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Spectral Problem Generated by the Equation of Smooth String with Piece-Wise Constant Friction

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In the paper, the spectral problem generated by the Sturm–Liouville equation

$$-y'' + q(x)y = (\lambda^2 - ip(x)\lambda)y,$$

where $q(x)$ is a real $L_2(0,a)$-function and $p(x)$ is a peace-wise constant, is considered with the Dirichlet boundary conditions at the ends of the interval $(0,a)$. The spectrum of the problem is compared with the spectra of auxiliary problems with the Dirichlet–Dirichlet and the Dirichlet–Neumann boundary conditions on the halves of the interval. Asymptotic formulas are obtained for the eigenvalues of this problem.

Key words: spectral problem, Sturm–Liouville equation, eigenvalues.

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1. Introduction

The equation for the transverse displacement $u(s,t)$ of an inhomogeneous string is of the form [1],

$$\frac{d^2u(s,t)}{ds^2} - \rho(s)\frac{d^2u(s,t)}{dt^2} = 0, \tag{1.1}$$

where $\rho(s)$ is the linear density of the string, s is a spatial coordinate, t is time. It should be noted that the most general case of the string is considered in [2].

Substituting $u(s,t) = v(\lambda,s)e^{i\lambda t}$ into equation (1.1), we obtain the equation for the amplitude function $v(\lambda,s)$,

$$\frac{d^2v}{ds^2} + \lambda^2\rho(s)v(\lambda,s) = 0, \tag{1.2}$$

where λ is the spectral parameter.

It is known that if the density $\rho(s)$ is twice differentiable (let, e.g., $\rho(s) \in W_2^2(0,l)$) and $\rho(s) \geq \varepsilon > 0$ for all $s \in [0,l]$, l is the length of the string, then applying the Liouville transformation [3]

$$x(s) = \int\limits_0^s \rho(s')^{1/2}\, ds',$$

$$y(\lambda, x) = \rho[x]^{1/2} v(\lambda, s(x)), \quad \rho[x] := \rho(s(x)),$$

equation (1.1) can be reduced to the Sturm–Liouville equation

$$-y'' + q(x)y = \lambda^2 y, \quad x \in (0,a), \tag{1.3}$$

where $q(x) = \rho[x]^{-1/4} \frac{d^2}{dx^2}(\rho[x]^{1/4})$, $a = \int\limits_0^l \rho(s)^{1/2}\, ds$.

Numerous amounts of literature is devoted to boundary problems generated by equation (1.3) (see, e.g., [1, 4–6], etc.).

Physically motivated is the consideration of the problem on the vibrations of a string subject to viscous friction (damping). Pioneering papers [7–9] dealt with the problem describing pointwise friction. Further the theory of such problems was developed in [10, 11].

In the case of a distributed friction we have

$$\frac{d^2u}{ds^2} + (\lambda^2 \rho(s) - i\lambda p(s))u = 0 \tag{1.4}$$

instead of (1.2) with real $\rho(s)$ and nonnegative $p(s)$.

In this case, if $\rho(s) \in W_2^2(0,l)$, $\rho(s) \geq \varepsilon > 0$ and $p(s) \in L_2(0,l)$, the Liouville transformation is used to obtain

$$-y'' + q(x)y = (\lambda^2 - i\lambda p(x))y. \tag{1.5}$$

The problem generated by equation (1.5) on the semi-axis was considered in [12]. The corresponding inverse problem, namely, the problem on finding a pair q and p by using the scattering data, appeared to be more complicated than the classical inverse Sturm–Liouville problem on the semi-axis. It is natural, since one of the operators included in the corresponding operator pencil is antisymmetric (skew-symmetric).

In parallel, there was developed the theory of direct and inverse problems generated by the diffusion equation

$$-y'' + q(x)y = (\lambda^2 - \lambda p(x))y, \tag{1.6}$$

where $p(x)$ and $q(x)$ are real. In this case the corresponding quadratic operator pencil contains only self-adjoint and symmetric operators and the problem turns out to be easier. Here the papers [13–15] should be mentioned.

The direct problems generated by equation (1.5) on the finite interval were considered in [16], where the separability of the spectrum from the real axis was analyzed. However, not putting essential restrictions on $p(x)$, we can obtain only rough asymptotics of eigenvalues.

The present paper studies the direct boundary value problem generated by equation (1.5) with

$$p(x) = \begin{cases} p_1 = \text{const}, & x \in [0, a/2]; \\ p_2 = \text{const}, & x \in [a/2, a], \end{cases}$$

where $p_1 > 0$, $p_2 > 0$, a is the length of the interval, and by the Dirichlet–Dirichlet boundary conditions

$$y(\lambda, 0) = 0, \qquad y(\lambda, a) = 0.$$

In Sec. 2, the boundary value problem generated by equation (1.6) with Dirichlet–Dirichlet conditions at both ends (the main problem) is given. It is natural to consider two pairs of the auxiliary problems on half-intervals with the Dirichlet–Neumann and Dirichlet–Dirichlet boundary conditions at the ends to compare their spectra with the spectrum of the main problem.

In Sec. 3, the operator theoretical approach to the problem is given. It is proved that in certain domains the eigenvalues of the main problem interlace with the elements of the union of spectra of the auxiliary problems.

In Sec. 4, we derive asymptomatic formulas for the eigenvalues of the main problem.

2. Statement of the Problems

For the sake of convenience, let us divide the interval $[0, a]$ into two parts and measure the distance on the left side of the interval from the left to the right, and on the right side from the right to the left. Then our boundary value problem takes the form

$$-y_1'' + q_1(x)y_1 = (\lambda^2 - ip_1\lambda)y_1, \qquad \text{for } x \in [0, a/2]; \tag{2.1}$$

$$-y_2'' + q_2(x)y_2 = (\lambda^2 - ip_2\lambda)y_2, \qquad \text{for } x \in [0, a/2]; \tag{2.2}$$

$$y_1(\lambda, 0) = 0, \qquad y_2(\lambda, 0) = 0, \tag{2.3}$$

where the matching conditions at the midpoint $a/2$ of the interval are

$$y_1(\lambda, a/2) = y_2(\lambda, a/2), \tag{2.4}$$

$$y_1'(\lambda, a/2) = -y_2'(\lambda, a/2), \tag{2.5}$$

and $q_j(x) \in L_2[0, a/2]$, $p_j > 0$ for $j = 1, 2$. For definiteness, we will assume that $p_1 < p_2$.

