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Optimality of Estimates for the Width of Support Layers of the Isoperimetrix in the Minkowski Geometry

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For the unit ball B of the Minkowski space M^n in the sense of H. Busemann there are proved necessary and sufficient conditions when the inequalities

$$\Delta_B(Q_I(\bar{u})) \geq \frac{4v_{n-1}}{nv_n}, \quad \Delta_B(I) \geq \frac{4v_{n-1}}{nv_n}, \quad q(I,B) \geq \frac{2v_{n-1}}{nv_n},$$

$$\Delta_B(Q_I(\bar{u})) \leq \frac{4v_{n-1}}{v_n}, \quad D_B(I) \leq \frac{4v_{n-1}}{v_n}, \quad Q(I,B) \leq \frac{2v_{n-1}}{v_n}$$

turn into equalities.

All the estimates are optimal since for every estimate there exists the corresponding B. Here $\Delta_B(I)$ and $D_B(I)$ denote the width and the diameter of the isoperimetrix I of M^n, $q(I,B)$ stands for the capacity coefficient of B with respect to I, $Q(I,B)$ is the inclusion coefficient of I with respect to B, and $\Delta_B(Q_I(\bar{u}))$ denotes the width of the support layer of I orthogonal to a vector $\bar{u}$. The estimates $q(I,B) \geq \frac{2v_{n-1}}{nv_n}$, $Q(I,B) \leq \frac{2v_{n-1}}{v_n}$ were obtained in [5, p. 196]. As shown in Remark 3, the statements on their optimality given in [5, p. 196] differ from our results.

Key words: isoperimetrix, Minkowski geometry, support layer.

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The article is devoted to the Minkowski geometry. Let us recall some basic concepts.

A *convex body* in the n-dimensional Euclidean space R^n, $n \geq 2$, is a convex compact set with a nonempty interior. Let B be a convex body in R^n, and o be a point inside B. For an arbitrary point $x \neq o$, consider a ray in R^n starting from o and passing through x. Denote by x_0 a point at which this ray intersects the boundary of B.

H. Minkowski [1, p. 26] introduces a distance function in R^n by the formula $g(\bar{x}) = \frac{|\bar{x}|}{|\bar{x}_0|}$, $g(\bar{o}) = 0$, where $\bar{x}$ is a position vector of x with respect to o.

Using the pair (B, o), H. Minkowski defines a new distance $\rho_B(x, y) = g(\bar{y} - \bar{x})$ evaluated for any ordered pair of points x, y in R^n [1, p. 28] and shows that this distance in R^n generically possesses all the standard properties of the metric except the property of symmetry.

The geometry of the space obtained by equipping R^n with the new distance $\rho_B(x, y)$ is called *the Minkowski geometry*. The body B is called *the unit ball*, the boundary of B is called *the etalon surface of measure* in the Minkowski geometry [1, p. 28].

Given a compact A, its surface area $S(A, B)$ is defined in the relative differential geometry, corresponding to the Minkowski geometry, by the following formula [1, p. 80]:

$$S(A, B) = \lim_{\lambda \to 0} \frac{V(A + \lambda B) - V(A)}{\lambda} = nV_1(A, B), \tag{1}$$

here $A + \lambda B$, $\lambda \geq 0$ stands for a linear combination of A and B in the sense of Minkowski, $V(A)$ is the volume of A in R^n, $V_1(A, B)$ denotes the first mixed volume of A and B in R^n.

Clearly, the equality $\rho_B(x, y) = \rho_B(y, x)$ holds for any pair of points x, $y \in R^n$ if and only if B is centrally symmetric and o is the center of symmetry of B. In this case $\rho_B(x, y)$ possesses all the standard properties of the metric. Such a measure is used by H. Minkowski in the number theory [1, p. 28].

Now let B be a centrally symmetric convex body in the n-dimensional affine space ($n \geq 2$), and o be the center of symmetry of B. Consider a Minkowski metric in A^n setting $\rho_B(x, y) = g(\bar{y} - \bar{x})$, where $g(\bar{x}) = \frac{\bar{x}}{\bar{x}_0}$. Thus, we obtain an n-dimensional Minkowski space M^n, $n \geq 2$ [2, p. 114] with the unit ball B.

