

Good Measures on Locally Compact Cantor Sets

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Received March 28, 2012

We study the set $M(X)$ of full non-atomic Borel measures μ on a non-compact locally compact Cantor set X . The set $\mathfrak{M}_\mu = \{x \in X : \text{for any compact open set } U \ni x \text{ we have } \mu(U) = \infty\}$ is called defective. μ is non-defective if $\mu(\mathfrak{M}_\mu) = 0$. The set $M^0(X) \subset M(X)$ consists of probability and infinite non-defective measures. We classify the measures from $M^0(X)$ with respect to a homeomorphism. The notions of goodness and the compact open values set $S(\mu)$ are defined. A criterion when two good measures are homeomorphic is given. For a group-like set D and a locally compact zero-dimensional metric space A we find a good non-defective measure μ on X such that $S(\mu) = D$ and \mathfrak{M}_μ is homeomorphic to A . We give a criterion when a good measure on X can be extended to a good measure on the compactification of X .

Key words: Borel measures, locally compact Cantor set, compactification, invariant measures.

Mathematics Subject Classification 2000: 37A05, 37B05 (primary); 28D05, 28C15 (secondary).

1. Introduction

The problem of classification of Borel finite or infinite measures on topological spaces has a long history. Two measures μ and ν defined on the Borel subsets of a topological space X are called *homeomorphic* if there exists a self-homeomorphism h of X such that $\mu = \nu \circ h$, i.e., $\mu(E) = \nu(h(E))$ for every Borel subset E of X . The topological properties of the space X are important for the classification of measures up to a homeomorphism. For instance, Oxtoby and Ulam [1] gave a criterion for a Borel probability measure on the finite-dimensional cube to be homeomorphic to the Lebesgue measure. Similar results were obtained for various manifolds (see [2, 3]).

A Cantor set (or a Cantor space) is a non-empty zero-dimensional compact perfect metric space. For Cantor sets the situation is much more complicated than for connected spaces. During the last decade, in the papers [4–8] Borel probability measures on Cantor sets were studied. In [9] infinite Borel measures on Cantor sets were considered. For many applications in dynamical systems the state space is only locally compact. In this paper, we study Borel both finite and infinite measures on non-compact locally compact Cantor sets.

It is possible to construct uncountably many full (the measure of every non-empty open set is positive) non-atomic measures on a Cantor set X which are pairwise non-homeomorphic (see [10]). This is due to the existence of a countable base of clopen subsets of a Cantor set. The *clopen values set* $S(\mu)$ is the set of finite values of a measure μ on all clopen subsets of X . The set provides an invariant for homeomorphic measures, although it is not a complete invariant.

For the class of the so-called *good* probability measures, $S(\mu)$ is a complete invariant. By definition, a full non-atomic probability or non-defective measure μ is good if, whenever U, V are clopen sets with $\mu(U) < \mu(V)$, there exists a clopen subset W of V such that $\mu(W) = \mu(U)$ (see [9, 11]). Good probability measures are exactly invariant measures of uniquely ergodic minimal homeomorphisms of Cantor sets (see [11, 12]). For an infinite Borel measure μ on a Cantor set X , denote by \mathfrak{M}_μ the set of all points in X whose clopen neighbourhoods have only infinite measures. The full non-atomic infinite measures μ such that $\mu(\mathfrak{M}_\mu) = 0$ are called *non-defective*. These measures arise as ergodic invariant measures for homeomorphisms of a Cantor set and the theory of good probability measures can be extended to the case of non-defective measures (see [9]).

In Sec. 2, we define a good probability measure and a good non-defective measure on a non-compact locally compact Cantor set X and extend the results concerning good measures on Cantor sets to non-compact locally compact Cantor sets. For a Borel measure μ on X , the set $S(\mu)$ is defined as a set of all finite values of μ on the compact open sets. The defective set \mathfrak{M}_μ is the set of all points x in X such that every compact open neighbourhood of x has an infinite measure. We prove the criterion when two good measures on non-compact locally compact Cantor sets are homeomorphic. For every group-like subset $D \subset [0, 1)$ we find a good probability measure μ on a non-compact locally compact Cantor set such that $S(\mu) = D$. For every group-like subset $D \subset [0, \infty)$ and any locally compact zero-dimensional metric space A (including $A = \emptyset$) we find a good non-defective measure μ on a non-compact locally compact Cantor set such that $S(\mu) = D$ and \mathfrak{M}_μ is homeomorphic to A .

In Sec. 3, the compactifications of non-compact locally compact Cantor sets are studied. We investigate whether a compactification can be used to classify the measures on non-compact locally compact Cantor sets. We consider only the compactifications which are Cantor sets and extend the measure μ by giving

the remainder of compactification a zero measure. It turns out that in some cases a good measure can be extended to a good measure on a Cantor set, while in other cases the extension always produces a measure which is not good. The extensions of a non-good measure are always non-good. After compactification of a non-compact locally compact Cantor set, new compact open sets are obtained. We study how the compact open values set changes. Basing on this study, we give a criterion when a good measure on a non-compact locally compact Cantor set remains good after the compactification.

Section 4 contains the examples to the results of Secs. 2 and 3. For instance, the Haar measure on the set of p -adic numbers and the invariant measure for (C, F) -construction are good. We give the examples of good ergodic invariant measures on the generating open dense subset of a path space of stationary Bratteli diagrams such that any compactification gives a non-good measure.

2. Measures on Locally Compact Cantor Sets

Let X be a non-compact locally compact metrizable space with no isolated points and with a (countable) basis of compact and open sets. Hence X is totally disconnected. The set X is called a *non-compact locally compact Cantor set*. Every two non-compact locally compact Cantor sets are homeomorphic (see [13]). Take a countable family of the compact open subsets $O_n \subset X$ such that $X = \bigcup_{n=1}^{\infty} O_n$. Denote $X_1 = O_1$, $X_2 = O_2 \setminus O_1$, $X_3 = O_3 \setminus (O_1 \cup O_2), \dots$. The subsets X_n are compact open pairwise disjoint, and $X = \bigcup_{n=1}^{\infty} X_n$. Since X is non-compact, we may assume without loss of generality that all X_n are non-empty. Since X has no isolated points, every X_n has the same property. Thus, we represent X as a disjoint union of a countable family of the compact Cantor sets X_n .

Recall that a Borel measure on a locally compact Cantor space is called *full* if every non-empty open set has a positive measure. It is easy to see that for a non-atomic measure μ the support of μ in the induced topology is a locally compact Cantor set. We can consider the measures on their supports to obtain full measures. Denote by $M(X)$ the set of full non-atomic Borel measures on X . Then $M(X) = M_f(X) \sqcup M_{\infty}(X)$, where $M_f(X) = \{\mu \in M(X) : \mu(X) < \infty\}$ and $M_{\infty}(X) = \{\mu \in M(X) : \mu(X) = \infty\}$. For a measure $\mu \in M_{\infty}(X)$, denote $\mathfrak{M}_{\mu} = \{x \in X : \text{for any compact and open set } U \ni x \text{ we have } \mu(U) = \infty\}$. It will be shown that \mathfrak{M}_{μ} is a Borel set. Denote $M_{\infty}^0(X) = \{\mu \in M_{\infty}(X) : \mu(\mathfrak{M}_{\mu}) = 0\}$. Let $M^0(X) = M_f(X) \sqcup M_{\infty}^0(X)$. Throughout the paper we will consider only the measures from $M^0(X)$. We normalize the measures from $M_f(X)$ such that $\mu(X) = 1$ for any $\mu \in M_f(X)$.

Recall that $\mu \in M^0(X)$ is *locally finite* if every point of X has a neighbourhood of a finite measure. The properties of the measures from the class $M^0(X)$ are collected in the following proposition.

Proposition 2.1. *Let $\mu \in M^0(X)$. Then*

- (1) *The measure μ is locally finite if and only if $\mathfrak{M}_\mu = \emptyset$.*
- (2) *The set $X \setminus \mathfrak{M}_\mu$ is open. The set \mathfrak{M}_μ is F_σ .*
- (3) *For any compact open set U with $\mu(U) = \infty$ and any $a > 0$ there exists a compact open subset $V \subset U$ such that $a \leq \mu(V) < \infty$.*
- (4) *The set \mathfrak{M}_μ is nowhere dense.*
- (5) *$X = \bigsqcup_{i=1}^{\infty} V_i \sqcup \mathfrak{M}_\mu$, where each V_i is a compact open set of a finite measure and \mathfrak{M}_μ is a nowhere dense F_σ and is of zero measure. The measure μ is σ -finite.*
- (6) *μ is uniquely determined by its values on the algebra of the compact open sets.*

P r o o f. (1) The condition $\mathfrak{M}_\mu = \emptyset$ means that every point $x \in X$ has a compact open neighbourhood of a finite measure. Hence μ is locally finite and vice versa.

