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Eigenfunctions of the Cosine and Sine Transforms

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A description of the eigensubspaces of the cosine and sine operators is given. The spectrum of each of these two operators consists of two eigenvalues 1, -1 and their eigensubspaces are infinite–dimensional. There are many possible bases for these subspaces, but most popular are the ones constructed from the Hermite functions. We present other "bases" which are not discrete orthogonal sequences of vectors, but continuous orthogonal chains of vectors. Our work can be considered to be a continuation and further development of the results obtained by Hardy and Titchmarsh: "Self-reciprocal functions"(Quart. J. Math., Oxford, Ser. 1 (1930)).

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1. The cosine transform $\mathfrak{C}$ and the sine transform $\mathfrak{S}$ are defined by the formulas

$$(\mathfrak{C}x)(t) = \sqrt{\tfrac{2}{\pi}} \int\limits_{\mathbb{R}_+} \cos(t\xi)\, x(\xi)\, d\xi, \quad t \in \mathbb{R}_+\,, \tag{1a}$$

$$(\mathfrak{S}x)(t) = \sqrt{\tfrac{2}{\pi}} \int\limits_{\mathbb{R}_+} \sin(t\xi)\, x(\xi)\, d\xi, \quad t \in \mathbb{R}_+\,, \tag{1b}$$

where $\mathbb{R}_+$ is the positive half-axis, $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$.
For $x \in L^1(\mathbb{R}_+)$, the integrals in (1) are well defined as Lebesgue integrals. If $x(t) \in L^2(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$, then the Parseval equalities hold:

$$\int\limits_{\mathbb{R}_+} |(\mathfrak{C}x)(t)|^2 dt = \int\limits_{\mathbb{R}_+} |x(t)|^2 dt, \tag{2a}$$

$$\int\limits_{\mathbb{R}_+} |(\mathfrak{S}x)(t)|^2 dt = \int\limits_{\mathbb{R}_+} |x(t)|^2 dt. \tag{2b}$$

Thus, the transforms $\mathfrak{C}$ and $\mathfrak{S}$ can both be considered as the linear operators defined on the linear manifold $L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ of the Hilbert space $L^2(\mathbb{R}_+)$, mapping this linear manifold into $L^2(\mathbb{R}_+)$ *isometrically.* Since the set $L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ is dense in $L^2(\mathbb{R}_+)$, each of these operators can be extended to an operator defined on the *whole* space $L^2(\mathbb{R}_+)$, which maps $L^2(\mathbb{R}_+)$ into $L^2(\mathbb{R}_+)$ isometrically. We retain the notation $\mathfrak{C}$ and $\mathfrak{S}$ for the extended operators. In an even broader context, the transformation (1) can be considered for those x, for which the integrals on the right–hand sides are meaningful.

Considered as operators in the Hilbert space $L^2(\mathbb{R}_+)$, the operators $\mathfrak{C}$ and $\mathfrak{S}$ are self-adjoint operators which satisfy the equalities

$$\mathfrak{C}^2 = \mathfrak{I}, \quad \mathfrak{S}^2 = \mathfrak{I}, \tag{3}$$

where $\mathfrak{I}$ is the identity operator in $L^2(\mathbb{R}_+)$. Each of the spectra $\sigma(\mathfrak{C})$ and $\sigma(\mathfrak{S})$ of these operators consists of two points: $+1$ and -1. By $\mathcal{C}_\lambda$ and $\mathcal{S}_\lambda$ we denote the spectral subspaces of the operators $\mathfrak{C}$ and $\mathfrak{S}$, respectively, corresponding to the points $\lambda = 1$ and $\lambda = -1$ of their spectra. These spectral subspaces are eigensubspaces:

$$\mathcal{C}_1 = \{x \in L^2(\mathbb{R}_+) : \mathfrak{C}x = x\}, \quad \mathcal{C}_{-1} = \{x \in L^2(\mathbb{R}_+) : \mathfrak{C}x = -x\}; \tag{4a}$$

$$\mathcal{S}_1 = \{x \in L^2(\mathbb{R}_+) : \mathfrak{S}x = x\}, \quad \mathcal{S}_{-1} = \{x \in L^2(\mathbb{R}_+) : \mathfrak{S}x = -x\}. \tag{4b}$$

Moreover, two orthogonal decompositions hold

$$L^2(\mathbb{R}_+) = \mathcal{C}_1 \oplus \mathcal{C}_{-1}, \quad L^2(\mathbb{R}_+) = \mathcal{S}_1 \oplus \mathcal{S}_{-1}\,. \tag{5}$$

The spectra of the operators $\mathfrak{C}$ and $\mathfrak{S}$ are highly degenerated: the eigensubspaces $\mathcal{C}_\lambda$ and $\mathcal{S}_\lambda$ are infinite-dimensional. Many bases are possible in these subspaces. The best known are the bases formed by the Hermite functions $h_k(t)$ restricted onto $\mathbb{R}_+$.

The Hermite functions $h_k(t)$ are defined as

$$h_k(t) = e^{\frac{t^2}{2}}\,\frac{d^k\,(e^{-t^2})}{dt^k},\ t \in \mathbb{R}, \quad k = 0,\,1,\,2,\,\dots\ . \tag{6}$$

It is known that the system $\{h_k\}_{k=0,1,2,\dots}$ forms an orthogonal basis in the Hilbert space $L^2(\mathbb{R})$. The properties of the Hermite functions h_k as eigenfunctions of the Fourier transform were established by N. Wiener, [1, Ch. 1]. In [1], N. Wiener developed the L^2-theory of the Fourier transform which was based on these properties of the Hermite functions.