Introduce an auxiliary Euclidean structure in M^n. Fix a coordinate system in M^n, whose origin point coincides with o, and fix an inner product in M^n with the help of some positive determined symmetric bilinear form. This inner product generates an auxiliary Euclidean metric in M^n.

Following the H. Busemann ideas [2, p. 278], for an arbitrary convex compact A, which belongs to an m-dimensional plane M^m, $1 \leq m \leq n$, define its m-dimensional volume $V_m^B(A)$ in M^n by the following formula:

$$V_m^B(A) = \frac{V_m(A)}{V_m(B \cap M_0^m)} \, \upsilon_m, \tag{2}$$

where V_m stands for the m-dimensional Lebesgue measure with respect to the auxiliary Euclidean metric, M_0^m is the plane passing through o which is parallel to M^m, and υ_m denotes the volume of the standard Euclidean unit ball in R^m.

It follows from (2) that $V_n^B(B) = \upsilon_n$. If the auxiliary Euclidean metric in M^n is normalized in such a way that

$$V_n(B) = \upsilon_n, \tag{3}$$

then the volume $V_n^B(A)$ is equal to the Euclidean volume $V_n(A)$ for any convex compact A in M^n.

To define the surface area $S^B(A)$ of a convex compact A in M^n, H. Busemann considers a convex body I in M^n, namely the isoperimetrix of M^n. This body is defined via its support function in the auxiliary Euclidean metric satisfying (3). The support function $h_I(\bar{u})$ of I is represented by the formula

$$h_I(\bar{u}) = \frac{\upsilon_{n-1}}{V_{n-1}(B \cap T_0(\bar{u}))}, \quad \bar{u} \in \Omega, \tag{4}$$

where Ω is the unit (with respect to the auxiliary Euclidean metric) sphere in R^n with the center o, $T_0(\bar{u})$ is the hyperplane passing through o orthogonal to $\bar{u}$ [2, p. 180].

In [2, p. 182], H. Busemann shows that if $m = n-1$, then by (2) the surface area $S^B(A)$ of any compact A in M^n with respect to the auxiliary Euclidean metric verifying (3) satisfies the following equality:

$$S^B(A) = nV_1(A, I). \tag{5}$$

It is also shown in [2, p. 279] that I depends only on the unit ball B of M^n and does not depend on the auxiliary Euclidean metric satisfying (3).

Given a convex body A in M^n, let T_A be an arbitrary support hyperplane of A, T'_A be a support hyperplane of A parallel to T_A and not coinciding with T_A. The point set $Q(T_A) = \overline{T_A} \cap \overline{T'_A}$, where $\overline{T}_A$ denotes the closed support half-space of A bounded by T_A, is called *the support layer* corresponding to T_A. The width of $Q(T_A)$ is defined by the formula $\Delta_B(Q(T_A)) = 2q(Q(T_A), B)$, where $q(Q(T_A), B)$ stands for the capacity coefficient of B with respect to $Q(T_A)$, i.e. such a maximum of α that αB can be placed into $Q(T_A)$ by some translation. If M^n is provided with some auxiliary Euclidean metric, then it is natural to denote the support layer $Q(T_A)$ by $Q_A(\bar{u})$, where $\bar{u} \in \Omega$ is a unit vector orthogonal to the support hyperplanes T_A and T'_A of the body A.

The following statements on the support layers width were obtained in [3, pp. 390, 391].

Theorem 1. *If $\bar{u} \in \Omega$ is an arbitrary unit vector in M^n with an arbitrary auxiliary Euclidean metric, then the width $\Delta_B(Q_A(\bar{u}))$ of the corresponding support layer $Q_A(\bar{u})$ holds the equation*

$$\Delta_B(Q_A(\bar{u})) = 2q(Q_A(\bar{u}), B) = 2\frac{h_A(\bar{u}) + h_A(-\bar{u})}{h_B(\bar{u}) + h_B(-\bar{u})}, \tag{6}$$

where $h_A(\bar{u})$ is a support value of A corresponding to $\bar{u}$ in the used Euclidean metric.