(2) We have $X \setminus \mathfrak{M}_\mu = \{x \in X : \text{there exists a compact open set } U_x \ni x \text{ such that } \mu(U_x) < \infty\}$. Then for every point $x \in X \setminus \mathfrak{M}_\mu$ we have $U_x \subset X \setminus \mathfrak{M}_\mu$. Hence $X \setminus \mathfrak{M}_\mu$ is open. Therefore, for every $n \in \mathbb{N}$ the set $X_n \setminus \mathfrak{M}_\mu$ is open and $X_n \cap \mathfrak{M}_\mu$ is closed. Then $\mathfrak{M}_\mu = \bigsqcup_{n \in \mathbb{N}} (X_n \cap \mathfrak{M}_\mu)$ is F_σ set.

(3) Let U be a non-empty compact open subset of X such that $\mu(U) = \infty$. Since $\mu \in M^0(X)$, we have $\mu(U) = \mu(U \setminus \mathfrak{M}_\mu)$. Since U is open, the set $U \setminus \mathfrak{M}_\mu = U \cap (X \setminus \mathfrak{M}_\mu)$ is open. There are only countably many compact open subsets in X , hence the open set $U \setminus \mathfrak{M}_\mu$ can be represented as a disjoint union of the compact open subsets $\{U_i\}_{i \in \mathbb{N}}$ of a finite measure. We have $\mu(U) = \sum_{i=0}^{\infty} \mu(U_i) = \infty$, hence for every $a \in \mathbb{R}$ there is a compact open subset $V = \bigsqcup_{i=0}^N U_i$ such that $a \leq \mu(V) < \infty$.

(4) Let U be a compact open subset of X . It suffices to show that there exists a non-empty compact open subset $V \subset U$ such that $V \cap \mathfrak{M}_\mu = \emptyset$. If $\mu(U) < \infty$, then $U \cap \mathfrak{M}_\mu = \emptyset$. Otherwise, by (3), there exists a compact open subset $V \subset U$ such that $0 < \mu(V) < \infty$. Obviously, $V \cap \mathfrak{M}_\mu = \emptyset$.

(5) Follows from the proof of (3).

(6) Follows from (5). ■

For a measure $\mu \in M^0(X)$ define the *compact open values set* as the set of all finite values of the measure μ on the compact open sets:

$$S(\mu) = \{\mu(U) : U \text{ is compact open in } X \text{ and } \mu(U) < \infty\}.$$

For each measure $\mu \in M^0(X)$, the set $S(\mu)$ is a countable dense subset of the interval $[0, \mu(X))$. Indeed, the set $S(\mu)$ is dense in $[0, \mu(V)]$ for every compact open set V of a finite measure (see [10]). By Proposition 2.1, $S(\mu)$ is dense in $[0, \mu(X))$.

Let X_1, X_2 be two non-compact locally compact Cantor sets. It is said that the measures $\mu_1 \in M(X_1)$ and $\mu_2 \in M(X_2)$ are *homeomorphic* if there exists a

homeomorphism $h: X_1 \rightarrow X_2$ such that $\mu_1(E) = \mu_2(h(E))$ for every Borel subset $E \subset X_1$. Clearly, $S(\mu_1) = S(\mu_2)$ for any homeomorphic measures μ_1 and μ_2 . We call the two Borel infinite measures $\mu_1 \in M_\infty^0(X_1)$ and $\mu_2 \in M_\infty^0(X_2)$ *weakly homeomorphic* if there exists a homeomorphism $h: X_1 \rightarrow X_2$ and a constant $C > 0$ such that $\mu_1(E) = C\mu_2(h(E))$ for every Borel subset $E \subset X_1$. Then $S(\mu_1) = CS(\mu_2)$.

Let D be a dense countable subset of the interval $[0, a)$ where $a \in (0, \infty]$. Then D is called *group-like* if there exists an additive subgroup G of \mathbb{R} such that $D = G \cap [0, a)$. It is easy to see that D is group-like if and only if for any $\alpha, \beta \in D$ such that $\alpha \leq \beta$ we have $\beta - \alpha \in D$ (see [9, 11]).

Definition 2.2. *Let X be a locally compact Cantor space (either compact or non-compact) and $\mu \in M^0(X)$. A compact open subset V of X is called good for μ (or just good when the measure is understood) if for every compact open subset U of X with $\mu(U) < \mu(V)$ there exists a compact open set W such that $W \subset V$ and $\mu(W) = \mu(U)$. A measure μ is called good if every compact open subset of X is good for μ .*

If $\mu \in M^0(X)$ is a good measure and $\nu \in M^0(X)$ is (weakly) homeomorphic to μ , then, obviously, ν is good. It is easy to see that in the case of a compact Cantor set the definition of a good measure coincides with the one given in [11]. For a compact open subset $U \subset X$, let $\mu|_U$ be the restriction of the measure μ to the Cantor space U . Then the set U is good if and only if $S(\mu|_U) = S(\mu|_X) \cap [0, \mu(U)]$. Denote by $H_\mu(X)$ the group of all homeomorphisms of a space X preserving the measure μ . The action of $H_\mu(X)$ on X is called *transitive* if for every $x_1, x_2 \in X$ there exists $h \in H_\mu(X)$ such that $h(x_1) = x_2$. The action is called *topologically transitive* if there exists a dense orbit, i.e., there is $x \in X$ such that the set $O(x) = \{h(x) : h \in H_\mu(X)\}$ is dense in X .

We extend naturally the notion of partition basis introduced in [4]. A *partition basis* \mathcal{B} for a non-compact locally compact Cantor set X is a collection of the compact open subsets of X such that every non-empty compact open subset of X can be partitioned by the elements of \mathcal{B} .

The properties of good measures on non-compact locally compact Cantor sets are gathered in the following proposition. The proofs for the measures on compact Cantor spaces can be found in [4, 9, 11].

Proposition 2.3. *Let X be a locally compact Cantor space (either compact or non-compact). Let $\mu \in M^0(X)$. Then*

- (a) *If μ is good and $C > 0$, then $C\mu$ is good and $S(C\mu) = CS(\mu)$.*
- (b) *If μ is good and U is a non-empty compact open subset of X , then the measure $\mu|_U$ is good and $S(\mu|_U) = S(\mu) \cap [0, \mu(U)]$.*
- (c) *μ is good if and only if every compact open subset of a finite measure is good.*

(d) μ is good if and only if for every non-empty compact open subset U of a finite measure the measure $\mu|_U$ is good.

(e) If μ is good, then $S(\mu)$ is group-like.

(f) If a compact open set U admits a partition by good compact open subsets, then U is good.

(g) The measure μ is good if and only if there exists a partition basis \mathcal{B} consisting of the compact open sets which are good for μ .

(h) If μ is good, then the group $H_\mu(X)$ acts transitively on $X \setminus \mathfrak{M}_\mu$. In particular, the group $H_\mu(X)$ acts topologically transitively on X .

(i) If μ is a good measure on X and ν is the counting measure on $\{1, 2, \dots, n\}$, then $\mu \times \nu$ is a good measure on $X \times \{1, 2, \dots, n\}$.

P r o o f. (a), (b) are clear.

(c) Suppose that every compact open subset of finite measure is good. Let V be any compact open set with $\mu(V) = \infty$ and U be a compact open set with $\mu(U) < \infty$. By Proposition 2.1, there exists a compact open subset $W \subset V$ such that $\mu(U) \leq \mu(W) < \infty$. By assumption, W is good. Hence there exists a compact open set $W_1 \subset W$ with $\mu(W_1) = \mu(U)$ and V is good.

(d) Suppose that for every non-empty compact open subset U of finite measure, the measure $\mu|_U$ is good. We prove that every compact open subset of finite measure is good, then use (c). Let U, V be compact open sets with $0 < \mu(U) < \mu(V) < \infty$. Set $W = U \cup V$. Then W is a compact open set of finite measure. Since $\mu|_W$ is good, there exists $W_1 \subset V$ such that $\mu(W_1) = \mu(U)$.

(e) If μ is good, then for any $\alpha, \beta \in S(\mu)$ such that $\beta - \alpha \geq 0$ we have $\beta - \alpha \in S(\mu)$. Hence $S(\mu)$ is group-like.

