\pm}(x,t)$, and it follows from (14), (15) that the kernels of these equations generate the Hilbert-Schmidt type operators. Thus, the equations

$$F_{11}^{\pm}(x_0+t) + x_0 F_{12}^{\pm}(x_0+t) + \int\limits_{x_0}^{\infty} K_0^{\pm}(x_0,\xi)\, F_{11}^{\pm}(\xi+t)d\xi$$

$$+\int\limits_{x_0}^{\infty} K_1^{\pm}(x_0,\xi)\, F_{12}^{\pm}(\xi+t)d\xi - \overline{K_0^{\pm}(x_0,t)} = 0,$$

$$F_{21}^{\pm}(x_0+t) + x_0 F_{22}^{\pm}(x_0+t) + \int\limits_{x_0}^{\infty} K_0^{\pm}(x_0,\xi)\, F_{21}^{\pm}(\xi+t)d\xi \tag{16}$$

$$+\int\limits_{x_0}^{\infty} K_1^{\pm}(x_0,\xi)\, F_{22}^{\pm}(\xi+t)d\xi - \overline{K_1^{\pm}(x_0,t)} = 0$$

are of the Fredholm type.

It is directly verified that if the coefficients of equation (1) are real, then $\overline{K_i^{\pm}(x_0,t)} = K_i^{\mp}(x_0,t)$, $i = \overline{0,1}$.

Definition. *We call the matrix function* $S^{\pm}(\lambda) = \left\{S_{ij}^{\pm}(\lambda)\right\}_{i,j=1}^{2}$ *a scattering matrix of the bundle* L_λ*, and its elements* $S_{ij}^{\pm}(\lambda)$ *we call the scattering data of the bundle* L_λ*.*

Notice that the scattering data are completely determined by the asymptotics of eigenfunctions $\varphi_i^{\pm}(x,\lambda)$ of a continuous spectrum as $x \to \infty$.

Now we put a problem on the restoration of the bundle L_λ by a scattering matrix.

Theorem 3. *For each fixed* $x = x_0 \in [0;\infty)$ *the main system of integral equations (13) in the class of twice differentiable functions from* $L_2(x_0,\infty)$ *has a unique solution* $K_0^{\pm}(x_0,t)$ $K_1^{\pm}(x_0,t)$*.*

P r o o f. By the Fredholm property of the system (16) it suffices to show that the system of homogeneous equations

$$\begin{aligned}&\int\limits_{x_0}^{\infty}\left[f_0(\xi)\,F_{11}^{\pm}(\xi+t)+f_1(\xi)\,F_{12}^{\pm}(\xi+t)\right]d\xi-\overline{f_0(t)}=0,\\&\int\limits_{x_0}^{\infty}\left[f_0(\xi)\,F_{21}^{\pm}(\xi+t)+f_1(\xi)\,F_{22}^{\pm}(\xi+t)\right]d\xi-\overline{f_1(t)}=0\end{aligned}\tag{17}$$

has only a zero solution from $L_2(x_0,\infty)$.

Consider the contrary: the system (17) has nonzero solutions $f_0(t)$ and $f_1(t)$, possessing twice differentiable derivative from $L_2(x_0;\infty)$. Assume

$$F_j(\lambda)=\frac{1}{2\pi}\int\limits_{x_0}^{\infty}f_j(\xi)e^{i\lambda\xi}d\xi,\tag{18}$$

$$f_j(\xi)=\int\limits_{-\infty}^{\infty}F_j(\lambda)e^{-i\lambda\xi}d\xi,\ \overline{f_j(\xi)}=\int\limits_{-\infty}^{\infty}\overline{F_j(\lambda)}e^{i\lambda\xi}d\lambda.\tag{19}$$

Multiply (17) by the derivatives of the twice differential functions $g_0(t)$, $g_1(t)$ from $L_2(x_0;\infty)$ and integrate it with respect to $t\in[x_0;\infty)$. Then

$$\int\limits_{-\infty}^{\infty}\left[S_{11}^{+}(\lambda)\,F_0(\lambda)+S_{12}^{+}(\lambda)\,F_1(\lambda)\right]G_0(\lambda)d\lambda$$

$$-\int\limits_{-\infty}^{\infty}\left[a_{11}^{+}(\lambda)\,F_0(\lambda)+a_{12}^{+}(\lambda)\,F_1(\lambda)\right]G_0(\lambda)d\lambda-\int\limits_{-\infty}^{\infty}\overline{F_0(\lambda)}G_0(\lambda)\,d\lambda=0,$$

$$\int\limits_{-\infty}^{\infty}\left[S_{21}^{+}(\lambda)\,F_0(\lambda)+S_{22}^{+}(\lambda)\,F_1(\lambda)\right]G_1(\lambda)d\lambda$$

$$-\int\limits_{-\infty}^{\infty}\left[a_{21}^{+}(\lambda)\,F_0(\lambda)+a_{22}^{+}(\lambda)\,F_1(\lambda)\right]G_1(\lambda)d\lambda-\int\limits_{-\infty}^{\infty}\overline{F_1(\lambda)}G_1(\lambda)\,d\lambda=0,$$

where $G_j(\lambda)=\int\limits_{x_0}^{\infty}g_j(t)e^{i\lambda t}dt$, and a_{ij}^{+} determines the asymptotics of $S_{ij}^{+}(\lambda)$. Obviously,

$$\int\limits_{-\infty}^{\infty}a_{ij}^{+}F_s(\lambda)G_j(\lambda)\,d\lambda=0.$$