R e m a r k 1. It is shown in [3, p. 390] that the right-hand side of (6) is continuous and it has a maximum and a minimum in Ω. The maximal value of (6) denoted by $D_B(A)$ is called *the diameter* of the body A in M^n. The minimal value of (6) denoted by $\Delta_B(A)$ is called *the width* of the body A in M^n. The diameter $D_B(A)$ is proved [4, p. 220] to be the maximal distance between points of A in M^n.

R e m a r k 2. Since B and I are centrally symmetric convex bodies in M^n, with point o being a joint center of symmetry, then it follows from Theorem 1 that for any choice of the auxiliary Euclidean metric in M^n the width of support layers of the isoperimetrix I is represented by the following formula:

$$\Delta_B(Q_I(\bar{u})) = 2\frac{h_I(\bar{u})}{h_B(\bar{u})}.$$

Theorem 2. *The support layer width $\Delta_B(Q_I(\bar{u}))$ of the isoperimetrix I in M^n satisfies the following estimates:*

$$\frac{4\upsilon_{n-1}}{n\upsilon_n} \leq 2\frac{h_I(\bar{u})}{h_B(\bar{u})} \leq \frac{4\upsilon_{n-1}}{\upsilon_n}. \tag{7}$$

Theorem 3. *The width $\Delta_B(I)$ and the diameter $D_B(I)$ of the isoperimetrix I in M^n satisfy the following estimates:*

$$\frac{4\upsilon_{n-1}}{n\upsilon_n} \leq \Delta_B(I) \leq D_B(I) \leq \frac{4\upsilon_{n-1}}{\upsilon_n}. \tag{8}$$

Let $q(I,B)$ denote the capacity coefficient of the unit ball B with respect to the isoperimetrix I in M^n, i.e. such a maximal α that the body αB can be placed into I by some translation. Let $Q(I,B)$ denote the inclusion coefficient of the isoperimetrix I with respect to the unit ball B, i.e. such a minimal β that I can be placed into βB by some translation.

The aim of this article is to prove the following results.

Theorem 4. *The following equalities hold:*

$$\Delta_B(I) = 2q(I,B), \quad D_B(I) = 2Q(I,B).$$

Theorem 5. *Let M^n be provided with an auxiliary Euclidean metric satisfying (3). Then the inequality*

$$\frac{2\upsilon_{n-1}}{n\upsilon_n} \leq \frac{h_I(\bar{u})}{h_B(\bar{u})} \tag{9}$$

turns into equality for the unit ball B and a vector $\bar{u}_0 \in \Omega$ in the space M^n if and only if the Schwarz symmetrization [2, p. 224] of the unit ball B with respect to a straight line with direction $\bar{u}_0 \in \Omega$ results into a right circular bicone.

Theorem 6. *Let M^n be provided with an auxiliary Euclidean metric satisfying (3). Then the inequalities*

$$\frac{4\upsilon_{n-1}}{n\upsilon_n} \leq \Delta_B(I), \tag{10}$$

$$\frac{2\upsilon_{n-1}}{n\upsilon_n} \leq q(I, B) \tag{11}$$

turn into equalities for the unit ball B of M^n if and only if there exists a unit vector $\bar{u}_0 \in \Omega$ in M^n such that the Schwarz symmetrization of the unit ball B with respect to a straight line with direction $\bar{u}_0$ results into a right circular bicone.

Theorem 7. *Let M^n be provided with an auxiliary Euclidean metric satisfying (3). Then the inequality*

$$\frac{h_I(\bar{u})}{h_B(\bar{u})} \leq \frac{2\upsilon_{n-1}}{\upsilon_n} \tag{12}$$

turns into equality for the unit ball B and a vector $\bar{u}_0 \in \Omega$ in the space M^n if and only if the Schwarz symmetrization of the unit ball B with respect to a straight line with direction $\bar{u}_0 \in \Omega$ results into a right circular cylinder.

Theorem 8. *Let M^n be provided with an auxiliary Euclidean metric satisfying (3). Then the inequalities*

$$D_B(I) \leq \frac{4\upsilon_{n-1}}{\upsilon_n}, \tag{13}$$

$$Q(I, B) \leq \frac{2\upsilon_{n-1}}{\upsilon_n} \tag{14}$$

turn into equalities for the unit ball B of the space M^n if and only if there exists a unit vector $\bar{u}_0 \in \Omega$ in M^n such that the Schwarz symmetrization of the unit ball B with respect to a straight line with direction $\bar{u}_0 \in \Omega$ results into a right circular cylinder.