(f) See [4] for the case of the finite measure and [9] for the infinite measure.

(g) If there exists a partition basis \mathcal{B} consisting of compact open sets which are good for μ , then, by (f), every compact open set is good.

(h) For any $x, y \in X \setminus \mathfrak{M}_\mu$ there exists a compact open set U of finite measure such that $x, y \in U$. By (d), the measure $\mu|_U$ is a good finite measure on a Cantor space U . By Theorem 2.13 from [11], there exists a homeomorphism $h: U \rightarrow U$ which preserves μ and $h(x) = y$. Define $h_1 \in H_\mu(X)$ to be h on U and the identity on $X \setminus U$. For every $x \in X \setminus \mathfrak{M}_\mu$ we have $O(x) = X \setminus \mathfrak{M}_\mu$. By Proposition 2.1, the set $X \setminus \mathfrak{M}_\mu$ is dense in X . Hence $H_\mu(X)$ acts topologically transitively on X .

(i) The rectangular compact open sets $U \times \{z\}$, where U is compact open in X and $z \in \{1, 2, \dots, n\}$, form a partition basis for $X \times \{1, 2, \dots, n\}$. Since $\mu \times \nu(U \times \{z\}) = \mu(U)$, these sets are good. The measure μ is good by (g). ■

For G , an additive subgroup of \mathbb{R} we call a positive real number δ a *divisor* of G if $\delta G = G$. The set of all divisors of G is called $Div(G)$. By a full measure on a discrete countable topological space Y we mean a measure ν such that

$0 < \nu(\{y\}) < \infty$ for every $y \in Y$. We will use the following theorem for $Y = \mathbb{Z}$, but the proof remains correct for any discrete countable topological space Y .

Theorem 2.4. *Let μ be a good measure on a non-compact locally compact Cantor space X . Let ν be a full measure on \mathbb{Z} , where \mathbb{Z} is endowed with discrete topology. Let G be an additive subgroup of \mathbb{R} generated by $S(\mu)$. Then $\mu \times \nu$ is good on $X \times \mathbb{Z}$ if and only if there exists $C > 0$ such that $\nu(\{i\}) \in C \cdot Div(G)$ for every $i \in \mathbb{Z}$.*

P r o o f. Let us prove the “if” part. Suppose μ is good on X and $\nu(\{i\}) \in C \cdot Div(G)$ for some $C > 0$ and every $i \in \mathbb{Z}$. By Proposition 2.3 (g), it suffices to prove that a compact open set of the form $U \times \{i\}$ is good for any compact open $U \subset X$ and any $i \in \mathbb{Z}$. Thus, it suffices to show that $S(\mu \times \nu|_{U \times \{i\}}) = S(\mu \times \nu|_{X \times \mathbb{Z}}) \cap [0, \mu \times \nu(U \times \{i\})]$. The inclusion $S(\mu \times \nu|_{U \times \{i\}}) \subset S(\mu \times \nu|_{X \times \mathbb{Z}}) \cap [0, \mu \times \nu(U \times \{i\})]$ is always true, hence we have to prove the inverse inclusion. We have $S(\mu \times \nu|_{U \times \{i\}}) = \nu(\{i\})S(\mu|_U) = C\delta S(\mu|_U)$ for some $\delta \in Div(G)$. Since μ is good on X , we obtain $S(\mu|_U) = G \cap [0, \mu(U)]$. Hence $S(\mu \times \nu|_{U \times \{i\}}) = CG \cap [0, C\delta\mu(U)] = CG \cap [0, \mu \times \nu(U \times \{i\})]$. Note that $C\delta\mu(U) \in CG$ because $\delta \in Div(G)$. Therefore, it suffices to prove that $S(\mu \times \nu|_{X \times \mathbb{Z}}) \subset CG$. The set $S(\mu \times \nu|_{X \times \mathbb{Z}})$ consists of all finite sums $\sum_{i,j} \mu(U_i)\nu(\{j\})$, where each U_i is a compact open set in X and $j \in \mathbb{Z}$. We have $\sum_{i,j} \mu(U_i)\nu(\{j\}) = \sum_{i,j} \mu(U_i)C\delta_j \subset CG$, here $\delta_j \in Div(G)$. Hence $S(\mu \times \nu|_{U \times \{i\}}) \supset S(\mu \times \nu|_{X \times \mathbb{Z}}) \cap [0, \mu \times \nu(U \times \{i\})]$ and $U \times \{i\}$ is good.

Now we prove the “only if part”. Suppose that $\mu \times \nu$ is good on $X \times \mathbb{Z}$. Then for any $i \in \mathbb{Z}$ we have $S(\mu \times \nu|_{X \times \{i\}}) = S(\mu \times \nu|_{X \times \mathbb{Z}}) \cap [0, \mu \times \nu(X \times \{i\})]$. Note that $S(\mu \times \nu|_{X \times \{i\}}) = \nu(\{i\})S(\mu|_X)$. Denote by \tilde{G} the additive subgroup of \mathbb{R} generated by $S(\mu \times \nu|_{X \times \mathbb{Z}})$. Let $\alpha = \nu(\{i\})$. Then $\alpha G = \tilde{G}$. Let $j \in \mathbb{Z}$ and $\beta = \nu(\{j\})$. By the same arguments, we have $\beta G = \tilde{G}$. Then $\frac{\alpha}{\beta} \in Div(G)$. Indeed, $\frac{\alpha}{\beta} G = \frac{1}{\beta} \tilde{G} = G$. Hence $\alpha = \beta\delta$, where $\delta \in Div(G)$. Set $C = \nu(\{j\})$. Then for every $i \in \mathbb{Z}$ we have $\nu(\{i\}) = C\delta_i$, where $\delta_i = \frac{\nu(\{i\})}{\nu(\{j\})} \in Div(G)$. ■

Theorem 2.5. *Let X, Y be non-compact locally compact Cantor sets. If $\mu \in M^0(X), \nu \in M^0(Y)$ are good measures, then the product $\mu \times \nu$ is a good measure on $X \times Y$, and*

$$S(\mu \times \nu) = \left\{ \sum_{i=0}^N \alpha_i \cdot \beta_i : \alpha_i \in S(\mu), \beta_i \in S(\nu), N \in \mathbb{N} \right\} \cap [0, \mu(X) \times \nu(Y)].$$

P r o o f. Let $X = \bigsqcup_{m=1}^{\infty} X_m$ and $Y = \bigsqcup_{n=1}^{\infty} Y_n$, where each X_m, Y_n is a Cantor set. Then $X \times Y = \bigsqcup_{m,n=1}^{\infty} X_m \times Y_n$ and $\mu \times \nu|_{X_m \times Y_n} = \mu|_{X_m} \times \nu|_{Y_n}$. Since $\mu|_{X_m}$ and $\nu|_{Y_n}$ are good finite or non-defective measures on a Cantor set,

the measure $\mu \times \nu|_{X_m \times Y_n}$ is good by Theorem 2.8 [4], Theorem 2.10 [9]. By Proposition 2.3, $\mu \times \nu$ is good on $X \times Y$. ■

Theorem 2.6. *Let X, Y be non-compact locally compact Cantor spaces. Let $\mu \in M^0(X)$ and $\nu \in M^0(Y)$ be good measures. Let $S(\mu) = S(\nu)$. Let \mathfrak{M} be the defective set for μ and \mathfrak{N} be the defective set for ν . Assume that there is a homeomorphism $h: \mathfrak{M} \rightarrow \mathfrak{N}$ where the sets \mathfrak{M} and \mathfrak{N} are endowed with the induced topologies. Then there exists a homeomorphism $\tilde{h}: X \rightarrow Y$ which extends h such that $\mu = \nu \circ \tilde{h}$.*

Conversely, if $\mu \in M^0(X)$ and $\nu \in M^0(Y)$ are good homeomorphic measures, then $S(\mu) = S(\nu)$, and there is a homeomorphism $h: \mathfrak{M} \rightarrow \mathfrak{N}$.

P r o o f. The second part of the theorem is clear. We will prove the first part. Let $X = \bigsqcup_{i=1}^{\infty} X_i$ and $Y = \bigsqcup_{j=1}^{\infty} Y_j$ where X_i, Y_j are compact Cantor spaces.