The Hermite functions h_k are originally defined on the whole real axis $\mathbb{R}$. The restrictions $h_{k|\mathbb{R}_+}$ of the Hermite functions h_k onto $\mathbb{R}_+$ are considered as the vectors of the Hilbert space $L^2(\mathbb{R}_+)$. Each of two systems $\{h_{2k_{|\mathbb{R}_+}}\}_{k=0,1,2,\dots}$ and

$\{h_{2k+1}{}_{|\mathbb{R}_+}\}_{k=0,1,2,\dots}$ is an orthogonal basis in $L^2(\mathbb{R}_+)$. The systems $\{h_{4l}{}_{|\mathbb{R}_+}\}_{l=0,1,2,\dots}$, $\{h_{4l+2}{}_{|\mathbb{R}_+}\}_{l=0,1,2,\dots}$, $\{h_{4l+1}{}_{|\mathbb{R}_+}\}_{l=0,1,2,\dots}$, and $\{h_{4l+3}{}_{|\mathbb{R}_+}\}_{l=0,1,2,\dots}$ are orthogonal bases of the eigensubspaces $\mathcal{C}_1$, $\mathcal{C}_{-1}$, $\mathcal{S}_1$ and $\mathcal{S}_{-1}$, respectively. We present other "bases" which are not discrete orthogonal sequences of vectors, but continuous orthogonal chains of (generalized) vectors. This is the main goal of this paper. Our work may be considered as a further development of the results given in [2] by Hardy and Titchmarsh. (The contents of [2] and [3] were reproduced in the book [4].)

2. First we discuss the eigenfunctions of the transforms $\mathfrak{C}$ and $\mathfrak{S}$ in the broad sense. These transforms are of the form $x \to \mathfrak{K}x$, where

$$(\mathfrak{K}x)(t) = \int\limits_{\mathbb{R}_+} k(t\xi)x(\xi)d\xi, \tag{7}$$

and k is a function of *one* variable defined on $\mathbb{R}_+$. (It should be mentioned that some operational calculus related to the operators of the form (7) was developed in [5].)

R e m a r k 1. If the integral (7) does not exist as a Lebesgue integral, i.e., the function $k(t\xi)x(\xi)$ of the variable ξ is not summable, then a meaning can be attached to the integral (7) by some regularization procedure. We use the regularization procedure

$$\int\limits_{\mathbb{R}_+} k(t\xi)x(\xi)d\xi = \lim_{\varepsilon\to+0}\int\limits_{\mathbb{R}_+} e^{-\varepsilon\xi}k(t\xi)x(\xi)d\xi, \tag{8}$$

and the regularization procedure

$$\int\limits_{\mathbb{R}_+} k(t\xi)x(\xi)d\xi = \lim_{R\to+\infty}\int\limits_0^R k(t\xi)x(\xi)d\xi\,. \tag{9}$$

If for some $a \in \mathbb{C}$ both integrals

$$\int\limits_{\mathbb{R}_+} k(t\xi)\xi^{-a}d\xi \quad \text{and} \quad \int\limits_{\mathbb{R}_+} k(t\xi)\xi^{a-1}d\xi \tag{10}$$

have a meaning for every positive t, then, changing the variable $t\xi \to \xi$, we obtain

$$\mathfrak{K}\,t^{-a} = \varkappa(a)\,t^{a-1}, \quad \mathfrak{K}\,t^{a-1} = \varkappa(1-a)\,t^{-a}, \tag{11}$$

where

$$\varkappa(a) = \int\limits_{\mathbb{R}_+} k(\xi)\xi^{-a}d\xi, \quad \varkappa(1-a) = \int\limits_{\mathbb{R}_+} k(\xi)\xi^{a-1}d\xi. \tag{12}$$

Equalities (11) mean that the subspace (two-dimensional if $a \neq 1/2$) generated by the functions t^{-a} and t^{a-1} is invariant with respect to the transformation $\mathcal{K}$ and that the matrix of this operator in the basis t^{-a}, t^{a-1} is: $\left\| \begin{smallmatrix} 0 & \varkappa(1-a) \\ \varkappa(a) & 0 \end{smallmatrix} \right\|$. Thus, assuming that $\varkappa(a) \neq 0$, $\varkappa(1-a) \neq 0$, we obtain that the functions

$$\sqrt{\varkappa(1-a)}t^{-a} + \sqrt{\varkappa(a)}t^{a-1} \quad \text{and} \quad \sqrt{\varkappa(1-a)}t^{-a} - \sqrt{\varkappa(a)}t^{a-1} \tag{13}$$

are the eigenfunctions of the transform $\mathcal{K}$ corresponding to the eigenvalues

$$\lambda_+ = \sqrt{\varkappa(a)\varkappa(1-a)} \quad \text{and} \quad \lambda_- = -\sqrt{\varkappa(a)\varkappa(1-a)}, \tag{14}$$

respectively.

To find the eigenfunctions of the form (13) for cosine and sine transforms $\mathcal{C}$ and $\mathcal{S}$, we have to calculate the constants (12) corresponding to the functions

$$k_c(\tau) = \sqrt{\frac{2}{\pi}}\cos\tau \quad \textit{and} \quad k_s(\tau) = \sqrt{\frac{2}{\pi}}\sin\tau, \tag{15}$$

which generate the kernels of these integral transforms. This is accomplished in the following

Lemma 1. *Let ζ belong to the strip $0 < \operatorname{Re}\zeta < 1$. Then*

1.
$$\int\limits_0^\infty (\cos s)\, s^{\zeta-1}\, ds = \Big(\cos\, \frac{\pi}{2}\zeta\Big)\, \Gamma(\zeta)\,, \tag{16a}$$

$$\int\limits_0^\infty (\sin s)\, s^{\zeta-1}\, ds = \Big(\sin\, \frac{\pi}{2}\zeta\Big)\, \Gamma(\zeta)\,, \tag{16b}$$

where Γ is the Euler Gamma-function and the integrals in (16) *are understood in the sense*

$$\int\limits_0^\infty \begin{Bmatrix} \cos s \\ \sin s \end{Bmatrix} s^{\,\zeta-1}\, ds$$

$$= \lim_{R\to+\infty} \int\limits_0^R \begin{Bmatrix} \cos s \\ \sin s \end{Bmatrix} s^{\,\zeta-1}\, ds = \lim_{\varepsilon\to+0} \int\limits_0^{+\infty} e^{-\varepsilon s} \begin{Bmatrix} \cos s \\ \sin s \end{Bmatrix} s^{\,\zeta-1}\, ds\,.$$

2. *The above limits exist uniformly with respect to ζ from any fixed compact subset of the strip* $0 < \operatorname{Re}\zeta < 1$.