Theorem 9. *If the unit ball B of M^n, $n \geq 3$, is a parallelepiped and $V(B) = \upsilon_n$, then*

$$\Delta_B(I) > \frac{2\upsilon_{n-1}}{n\upsilon_n}.$$

Theorem 10. *If the unit ball B of M^n is centrally symmetric, $V_n(B) = \upsilon_n$, and A is a convex body in M^n, then*

$$\frac{2\upsilon_{n-1}}{n\upsilon_n} \leq \frac{S^B(A)}{S(A, B)} \leq \frac{2\upsilon_{n-1}}{\upsilon_n}, \tag{15}$$

$$2\upsilon_{n-1} \leq S^B(B) \leq 2n\upsilon_n. \tag{16}$$

R e m a r k 3. Let B be a unit ball of M^n in the sense of H. Busemann. In [5, p. 190], a convex body whose support function with respect to an auxiliary Euclidean metric has the form (4) is denoted by I_B and the isoperimetrix of M^n which satisfies $\widehat{I}_B = \frac{V(B)}{\upsilon_n} I_B$ is denoted by $\widehat{I}_B$. If the assumption (3) is satisfied, then $\widehat{I}_B$ just coincides with the isoperimetrix I of M^n as viewed in our article.

In [5, p. 190], for an arbitrary convex body K in M^n, an intrinsic radius $r(K)$ and an extrinsic radius $R(K)$ of K in M^n are defined as follows:

$$r(K) = \max\left\{\alpha : \exists\, x \in M^n \text{ such that } \alpha\widehat{I}_B \subseteq K + x\right\},$$

$$R(K) = \min\left\{\alpha : \exists\, x \in M^n \text{ such that } \alpha\widehat{I}_B \supseteq K + x\right\}.$$

Following our notations, one has $r(K) = q(K, I)$, $R(K) = Q(K, I)$.

Since the equality $q(A, C) = \frac{1}{Q(C,A)}$ holds for both convex bodies A and C in R^n [6, p.102], then

$$r(K) = \frac{1}{Q(I, K)},$$

$$R(K) = \frac{1}{q(I, K)}.$$

The following estimates for the intrinsic and extrinsic radiuses of B in M^n were obtained in [5, p. 196]:

$$r(B) \geq \frac{\upsilon_n}{2\upsilon_{n-1}}, \tag{17}$$

$$R(B) \leq \frac{n\upsilon_n}{2\upsilon_{n-1}}. \tag{18}$$

Comparing our results with those obtained in [5], one can see that the estimate (17) is equivalent to (14) since $r(B) = \frac{1}{Q(I,B)}$. Moreover, the estimate (18) is equivalent to (11) since $R(B) = \frac{1}{q(I,B)}$.

It is stated in [5, p. 196] that the estimate (17) is not optimal, whereas the estimate (18) turns into equality if and only if B is a parallelotop.

In fact, the estimate (17) is optimal since it is equivalent to (14). The conditions necessary and sufficient for the estimate (14) (as well as for (17)) to turn into equality are given in Theorem 8. For example, the equality in (17) holds if B is a cube in M^n and $\bar{u}_0$ is orthogonal to a cube face.

The estimate (18) is actually optimal. However, if B is a parallelepiped, as well as a parallelotop, then it follows from Theorem 9 that the estimate (18) does not turn into equality in the case of $n \geq 3$. The conditions necessary and sufficient for the estimate (18) to turn into equality are given in Theorem 6. For example, the estimate (18) turns into equality if B is a right circular bicone in M^n and $\bar{u}_0$ is orthogonal to the base of bicone.

P r o o f o f T h e o r e m 4. Let us show that

$$\Delta_B(I) \leq 2q(I, B). \tag{19}$$

From the definition of $\Delta_B(I)$ we have $\Delta_B(I) = \min\limits_{\bar{u}\in\Omega} 2\frac{h_I(\bar{u})}{h_B(\bar{u})}$. Therefore, $\Delta_B(I) \leq 2\frac{h_I(\bar{u})}{h_B(\bar{u})}$ for any $\bar{u} \in \Omega$. Hence, $\frac{1}{2}\Delta_B(I)h_B(\bar{u}) \leq h_I(\bar{u})$ for any $\bar{u} \in \Omega$. Taking into account the properties of the support functions (see [1, p. 30]), we obtain $\frac{1}{2}\Delta_B(I)B \subset I$. Thus, $\frac{1}{2}\Delta_B(I) \leq q(I, B)$.