First, consider the case when $\mathfrak{M} = \mathfrak{N} = \emptyset$, i.e., the measures μ, ν are either finite or infinite locally finite measures. Since $S(\mu) = S(\nu)$, we have $\mu(X_1) \in S(\nu)$. There exists $n \in \mathbb{N}$ such that $\nu(\bigsqcup_{j=1}^{n-1} Y_j) \leq \mu(X_1) < \nu(\bigsqcup_{j=1}^n Y_j)$. Since $S(\nu)$ is group-like, we see that $\mu(X_1) - \nu(\bigsqcup_{j=1}^{n-1} Y_j) \in S(\nu)$. Since ν is good, there exists a compact open subset $W \subset Y_n$ such that $\nu(W) = \mu(X_1) - \nu(\bigsqcup_{j=1}^{n-1} Y_j)$. Hence $Z = \bigsqcup_{j=1}^{n-1} Y_j \sqcup W$ is a compact Cantor set and $\mu(X_1) = \nu(Z)$. By Theorem 2.9 [11], there exists a homeomorphism $h_1: X_1 \rightarrow Z$ such that $\mu|_{X_1} = \nu|_Z \circ h_1$. Set $\tilde{h}|_{X_1} = h_1$. Consider $(Y_n \setminus W) \bigsqcup_{j=n+1}^{\infty} Y_j$ instead of Y , and $\bigsqcup_{i=2}^{\infty} X_i$ instead of X . Reverse the roles of X and Y . Proceed in the same way using $Y_n \setminus W$ instead of X_1 . Thus, we obtain countably many homeomorphisms $\{h_i\}_{i=1}^{\infty}$. Given $x \in X$, set $\tilde{h}(x) = h_i(x)$ for the corresponding h_i . Then $\tilde{h}: X \rightarrow Y$ is a homeomorphism which maps μ into ν .

Now, let $\mathfrak{M} \neq \emptyset$. If $\mu(X_1) < \infty$, we proceed as in the previous case. If $\mu(X_1) = \infty$, then $X_1 \cap \mathfrak{M} \neq \emptyset$. Then $h(X_1 \cap \mathfrak{M})$ is a compact open subset of \mathfrak{N} in the induced topology. Hence there exists a compact open set $W \subset Y$ such that $W \cap \mathfrak{N} = h(X_1 \cap \mathfrak{M})$. Then, by Theorem 2.11 [9], the sets X_1 and W are homeomorphic via the measure preserving homeomorphism h_1 and $h_1|_{X_1 \cap \mathfrak{M}} = h$. Since W is compact, there exists N such that $W \subset \bigsqcup_{n=1}^N Y_n$. Reverse the roles of X and Y and consider $\bigsqcup_{n=1}^N Y_n \setminus W$ instead of X_1 . ■

The corollary for weakly homeomorphic measures follows:

Corollary 2.7. *Let $\mu \in M_{\infty}^0(X)$ and $\nu \in M_{\infty}^0(Y)$ be good infinite measures on the non-compact locally compact Cantor sets X and Y . Let \mathfrak{M} be the defective set for μ and \mathfrak{N} be the defective set for ν . Then μ is weakly homeomorphic to ν if and only if the following conditions hold:*

- (1) *There exists $c > 0$ such that $S(\mu) = cS(\nu)$,*
- (2) *There exists a homeomorphism $h: \mathfrak{M} \rightarrow \mathfrak{N}$ where the sets \mathfrak{M} and \mathfrak{N} are endowed with the induced topologies.*

R e m a r k 1. Let $\mu \in M_\infty^0(X)$ be a good measure on a non-compact locally compact Cantor set X and V be any compact open subset of X with $\mu(V) < \infty$. Then μ on X is homeomorphic to μ on $X \setminus V$. Let $S(\mu) = G \cap [0, \infty)$. Then μ is homeomorphic to $c\mu$ if and only if $c \in Div(G)$.

Corollary 2.8. *Let μ be a good finite or non-defective measure on a non-compact locally compact Cantor set X . Let U, V be two compact open subsets of X such that $\mu(U) = \nu(V) < \infty$. Then there is $h \in H_\mu(X)$ such that $h(U) = V$.*

P r o o f. Set $Y = U \cup V$. Then Y is a Cantor set with $\mu(Y) < \infty$. By Proposition 2.11 from [11], there exists a self-homeomorphism h of Y such that $h(U) = V$ and h preserves μ . Set h to be the identity on $X \setminus Y$. ■

Corollary 2.9. *Let μ and ν be good non-defective measures on non-compact locally compact Cantor sets X and Y . Let \mathfrak{M} be the defective set for μ and \mathfrak{N} be the defective set for ν . If there exist compact open sets $U \subset X$ and $V \subset Y$ such that $\mu(U) = \nu(V) < \infty$ and $\mu|U$ is homeomorphic to $\nu|V$, then μ is homeomorphic to ν if and only if \mathfrak{M} and \mathfrak{N} (with the induced topologies) are homeomorphic.*

P r o o f. Let $\gamma = \mu(U) = \nu(V)$. Since $\mu|U$ is homeomorphic to $\nu|V$, we have $S(\mu|U) = S(\nu|V)$. Since μ and ν are good, we have $S(\mu) \cap [0, \gamma] = S(\nu) \cap [0, \gamma]$ by Proposition 2.3. Since $S(\mu)$ and $S(\nu)$ are group-like, we obtain $S(\mu) = S(\nu)$. ■

Theorem 2.10. *Let $\mu \in M^0(X)$ be a good measure on a non-compact locally compact Cantor set X . Then the compact open values set $S(\mu)$ is group-like and the defective set \mathfrak{M}_μ is a locally compact zero-dimensional metric space (including \emptyset).*

Conversely, for every countable dense group-like subset D of $[0, 1)$, there is a good probability measure μ on a non-compact locally compact Cantor set such that $S(\mu) = D$. For every countable dense group-like subset D of $[0, \infty)$ and any locally compact zero-dimensional metric space A (including $A = \emptyset$) there is a good non-defective measure μ on a non-compact locally compact Cantor set such that $S(\mu) = D$ and \mathfrak{M}_μ is homeomorphic to A .

P r o o f. The first part of the theorem follows from Propositions 2.1, 2.3.

We will prove the second part. First, consider the case of the finite measure. Let $D \subset [0, 1)$ be a countable dense group-like subset. Then there exists a strictly increasing sequence $\{\gamma_n\}_{n=1}^\infty \subset D$ such that $\lim_{n \rightarrow \infty} \gamma_n = 1$. For $n = 1, 2, \dots$,

set $\delta_n = \gamma_n - \gamma_{n-1}$. Denote $S_n = D \cap [0, \delta_n]$. Then $D_n = \frac{1}{\delta_n}(D \cap [0, \delta_n])$ is a group-like subset of $[0, 1]$ with $1 \in D_n$. In [11], it was proved that there exists a good probability measure μ_n on a Cantor set X_n such that $S(\mu_n|_{X_n}) = D_n$. The measure $\nu_n = \delta_n \mu_n$ is a good finite measure on X_n with $S(\nu_n|_{X_n}) = D \cap [0, \delta_n]$. Set $X = \bigsqcup_{n=1}^{\infty} X_n$ and let $\mu|_{X_n} = \nu_n$. Then μ is a good probability measure on a non-compact locally compact Cantor space X and $S(\mu|_X) = D$.

Now consider the case of the infinite measure. Let $\gamma \in D$. Since $D \subset [0, \infty)$ is group-like, we see that $\frac{1}{\gamma}D \cap [0, 1]$ is a group-like subset of $[0, 1]$. In [11], it was proved that there exists a good probability measure μ_1 on a Cantor space Y with $S(\mu_1) = \frac{1}{\gamma}D \cap [0, 1]$. Set $\mu = \gamma \mu_1$. Then μ is a good finite measure on Y and $S(\mu) = D \cap [0, \gamma]$. Endow the set \mathbb{Z} with discrete topology. Let ν be a counting measure on \mathbb{Z} . Set $X = Y \times \mathbb{Z}$ and $\tilde{\mu} = \mu \times \nu$. Then, by Theorem 2.4, $\tilde{\mu}$ is good with $S(\tilde{\mu}) = D$ and $\mathfrak{M}_{\tilde{\mu}} = \emptyset$.

Suppose A is a non-empty compact zero-dimensional metric space. Then, by Theorem 2.15 [9], there exists a good non-defective measure μ on a Cantor space Y such that $S(\mu) = D$ and \mathfrak{M}_{μ} is homeomorphic to A . By the above, there exists a good locally finite measure ν on a non-compact locally compact set X with $S(\nu) = D$ and $\mathfrak{M}_{\nu} = \emptyset$. Set $Z = Y \sqcup X$ and $\tilde{\mu}|_Y = \mu, \tilde{\mu}|_X = \nu$. Then $\tilde{\mu}$ is good on a non-compact locally compact Cantor set Z with $S(\tilde{\mu}) = D$ and $\mathfrak{M}_{\tilde{\mu}}$ is homeomorphic to A .