3. *Given $\delta > 0$, then for any ζ from the strip $\delta < \operatorname{Re}\zeta < 1-\delta$ and for any $R \in (0,+\infty)$, $\varepsilon \in (0,+\infty)$, the estimates*

$$\left|\int\limits_0^R \begin{Bmatrix}\cos s\\ \sin s\end{Bmatrix} s^{\zeta-1}ds\right| \le C(\delta)e^{\frac{\pi}{2}|\operatorname{Im}\zeta|}, \tag{17a}$$

$$\left|\int\limits_0^{+\infty} e^{-\varepsilon s}\begin{Bmatrix}\cos s\\ \sin s\end{Bmatrix} s^{\zeta-1}ds\right| \le C(\delta)e^{\frac{\pi}{2}|\operatorname{Im}\zeta|} \tag{17b}$$

hold, where $C(\delta) < \infty$ does not depend on ζ, R and ε.

We omit the proof of Lemma 1. This lemma can be proved by a standard method using integration in the complex plane. ■

According to Lemma 1, the integrals

$$\int\limits_0^\infty \begin{Bmatrix}k_c(s)\\ k_s(s)\end{Bmatrix} s^{-a}\,ds \quad \text{and} \quad \int\limits_0^\infty \begin{Bmatrix}k_c(s)\\ k_s(s)\end{Bmatrix} s^{a-1}\,ds,$$

where k_c and k_s, (15), are the functions generating the kernels of the integral transformations $\mathfrak{C}$ and $\mathfrak{S}$, exist for every a such that $0 < \operatorname{Re} a < 1$, or, amounting to the same, $0 < \operatorname{Re}(1-a) < 1$.

The constants $\varkappa_c(a)$ and $\varkappa_c(1-a)$, corresponding to the function $k_c(\tau) = \sqrt{\frac{2}{\pi}}\cos\tau$, are:

$$\varkappa_c(a) = \sqrt{\frac{2}{\pi}}\sin\frac{\pi a}{2}\,\Gamma(1-a), \quad \varkappa_c(1-a) = \sqrt{\frac{2}{\pi}}\cos\frac{\pi a}{2}\,\Gamma(a). \tag{18a}$$

The constants $\varkappa_s(a)$ and $\varkappa_s(1-a)$, corresponding to the function $k_s(\tau) = \sqrt{\frac{2}{\pi}}\sin\tau$, are:

$$\varkappa_s(a) = \sqrt{\frac{2}{\pi}}\cos\frac{\pi a}{2}\,\Gamma(1-a), \quad \varkappa_s(1-a) = \sqrt{\frac{2}{\pi}}\sin\frac{\pi a}{2}\,\Gamma(a). \tag{18b}$$

3. Later, we will have to transform the expression (18) for the constants $\varkappa_c$

and $\varkappa_s$ using the following identities for the Euler Gamma-function $\Gamma(\zeta)$:

$$\Gamma(\zeta+1)=\zeta\Gamma(\zeta)\,, \qquad \text{see [6], } \mathbf{12.12}, \tag{19a}$$

$$\Gamma(\zeta)\Gamma(1-\zeta)=\frac{\pi}{\sin\pi\zeta}\,, \qquad \text{see [6], } \mathbf{12.14}, \tag{19b}$$

$$\Gamma(\zeta)\Gamma\Big(\zeta+\frac{1}{2}\Big)=2\sqrt{\pi}\,2^{-2\zeta}\Gamma(2\zeta), \qquad \text{see [6], } \mathbf{12.15}. \tag{19c}$$

Lemma 2. *The following identities hold:*

$$\sqrt{\frac{2}{\pi}}\,\Big(\cos\frac{\pi}{2}\zeta\Big)\,\Gamma(\zeta)=2^{\zeta-\frac{1}{2}}\,\frac{\Gamma\big(\frac{\zeta}{2}\big)}{\Gamma\big(\frac{1}{2}-\frac{\zeta}{2}\big)}\,, \tag{20a}$$

$$\sqrt{\frac{2}{\pi}}\,\Big(\sin\frac{\pi}{2}\zeta\Big)\,\Gamma(\zeta)=2^{\zeta-\frac{1}{2}}\,\frac{\Gamma\big(\frac{1}{2}+\frac{\zeta}{2}\big)}{\Gamma\big(1-\frac{\zeta}{2}\big)}\,. \tag{20b}$$

P r o o f. From (19b) it follows that

$$\cos\frac{\pi}{2}\zeta=\frac{\pi}{\Gamma\big(\frac{1}{2}-\frac{\zeta}{2}\big)\,\Gamma\big(\frac{1}{2}+\frac{\zeta}{2}\big)}\,.$$

From (19c) it follows that

$$\Gamma(\zeta)=\pi^{-\frac{1}{2}}\,\Gamma\big(\tfrac{\zeta}{2}\big)\,\Gamma\big(\tfrac{1}{2}+\tfrac{\zeta}{2}\big)\,2^{\zeta-1}\,.$$

Combining the last two formulas, we obtain (20a). Combining the last formula with the formula

$$\sin\frac{\pi}{2}\zeta=\frac{\pi}{\Gamma\big(\frac{\zeta}{2}\big)\,\Gamma\big(1-\frac{\zeta}{2}\big)}\,,$$

we obtain (20b). ■

Lemma 3. *The values* $\varkappa_c(a)$, $\varkappa_c(1-a)$, $\varkappa_s(a)$, $\varkappa_s(1-a)$, *which appear as the coefficients of the linear combinations* (13), *are:*

$$\varkappa_c(a)=2^{\frac{1}{2}-a}\frac{\Gamma\big(\frac{1}{2}-\frac{a}{2}\big)}{\Gamma\big(\frac{a}{2}\big)}, \qquad \varkappa_c(1-a)=2^{a-\frac{1}{2}}\,\frac{\Gamma\big(\frac{a}{2}\big)}{\Gamma\big(\frac{1}{2}-\frac{a}{2}\big)}, \tag{21a}$$

$$\varkappa_s(a)=2^{\frac{1}{2}-a}\frac{\Gamma\big(1-\frac{a}{2}\big)}{\Gamma\big(\frac{1}{2}+\frac{a}{2}\big)}, \qquad \varkappa_s(1-a)=2^{a-\frac{1}{2}}\,\frac{\Gamma\big(\frac{1}{2}+\frac{a}{2}\big)}{\Gamma\big(1-\frac{a}{2}\big)}. \tag{21b}$$