Now let us take $q = q(I, B)$ and place qB into I by some translation, so $qB_1 \subset I$. Then the inequality

$$\Delta_B(Q_{qB_1}(\bar{u})) \leq \Delta_B(Q_I(\bar{u})) \tag{20}$$

holds for any $\bar{u} \in \Omega$. Since the width of the support layer orthogonal to $\bar{u} \in \Omega$ is constant under translation, then it follows from (20) that the inequality

$$2q \leq 2\frac{h_I(\bar{u})}{h_B(\bar{u})} \tag{21}$$

holds, here $2q$ is the width of the support layer of qB orthogonal to $\bar{u}$, as well $2\frac{h_I(\bar{u})}{h_B(\bar{u})}$ stands for the width of the support layer of I orthogonal to $\bar{u}$ in M^n. Finally, (21) leads to the following inequality:

$$2q(I, B) \leq \min_{\bar{u}\in\Omega} 2\frac{h_I(\bar{u})}{h_B(\bar{u})} = \Delta_B(I).$$

This inequality and (19) imply the desired equality stated in Theorem 4, q.e.d.

P r o o f o f T h e o r e m 5. Let M^n be provided with an auxiliary Euclidean metric satisfying (3). Chose an orthonormal vector basis and denote by $ox_1 \ldots x_n$ the corresponding Cartesian coordinate system whose origin o is the center of symmetry of B. The hyperplane $x_n = 0$ divides B into two parts $B_1 = B \cap (x_n \geq 0)$ and $B_2 = B \cap (x_n \leq 0)$. Clearly, $V(B_1) = V(B_2) = \frac{v_n}{2}$. Let $[-b, b]$, $b > 0$, be the image of B under the orthogonal projection into ox_n. Then $h_B(\bar{u}) = h_{B_1}(\bar{u}) = b$ for the unit vector $\bar{u} \in \Omega$ along the axis ox_n.

Apply to B the Schwarz symmetrization [2, p. 224] with respect to the axis ox_n and denote the result by $\tilde{B}$. The orthogonal projection of $\tilde{B}$ onto the axis ox_n is the segment $[-b, b]$. The intersection $\tilde{B} \cap (x_n = c)$, $-b \leq c \leq b$, represents an $(n-1)$-dimensional ball centered at the point $x_n = c$ on the axis ox_n, its $(n-1)$-dimensional volume is equal to $V_{n-1}(\tilde{B} \cap (x_n = c)) = V_{n-1}(B \cap (x_n = c))$.

The bodies $\tilde{B}_1 = \tilde{B} \cap (x_n \geq 0)$ and $\tilde{B}_2 = \tilde{B} \cap (x_n \leq 0)$ are obtained by the symmetrization of the bodies B_1 and B_2, respectively. Moreover, one has $V(\tilde{B}) = V(B) = \upsilon_n$, $V(\tilde{B}_1) = V(B_1) = \frac{\upsilon_n}{2}$, $h_B(\bar{u}) = h_{B_1}(\bar{u}) = h_{\tilde{B}_1}(\bar{u}) = h_{\tilde{B}}(\bar{u}) = b$.

It is shown in [3, p. 392] that $V_{n-1}(B \cap (x_n = \alpha)) = V_{n-1}(B \cap (x_n = -\alpha)) \leq V_{n-1}(B \cap (x_n = 0))$, $0 \leq \alpha \leq b$. Hence, $\tilde{B}$ is symmetric with respect to the hyperplane $x_n = 0$.

Given B_1 and $\tilde{B}_1$, consider the right cone K whose base is $\tilde{B}_1 \cap (x_n = 0))$ and whose height is the segment $[0, b]$ of the axis ox_n.