In parallel with the main problem we will consider the pairs of the boundary value problem related to it:

I. Dirichlet–Neumann problem (the Dirichlet condition on the left end and the Neumann condition on the right end) on the half-intervals

$$-y_1'' + q_1(x)y_1 = (\lambda^2 - ip_1\lambda)y_1, \tag{2.6}$$

$$y_1(\lambda, 0) = 0, \qquad y_1'(\lambda, a/2) = 0; \tag{2.7}$$

and

$$-y_2'' + q_2(x)y_2 = (\lambda^2 - ip_2\lambda)y_2, \tag{2.8}$$

$$y_2(\lambda, 0) = 0, \qquad y_2'(\lambda, a/2) = 0. \tag{2.9}$$

II. Dirichlet–Dirichlet problem (the Dirichlet conditions at both ends) on the half-intervals

$$-y_1'' + q_1(x)y_1 = (\lambda^2 - ip_1\lambda)y_1, \tag{2.10}$$

$$y_1(\lambda, 0) = 0, \qquad y_1(\lambda, a/2) = 0; \tag{2.11}$$

and

$$-y_2'' + q_2(x)y_2 = (\lambda^2 - ip_2\lambda)y_2, \tag{2.12}$$

$$y_2(\lambda, 0) = 0, \qquad y_2(\lambda, a/2) = 0. \tag{2.13}$$

Let us look for the solution of problem (2.1)–(2.5) in the form $y_j(\lambda, x) = M_j S_j(\sqrt{\lambda^2 - ip_j\lambda}, x)$, where M_j are constants, $S_j(\sqrt{\lambda^2 - ip_j\lambda}, x)$ are the solutions of equations (2.1) and (2.2) satisfying the following initial conditions:

$$S_j(\sqrt{\lambda^2 - ip_j\lambda}, 0) = 0, \quad S_j'(\sqrt{\lambda^2 - ip_j\lambda}, 0) = 1, \quad j = 1, 2. \tag{2.14}$$

Matching conditions (2.4), (2.5) at the midpoint of the interval imply

$$\begin{aligned} M_1 S_1(\sqrt{\lambda^2 - ip_1\lambda}, a/2) &= M_2 S_2(\sqrt{\lambda^2 - ip_2\lambda}, a/2), \\ M_1 S_1'(\sqrt{\lambda^2 - ip_1\lambda}, a/2) &= -M_2 S_2'(\sqrt{\lambda^2 - ip_2\lambda}, a/2). \end{aligned} \tag{2.15}$$

The system of equations (2.15) with respect to the unknown M_1 and M_2 possesses a nontrivial solution if its determinant, which is said to be the characteristic function $\varphi(\lambda)$ of problem (2.1)–(2.5), equals zero

$$\begin{aligned} \varphi(\lambda) &= S_1(\sqrt{\lambda^2 - ip_1\lambda}, a/2) \cdot S_2'(\sqrt{\lambda^2 - ip_2\lambda}, a/2) + \\ &+ S_2(\sqrt{\lambda^2 - ip_2\lambda}, a/2) \cdot S_1'(\sqrt{\lambda^2 - ip_1\lambda}, a/2) = 0. \end{aligned} \tag{2.16}$$

It is clear that the set of zeros of the function $\varphi(\lambda)$ is the spectrum of problem (2.1)–(2.5).

3. Operator Theoretical Approach to the Problem

Here and further in the paper we will use the following definitions:

Definition 1 (see, e.g., [17]).

1) *A number* $\lambda_0 \in \mathbb{C}$ *is said to be an eigenvalue of the pencil* $L(\lambda)$ *if there exists a vector* $y_0 \in D(L)$ *(called an eigenvector of* $L(\lambda)$*) such that* $y_0 \neq 0$ *and* $L(\lambda_0)y_0 = 0$.

2) *The vectors* $y_1, y_2, \dots, y_{m-1}$ *are called associated to* y_0 *if*

$$\sum_{s=1}^{k} \frac{1}{s!} \left. \frac{d^s L(\lambda)}{d\lambda^s} \right|_{\lambda=\lambda_0} y_{k-s} = 0, \qquad k = \overline{1, m-1}.$$

The number m *is said to be the length of the chain composed of the eigen- and associated vectors.*

3) *The geometric multiplicity of an eigenvalue is defined to be the number of the corresponding linearly independent eigenvectors.*

4) *The algebraic multiplicity of an eigenvalue is defined to be the greatest value of the sum of the lengths of chains corresponding to linearly independent eigenvectors.*

5) *If algebraic and geometric multiplicity of an eigenvalue coincide, we call it semisimple.*

6) *An eigenvalue is said to be isolated if it has some deleted neighbourhood contained in the resolvent set. An isolated eigenvalue* λ_0 *of finite algebraic multiplicity is said to be normal if the image* $\mathrm{Im}L(\lambda_0)$ *is closed.*

Let us introduce the operators A and K acting in the Hilbert space $H = L_2(0, a/2) \oplus L_2(0, a/2)$,

$$A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_1'' + q_1(x) y_1 \\ -y_2'' + q_2(x) y_2 \end{pmatrix},$$

$$D(A) = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : \begin{array}{ll} y_1, y_2 \in W_2^2(0, a/2), & y_1(0) = y_2(0) = 0; \\ y_1(a/2) = y_2(a/2), & y_1'(a/2) = -y_2'(a/2), \end{array} \right\},$$

where $W_2^2(0, a/2)$ is the corresponding Sobolev space, and

$$K = \begin{pmatrix} p_1 I & 0 \\ 0 & p_2 I \end{pmatrix}, \qquad D(K) = H,$$

where I is the identity operator in $L_2(0, a/2)$.

It is obvious that the operator K is strictly positive, i.e., $K \geq p_1 I > 0$.

Let us denote

$$\mathbf{I} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

and consider the operator pencil

$$L(\lambda) = \lambda^2 \mathbf{I} - i\lambda K - A,$$

acting in the space H with the domain

$$D(L) = D(I) \cap D(K) \cap D(A) = D(A)$$

not depending on spectral parameter. It is natural to identify the spectrum of the main problem with the spectrum of $L(\lambda)$.