Therefore

$$\int_{-\infty}^{\infty}\left[S_{11}^{+}(\lambda)\,F_0(\lambda)+S_{12}^{+}(\lambda)\,F_1(\lambda)-\overline{F_0(\lambda)}\right]G_0(\lambda)d\lambda=0,$$

$$\int_{-\infty}^{\infty}\left[S_{21}^{+}(\lambda)\,F_0(\lambda)+S_{22}^{+}(\lambda)\,F_1(\lambda)-\overline{F_1(\lambda)}\right]G_1(\lambda)d\lambda=0.$$

It follows from the relations

$$\int_{-\infty}^{\infty} f_i'(x)\,e^{-i\lambda x}dx=f_i(x)\,e^{-i\lambda x}\Big|_{-\infty}^{\infty}+i\lambda\int_{-\infty}^{\infty} f_i(x)\,e^{-i\lambda x}dx=i\lambda F_i(\lambda)$$

that $F_i(\lambda)=O\left(\frac{1}{\lambda}\right)$. Then there exist the functions $\gamma_0(\xi)$ and $\gamma_1(\xi)$ from $L_2(x_0;\infty)$ such that

$$S_{11}^{+}(\lambda)\,F_0(\lambda)+S_{12}^{+}(\lambda)\,F_1(\lambda)-\overline{F_0(\lambda)}=\int_{-x_0}^{\infty}\gamma_0(\xi)\,e^{i\lambda\xi}d\xi=H_0(\lambda),$$

$$S_{21}^{+}(\lambda)\,F_0(\lambda)+S_{22}^{+}(\lambda)\,F_1(\lambda)-\overline{F_1(\lambda)}=\int_{-x_0}^{\infty}\gamma_1(\xi)\,e^{i\lambda\xi}d\xi=H_1(\lambda).$$

In a matrix notation

$$S^{+}(\lambda)\,F(\lambda)-\overline{F(\lambda)}=H(\lambda)\,, \tag{20}$$

where

$$S^{+}(\lambda)=\begin{pmatrix} S_{11}^{+}(\lambda) & S_{12}^{+}(\lambda)\\ S_{21}^{+}(\lambda) & S_{22}^{+}(\lambda)\end{pmatrix},\ F(\lambda)=\begin{pmatrix} F_0(\lambda)\\ F_1(\lambda)\end{pmatrix},\ H(\lambda)=\begin{pmatrix} H_0(\lambda)\\ H_1(\lambda)\end{pmatrix}.$$

The definition of $S^{+}(\lambda)$ yields $S^{+}(\lambda)\,\overline{S^{+}(\lambda)}=E$. Therefore, from (20) we get

$$\overline{S^{+}(\lambda)F(\lambda)}-F(\lambda)=\overline{H(\lambda)}$$

that is equivalent to

$$\overline{F(\lambda)}-S^{+}(\lambda)\,F(\lambda)=S^{+}(\lambda)\,\overline{H(\lambda)}. \tag{21}$$

By (19) and (20), $S^{+}(\lambda)\,\overline{H(\lambda)}=-H(\lambda)\,.$

Consider the case of $x_0=0$. It is directly verified that $H(\lambda)$ and $\overline{H(\lambda)}$ are holomorphic functions in the upper and lower half-planes, respectively, and that they decrease. The functions $F_1^+(0,\lambda)$, $F_1^-(0,\lambda)$ and $F_1^-(0,\lambda)\overline{H(\lambda)} = -F_1^+(0,\lambda)H(\lambda)$ are also holomorphic. Thus, the function

$$M(\lambda) = \begin{cases} F_1^+(0,\lambda)H(\lambda), & \mathrm{Im}\lambda > 0, \\ F_1^-(0,\lambda)H(\lambda), & \mathrm{Im}\lambda < 0, \end{cases}$$

is an entire function of λ on a real axis $M(\lambda) \to 0$ as $|\lambda| \to \infty$. From the well-known Privalov's theorem we conclude that $M(\lambda) \equiv 0$. Then $H(\lambda) \equiv 0$, and therefore $S^+(\lambda)F(\lambda) - \overline{F(\lambda)} \equiv 0$. From hence, after simple transformations we get $F(\lambda) \equiv 0$. Then $f_0(t) \equiv f_1(t) \equiv 0$, but this is a contradiction. The theorem is proved for the case of $x_0 = 0$.