Let us show that the following equality holds:

$$\frac{h_I(\bar{u})}{h_B(\bar{u})} V(K) = \frac{\upsilon_{n-1}}{n}. \tag{22}$$

Express $V(K)$ via $\frac{h_I(\bar{u})}{h_B(\bar{u})}$ by using (4). One has

$$V(K) = \frac{1}{n} b V_{n-1}(\tilde{B}_1 \cap (x_n = 0)) = \frac{h_B(\bar{u})\upsilon_{n-1}}{\dfrac{n\upsilon_{n-1}}{V_{n-1}(B \cap (x_n = 0))}} = \frac{h_B(\bar{u})\upsilon_{n-1}}{n h_I(\bar{u})},$$

which leads to (22).

Now, if (9) turns into equality for some $\bar{u} = \bar{u}_0$, i.e., $\frac{h_I(\bar{u}_0)}{h_B(\bar{u}_0)} = \frac{2\upsilon_{n-1}}{n\upsilon_n}$, then (22) leads to $V(K) = \frac{\upsilon_n}{2}$. Hence, $V(K) = V(\tilde{B}_1)$. Since $K \subset \tilde{B}_1$ and the interior of K is nonempty, then $K = \tilde{B}_1$, so $\tilde{B}$ is a right circular bicone.

On the other hand, if the Schwarz symmetrization of B with respect to the straight line parallel to the vector $\bar{u} = \bar{u}_0$ results in a bicone, then $V(K) = V(\tilde{B}_1) = \frac{\upsilon_n}{2}$. Hence it follows from (22) that (9) turns into equality for $\bar{u} = \bar{u}_0$, q.e.d.

P r o o f o f T h e o r e m 6. By definition, the width $\Delta_B(I)$ of the isoperimetrix I is equal to $\Delta_B(I) = \min\limits_{\bar{u} \in \Omega} 2\frac{h_I(\bar{u})}{h_B(\bar{u})}$.

It follows from Remark 1 that $\frac{h_I(\bar{u})}{h_B(\bar{u})}$ is defined and continuous in Ω. Hence it attains its minimum at some $\bar{u}_0 \in \Omega$. Then the equality $\frac{4\upsilon_{n-1}}{n\upsilon_n} = \Delta_B(I)$ is equivalent to the equality $\frac{2\upsilon_{n-1}}{n\upsilon_n} = \frac{h_I(\bar{u}_0)}{h_B(\bar{u}_0)}$. Now the statement of Theorem 6 about the optimality of (10) follows directly from Theorem 5. As for the optimality of (11), Theorem 6 follows from Theorem 4.

P r o o f o f T h e o r e m 7. Given B_1 and $\tilde{B}_1$, consider the right circular cylinder Π, whose base is $\tilde{B}_1 \cap (x_n = 0)$ and whose height is the segment $[0, b]$ of the axis ox_n. Since Π and K have the same base and height, one has $V(\Pi) = nV(K)$. Then (22) leads to the following equality:

$$\frac{h_I(\bar{u})}{h_B(\bar{u})} V(\Pi) = \upsilon_{n-1}. \tag{23}$$

If (12) turns into equality for some $\bar u = \bar u_0 \in \Omega$, i.e., $\frac{h_I(\bar u_0)}{h_B(\bar u_0)} = \frac{2\upsilon_{n-1}}{\upsilon_n}$, then it follows from (23) that $V(\Pi) = \frac{\upsilon_n}{2}$. Hence, $V(\Pi) = V(\tilde B_1)$. Since $\tilde B_1 \subset \Pi$ and the interior of $\tilde B_1$ is nonempty, then $\tilde B_1 = \Pi$. So, $\tilde B_1$ is a right circular cylinder and consequently $\tilde B$ is a right circular cylinder.

On the other hand, if $\tilde B_1$ is a right circular cylinder, then $V(\Pi) = V(\tilde B_1)$. Hence $V(\tilde B_1) = V(\Pi) = \frac{\upsilon_n}{2}$, so (23) leads to $\frac{h_I(\bar u_0)}{h_B(\bar u_0)} = \frac{2\upsilon_{n-1}}{\upsilon_n}$, q.e.d.

P r o o f o f T h e o r e m 8 is similar to the proof of Theorem 6.