It is known that the operator A is self-adjoint and bounded below, i.e., there exists a number $\beta > 0$ such that the inequality $A + \beta I \geq 0$ is true and for some $\beta_1 > \beta$ the operator $A + \beta_1 I$ possesses a compact resolvent. This implies that the number of negative eigenvalues of the operator A is finite. Since the operators K and I are bounded, the spectrum of the pencil L consists of normal eigenvalues which accumulate only to $+\infty$.

Theorem 1.
1) *The eigenvalues of problem* (2.1)–(2.5) *lie in the strip* $\mathrm{Im}\lambda \in \left[\frac{p_1}{2}; \frac{p_2}{2}\right]$ *and on the intervals* $(-i\infty; \frac{ip_1}{2})$ *or* $(\frac{ip_2}{2}; +i\infty)$ *of the imaginary axis.*
2) *Non-pure imaginary eigenvalues of problem* (2.1)–(2.5) *are symmetrical with respect to the imaginary axis, i.e.,* $\lambda_k = -\overline{\lambda}_k$*, and algebraic multiplicities of the symmetrically located eigenvalues coincide.*
3) *The number of pure imaginary eigenvalues is finite.*

P r o o f. 1) Let us transform the spectral parameter $\tau = \lambda - \frac{ip_1}{2}$. Then we obtain the quadratic operator pencil

$$L_1(\tau) \quad = \quad \tau^2 \mathbf{I} - i\tau K_1 - A_1,$$

where

$$K_1 = \begin{pmatrix} 0 & 0 \\ 0 & p_2 - p_1 \end{pmatrix}, \qquad A_1 = A - \begin{pmatrix} \frac{p_1^2}{4} I & 0 \\ 0 & (\frac{p_1^2}{4} - \frac{p_1(p_1 - p_2)}{2})I \end{pmatrix}.$$

It is clear that $K_1 \geq 0$, and A_1 is a self-adjoint operator bounded below. We apply Lemma 2.2 from [17] to the pencil $L_1(\tau)$ and conclude that the spectrum of the pencil $L_1(\tau)$ can lie only in the half-plane $\mathrm{Im}\tau \geq 0$ and on the imaginary axis. As far as $\tau = \lambda - \frac{ip_1}{2}$, the spectrum of the pencil $L(\lambda)$ can lie only in the half-plane $\mathrm{Im}\lambda \geq p_1/2$ and on the imaginary axis.

To prove that non-pure imaginary eigenvalues lie in the half-plane $\mathrm{Im}\lambda \leq \frac{p_2}{2}$, we apply the transformation of the spectral parameter $\tau = -\lambda + \frac{ip_2}{2}$. Then we obtain the quadratic pencil

$$L_2(\tau) \quad = \quad \tau^2 \mathbf{I} - i\tau K_2 - A_2,$$

where

$$K_2 = \begin{pmatrix} p_2 - p_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_2 = A - \begin{pmatrix} (-\frac{p_2^2}{4} + \frac{p_2(p_2-p_1)}{2})I & 0 \\ 0 & -\frac{p_2^2}{4}I \end{pmatrix}.$$

Evidently, $K_2 \geq 0$ and A_2 is a self-adjoint operator bounded below. We apply Lemma 2.2 from [17] to the pencil $L_2(\tau)$ and conclude that the spectrum of this pencil $L_2(\tau)$ can lie only in the half-plane $\mathrm{Im}\tau \geq 0$ and on the imaginary axis. Since $\tau = -\lambda + \frac{ip_2}{2}$, the spectrum of the pencil $L(\lambda)$ can lie only in the half-plane $\mathrm{Im}\lambda \leq p_2/2$ and on the imaginary axis.

We have proved that the spectrum of the pencil $L(\lambda)$ can lie only in the strip $p_1/2 \leq \mathrm{Im}\lambda \leq p_2/2$ and on the imaginary axis.

Let us prove assertion 2).

Since the functions $S_j(\sqrt{\lambda^2 - ip_j\lambda}, x)$ and $S_j'(\sqrt{\lambda^2 - ip_j\lambda}, x)$ are even functions of the first argument, i.e., they are entire functions of $\lambda^2 - ip_j\lambda$, and the identity $(-\overline{\lambda})^2 - i(-\overline{\lambda})p_j = \overline{\lambda^2 - i\lambda p_j}$ is true, then the function $\varphi(\lambda)$ possesses the symmetry property $\varphi(-\overline{\lambda}) = \overline{\varphi(\lambda)}$. This implies assertion 2).

Let us prove assertion 3).

As it was mentioned above, the number of negative eigenvalues of the operator A is finite and thereby the number of negative eigenvalues of the operator A_1 is also finite. Then, applying Theorem 2.2 from [17], we obtain that the number of eigenvalues of the pencil $L_1(\tau)$ lying on the negative imaginary half-axis $(-i\infty, 0)$ is finite and coincides with the number of negative eigenvalues of the operator A_1. Similarly, the number of eigenvalues of the pencil $L_2(\tau)$, which lie on the negative imaginary axis $(-i\infty, 0)$, is finite and equals the number of negative eigenvalues of the operator A_2. This implies assertion 3).

Consequently, the pure imaginary eigenvalues of problem (2.1)–(2.5) are on the finite interval and do not accumulate to a point of the imaginary axis. The theorem is proved.

In the sequel we will use

Definition 2. *The function $f(z)$, $z \in \mathbb{C}\backslash\mathbb{R}$, is said to be a Nevalinna function if it maps the open upper half-plane into the closed upper half-plane and takes complex conjugate values at non-real points symmetric with respect to the real axis ($f(z) = \overline{f(\overline{z})}$).*

Now we consider the pure imaginary eigenvalues of problem (2.1)–(2.5). For conveniency, we compare their location with the location of pure imaginary eigenvalues of problems (2.6)–(2.7), (2.8)–(2.9).