4. From the expressions (21) we can see that the products $\varkappa_c(a)\varkappa_c(1-a)$ and $\varkappa_s(a)\varkappa_s(1-a)$ do not depend on a,

$$\varkappa_c(a)\varkappa_c(1-a)=1, \quad \varkappa_s(a)\varkappa_s(1-a)=1 \quad 0<\operatorname{Re}a<1.$$

Theorem 1. *Let $a \in \mathbb{C}$, $0 < \operatorname{Re} a < 1$, $a \neq \frac{1}{2}$, and $\varkappa_c(a)$, $\varkappa_c(1-a)$, $\varkappa_s(a)$, $\varkappa_s(1-a)$ be the values appeared in* (21).
Then:

1. *The functions*

$$E_c^+(t,a) = \sqrt{\varkappa_c(1-a)}\,t^{-a} + \sqrt{\varkappa_c(a)}\,t^{a-1}, \tag{22a}$$
$$E_c^-(t,a) = \sqrt{\varkappa_c(1-a)}\,t^{-a} - \sqrt{\varkappa_c(a)}\,t^{a-1} \tag{22b}$$

of the variable $t \in \mathbb{R}_+$ are the eigenfunctions (in the broad sense) of the cosine transform $\mathfrak{C}$ corresponding to the eigenvalues $+1$ and -1, respectively,

$$E_c^+(t,a) = \lim_{R\to\infty} \sqrt{\tfrac{2}{\pi}} \int\limits_0^R \cos(t\xi)\, E_c^+(\xi,a)\, d\xi,$$
$$E_c^-(t,a) = -\lim_{R\to\infty} \sqrt{\tfrac{2}{\pi}} \int\limits_0^R \cos(t\xi)\, E_c^-(\xi,a)\, d\xi.$$

2. *The functions*

$$E_s^+(t,a) = \sqrt{\varkappa_s(1-a)}\,t^{-a} + \sqrt{\varkappa_s(a)}\,t^{a-1}, \tag{23a}$$
$$E_s^-(t,a) = \sqrt{\varkappa_s(1-a)}\,t^{-a} - \sqrt{\varkappa_s(a)}\,t^{a-1} \tag{23b}$$

of the variable $t \in \mathbb{R}_+$ are the eigenfunctions (in the broad sense) of the sine transform $\mathfrak{S}$ corresponding to the eigenvalues $+1$ and -1, respectively,

$$E_s^+(t,a) = \lim_{R\to\infty} \sqrt{\tfrac{2}{\pi}} \int\limits_0^R \sin(t\xi)\, E_s^+(\xi,a)\, d\xi, \tag{24}$$
$$E_s^-(t,a) = -\lim_{R\to\infty} \sqrt{\tfrac{2}{\pi}} \int\limits_0^R \sin(t\xi)\, E_s^-(\xi,a)\, d\xi. \tag{25}$$

For the fixed $t \in (0,\infty)$, the limits exist uniformly with respect to a, from any compact subset of the strip $0 < \operatorname{Re} a < 1$.

R e m a r k 2. In (22) and (23), the values of the square roots $\sqrt{\varkappa(a)}$ and $\sqrt{\varkappa(1-a)}$ should be chosen such that their products equal 1.

R e m a r k 3. For $a = \frac{1}{2}$, there is only one eigenfunction

$$E\big(t, \tfrac{1}{2}\big) = 2t^{-\frac{1}{2}}.$$

R e m a r k 4. Since

$$E_c^+(t,a) = E_c^+(t,1-a), \quad E_c^-(t,a) = -E_c^-(t,1-a), \tag{26a}$$

$$E_s^+(t,a) = E_s^+(t,1-a), \quad E_s^-(t,a) = -E_s^-(t,1-a), \tag{26b}$$

each eigenfunction appears in the family $\{E_{c,s}^{\pm}(t,a)\}_{0<\operatorname{Re} a<1}$ twice. To avoid this redundancy, we should consider the family where only one of the points a or $1-a$ appears.

5. If $0 < \operatorname{Re} a < 1$ and $x(t)$ is any of the eigenfunctions of the form either (22), or (23), then the integral $\int\limits_{\mathbb{R}_+} |x(t)|^2$ diverges. Thus, none of these eigenfunctions belong to $L^2(\mathbb{R}_+)$. This integral diverges both at the points $t = +0$ and $t = +\infty$. However, this integral diverges variously for a with $\operatorname{Re} a = \frac{1}{2}$ and for a with $\operatorname{Re} a \neq \frac{1}{2}$. If $\operatorname{Re} a = \frac{1}{2}$, then the integrals diverge *logarithmically* both at $t = +0$ and $t = +\infty$. If $\operatorname{Re} a \neq \frac{1}{2}$, then the integrals diverge more strongly: *powerwisely.* We try to construct the eigenfunctions of the operator $\mathfrak{C}$ (of the operator $\mathfrak{S}$) from L^2 as the continuous combinations of the eigenfunctions of the form (22) (of the form (23)). Our hope is that the singularities of the "*continuous linear combinations*" of eigenfunctions, which are in some sense an *averaging* of the eigenfunctions of the family, are weaker than the singularities of individual eigenfunctions. These continuous linear combinations should *not* include the eigenfunctions of the form (22) and (23) with $a:\ \operatorname{Re} a \neq \frac{1}{2}$. The singularities of eigenfunctions with $a:\ \operatorname{Re} a \neq \frac{1}{2}$ are too strong and can not disappear by averaging. Thus, we have to restrict ourselves to a's of the form $a = \frac{1}{2} + i\tau$, $\tau \in \mathbb{R}$.