P r o o f o f T h e o r e m 9. Fix a coordinate system so that the origin o is a vertex of the parallelepiped B and the basis vectors are directed along the edges of B erected at o. Instead of the initial auxiliary Euclidean metric satisfying (3), when B is viewed as a a parallelepiped, let us consider a new auxiliary Euclidean metric satisfying (3), which views B as a cube. It is shown in [2, p. 279] that the changing of metrics does not change the isoperimetrix, i.e. the isoperimetrix I of the parallelepiped coincides with the isoperimetrix I of the cube.

The edges of the cube B erected at o are pairwise orthogonal, they have the same length α, moreover, $\alpha^n = \upsilon_n$. Consider a new Cartesian coordinate system, whose origin o_1 is the center of symmetry of the cube B and whose axes are parallel to the edges of B. Evidently, the most distant points of B with respect to o_1 are the vertices of B. Hence for any $\bar u \in \Omega$ one has $h_B(\bar u) \le \frac{\alpha\sqrt{n}}{2}$, where $\alpha\sqrt{n}$ denotes the diagonal of cube.

In [7], K. Ball proved the maximal $(n-1)$-dimensional volume for the intersections of the unit cube with hyperplanes in R^n $(n \ge 2)$ to be equal to $\sqrt{2}$. Therefore, $V_{n-1}(B \cap T_{o_1}(\bar u)) \le \sqrt{2}\alpha^{n-1}$ for any $\bar u \in \Omega$. Hence, for the width $\Delta_B(Q_I(\bar u)) = 2\frac{h_I(\bar u)}{h_B(\bar u)}$, $\bar u \in \Omega$, of support layers of the isoperimetrix I of M^n, $n \ge 3$, one has the following estimate:

$$2\frac{h_I(\bar u)}{h_B(\bar u)} = 2\frac{\upsilon_{n-1}}{h_B(\bar u)\, V_{n-1}(B \cap T_{o_1}(\bar u))} \ge \frac{2\upsilon_{n-1}}{\frac{\sqrt{n}}{2}\alpha\sqrt{2}\alpha^{n-1}} = \frac{4\upsilon_{n-1}}{\sqrt{2n}\alpha^n} > \frac{4\upsilon_{n-1}}{n\upsilon_n},$$

since $\sqrt{2n} < n$ for $n \ge 3$. By the continuity of $\frac{h_I(\bar u)}{h_B(\bar u)}$ at $\bar u \in \Omega$, one has $\Delta_B(I) = \min\limits_{\bar u \in \Omega} 2\frac{h_I(\bar u)}{h_B(\bar u)} > \frac{4\upsilon_{n-1}}{n\upsilon_n}$ for $n \ge 3$.

P r o o f o f T h e o r e m 10. Rewrite the inequalities (7) as follows:

$$\frac{2\upsilon_{n-1}}{n\upsilon_n} h_B(\bar u) \le h_I(\bar u) \le \frac{2\upsilon_{n-1}}{\upsilon_n} h_B(\bar u).$$

The above and the properties of support functions [1, p. 30] result

$$\frac{2\upsilon_{n-1}}{n\upsilon_n} B \subset I \subset \frac{2\upsilon_{n-1}}{\upsilon_n} B. \tag{24}$$

Let us use an auxiliary Euclidean metric satisfying (3) in M^n. Due to the homogeneity and monotonicity of the mixed volume [1, p. 49], the inclusions (24) lead to the following inequalities:

$$\frac{2\upsilon_{n-1}}{n\upsilon_n} V_1(A,B) \le V_1(A,I) \le \frac{2\upsilon_{n-1}}{\upsilon_n} V_1(A,B). \tag{25}$$

Then take into account (1) and (5) to see that the estimates (15) for $\frac{S^B(A)}{S(A,B)}$ hold for an arbitrary centrally symmetric convex body B and an arbitrary convex body A in M^n.

Set $A = B$ in (15). Then $S(B,B) = nV_1(B,B) = nV(B) = n\upsilon_n$, here $S^B(B)$ is the surface area of the unit ball B in M^n in the sense of H. Busemann (5). Substituting these expressions into (15), one obtains the following known Busemann–Petty estimates for $S^B(B)$ in M^n [8, p. 242]:

$$2\upsilon_{n-1} \le S^B(B) \le 2n\upsilon_{n-1}.$$

The right-hand side estimate turns into equality if and only if B is a parallelotop [8, p. 242].

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