Suppose that A is a non-empty non-compact, locally compact zero-dimensional metric space. Then $A = \bigsqcup_{n=1}^{\infty} A_n$ where each A_n is a non-empty compact zero-dimensional metric space. By Theorem 2.15 [9], for every $n = 1, 2, \dots$ there exists a good non-defective measure μ_n on a Cantor set Y_n such that $S(\mu_n) = D$ and \mathfrak{M}_{μ_n} is homeomorphic to A_n . Set $X = \bigsqcup_{n=1}^{\infty} Y_n$ and $\mu|_{Y_n} = \mu_n$. Then μ is good on a non-compact locally compact Cantor set X with $S(\mu) = D$, and \mathfrak{M}_{μ} is homeomorphic to A . ■

Corollary 2.11. *Let D be a countable dense group-like subset of $[0, \infty)$. Then there exists an aperiodic homeomorphism of a non-compact locally compact Cantor set with good non-defective invariant measure $\tilde{\mu}$ such that $S(\tilde{\mu}) = D$.*

P r o o f. We use the construction similar to the one in the proof of Theorem 2.10. Let μ be a good measure on a Cantor set Y with $S(\mu) = D \cap [0, \gamma]$ for some $\gamma \in D$. Let ν be a counting measure on \mathbb{Z} . Set $\tilde{\mu} = \mu \times \nu$ on $X = Y \times \mathbb{Z}$. Then $\tilde{\mu}$ is a good non-defective measure on a non-compact locally compact Cantor set X with $S(\tilde{\mu}) = D$. Since the measure μ is a good finite measure on Y , there exists a minimal homeomorphism $T: Y \rightarrow Y$ such that μ is invariant for T (see [11]). Let $T_1(x, n) = (Tx, n + 1)$. Then T_1 is aperiodic homeomorphism of X . The measure $\tilde{\mu}$ is invariant for T_1 . ■

R e m a r k 2. The measure $\tilde{\mu}$ built in Corollary 2.11 is invariant for any skew product with the base (Y, T) and the cocycle acting on \mathbb{Z} .

Theorem 2.12. *Let X be a non-compact locally compact Cantor set. Then there exist continuum distinct classes of homeomorphic good measures in $M_f(X)$. There also exist continuum distinct classes of weakly homeomorphic good measures in $M_\infty^0(X)$.*

P r o o f. There exist uncountably many distinct group-like subsets $\{D_\alpha\}_{\alpha \in \Lambda}$ of $[0, 1]$. By Theorem 2.10, for each D_α there exists a good probability measure μ_α on X such that $S(\mu_\alpha) = D_\alpha$. By Theorem 2.6, the measures $\{\mu_\alpha\}_{\alpha \in \Lambda}$ are pairwise non-homeomorphic.

Let Y be a compact Cantor set. Let μ be a non-defective measure on Y . Denote by $[\mu]$ the class of weak equivalence of μ in the set of all non-defective measures on Y . There exist the continuum distinct classes $[\mu_\alpha]$ of weakly homeomorphic good non-defective measures on a Cantor set Y (see Theorem 2.18 in [9]). Moreover, if there exists $C > 0$ such that $G(S(\mu_\alpha)) = CG(S(\mu_\beta))$, then $\mu_\beta \in [\mu_\alpha]$. Let ν be a counting measure on \mathbb{Z} . Then, by Theorem 2.4, $\mu_\alpha \times \nu$ is a good measure on a non-compact locally compact Cantor set $Y \times \mathbb{Z}$, and $G(S(\mu_\alpha \times \nu)) = G(S(\mu_\alpha))$. Hence, by Corollary 2.7, the measures $\mu_\alpha \times \nu$ and $\mu_\beta \times \nu$ are weakly homeomorphic if and only if $\mu_\beta \in [\mu_\alpha]$. ■

Proposition 2.13. *If μ is a Haar measure for some topological group structure on a non-compact locally compact Cantor space X , then μ is a good measure on X .*

P r o o f. The ball B centered at the identity in the invariant ultrametric is a compact open subgroup of X . Since μ is translation-invariant, by Proposition 2.3, it suffices to show that $\mu|_B$ is good for every such ball B . Since the restriction of μ on B is a Haar measure on a compact Cantor space, $\mu|_B$ is good by Proposition 2.4 in [4]. ■

3. From Measures on Non-Compact Spaces to Measures on Compact Spaces and Back

Let X be a non-compact locally compact Cantor space. A *compactification* of X is a pair (Y, c) where Y is a compact space and $c: X \rightarrow Y$ is a homeomorphic embedding of X into Y (i.e., $c: X \rightarrow c(X)$ is a homeomorphism) such that $\overline{c(X)} = Y$, where $\overline{c(X)}$ is the closure of $c(X)$. In the paper, by compactification we will mean not only a pair (Y, c) but also the compact space Y in which X can be embedded as a dense subset. We will denote the compactifications of a space X by the symbols $cX, \omega X$, etc., where c, ω are the corresponding homeomorphic embeddings.

Let $\mu \in M^0(X)$. We will consider only the compactifications cX where cX is a Cantor set. Since c is a homeomorphism, the measure μ on X passes to a homeomorphic measure on $c(X)$. Since we are interested in the classification of measures up to homeomorphisms, we can identify the set $c(X)$ with X . Hence X can be considered as an open dense subset of cX . The set $cX \setminus X$ is called the *remainder* of compactification. As far as X is locally compact, the remainder $cX \setminus X$ is closed in cX for every compactification cX (see [14]). Since $\bar{X} = cX$, the set $cX \setminus X$ is a closed nowhere dense subset of cX .

The compactifications c_1X and c_2X of a space X are *equivalent* if there exists a homeomorphism $f: c_1X \rightarrow c_2X$ such that $fc_1(x) = c_2(x)$ for every $x \in X$. We will identify the equivalent compactifications. For any space X one can consider the family $\mathcal{C}(X)$ of all compactifications of X . The order relation on $\mathcal{C}(X)$ is defined as follows: $c_2X \leq c_1X$ if there exists a continuous map $f: c_1X \rightarrow c_2X$ such that $fc_1 = c_2$. Then we have $f(c_1(X)) = c_2(X)$ and $f(c_1X \setminus c_1(X)) = c_2X \setminus c_2(X)$.

Theorem 3.1 (The Alexandroff compactification theorem). *Every non-compact locally compact space X has a compactification ωX with one-point remainder. This compactification is the smallest element in the set of all compactifications $\mathcal{C}(X)$ with respect to the order \leq .*

The topology on ωX is defined as follows. Denote by $\{\infty\}$ the point $\omega X \setminus X$. The open sets in ωX are the sets of the form $\{\infty\} \cup (X \setminus F)$, where F is a compact subspace of X , together with all sets that are open in X .

For any Borel measure ν on the set $cX \setminus X$ with the induced topology, $\tilde{\mu} = \mu + \nu$ is a Borel measure on cX such that $\tilde{\mu}|_X = \mu$. Since the aim of compactification is the study of a measure μ on a locally compact set X , we will consider only the extensions $\tilde{\mu}$ on cX where $\tilde{\mu}(cX \setminus X) = 0$.

Lemma 3.2. *Let X be a non-compact locally compact Cantor set and $\mu \in M^0(X)$. Let c_1X, c_2X be the compactifications of X such that $c_1X \leq c_2X$. Denote by μ_1 the extension of μ on c_1X , and by μ_2 the extension of μ on c_2X . Then $S(\mu) \subseteq S(\mu_1) \subseteq S(\mu_2)$.*

P r o o f. Since $c_1X \leq c_2X$, there exists a continuous map $f: c_2X \rightarrow c_1X$ such that $f(c_2X \setminus X) = c_1X \setminus X$ and $fc_2(x) = c_1(x)$ for any $x \in X$. Since f is continuous, it suffices to prove that f preserves measure, that is, $\mu_1(V) = \mu_2(f^{-1}(V))$ for any compact open $V \subset X$. Recall that we can identify $c_i(X)$ with X . Hence we can consider f as an identity on $X \subset c_iX$ and f preserves measure. That is, for every compact open subset U of X we have $\mu(U) = \mu_1(U) = \mu_2(U)$. Hence $S(\mu) \subseteq S(\mu_1)$. Since $\mu(c_iX \setminus X) = 0$, the measure of any clopen subset of c_iX is the sum of the measures of the compact open subsets of X . Hence the measures of all clopen sets are preserved. Thus, $S(\mu_1) \subseteq S(\mu_2)$. ■

R e m a r k 3. We can consider the homeomorphic embedding of a set X into a non-compact locally compact Cantor set Y such that $\mu(Y \setminus X) = 0$. Then, by the same arguments as above, the inclusion $S(\mu|_X) \subseteq S(\mu|_Y)$ holds.