Before proving the solvability of the main equations for $x_0 > 0$ we should notice that the dependence of the transient functions $F_{kj}^+(x,t)$ for $x, t \geq x_0$ on the values of coefficients of the functions $r(x)$, $p(x)$ and $q(x)$ for $x \geq x_0$ is revealed from the main integral equations. Therefore, in $L_2(x_0;\infty)$ we consider a new boundary value problem

$$\left(\frac{d^2}{dx^2} + \lambda^2\right)^2 y + r(x)y' + (\lambda p(x) + q(x))y = 0,$$

$$y(x_0) = y'(x_0) = 0.$$

This problem has its own scattering matrix $\tilde{S}(x_0,\lambda)$, and in the main equation the transition function is determined as follows:

$$F_{kj}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\tilde{S}_{kj}(x_0,\lambda) - a_{kj}(x_0)e^{-2i\lambda x_0}\right\}e^{i\lambda(x+t)}d\lambda,\ x, t \geq x_0.$$

Thus, in the main equations for to study their solvability we can use this formula for $F_{kj}(x,t)$. Then the reasonings cited for $x_0 = 0$ should be completely repeated. The theorem is proved.

We introduce the denotations:

$$F(x_0+t) = \begin{pmatrix} F_{11}^+(x_0+t) + x_0F_{12}^+(x_0+t) \\ F_{21}^+(x_0+t) + x_0F_{22}^+(x_0+t) \\ F_{11}^-(x_0+t) + x_0F_{12}^-(x_0+t) \\ F_{21}^-(x_0+t) + x_0F_{22}^-(x_0+t) \end{pmatrix},\ K(x_0,t) = \begin{pmatrix} K_0^-(x_0,t) \\ K_1^-(x_0,t) \\ K_0^+(x_0,t) \\ K_1^+(x_0,t) \end{pmatrix},$$

$$H(\xi+t) = \begin{pmatrix} F_{11}^+(\xi+t) & F_{12}^+(\xi+t) & 0 & 0 \\ F_{21}^+(\xi+t) & F_{22}^+(\xi+t) & 0 & 0 \\ 0 & 0 & F_{11}^-(\xi+t) & F_{12}^-(\xi+t) \\ 0 & 0 & F_{21}^-(\xi+t) & F_{22}^-(\xi+t) \end{pmatrix}.$$

Then the system of equations (16) can be written in the form

$$K(x_0,t) = F(x_0+t) + \int\limits_x^\infty K(x_0,\xi)\, H(\xi+t)d\xi,$$

or $K = F+HK$, where H is a linear integral operator in the space $L_2\left([x_0;\infty);E^4\right)$; E^4 is a complex four-dimensional space, $Hf = \int\limits_x^\infty H(\xi+t)f(t)dt$. Here f is any function from $L_2\left([x_0;\infty);E^4\right)$, $f = (f_0, f_1, f_2, f_3)$, $f_i^{(k)}(x) \in L_2\left([x_0;\infty);E^4\right)$, $k=\overline{1,2}$, $i=\overline{0,3}$. A scalar product in this space is determined as

$$(f,g) = \sum_{j=0}^{3} \int\limits_{x_0}^\infty f_j(x)\overline{g_j(x)}dx.$$

From the introduced norm of the space $L_2\left([x_0;\infty);E^4\right)$ it is easy to get $\|H\| < 1$, and therefore the system of the integral equations is solved by a successive approximation method.

Considering that conditions (6) allow to determine uniquely all the coefficients of equation (1) with respect to $K_j^{\pm}(x,t)$, $j=0,1$, we arrive at the main theorem.

Theorem 4. *If the bundle L_λ has neither eigenvalues nor spectral properties, then by the scattering matrix $S^{\pm}(\lambda)$ equation (1) is determined in a unique way.*

R e m a r k. The above stated inverse problem is overdetermined. There are definite ties between the scattering data. We can find relations between them if for $-\infty < \lambda < \infty$ we can write the representations $\varphi_i^-(x,\lambda) = A_i(\lambda)\varphi_1^+(x,\lambda) + B_i(\lambda)\varphi_2^+(x,\lambda)$, and then, taking into account a linear independence of the system $x^j e^{\pm i\lambda x}$, $j=\overline{0,1}$, from the last equalities as $x \to \infty$ we find the following relations:

$$S_{11}^- = -A_1(\lambda),\ S_{12}^- = B_1(\lambda),\ S_{11}^-S_{11}^+ - S_{12}^-S_{21}^+ = 1,\ S_{11}^-S_{12}^+ - S_{12}^-S_{22}^+ = 0,$$

$$S_{21}^- = -A_2(\lambda),\ S_{22}^- = B_2(\lambda),\ S_{21}^-S_{12}^+ - S_{22}^-S_{22}^+ = 1,\ S_{21}^-S_{11}^+ - S_{22}^-S_{21}^+ = 0.$$

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