Considering the case $\operatorname{Re} a = \frac{1}{2}$ in more detail, we introduce special notation for the eigenfunctions $E_{c,s}^{\pm}(t, \frac{1}{2} + i\tau)\}$:

$$e_c^+(t,\tau) = \tfrac{1}{2\sqrt{\pi}} E_c^+(t, \tfrac{1}{2} + i\tau), \quad e_c^-(t,\tau) = \tfrac{1}{2i\sqrt{\pi}} E_c^-(t, \tfrac{1}{2} + i\tau), \tag{27a}$$

$$e_s^+(t,\tau) = \tfrac{1}{2\sqrt{\pi}} E_s^+(t, \tfrac{1}{2} + i\tau), \quad e_s^-(t,\tau) = \tfrac{1}{2i\sqrt{\pi}} E_s^-(t, \tfrac{1}{2} + i\tau). \tag{27b}$$

(We include the normalizing factor $\frac{1}{2\sqrt{\pi}}$ in the definition of the functions $e_{c,s}^{\pm}$.) According to (21), (22), the functions $e_{c,s}^{\pm}(t,\tau)$ can be expressed as

$$e_c^+(t,\tau) = \tfrac{1}{2\sqrt{\pi}} \Big(t^{-\frac{1}{2} - i\tau} c(\tau) + t^{-\frac{1}{2} + i\tau} c(-\tau) \Big), \tag{28a}$$

$$e_c^-(t,\tau) = \tfrac{1}{2i\sqrt{\pi}} \Big(t^{-\frac{1}{2} - i\tau} c(\tau) - t^{-\frac{1}{2} + i\tau} c(-\tau) \Big), \tag{28b}$$

$$e_s^+(t,\tau) = \tfrac{1}{2\sqrt{\pi}}\Big(t^{-\frac{1}{2}-i\tau}\,s(\tau) + t^{-\frac{1}{2}+i\tau}\,s(-\tau)\Big), \tag{29a}$$

$$e_s^-(t,\tau) = \tfrac{1}{2i\sqrt{\pi}}\Big(t^{-\frac{1}{2}-i\tau}\,s(\tau) - t^{-\frac{1}{2}+i\tau}\,s(-\tau)\Big), \tag{29b}$$

where $c(\tau)$, $s(\tau)$ are the "phase factors"

$$c(\tau) = 2^{i\frac{\tau}{2}} \exp\big\{i \arg\Gamma(\tfrac{1}{4} + i\tfrac{\tau}{2})\big\}, \quad -\infty < \tau < \infty, \tag{30a}$$

$$s(\tau) = 2^{i\frac{\tau}{2}} \exp\big\{i \arg\Gamma(\tfrac{3}{4} + i\tfrac{\tau}{2})\big\}, \quad -\infty < \tau < \infty. \tag{30b}$$

In (30), $\exp\{i\arg\Gamma(\zeta)\} = \frac{\Gamma(\zeta)}{|\Gamma(\zeta)|}$.

Since $c(\tau) = \overline{c(-\tau)}$, $s(\tau) = \overline{s(-\tau)}$ for real τ, the values of the functions $e_c^+(t,\tau)$, $e_c^-(t,\tau)$, $e_s^+(t,\tau)$, $e_s^-(t,\tau)$ are *real* for $t \in (0,\infty)$, $\tau \in (0,\infty)$.

R e m a r k 5. The parameter τ, which enumerates the families $\{e_c^\pm(t,\tau\}$, $\{e_s^\pm(t,\tau\}$, runs over the interval $(0,\infty)$. There is no need to consider negative τ. (See Remark 4).

6. Let us introduce four integral transforms $\mathfrak{T}_c^+$, $\mathfrak{T}_c^-$, $\mathfrak{T}_s^+$, $\mathfrak{T}_s^-$.
For $\phi(t) \in L^1(\mathbb{R}_+)$ and $t > 0$, let us define

$$(\mathfrak{T}_c^+\phi)(t) = \int\limits_{\mathbb{R}_+} e_c^+(t,\tau)\phi(\tau)\,d\tau, \quad (\mathfrak{T}_c^-\phi)(t) = \int\limits_{\mathbb{R}_+} e_c^-(t,\tau)\phi(\tau)\,d\tau, \tag{31a}$$

$$(\mathfrak{T}_s^+\phi)(t) = \int\limits_{\mathbb{R}_+} e_s^+(t,\tau)\phi(\tau)\,d\tau, \quad (\mathfrak{T}_s^-\phi)(t) = \int\limits_{\mathbb{R}_+} e_s^-(t,\tau)\phi(\tau)\,d\tau. \tag{31b}$$

Lemma 4. *If $\phi(\tau) \in L^1(\mathbb{R}_+)$, and $x(t) = (\mathfrak{T}\boldsymbol{\phi})(t)$, where $\mathfrak{T}$ is any of the above–introduced four transformations $\mathfrak{T}_{c,s}^\pm$, then the function $x(t)$ is continuous on the interval $(0,\infty)$ and the estimate*

$$|x(t)| \leq \tfrac{1}{\sqrt{\pi}}\|\phi\|_{L^1(\mathbb{R}_+)} \cdot t^{-\frac{1}{2}}, \quad 0 < t < \infty, \tag{32}$$

holds.