Theorem 3.3. *Let X be a non-compact locally compact Cantor set and $\mu \in M^0(X)$ be a good measure. Let cX be any compactification of X . Then μ is good on cX if and only if $S(\mu|_{cX}) \cap [0, \mu(X)) = S(\mu|_X)$.*

P r o o f. First, we prove the “if” part. Let V be a clopen set in cX . Consider two cases. Let $V \cap (cX \setminus X) = \emptyset$. Then V is a compact open subset of X . Since μ is good on X and $S(\mu|_{cX}) \cap [0, \mu(X)) = S(\mu|_X)$, we see that V remains good in cX . Now suppose that $V \cap (cX \setminus X) \neq \emptyset$. Then $V \cap X$ is an open set and $\mu(V) = \mu(V \cap X) = \mu(\bigsqcup_{n=1}^{\infty} V_n)$ where each V_n is a compact open set in X . Let U be a compact open subset of X with $\mu(U) < \mu(V)$. Then there exists $N \in \mathbb{N}$ such that $\mu(U) < \mu(\bigsqcup_{n=1}^N V_n)$. The set $Z = \bigsqcup_{n=1}^N V_n$ is a compact open subset of X . Since $S(\mu|_{cX}) \cap [0, \mu(X)) = S(\mu|_X)$, we have $\mu(U) \in S(\mu|_X)$. Since μ is good on X , there exists a compact open subset $W \subset Z$ such that $\mu(W) = \mu(U)$.

Now we prove the “only if” part. Assume the converse. Suppose that μ is a good measure and the equality does not hold. Then there exists $\gamma \in (0, \mu(X))$ such that $\gamma \in S(\mu|_{cX}) \setminus S(\mu|_X)$. Since $S(\mu|_X)$ is dense in $(0, \mu(X))$, there exists a compact open subset $U \subset X$ such that $\mu(U) > \gamma$. Hence $\gamma \in S(\mu|_{cX}) \cap [0, \mu(U)]$ and $\gamma \notin S(\mu|_U)$. Thus U is not good and we get a contradiction. ■

R e m a r k 4. By Proposition 2.1, the set $X \setminus \mathfrak{M}_\mu$ is a non-compact locally compact Cantor set and $\overline{X \setminus \mathfrak{M}_\mu} = X$. Thus, the set $X \setminus \mathfrak{M}_\mu$ can be homeomorphically embedded into X and then into some compactification cX . After embedding $X \setminus \mathfrak{M}_\mu$ into X , we add only compact open sets of infinite measure. Hence if μ was good on $X \setminus \mathfrak{M}_\mu$, it remains good on X and $S(\mu|_{X \setminus \mathfrak{M}_\mu}) = S(\mu|_X)$. We can consider X as a step towards compactification of $X \setminus \mathfrak{M}_\mu$ and include \mathfrak{M}_μ into $cX \setminus X$. The measure $\mu \in M^0(X)$ is locally finite on $X \setminus \mathfrak{M}_\mu$, so we can consider only locally finite measures among infinite ones.

If μ is not good on a locally compact Cantor set X , then clearly μ is not good on any compactification cX .

Corollary 3.4. *Let μ be a good infinite locally finite measure on a non-compact locally compact Cantor set X . Then μ is good on ωX .*

P r o o f. By definition of topology on ωX , the “new” open sets have a compact complement. Since μ is locally finite on X , the measure of compact subsets of X is finite. Hence the measure of each new clopen set is infinite. By Theorem 3.3, μ is good on ωX . ■

Theorem 3.5. *Let μ be a good measure on a non-compact locally compact Cantor set X . Then for any $\gamma \in [0, \mu(X))$ there exists a compactification cX such that $\gamma \in S(\mu|_{cX})$.*

P r o o f. The set $S(\mu|_X)$ is dense in $[0, \mu(X))$. Hence for every $\gamma \in [0, \mu(X))$ there exist $\{\gamma_n\}_{n=1}^\infty \subset S(\mu|_X)$ such that $\lim_{n \rightarrow \infty} \gamma_n = \gamma$. Since μ is good, there exist the disjoint compact open subsets $\{U_n\}_{n=1}^\infty$ such that $\mu(U_n) = \gamma_n$. Then $U = \bigsqcup_{n=1}^\infty U_n$ is a non-compact locally compact Cantor set. Consider the compactification $cX = \omega U \sqcup c(X \setminus U)$, where $c(X \setminus U)$ is any compactification of $X \setminus U$. Then ωU is a clopen set in cX , and $\mu(\omega U) = \gamma \in S(\mu|_{cX})$. ■

From Theorems 3.3, 3.5 the corollary follows:

Corollary 3.6. *For any measure μ on a non-compact locally compact Cantor space X there exists a compactification cX such that μ is not good on cX .*

If a measure $\mu \in M^0(X)$ is a good probability measure, then, by Theorem 3.3, the measure μ is good on cX if and only if $S(\mu|_{cX}) = S(\mu|_X) \cup \{1\}$.

Proposition 3.7. *Let X be a non-compact locally compact Cantor set and $\mu \in M_f(X)$. If there exists a compactification cX such that $S(\mu|_{cX}) = S(\mu|_X) \cup \{1\}$, then $1 \in G(S(\mu|_X))$.*

P r o o f. Let $\gamma \in S(\mu|_{cX}) \cap (0, 1)$. Since the complement of a clopen set is a clopen set, we have $1 - \gamma \in S(\mu|_{cX})$. Since $S(\mu|_{cX}) = S(\mu|_X) \cup \{1\}$, we have $\gamma, 1 - \gamma \in S(\mu|_X)$. Hence $1 \in G(S(\mu|_X))$. ■

Thus, if $1 \notin G(S(\mu|_X))$, then for any compactification cX the set $S(\mu|_X)$ cannot be preserved after the extension. The examples are given in the last section.

The corollary follows from Proposition 3.7 and Theorem 3.3.

Corollary 3.8. *Let μ be a probability measure on a non-compact locally compact Cantor set X , and $1 \notin G(S(\mu|_X))$. Then for any compactification cX of X , μ is not good on cX .*

Theorem 3.9. *Let μ be a good probability measure on a non-compact locally compact Cantor set X . Then μ is good on the Alexandroff compactification ωX if and only if $1 \in G(S(\mu|_X))$.*

P r o o f. By Proposition 3.7 and Theorem 3.3, if μ is good on ωX , then $1 \in G(S(\mu|_X))$.

Suppose μ is good on X and $1 \in G(S(\mu|_X))$. Since μ is good, any compact open subset of μ is good, hence for every compact open $U \subset X$ we have $\mu(U) =$

$G(S(\mu|_X)) \cap [0, \mu(U)] = S(\mu) \cap [0, \mu(U)]$. Every clopen subset of ωX has a compact open subset of X as a complement. Hence for every clopen $V \subset \omega X$ we have $\mu(V) = 1 - \mu(X \setminus V) \in G(S(\mu|_X)) \cap (0, 1) = S(\mu|_X)$. So, $S(\mu|_{\omega X}) = S(\mu|_X) \cup \{1\}$. Hence μ is good on ωX by Theorem 3.3. ■

For a Cantor set Y , denote by $M^0(Y)$ the set of all either finite or non-defective measures on Y (see [9]). Since an open dense subset of a Cantor set is a locally compact Cantor set, the corollary follows:

Corollary 3.10. *Let Y be a (compact) Cantor set and the measure $\mu \in M^0(Y)$. Let $X \subset Y$ be an open dense subset of Y of full measure. If μ is good on Y , then μ is good on X .*

P r o o f. The set X is a locally compact Cantor set, and Y is a compactification of X . Any compact open subset U of X is a clopen subset of Y and all clopen subsets of U are compact open sets. Thus, a $\mu|_Y$ -good compact open set in X is, a fortiori, $\mu|_X$ -good. ■

Thus, the extensions of a non-good measure are always non-good. The corollary follows from Lemma 3.2, Theorem 3.3 and Corollary 3.10.