Let us denote by $\{\lambda_k\}$, $\{\mu_k^{(1)}\}$ and $\{\mu_k^{(2)}\}$ the eigenvalues of problems (2.1)–(2.5), (2.6)–(2.7) and (2.8)–(2.9), respectively, lying on the union of the intervals

$(-i\infty; ip_1/2) \cup (ip_2/2; +i\infty)$; and let $\{\zeta_k\} = \{\mu_k^{(1)}\} \cup \{\mu_k^{(2)}\}$. We enumerate them in the following way:

on the interval $(-i\infty; ip_1/2)$: $|\zeta_{-1} - ip_1/2| \geq |\zeta_{-2} - ip_1/2| \geq \ldots$, $|\lambda_{-1} - ip_1/2| \geq |\lambda_{-2} - ip_1/2| \geq \ldots$,

and on the interval $(ip_2/2; +i\infty)$: $|\zeta_1| \geq |\zeta_2| \geq \ldots$, $|\lambda_1| \geq |\lambda_2| \geq \ldots$.

Theorem 2.

1) *The elements of the sequences* $\{\zeta_k\}$ *and* $\{\lambda_k\}$, *which lie on the intervals* $(-i\infty; ip_1/2)$ *and* $(ip_2/2; +i\infty)$, *interlace as follows:*

$$|\zeta_1| \geq |\lambda_1| \geq |\zeta_2| \geq |\lambda_2| \geq ...$$

$$|\zeta_{-1} - ip_1/2| \geq |\lambda_{-1} - ip_1/2| \geq |\zeta_{-2} - ip_1/2| \geq |\lambda_{-2} - ip_1/2| \geq ...$$

2) *The multiplicities of* ζ_k *do not exceed* 2.

3) *The equality* $\lambda_k = \zeta_k$ *for* $k > 0$ *is valid if and only if* $\lambda_k = \zeta_{k+1}$, *and for* $k < 0$, *if and only if* $\lambda_k = \zeta_{k-1}$.

P r o o f. Let us introduce the notation $\xi_j = \lambda^2 - ip_j\lambda$ and consider the functions $\Phi_j(\sqrt{\xi_j}) = \frac{S_j(\sqrt{\xi_j}, a/2)}{S'_j(\sqrt{\xi_j}, a/2)}$ and $\Phi_j(\sqrt{\lambda^2 - ip_j\lambda}) = \frac{S_j(\sqrt{\lambda^2 - ip_j\lambda}, a/2)}{S'_j(\sqrt{\lambda^2 - ip_j\lambda}, a/2)}$, $j = 1, 2$.

Since $S_j(\sqrt{\xi_j}, a/2)$ and $S'_j(\sqrt{\xi_j}, a/2)$ are entire functions, $\Phi_j(\sqrt{\xi_j})$ is meromorphic.

Using the standard method (see, e.g., [2, p. 661–662], we can show that

$$\operatorname{Im} \frac{S_j(\sqrt{\xi_j}, a/2)}{S'_j(\sqrt{\xi_j}, a/2)} = \operatorname{Im} \xi_j \cdot \frac{\int\limits_0^{a/2} |S_j^2(\sqrt{\xi_j}, x)|\, dx}{|S'_j(\sqrt{\xi_j}, a/2)|^2}.$$

It is clear that if $\operatorname{Im} \xi_j > 0$, then $\operatorname{Im} \frac{S_j(\sqrt{\xi_j}, a/2)}{S'_j(\sqrt{\xi_j}, a/2)} > 0$. Consequently, the function $\Phi_j(\sqrt{\xi_j})$ maps the open upper half-plane into the open upper half-plane, and $\overline{\Phi_j(\sqrt{\xi_j})} = \Phi_j(\sqrt{\overline{\xi_j}})$. Thus they are Nevalinna functions.

From meromorphic and Nevalinna properties of the function $\Phi_j(\sqrt{\xi_j})$ it follows (see [18, p. 399]) that in the half-planes $\operatorname{Im} \xi_j > 0$ and $\operatorname{Im} \xi_j < 0$, $\Phi_j(\sqrt{\xi_j})$ it does not have zeros and poles, i.e., all its zeros and poles lie on the real axis and strictly interlace. Moreover, the function $\Phi_j(\sqrt{\xi_j})$ is monotonically increasing on its intervals of continuity.

Using the integral representations of the functions $S_j(\sqrt{\xi_j}, a/2)$ and $S'_j(\sqrt{\xi_j}, a/2)$ (see (1.2.11) in [19, p. 18]), we can state that $\Phi_j(\sqrt{\xi_j}) \longrightarrow +0$

as $\xi_j \longrightarrow -\infty$, i.e., the most left element of the interlaced sequence of zeros and poles of the function $\Phi_j(\sqrt{\xi_j})$ is its pole. Hence, the number of its negative zeros and poles is finite.

Since $\mathrm{Im}\,\xi_j = \mathrm{Re}\,\lambda(2\mathrm{Im}\,\lambda - p_j)$, the function $\Phi_j(\sqrt{\lambda^2 - ip_j\lambda})$ $(j = 1, 2)$ maps the quadrants $D_1^{(j)} = \{\lambda : \mathrm{Re}\,\lambda > 0, \mathrm{Im}\,\lambda > p_j/2\}$ and $D_2^{(j)} = \{\lambda : \mathrm{Re}\,\lambda < 0, \mathrm{Im}\,\lambda < p_j/2\}$ into the open upper half-plane, and the quadrants $D_3^{(j)} = \{\lambda : \mathrm{Re}\,\lambda < 0, \mathrm{Im}\,\lambda > p_j/2\}$ and $D_4^{(j)} = \{\lambda : \mathrm{Re}\,\lambda > 0, \mathrm{Im}\,\lambda < p_j/2\}$ into the open lower half-plane.

Consequently, the function $\Phi_j(\sqrt{\lambda^2 - ip_j\lambda})$ $(j = 1, 2)$ does not have zeros and poles in the quadrants $D_1^{(j)}$, $D_2^{(j)}$, $D_3^{(j)}$ and $D_4^{(j)}$ because its zeros and poles are on the imaginary axis and on the straight line $\mathrm{Im}\,\lambda = p_j/2$ and are interlaced.