P r o o f. Let $e(t,\tau)$ be any of the four above–introduced functions $e_c^+(t,\tau)$, $e_c^-(t,\tau)$, $e_s^+(t,\tau)$, $e_s^-(t,\tau)$. The function $e(t,\tau)$ is continuous with respect to t at each $t > 0, \tau > 0$ and satisfies the estimate

$$|e(t,\tau| \leq \tfrac{1}{\sqrt{\pi}} t^{-\frac{1}{2}}, \quad 0 < t < \infty, 0 < \tau < \infty. \tag{33}$$

Now Lemma 4 is a consequence of the standard results of the Lebesgue integration theory. ∎

Theorem 2. *Let $\phi(\tau)$ be a function satisfying the condition*

$$\int\limits_0^\infty |\phi(\tau)| e^{\frac{\pi}{2}\tau}\, d\tau < \infty, \tag{34}$$

and

$$x_c^+(t) = (\mathfrak{T}_c^+\phi)(t), \quad x_c^-(t) = (\mathfrak{T}_c^-\phi)(t), \tag{35}$$
$$x_s^+(t) = (\mathfrak{T}_s^+\phi)(t), \quad x_s^-(t) = (\mathfrak{T}_c^-\phi)(t). \tag{36}$$

Then the functions $x_c^+(t)$, $x_c^-(t)$ are the eigenfunctions (in the broad sense) of the cosine transform $\mathfrak{C}$, and the functions $x_s^+(t)$, $x_s^-(t)$ are the eigenfunctions (in the broad sense) of the sine transform $\mathfrak{S}$, i.e.,

$$x_c^+(t) = \lim_{R\to\infty} \sqrt{\tfrac{2}{\pi}} \int\limits_0^R \cos(t\xi)\, x_c^+(\xi)\, d\xi, \tag{37a}$$

$$x_c^-(t) = -\lim_{R\to\infty} \sqrt{\tfrac{2}{\pi}} \int\limits_0^R \cos(t\xi)\, x_c^-(\xi)\, d\xi \tag{37b}$$

and

$$x_s^+(t) = \lim_{R\to\infty} \sqrt{\tfrac{2}{\pi}} \int\limits_0^R \sin(t\xi)\, x_s^+(\xi)\, d\xi, \tag{38a}$$

$$x_s^-(t) = -\lim_{R\to\infty} \sqrt{\tfrac{2}{\pi}} \int\limits_0^R \sin(t\xi)\, x_s^-(\xi)\, d\xi \tag{38b}$$

for every $t \in (0,\infty)$. In particular, in (37), (38) the limits exist.

P r o o f. According to Theorem 1 and (27),

$$e_c^+(t,\tau) = \lim_{R\to\infty} \sqrt{\tfrac{2}{\pi}} \int\limits_0^R \cos(t\xi) e_c^+(\xi,\tau)\, d\xi \quad \text{for every } t,\ \tau.$$

Multiplying by $\phi(\tau)$ and integrating with respect to τ, we obtain

$$x_c^+(t) = \sqrt{\tfrac{2}{\pi}} \int\limits_0^\infty \Bigg(\lim_{R\to\infty} \int\limits_0^R \cos(t\xi) e_c^+(\xi,\tau)\, d\xi \Bigg) \phi(\tau)\, d\tau.$$

From (17) we obtain the estimate

$$\Bigl|\int\limits_0^R \cos(t\xi)e_c^+(\xi,\tau)\,d\xi\Bigr| \le Ct^{-\frac{1}{2}}e^{\frac{\pi}{2}\tau}, \quad \forall\; R<\infty,\, \tau\in\mathbb{R}_+,\, t\in\mathbb{R}_+,$$

where the value $C<\infty$ does not depend on R, τ, t. This estimate and condition (34) for the function $\phi(t)$ allow us to apply the Lebesgue dominated convergence theorem

$$\int\limits_0^\infty \Bigl(\lim_{R\to\infty}\int\limits_0^R \cos(t\xi)e_c^+(\xi,\tau)\,d\xi\Bigr)\phi(\tau)\,d\tau$$
$$= \lim_{R\to\infty}\int\limits_0^\infty \Bigl(\int\limits_0^R \cos(t\xi)e_c^+(\xi,\tau)\,d\xi\Bigr)\phi(\tau)\,d\tau. \quad (39)$$

Thus,

$$x_c^+(t) = \lim_{R\to\infty}\sqrt{\tfrac{2}{\pi}}\int\limits_0^\infty \Bigl(\int\limits_0^R \cos(t\xi)e_c^+(\xi,\tau)\,d\xi\Bigr)\phi(\tau)\,d\tau.$$

On the other hand, using estimate (33) for $e_c^+(\xi,\tau)$, we can justify the change of order of integration in the series integral which appears on the right–hand side of the above equality. For any finite R,

$$\int\limits_0^\infty \Bigl(\int\limits_0^R \cos(t\xi)e_c^+(\xi,\tau)\,d\xi\Bigr)\phi(\tau)\,d\tau$$
$$= \int\limits_0^R \cos(t\xi)\Bigl(\int\limits_0^\infty e_c^+(\xi,\tau)\phi(\tau)\,d\tau\Bigr)d\xi = \int\limits_0^R \cos(t\xi)x_c^+(\xi)\,d\xi.$$

Finally, we obtain the equality $x_c^+(t) = \lim_{R\to\infty}\int\limits_0^R \cos(t\xi)x_c^+(\xi)\,d\xi$, i.e., equality (37a) for the function x_c^+. Equality (37b) for the function x_c^- and equalities (38) for the functions x_s^+, x_s^+ can be obtained analogously. ∎

R e m a r k 6. In Theorem 2 we assume that the function ϕ satisfies condition (34). Assuming only that $\int\limits_0^\infty |\phi(\tau)|\,d\tau<\infty$, we can not justify equality (39). To apply the Lebesgue dominated convergence theorem, we need the estimate

$$\sup_{\substack{R\in(0,\infty)\\ \tau\in(-\infty,\infty)}} \left| \int\limits_0^R (\cos\xi)\cdot \xi^{-\frac{1}{2}+i\tau}\, d\xi \right| < \infty.$$

We are, however, able to establish (17), but this estimate is not strong enough.