Corollary 3.11. *Let X be a non-compact locally compact Cantor set and $\mu \in M^0(X)$. Let c_1X, c_2X be compactifications of X such that $c_1X \geq c_2X$. Let μ be good on c_1X . Then μ is good on c_2X . Moreover, if $\mu \in M_f(X)$, then $\mu|_{c_1X}$ is homeomorphic to $\mu|_{c_2X}$.*

R e m a r k 5. Recall that the Alexandroff compactification ωX is the smallest element in the set of all compactifications of X . Hence, if μ is not good on ωX , then μ is not good on any compactification cX of X .

The following theorem can be proved using the results of Akin [11] for measures on compact sets.

Theorem 3.12. *Let X, Y be non-compact locally compact Cantor spaces, and $\mu \in M_f^0(X), \nu \in M_f^0(Y)$ be good measures such that their extensions to $\omega X, \omega Y$ are good. Then $\mu|_X$ and $\nu|_Y$ are homeomorphic if and only if $S(\mu|_X) = S(\nu|_Y)$.*

P r o o f. The “only if” part is trivial, we prove the “if” part. Since $\mu|_{\omega X}$ and $\nu|_{\omega Y}$ are good by Theorem 3.3, we have $S(\mu|_{\omega X}) = S(\nu|_{\omega Y})$. Denote by $x_0 = \omega X \setminus X$ and $y_0 = \omega Y \setminus Y$. By Theorem 2.9 [11], there exists a homeomorphism $f: \omega X \rightarrow \omega Y$ such that $f_*\mu = \nu$ and $f(x_0) = y_0$. Hence $f(X) = Y$ and the theorem is proved. ■

In Example 1, we present a class of good measures on non-compact locally compact Cantor sets such that these measures are not good on the Alexandroff compactifications. Thus, these measures are not good on any compactification of the corresponding non-compact locally compact Cantor sets.

4. Examples

Example 1. (Ergodic invariant measures on stationary Bratteli diagrams). Let B be a non-simple stationary Bratteli diagram with the matrix A transpose to the incidence matrix. Let μ be an ergodic \mathcal{R} -invariant measure on B (see [6, 9, 15]). Let α be the class of vertices that defines μ . Then μ is good as a measure on a non-compact locally compact set X_α . The measure μ on X_α can be either finite or infinite, but it is always locally finite. The set X_B is a compactification of X_α . Since μ is ergodic, we have $\mu(X_B \setminus X_\alpha) = 0$. In [6, 9], the criteria of goodness for probability or non-defective measure μ on X_B were proved in terms of the Perron–Frobenius eigenvalue and eigenvector of A corresponding to μ (see Theorem 3.5 [6] for probability measures and Corollary 3.4 [9] for infinite measures). It is easy to see that these criteria are particular cases of Theorem 3.3.

We consider now a class of the stationary Bratteli diagrams and give a criterion when a measure μ from this class is good on the Alexandroff compactification ωX_α . Fix an integer $N \geq 3$ and let

$$F_N = \begin{pmatrix} 2 & 0 & 0 \\ 1 & N & 1 \\ 1 & 1 & N \end{pmatrix}$$

be the incidence matrix of the Bratteli diagram B_N . For $A_N = F_N^T$ we can easily find the Perron–Frobenius eigenvalue $\lambda = N + 1$ and the corresponding probability eigenvector

$$x = \left(\frac{1}{N}, \frac{N-1}{2N}, \frac{N-1}{2N} \right)^T.$$

Let μ_N be the measure on B_N determined by λ and the eigenvector x . The measure μ_N is good on ωX_α if and only if there exists $R \in \mathbb{N}$ such that $\frac{2(N+1)^R}{N-1}$ is an integer. This is possible if and only if $N = 2^k + 1$, $k \in \mathbb{N}$. For instance, the measure μ_N is good on ωX_α for $N = 3, 5$ but is not good for $N = 4$. Note that the criterion for goodness on ωX_α here is the same as for goodness on X_B . This example is a particular case of more general result (the notation from [6] is used below):

Proposition 4.1. *Let B be a stationary Bratteli diagram defined by a distinguished eigenvalue λ of the matrix $A = F^T$. Denote by $x = (x_1, \dots, x_n)^T$ the corresponding reduced vector. Let the vertices $2, \dots, n$ belong to the distinguished class α corresponding to μ . Then μ is good on X_B if and only if μ is good on ωX_α .*

Proof. By Theorem 3.1 and Corollary 3.11, if μ is good on X_B , then μ is good on ωX_α . We prove the converse. By Theorem 3.5 in [6] and Theorem 3.9,

it suffices to prove that $1 \in G(S(\mu|_{X_\alpha}))$ only if there exists $R \in \mathbb{N}$ such that $\lambda^R x_1 \in H(x_2, \dots, x_n)$. Note that $G(S(\mu|_{X_\alpha})) = (\bigcup_{N=0}^\infty \frac{1}{\lambda^N} H(x_2, \dots, x_n))$, where $H(x_2, \dots, x_n)$ is an additive group generated by x_2, \dots, x_n . Suppose that $1 \in G(S(\mu|_{X_\alpha}))$. Since $\sum_{i=1}^n x_i = 1$, we see that $x_1 \in G(S(\mu|_{X_\alpha}))$, hence there exists $R \in \mathbb{N}$ such that $\lambda^R x_1 \in H(x_2, \dots, x_n)$. ■

Return to a general case of ergodic invariant measures on the stationary Bratteli diagrams. If μ is a probability measure on X_α and $S(\mu|_{X_\alpha}) \cup \{1\} = S(\mu|_{X_B})$, then, by Lemma 3.2, we have $S(\mu|_{\omega X_\alpha}) = S(\mu|_{X_B})$. By Theorem 3.3, the measure μ is good on ωX_α . Hence $\mu|_{\omega X_\alpha}$ is homeomorphic to $\mu|_{X_B}$ (see [11]). If μ is infinite, then the measures $\mu|_{\omega X_\alpha}$ and $\mu|_{X_B}$ are not homeomorphic since $\mathfrak{M}_{\mu|_{\omega X_\alpha}}$ is one point and $\mathfrak{M}_{\mu|_{X_B}}$ is a Cantor set (see [9]).

Example 2. Let X be a Cantor space and μ be a good probability measure on X with $S(\mu) = \{\frac{m}{2^n} : m \in \mathbb{N} \cap [0, 2^n], n \in \mathbb{N}\}$ (for example a Bernoulli measure $\beta(\frac{1}{2}, \frac{1}{2})$). Clearly, $\mu_n = \frac{1}{2^n} \mu$ is a good measure for $n \in \mathbb{N}$ with $S(\mu_n) = \frac{1}{2^n} S(\mu) \subset S(\mu)$. Let $\{X_n, \mu_n\}_{n=1}^\infty$ be a sequence of Cantor spaces with measures μ_n . Let $A = \bigsqcup_{n=1}^\infty X_n$ be the disjoint union of X_n . Denote by ν a measure on A such that $\nu|_{X_n} = \mu_n$. Then ν is a good measure on a locally compact Cantor space A with $S(\nu) = S(\mu) \cap [0, 1)$.

Consider the one-point compactification ωA and the extension ν_1 of ν to ωA . We add to $S(\nu)$ the measures of sets which contain $\{\infty\}$ and have a compact open complement. Hence we add the set $\Gamma = \{1 - \gamma : \gamma \in S(\nu)\}$. Since $\Gamma \subset S(\nu) \cup \{1\}$, we have $S(\nu_1) = S(\nu) \cup \{1\}$. By Theorem 3.3, the measure ν_1 is good on ωA .

Consider the two-point compactification of A . Let $A = A_1 \sqcup A_2$ where $A_1 = \bigsqcup_{k=1}^\infty X_{2k-1}$ and $A_2 = \bigsqcup_{j=1}^\infty X_{2j}$. Then $cA = \omega A_1 \sqcup \omega A_2$ is a two-point compactification of A . Let ν_2 be the extension of ν to cA . Then $\nu_2(A_1) = \frac{2}{3} \notin S(\nu)$. Hence, by Theorem 3.3, the measure ν_2 is not good on cA .

In the same example, we can make a two-point compactification which preserves $S(\nu|_A)$. Since μ_n is good for $n \in \mathbb{N}$, there is a compact open partition $X_n^{(1)} \sqcup X_n^{(2)} = X_n$ such that $\mu_n(X_n^{(i)}) = \frac{1}{2^{n+1}}$ for $i = 1, 2$. Let $B_i = \bigsqcup_{n=1}^\infty X_n^{(i)}$ for $i = 1, 2$. Consider $\tilde{c}A = \omega B_1 \sqcup \omega B_2$. Then it can be proved in the same way as above that $S(\nu|_{\tilde{c}A}) = S(\nu|_A) \cup \{1\}$.