The function $\Phi_1(\sqrt{\lambda^2 - ip_1\lambda}) + \Phi_2(\sqrt{\lambda^2 - ip_2\lambda})$ maps the quadrants $D_1^+ = \{\lambda : \mathrm{Re}\lambda > 0,\ \mathrm{Im}\lambda > p_2/2\}$ and $D_2^+ = \{\lambda : \mathrm{Re}\lambda < 0,\ \mathrm{Im}\lambda < p_1/2\}$ into the upper half-plane, that is, for all $\lambda \in D_1^+ \cup D_2^+$: $0 < \arg(\Phi_1(\sqrt{\lambda^2 - ip_1\lambda}) + {}$ ${}+ \Phi_2(\sqrt{\lambda^2 - ip_2\lambda})) < \pi$, and the quadrants $D_3^+ = \{\mathrm{Re}\lambda > 0,\ \mathrm{Im}\lambda < p_1/2\}$ and $D_4^+ = \{\mathrm{Re}\lambda < 0,\ \mathrm{Im}\lambda > p_2/2\}$ into the lower half-plane, that is, for all $\lambda \in D_3^+ \cup D_4^+$: $-\pi < \arg(\Phi_1(\sqrt{\lambda^2 - ip_1\lambda}) + \Phi_2(\sqrt{\lambda^2 - ip_2\lambda})) < 0$.

The functions $\Phi_1(\sqrt{\lambda^2 - ip_1\lambda})$ and $\Phi_2(\sqrt{\lambda^2 - ip_2\lambda})$ do not have poles in these quadrants, therefore the sum of these functions does not have them as well.

Let us show that the function $\Phi_1(\sqrt{\lambda^2 - ip_1\lambda}) + \Phi_2(\sqrt{\lambda^2 - ip_2\lambda})$ also does not have zeros in the quadrants D_1^+, D_2^+ and D_3^+, D_4^+.

Let us assume that the function $\Phi_1(\sqrt{\lambda^2 - ip_1\lambda}) + \Phi_2(\sqrt{\lambda^2 - ip_2\lambda})$ has a zero, for example, in the quadrant D_1^+, and let Γ be a simple closed contour completely located in the quadrant D_1^+ circling this zero. Then the increment of the argument of the function $\Phi_1(\sqrt{\lambda^2 - ip_1\lambda}) + \Phi_2(\sqrt{\lambda^2 - ip_2\lambda})$ along the contour Γ is not less than 2π, which is impossible because the whole contour lies in the open half-plane.

It is obvious that $\Phi_j(\sqrt{\lambda^2 - ip_j\lambda}) \longrightarrow +0$ as $\lambda \longrightarrow \pm i\infty$. If λ moves along the imaginary axis towards $ip_j/2$, then the function $\Phi_j(\sqrt{\lambda^2 - ip_j\lambda})$ increases monotonically on each of the intervals of its continuity.

To prove that the zeros and the poles of $\Phi_1(\sqrt{\lambda^2 - ip_1\lambda}) + \Phi_2(\sqrt{\lambda^2 - ip_2\lambda})$ on the intervals $(-i\infty; ip_1/2)$ and $(ip_2/2; +i\infty)$ are interlaced, we consider a contour surrounding exactly two poles which lie both on $(-i\infty; ip_1/2)$ or on $(ip_2, i\infty)$. If there is no zero between them, then the increment of the argument of the function $\Phi_1(\sqrt{\lambda^2 - ip_1\lambda}) + \Phi_2(\sqrt{\lambda^2 - ip_2\lambda})$ along the contour is 4π, which is impossible because $\mathrm{Im}\xi_j \cdot \mathrm{Im}\Phi_j(\xi_j) > 0$.

Since the monotonically increasing functions $\Phi_j(\sqrt{\xi_j}) \longrightarrow +0$ $(j = 1, 2)$ as $\xi_j \longrightarrow -\infty$, their sum $\Phi_1(\sqrt{\xi_1}) + \Phi_2(\sqrt{\xi_2}) \longrightarrow +0$ as $\xi_1 \longrightarrow -\infty$ and $\xi_2 \longrightarrow -\infty$.

When solving the equation $\lambda^2 - ip_j\lambda = \xi_j$ with respect to λ, we obtain $\lambda_{1,2} = \frac{-ip_j \pm \sqrt{4\xi_j - p_j^2}}{2}$. It is clear that $\lambda \longrightarrow \pm i\infty$, as $\xi_j \longrightarrow -\infty$, that is, the pure imaginary zeros and poles of the functions $\Phi_j(\sqrt{\lambda^2 - ip_j\lambda})$ lie on the finite interval of an imaginary axis and their number is also finite.

Let us divide both sides of (2.16) by $S_1'(\sqrt{\lambda^2 - ip_1\lambda}, a/2) \cdot S_2'(\sqrt{\lambda^2 - ip_1\lambda}, a/2)$:

$$\frac{\varphi(\lambda)}{S_1'(\sqrt{\lambda^2 - ip_1\lambda}, a/2) \cdot S_2'(\sqrt{\lambda^2 - ip_2\lambda}, a/2)} =$$
$$= \frac{S_1(\sqrt{\lambda^2 - ip_1\lambda}, a/2)}{S_1'(\sqrt{\lambda^2 - ip_1\lambda}, a/2)} + \frac{S_2(\sqrt{\lambda^2 - ip_2\lambda}, a/2)}{S_2'(\sqrt{\lambda^2 - ip_1\lambda}, a/2)}.$$

It was shown above that the zeros of each of the summands in the right-hand side of the last equation are strictly interlaced with its poles on the intervals $(-i\infty; ip_1/2)$ and $(ip_2/2; +i\infty)$. This implies that the zeros of the numerator in the left-hand side interlace with the zeros of the denominator on these intervals.

Concluding from the above, the zeros of the characteristic function $\varphi(\lambda)$ are the eigenvalues λ_k of problem (2.1)–(2.5), and the zeros of the functions $S_1'(\sqrt{\lambda^2 - ip_1\lambda}, a/2)$ and $S_2'(\sqrt{\lambda^2 - ip_2\lambda}, a/2)$ are the eigenvalues $\{\mu_k^{(1)}\}$ and $\{\mu_k^{(2)}\}$ of problems (2.6)–(2.7) and (2.8)–(2.9), respectively. Whence assertions 1) follows.

2) The eigenvalues $\{\mu_k^{(1)}\}$ and $\{\mu_k^{(2)}\}$ of problems (2.6)–(2.7) and (2.8)–(2.9) are simple, that is, their multiplicities equal 1. Thus the multiplicity of each $\zeta_k \in \in \{\mu_k^{(1)}\} \cup \{\mu_k^{(2)}\}$ does not exceed 2.