The question of whether equalities (37), (38) hold under the assumption $\int\limits_0^\infty |\phi(\tau)|\, d\tau < \infty$ *remains open.*

7. Our considerations in the context of the L^2-theory on the operators $\mathfrak{C}$ and $\mathfrak{S}$ are based on the L^2-theory for the Melline transform. (See the article "Melline Transform" on page 192 of [7, Vol. 6] and references there.) The Melline transform $\mathfrak{M}$ is defined by

$$(\mathfrak{M}f)(\zeta) = \int\limits_0^\infty f(t)t^{\zeta-1}\, dt.$$

If the function $f(t) \in L^2(\mathbb{R}_+)$ is *compactly* supported in the *open* interval $(0,\infty)$, then the function $\Phi(\zeta) = (\mathfrak{M}f)(\zeta)$ of the variable ζ is defined in the whole complex ζ-plane and is holomorphic. The function $f(t)$ can be recovered from the function $\Phi = \mathfrak{M}f$ by the formula

$$f(t) = \tfrac{1}{2\pi i} \int\limits_{\operatorname{Re}\zeta = c} \Phi(\zeta)\, t^{-\zeta}\, d\zeta,$$

where c is an arbitrary real number. Moreover, the Parseval equality

$$\int\limits_0^\infty |f(t)|^2 dt = \frac{1}{2\pi} \int\limits_{\operatorname{Re}\zeta=\frac{1}{2}} |\Phi(\zeta)|^2\, |d\zeta|$$

holds (from which we recognize the significance of the vertical line $\operatorname{Re}\zeta = \frac{1}{2}$). Thus the Melline transform $\mathfrak{M}$ generates the linear operator defined on the set of all compactly supported functions f from $L^2(\mathbb{R}_+)$ which maps this set isometrically into the space $L^2\big(\operatorname{Re}\zeta = \frac{1}{2}\big)$ of the functions defined on the vertical line $\operatorname{Re}\zeta = \frac{1}{2}$ and which are square-integrable. Since the set of all compactly supported functions f is dense in $L^2(\mathbb{R}_+)$, this operator can be extended to an isometrical operator defined on the whole $L^2(\mathbb{R}_+)$ which maps $L^2(\mathbb{R}_+)$ isometrically into $L^2\big(\operatorname{Re}\zeta = \frac{1}{2}\big)$. We will continue to denote this extended operator by $\mathfrak{M}$.

It turns out that the operator $\mathfrak{M}$ maps the space $L^2(\mathbb{R}_+)$ *onto* the whole space $L^2\big(\operatorname{Re}\zeta = \frac{1}{2}\big)$. The inverse operator $\mathfrak{M}^{-1}$ is defined *everywhere* on $L^2\big(\operatorname{Re}\zeta = \frac{1}{2}\big)$. If $\Phi \in L^2\big(\operatorname{Re}\zeta = \frac{1}{2}\big)$, then the function

$$f(t) = (\mathfrak{M}^{-1}\Phi)(t)$$

is defined as an $L^2(\mathbb{R}_+)$-function and can be expressed as

$$f(t) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \Phi\big(\tfrac{1}{2} + i\tau\big)\, t^{-\frac{1}{2}-i\tau} d\tau, \quad 0 < t < \infty. \tag{40a}$$

Furthermore, the function

$$\Phi\big(\tfrac{1}{2} + i\tau\big) = (\mathfrak{M}f)\big(\tfrac{1}{2} + i\tau\big)$$

can be expressed as

$$\Phi\big(\tfrac{1}{2} + i\tau\big) = \int_0^{\infty} f(t)\, t^{-\frac{1}{2}+i\tau} dt, \quad -\infty < \tau < \infty. \tag{40b}$$

The pair of formulas (40a) and (40b) together with the Parseval equality

$$\int_0^{\infty} |f(t)|^2 dt = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \big|\Phi\big(\tfrac{1}{2} + i\tau\big)\big|^2\, d\tau \tag{40c}$$

make up the most import part of the L^2-theory of the Melline transform.

8. Developing the L^2-theory of the cosine and sine transforms, we first of all prove

Lemma 5. *Let $\phi(t) \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. Then*

$$\int_{\mathbb{R}_+} |(\mathfrak{T}\phi)(t)|^2 dt = \int_{\mathbb{R}_+} |\phi(\tau)|^2 d\tau\,, \tag{41}$$

where $\mathfrak{T}$ is any of the above–introduced (see (31)) four transformations $\mathfrak{T}^{\pm}_{c,s}$.

P r o o f. The proof is based on the Parseval equality for the Melline transform. We present the transformations $\mathfrak{T}^{\pm}_{c,s}$ as the inverse Melline transforms. Given a function $\phi(\tau)$ defined for $\tau \in (0, \infty)$, we introduce the functions

$$\Phi^{+}_{c}\big(\tfrac{1}{2} + i\tau\big) = \sqrt{\pi}\, c(\tau)\, \phi(|\tau|), \tag{42a}$$

$$\Phi^{-}_{c}\big(\tfrac{1}{2} + i\tau\big) = \tfrac{1}{i}\operatorname{sign}(\tau)\sqrt{\pi}\, c(\tau)\, \phi(|\tau|), \tag{42b}$$

and

$$\Phi_s^+\left(\tfrac{1}{2}+i\tau\right) = \qquad\qquad \sqrt{\pi}\, s(\tau)\, \phi(|\tau|), \tag{43a}$$
$$\Phi_s^-\left(\tfrac{1}{2}+i\tau\right) = \tfrac{1}{i}\,\mathrm{sign}\,(\tau)\sqrt{\pi}\, s(\tau)\, \phi(|\tau|), \tag{43b}$$

which are defined for $\tau \in (-\infty, \infty)$. Here $c(\tau)$, $s(\tau)$ are the "phase factors" introduced in (30). It is clear that $\left|\Phi\left(\tfrac{1}{2}+i\tau\right)\right| = \sqrt{\pi}\phi(|\tau|)|$, thus

$$\int\limits_{-\infty}^{\infty} \left|\Phi\left(\tfrac{1}{2}+i\tau\right)\right|^2 d\tau = 2\pi \int\limits_{0}^{\infty} |\phi(\tau)|^2 d\tau,$$

where Φ is any of the four functions Φ_c^+, Φ_c^-, Φ_s^+, Φ_s^-. Comparing (31a), (28a) and (42a), we can see that the function $(\mathfrak{T}_c^+\phi)(t)$ can be interpreted as the inverse Melline transform of the function Φ_c^+,

$$(\mathfrak{T}_c^+\phi)(t) = \tfrac{1}{2\pi} \int\limits_{-\infty}^{\infty} t^{-\frac{1}{2}-i\tau}\Phi_c^+\left(\tfrac{1}{2}+i\tau\right) d\tau. \tag{44a}$$