Example 3. Let $\mu = \beta(\frac{1}{3}, \frac{2}{3})$ be a Bernoulli (product) measure on the Cantor space $Y = \{0, 1\}^\mathbb{N}$ generated by the initial distribution $p(0) = \frac{1}{3}$, $p(1) = \frac{2}{3}$. Then μ is not good but $S(\mu) = \{\frac{a}{3^n} : a \in \mathbb{N} \cap [0, 3^n], n \in \mathbb{N}\}$ is group-like (see [10]). Let X be an open dense subset of Y such that $\mu(Y \setminus X) = 0$. Thus, Y is a compactification of a non-compact locally compact Cantor space X and μ extends from X to Y . Then μ is not good on X .

The compact open subsets of X are exactly the clopen subsets of Y that lie in X . The compact open subset of X is a union of the finite number of the compact open cylinders. Consider any compact open cylinder $U = \{a_0, \dots, a_n, *\}$

which consists of all points in $z \in Y$ such that $z_i = a_i$ for $0 \leq i \leq n$. Then U is a disjoint union of two subcylinders $V_1 = \{a_0, \dots, a_n, 0, *\}$ and $V_2 = \{a_0, \dots, a_n, 1, *\}$ with $\mu(V_2) = 2\mu(V_1)$. Let the numerator of the fraction $\mu(V_1)$ be 2^k . Then the numerator of the fraction $\mu(V_2)$ is 2^{k+1} . Moreover, for any compact open $W \subset V_2$ the numerator of the fraction $\mu(W)$ will be divisible by 2^{k+1} . Since $S(\mu)$ contains only finite sums of the measures of the cylinder compact open sets and the denominators of the elements of $S(\mu)$ are the powers of 3, there is no compact open subset $W \subset V_2$ such that $\mu(W) = \mu(V_1)$. Hence μ is not good on X .

Moreover, let $x = \{00\dots\}$ be a point in Y which consists only of zeroes. Then $S(\mu|_{Y \setminus \{x\}}) \subsetneq S(\mu|_Y)$ while $S(\mu|_{Y \setminus \{y\}}) = S(\mu|_Y)$ for any $y \neq x$.

Consider the case $y \neq x$. Let, for instance, $y = \{111\dots\}$, all other cases are proved in the same way. Let $U_n = \{z \in Y : z_0 = \dots = z_{n-1} = 1, z_n = 0\}$. Then $Y \setminus \{y\} = \bigsqcup_{n=1}^{\infty} U_n \sqcup \{0*\}$. Denote $S_N = \mu(\bigsqcup_{n=1}^N U_n)$ and $S_0 = 0$. Then $\lim_{N \rightarrow \infty} S_N = \frac{2}{3}$. Let $G = \{\frac{a}{3^n} : a \in \mathbb{Z}, n \in \mathbb{N}\}$. Then G is an additive subgroup of reals, and $S(\mu|_Y) = G \cap [0, 1]$. We prove that $S(\mu|_{Y \setminus \{y\}}) = G \cap [0, 1]$, i.e., for every $n \in \mathbb{N}$ and $a = 0, \dots, 3^n - 1$ there exists a compact open set W in $Y \setminus \{y\}$ such that $\mu(W) = \frac{a}{3^n}$. Indeed, we have $S(\mu|_{\{0*\}}) = G \cap [0, \frac{1}{3}]$ and $[0, 1] = \cup_{n=0}^{\infty} [S_N, S_N + \frac{1}{3}]$. Hence $G \cap [0, 1] = \cup_{n=0}^{\infty} (G \cap [S_N, S_N + \frac{1}{3}])$. For every $\gamma \in G$ there exists $N \in \mathbb{N}$ such that $\gamma \in [S_N, S_N + \frac{1}{3}]$. There exists a compact open subset W_0 of $\{0*\}$ such that $\mu(W_0) = \gamma - S_N$. Set $W = U_N \sqcup W_0$. Then W is a compact open subset of $Y \setminus \{y\}$ and $\mu(W) = \gamma$.

Now consider the set $Y \setminus \{x\}$. Every cylinder that lies in $Y \setminus \{x\}$ has an even numerator, hence $S(\mu|_{Y \setminus \{x\}}) \subsetneq S(\mu|_Y)$. It can be proved in the same way as above that $S(\mu|_{Y \setminus \{x\}}) = \{\frac{2^k}{3^n} : k \in \mathbb{N}\} \cap [0, 1)$.

E x a m p l e 4. ((C, F)-construction). Denote by $|A|$ the cardinality of a set A . Given two subsets $E, F \subset \mathbb{Z}$, by $E + F$ we mean $\{e + f | e \in E, f \in F\}$ (for more details see [13, 16]). Let $\{F_n\}_{n=1}^{\infty}, \{C_n\}_{n=1}^{\infty} \subset \mathbb{Z}$ such that for each n :

- (1) $|F_n| < \infty, |C_n| < \infty$,
- (2) $|C_n| > 1$,
- (3) $F_n + C_n + \{-1, 0, 1\} \subset F_{n+1}$,
- (4) $(F_n + c) \cap (F_n + c') = \emptyset$ for all $c \neq c' \in C_{n+1}$.

Set $X_n = F_n \times \prod_{k>n} C_k$ and endow X_n with a product topology. By (1), (2), each X_n is a Cantor space.

For each n , define a map $i_{n,n+1}: X_n \rightarrow X_{n+1}$ such that

$$i_{n,n+1}(f_n, c_{n+1}, c_{n+2}, \dots) = (f_n + c_{n+1}, c_{n+2}, \dots).$$

By (1), (2) each $i_{n,n+1}$ is a well defined injective continuous map. Since X_n is compact, we see that $i_{n,n+1}$ is a homeomorphism between X_n and $i_{n,n+1}(X_n)$. Thereby the embedding $i_{n,n+1}$ preserves topology. The set $i_{n,n+1}(X_n)$ is a clopen subset of X_{n+1} . Let $i_{m,n}: X_m \rightarrow X_n$ such that $i_{m,n} = i_{n,n-1} \circ i_{n-1,n-2} \circ \dots \circ i_{m+1,m}$

for $m < n$ and $i_{n,n} = id$. Denote by X the topological inductive limit of the sequence $(X_n, i_{n,n+1})$. Then $X = \bigcup_{n=1}^{\infty} X_n$. Since $i_{m,n} = i_{n,n-1} \circ i_{n-1,n-2} \circ \dots \circ i_{m+1,m}$ for $m < n$, we can write $X_1 \subset X_2 \subset \dots$. The set X is a non-compact locally compact Cantor set. The Borel σ -algebra on X is generated by the cylinder sets $[A]_n = \{x \in X : x = (f_n, c_{n+1}, c_{n+2}, \dots) \in X_n \text{ and } f_n \in A\}$. There exists a canonical measure on X . Let κ_n stand for the equidistribution on C_n and let $\nu_n = \frac{|F_n|}{|C_1| \dots |C_n|}$ on F_n . The product measure on X_n is defined as $\mu_n = \nu_n \times \kappa_{n+1} \times \kappa_{n+2} \times \dots$, and a σ -finite measure μ on X is defined by the restrictions $\mu|_{X_n} = \mu_n$. The measure μ is a unique up to the scaling ergodic locally finite invariant measure for a minimal self-homeomorphism of X (see [13, 16]). For every two compact open subsets $U, V \subset X$ there exists $n \in \mathbb{N}$ such that $U, V \subset X_n$. The measure μ is obviously good, since the restriction of μ onto X_n is just the infinite product of the equidistributed measures on F_n and C_m , $m > n$. We have $S(\mu) = \{\frac{a}{|C_1| \dots |C_n|} : a, n \in \mathbb{N}\} \cap [0, \mu(X))$.

Example 5. Let p be a prime number and \mathbb{Q}_p be the set of p -adic numbers. Endowed with the p -adic norm, the set \mathbb{Q}_p is a non-compact locally compact Cantor space. Thus the Haar measure μ on \mathbb{Q}_p is good and $S(\mu) = \{np^\gamma | n \in \mathbb{N}, \gamma \in \mathbb{Z}\}$.

Acknowledgement. I am grateful to Prof. S. Bezuglyi for giving me the idea of this work and for many helpful discussions of this paper.

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