3) If $\lambda_k = \zeta_k = \mu_p^{(1)}$, then $\varphi(\lambda_k) = 0$ and $S_1'(\sqrt{{\mu_p^{(1)}}^2 - ip_1\mu_p^{(1)}}, a/2) = 0$. Consequently, the second term of (2.16) is

$$S_2'(\sqrt{{\mu_p^{(1)}}^2 - ip_1\mu_p^{(1)}}, a/2) \cdot S_1(\sqrt{{\mu_p^{(1)}}^2 - ip_1\mu_p^{(1)}}, a/2) = 0.$$

The function $S_1(\sqrt{{\mu_p^{(1)}}^2 - ip_1\mu_p^{(1)}}, x)$ and its derivative $S_1'(\sqrt{{\mu_p^{(1)}}^2 - ip_1\mu_p^{(1)}}, x)$ do not vanish simultaneously, consequently, $S_1(\sqrt{{\mu_p^{(1)}}^2 - ip_1\mu_p^{(1)}}, a/2) \neq 0$.

It means that $S_2'(\sqrt{{\mu_p^{(1)}}^2 - ip_1\mu_p^{(1)}}, a/2) = 0$. So, $\mu_p^{(1)}$ is an eigenvalue of problem (2.8)–(2.9).

If eigenvalues of the auxiliary problems (2.6)–(2.7) and (2.8)–(2.9) on the interval $(ip_2/2; +i\infty)$ (on the interval $(-i\infty; ip_1/2)$) coincide, then they are numbered as ζ_k and ζ_{k+1} (ζ_{k-1} and ζ_k). The theorem is proved.

Now let us compare the location of the pure imaginary eigenvalues of problem (2.1)–(2.5) with the location of the pure imaginary eigenvalues of problems (2.10)–(2.11), (2.12)–(2.13).

We denote by $\{\lambda_k\}$, $\{\nu_k^{(1)}\}$ and $\{\nu_k^{(2)}\}$ the eigenvalues of problems (2.1)–(2.5), (2.10)–(2.11) and (2.12)–(2.13), respectively, lying on the intervals of the imaginary axis $(-i\infty; ip_1/2)$ and $(ip_2/2; +i\infty)$; let $\{\xi_k\} = \{\nu_k^{(1)}\} \cup \{\nu_k^{(2)}\}$, i.e., $\lambda_k, \zeta_k \in (-i\infty; ip_1/2) \cup (ip_2/2; +i\infty)$.

Let us enumerate them in the following way:

on the interval $(-i\infty; ip_1/2)$: $\quad |\xi_{-1} - ip_1/2| \le |\xi_{-2} - ip_1/2| \le \ldots,$
$\quad |\lambda_{-1} - ip_1/2| \le |\lambda_{-2} - ip_1/2| \le \ldots,$

and on the interval $(ip_2/2; +i\infty)$: $\quad |\xi_1| \ge |\xi_2| \ge \ldots,$
$\quad |\lambda_1| \ge |\lambda_2| \ge \ldots.$

Theorem 3.

1) *The sequences* $\{\xi_k\}$ *and* $\{\lambda_k\}$ *are interlaced on the intervals* $(-i\infty; ip_1/2)$ *and* $(ip_2/2; +i\infty)$,

$$|\lambda_1| \ge |\xi_1| \ge |\lambda_2| \ge \ldots,$$

$$|\lambda_{-1} - ip_1/2| \ge |\xi_{-2} - ip_1/2| \ge |\lambda_{-2} - ip_1/2| \ge \ldots$$

2) *The multiplicity of each* ξ_k *does not exceed* 2.

3) *The equation* $\lambda_k = \xi_k$ *is valid for* $k > 0$ *if and only if* $\lambda_k = \xi_{k-1}$, *and for* $k < 0$ *if and only if* $\lambda_k = \xi_{k+1}$.

The proof of Theorem 3 is similar to that of Theorem 2.

4. Asymptotics of Eigenvalues

Theorem 4. *The eigenvalues* λ_n *of problem* (2.1)–(2.5) *behave asymptotically as follows:*

$$\lambda_n = \frac{\pi n}{a} + \frac{i(p_1+p_2)a}{4} + \frac{B_n}{n} + o\Big(\frac{1}{n}\Big), \tag{4.1}$$

where $B_n = \frac{1}{\pi a}\Big[(-1)^n \frac{p_2-p_1}{4} \operatorname{sh}\frac{(p_2-p_1)a}{4} - (Q_1+Q_2)\Big]$, $Q_j = \dfrac{1}{2}\displaystyle\int_0^{a/2} q_j(t)\,dt$, $j = 1, 2$.

P r o o f. Let us introduce $(\tau^{(j)})^2 = \lambda^2 - ip_j\lambda$. Under this transformation equations (2.1), (2.2) can be rewritten as

$$-y_j'' + q_j(x)y_j = (\tau^{(j)})^2 y_j, \quad j = 1, 2. \tag{4.2}$$

There exist (see formula (1.2.11), [19, p. 18]) the solutions of equations (4.2) satisfying the initial conditions $S_j(\tau^{(j)}, 0) = 0$, $S_j'(\tau^{(j)}, 0) = 1$, and these solutions are of the form

$$S_j(\tau^{(j)}, x) = \frac{\sin \tau^{(j)} x}{\tau^{(j)}} + \int_0^x K_j(x, t, \infty) \frac{\sin \tau^{(j)} x}{\tau^{(j)}}\,dt, \tag{4.3}$$

where $K_j(x,t,\infty) = K_j(x,t) - K_j(x,-t)$, $K_j(x,t)$ is the solution of the integral equation (see (1.2.18), [19, p. 22])

$$K_j(x,t) = \frac{1}{2}\int\limits_0^{\frac{x+t}{2}} g_j(s)\,ds + \int\limits_0^{\frac{x+t}{2}} ds \int\limits_0^{\frac{x-t}{2}} g_j(s+p)\,K_j(s+p,s-p)dp,$$

and $K_j(x,x) = \frac{1}{2}\int\limits_0^x g_j(t)\,dt, \quad K_j(x,-x) = 0.$

The functions $q_j(x) \in L_2(0,a/2)$, consequently, there exist the derivatives $\frac{\partial K_j(x,t,\infty)}{\partial x}$, $\frac{\partial K_j(x,t,\infty)}{\partial t}$, which belong to $L_2(0,a/2)$ as functions of each of its variables.

Integrating by parts, we modify the right-hand side of (4.3) to the form

$$S_j(\tau^{(j)},x) = \frac{\sin\tau^{(j)}x}{\tau^{(j)}} - \frac{1}{2}\int\limits_0^x q_j(t)dt\cdot\frac{\cos\tau^{(j)}x}{(\tau^{(j)})^2} + \int\limits_0^x \frac{\partial K_j(x,t,\infty)}{\partial t}\,\frac{\cos\tau^{(j)}t}{(\tau^{(j)})^2}\,dt. \tag{4.4}$$

For the derivatives of $S_j(\tau^{(j)},x)$ with respect to x, we have

$$S_j'(\tau^{(j)},x) = \cos\tau^{(j)}x + K_j(x,x,\infty)\cdot\frac{\sin\tau^{(j)}x}{\tau^{(j)}} + \int\limits_0^x \frac{\partial K_j(x,t,\infty)}{\partial t}\,\frac{\sin\tau^{(j)}t}{\tau^{(j)}}\,dt. \tag{4.5}$$

Substituting (4.4) and (4.5) into the equation

$$\varphi(\lambda) = S_1(\tau^{(1)},a/2)\,S_2'(\tau^{(2)},a/2) + S_2(\tau^{(2)},a/2)\,S_1'(\tau^{(1)},a/2) = 0,$$

we obtain

$$\varphi(\lambda) = \frac{\sin(\tau^{(1)}+\tau^{(2)})\frac{a}{2}}{\tau^{(1)}} + \Theta(\lambda), \tag{4.6}$$

where

$$\begin{aligned}\Theta(\lambda) = {} & \frac{\tau^{(1)}-\tau^{(2)}}{\tau^{(1)}\tau^{(2)}}\sin\tau^{(2)}\frac{a}{2}\cos\tau^{(1)}\frac{a}{2} + (Q_1+Q_2)\frac{\sin\tau^{(1)}\frac{a}{2}\sin\tau^{(1)}\frac{a}{2}}{\tau^{(1)}\tau^{(2)}}\\ & + \frac{1}{\tau^{(1)}\tau^{(2)}}\Big(\sin\tau^{(1)}\frac{a}{2}K_s(\tau^{(2)}) + \sin\tau^{(2)}\frac{a}{2}K_s(\tau^{(1)})\Big)\\ & - \cos\tau^{(1)}\frac{a}{2}\cos\tau^{(2)}\frac{a}{2}\Big(\frac{Q_1}{(\tau^{(1)})^2} + \frac{Q_2}{(\tau^{(2)})^2}\Big) + \frac{1}{(\tau^{(1)})^2}\cos\tau^{(2)}\frac{a}{2}K_c(\tau^{(1)})\\ & + \frac{1}{(\tau^{(2)})^2}\cos\tau^{(1)}\frac{a}{2}K_c(\tau^{(2)}) - Q_1Q_2\Big(\frac{\cos\tau^{(1)}\frac{a}{2}\sin\tau^{(2)}\frac{a}{2}}{(\tau^{(1)})^2\tau^{(2)}}\\ & + \frac{\cos\tau^{(2)}\frac{a}{2}\sin\tau^{(1)}\frac{a}{2}}{\tau^{(1)}(\tau^{(2)})^2}\Big) + \frac{Q_2K_c(\tau^{(1)})\sin\tau^{(2)}\frac{a}{2}}{(\tau^{(1)})^2\tau^{(2)}} + \frac{Q_1K_c(\tau^{(2)})\sin\tau^{(1)}\frac{a}{2}}{\tau^{(1)}(\tau^{(2)})^2}\end{aligned}$$

$$+\frac{K_c(\tau^{(1)})K_s(\tau^{(2)})}{(\tau^{(1)})^2\tau^{(2)}}+\frac{K_s(\tau^{(1)})K_c(\tau^{(2)})}{\tau^{(1)}(\tau^{(2)})^2}-\frac{Q_1K_s(\tau^{(2)})\cos\tau^{(1)}\frac{a}{2}}{(\tau^{(1)})^2\tau^{(2)}}$$
$$-\frac{Q_2K_s(\tau^{(1)})\cos\tau^{(2)}\frac{a}{2}}{\tau^{(1)}(\tau^{(2)})^2},$$

$$K_c(\tau^{(j)})=\int\limits_0^{a/2}\frac{\partial K_j(\frac{a}{2},t,\infty)}{\partial t}\cos\tau^{(j)}t\,dt,\ \ K_s(\tau^{(j)})=\int\limits_0^{a/2}\frac{\partial K_j(\frac{a}{2},t,\infty)}{\partial t}\sin\tau^{(j)}t\,dt.$$

To find the asymptotic formulas for eigenvalues, we use the Rouché theorem. Let us denote

$$\Phi(\lambda)=\frac{\sin(\lambda a-\frac{i(p_1+p_2)a}{4})}{\lambda},\tag{4.7}$$

$$\Psi(\lambda)=\varphi(\lambda)-\Phi(\lambda)=\frac{\sin(\tau^{(1)}+\tau^{(2)})\frac{a}{2}}{\tau^{(1)}}-\frac{\sin(\lambda a-\frac{i(p_1+p_2)a}{4})}{\lambda}+\Theta(\lambda).\tag{4.8}$$

Let us consider the contours Γ_n which are the circles centered at $\frac{\pi n}{a}+\frac{i(p_1+p_2)}{4}$ with the radii r, $|r|<\frac{\pi}{2a}$.

For $\lambda\in\Gamma_n$, we have $\lambda=\frac{\pi n}{a}+\frac{i(p_1+p_2)}{4}+re^{i\theta}$.

Let us multiply (4.6) by λ and prove that for all $\lambda\in\Gamma_n$

$$|\lambda\Phi(\lambda)|=\Big|\sin(\lambda a-\frac{i(p_1+p_2)a}{4})\Big|\ge C_1(r),\tag{4.9}$$

where C_1 depends only on r.

The function $|\lambda\Phi(\lambda)|$ is continuous on each Γ_n and therefore it is bounded on Γ_n. Since this function is periodic, for all n we have

$$\min_{\lambda\in\Gamma_1}|\lambda\Phi(\lambda)|=\min_{\lambda\in\Gamma_n}|\lambda\Phi(\lambda)|\le|\lambda\Phi(\lambda)|\le\max_{\lambda\in\Gamma_n}|\lambda\Phi(\lambda)|=\max_{\lambda\in\Gamma_1}|\lambda\Phi(\lambda)|.$$

We denote $C_1(r)=\min\limits_{\lambda\in\Gamma_1}|\lambda\Phi(\lambda)|$ and obtain inequality (4.9).

Let us show now that there exists a constant $C_2(r)$ such that for n large enough ($n\ge N(r)$) the inequality

$$|\lambda\Psi(\lambda)|\le\frac{C_2(r)}{n}\tag{4.10}$$

holds on the circles Γ_n.

The moduli of summands in (4.8), which are the products of the constants Q_1, Q_2 and functions $\sin\tau^{(j)}\frac{a}{2}$, $\cos\tau^{(j)}\frac{a}{2}$, $K_s(\tau^{(j)})$ and $K_c(\tau^{(j)})$, are bounded in the strip $\mathrm{Im}\lambda\in[\frac{(p_1+p_2)}{4}-r,\frac{(p_1+p_2)}{4}+r]$. The factors of the form $\frac{\lambda}{(\tau^{(1)})^\alpha(\tau^{(2)})^\beta}$

($\alpha, \beta = 0, 1, 2,\ \alpha + \beta \geq 2$) satisfy $\frac{\lambda}{(\tau^{(1)})^{\alpha}(\tau^{(2)})^{\beta}} = O\big(\frac{1}{\lambda}\big)$. Therefore, inequality (4.10) will be proved if we can show that

$$\Big|\sin(\tau^{(1)} + \tau^{(2)})\frac{a}{2} - \sin(\lambda a - \frac{i(p_1 + p_2)a}{4})\Big| = O\Big(\frac{1}{n}\Big). \tag{4.11}$$

Since $(\tau^{(j)})^2 = \lambda^2 - ip_j\lambda$, we have

$$\tau^{(j)} = \lambda\sqrt{1 - \frac{ip_j}{\lambda}} = \lambda\Big(1 - \frac{ip_j}{2\lambda} + O\Big(\frac{1}{\lambda^2}\Big)\Big) = \lambda - \frac{ip_j}{2} + O\Big(\frac{1}{\lambda}\Big),$$

$$(\tau^{(1)} + \tau^{(2)})\frac{a}{2} = \lambda a - \frac{ia(p_1 + p_2)}{4} + O\Big(\frac{1}{\lambda}\Big).$$

For $\lambda \in \Gamma_n$,

$$\Big|\sin(\lambda a - \frac{ia(p_1 + p_2)}{4}) - \sin(\tau^{(1)} + \tau^{(2)})\frac{a}{2}\Big|$$

$$= \Big|2\cos\frac{\lambda a - \frac{ia(p_1+p_2)}{4}) + \Big(\lambda a - \frac{ia(p_1+p_2)}{4}) + O\Big(\frac{1}{\lambda}\Big)\Big)}{2}$$

$$\times \sin\frac{\lambda a - \frac{ia(p_1+p_2)}{4}) - \Big(\lambda a - \frac{ia(p_1+p_2)}{4}) + O\Big(\frac{1}{\lambda}\Big)\Big)}{2}\Big|$$

$$= 2\Big|\cos\Big(\lambda a - \frac{ia(p_1 + p_2)}{4} + O\Big(\frac{1}{\lambda}\Big)\Big)\Big|\Big|\sin O\Big(\frac{1}{\lambda}\Big)\Big| \leq C(r)\Big|O\Big(\frac{1}{|\lambda|}\Big)\Big| \leq \frac{C_2(r)}{n}.$$

Taking into account (4.9) and (4.10), we obtain

$$|\lambda\varphi(\lambda)| = |\lambda|\left(|\Phi(\lambda) + \Psi(\lambda)|\right) \geq |\lambda|\left(|\Phi(\lambda)| - |\Psi(\lambda)|\right) \leq C_1(r) - \frac{C_2(r)}{n}.$$

It is clear that for n large enough the equality $C_1(r) - \frac{C_2(r)}{n} > 0$ holds.

The function $\lambda\Phi(\lambda)$ has exactly 1 simple zero ($\lambda_n = \frac{\pi n}{a} + \frac{ia(p_1+p_2)}{4}$) inside each of the contours Γ_n. Therefore, by the Rouche theorem, the function $\lambda\varphi(\lambda)$ also has exactly 1 simple zero inside each of the counters Γ_n for n large enough. Since the radius r of the contours Γ_n can be taken arbitrarily small, for the eigenvalues we have the following asymptomatic expansion:

$$\lambda_n = \frac{\pi n}{a} + \frac{ia(p_1 + p_2)}{4} + \Delta_n, \quad \Delta_n \longrightarrow 0 \quad \text{for} \quad n \longrightarrow \infty. \tag{4.12}$$

Let us substitute (4.12) into (4.6). Taking into account that

$$K_c(\tau^{(j)}) = \int_0^{a/2} \frac{\partial K_j(a/2, t, \infty)}{\partial t} \cos\tau^{(j)}t\,dt \underset{\lambda \longrightarrow \pm\infty}{=} 0,$$

$$K_s(\tau^{(j)}) = \int\limits_0^{a/2} \frac{\partial K_j(a/2, t, \infty)}{\partial t} \sin \tau^{(j)} t \, dt \underset{\lambda \longrightarrow \pm\infty}{=} 0$$

(Lemma 1.4.3 [19, p. 61]) and, expanding in power series in n the right-hand side of (4.6), we obtain the coefficient

$$B_n = \frac{1}{\pi a} \Big[(-1)^n \frac{p_2 - p_1}{4} \operatorname{sh} \frac{(p_2 - p_1)a}{4} - (Q_1 + Q_2) \Big]. \tag{4.13}$$

The theorem is proved.

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