The Parseval equality transform, as applied to the inverse Melline transform of the function $\varphi_c^+(\tau)$, yields

$$\int\limits_{0}^{\infty} |(\mathfrak{T}_c^+\phi)(t)|^2 dt = \tfrac{1}{2\pi} \int\limits_{-\infty}^{\infty} \left|\Phi_c^+\left(\tfrac{1}{2}+i\tau\right)\right|^2 d\tau = \int\limits_{0}^{\infty} |\phi(\tau)|^2 d\tau.$$

This is equality (41) for the transform $\mathfrak{T}_c^+$.

The functions $\mathfrak{T}_c^-\phi$, $\mathfrak{T}_s^+\phi$, $\mathfrak{T}_s^-\phi$ can also be interpreted as the inverse Melline transforms:

$$(\mathfrak{T}_c^-\phi)(t) = \tfrac{1}{2\pi} \int\limits_{-\infty}^{\infty} t^{-\frac{1}{2}-i\tau}\Phi_c^-\left(\tfrac{1}{2}+i\tau\right) d\tau \tag{44b}$$

and

$$(\mathfrak{T}_s^+\phi)(t) = \tfrac{1}{2\pi} \int\limits_{-\infty}^{\infty} t^{-\frac{1}{2}-i\tau}\Phi_s^+\left(\tfrac{1}{2}+i\tau\right) d\tau, \tag{45a}$$

$$(\mathfrak{T}_s^-\phi)(t) = \tfrac{1}{2\pi} \int\limits_{-\infty}^{\infty} t^{-\frac{1}{2}-i\tau}\Phi_s^-\left(\tfrac{1}{2}+i\tau\right) d\tau. \tag{45b}$$

The Parseval equalities, as applied to the inverse Melline transform of the functions Φ_c^-, Φ_s^+ and Φ_s^-, yield equalities (41) for the transforms $\mathfrak{T}_c^-$, $\mathfrak{T}_s^+$ and $\mathfrak{T}_s^-$, respectively. ■

9. According to Lemma 5, the operators $\mathfrak{T}_c^+$, $\mathfrak{T}_c^-$, $\mathfrak{T}_s^+$, $\mathfrak{T}_s^-$ are linear operators each of which is defined on the linear manifold $L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ of the Hilbert space $L^2(\mathbb{R}_+)$ and which maps this linear manifold into $L^2(\mathbb{R}_+)$ *isometrically.* Since the set $L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ is dense in $L^2(\mathbb{R}_+)$, each of these operators can be extended to an operator defined on the whole space $L^2(\mathbb{R}_+)$, which maps $L^2(\mathbb{R}_+)$ into $L^2(\mathbb{R}_+)$ isometrically. We will continue to write $\mathfrak{T}_c^+$, $\mathfrak{T}_c^-$, $\mathfrak{T}_s^+$ and $\mathfrak{T}_s^-$ for the extended operators.

We now consider the operators $\mathfrak{T}_c^+$, $\mathfrak{T}_c^-$, $\mathfrak{T}_s^+$, $\mathfrak{T}_s^-$ as the operators defined on *all* of $L^2(\mathbb{R}_+)$, mapping $L^2(\mathbb{R}_+)$ into $L^2(\mathbb{R}_+)$ isometrically and acting on the functions $\phi(t) \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ according to (31).

Theorem 3.

1. *The range of values of the operator $\mathfrak{T}_c^+$ is the eigensubspace $\mathcal{C}_{+1}$ of the operator $\mathfrak{C}$;*
2. *The range of values of the operator $\mathfrak{T}_c^-$ is the eigensubspace $\mathcal{C}_{-1}$ of the operator $\mathfrak{C}$;*
3. *The range of values of the operator $\mathfrak{T}_s^+$ is the eigensubspace $\mathcal{S}_{+1}$ of the operator $\mathfrak{S}$;*
4. *The range of values of the operator $\mathfrak{T}_s^-$ is the eigensubspace $\mathcal{S}_{-1}$ of the operator $\mathfrak{S}$.*

Remark 7. Since the operators $\mathfrak{T}_c^+, \mathfrak{T}_c^-$, $\mathfrak{T}_s^+, \mathfrak{T}_s^-$ act isometrically from $L^2(\mathbb{R}_+)$ into $L^2(\mathbb{R}_+)$, the equalities

$$(\mathfrak{T}_c^+)^*\mathfrak{T}_c^+ = \mathfrak{I}, \quad \mathfrak{T}_c^+(\mathfrak{T}_c^+)^* = \mathfrak{P}_c^+, \quad \mathfrak{C}\mathfrak{T}_c^+ = \mathfrak{T}_c^+; \tag{46a}$$
$$(\mathfrak{T}_c^-)^*\mathfrak{T}_c^- = \mathfrak{I}, \quad \mathfrak{T}_c^-(\mathfrak{T}_c^-)^* = \mathfrak{P}_c^-, \quad \mathfrak{C}\mathfrak{T}_c^- = -\mathfrak{T}_c^- \tag{46b}$$

and

$$(\mathfrak{T}_s^+)^*\mathfrak{T}_s^+ = \mathfrak{I}, \quad \mathfrak{T}_s^+(\mathfrak{T}_s^+)^* = \mathfrak{P}_s^+, \quad \mathfrak{S}\mathfrak{T}_s^+ = \mathfrak{T}_s^+; \tag{47a}$$
$$(\mathfrak{T}_s^-)^*\mathfrak{T}_s^- = \mathfrak{I}, \quad \mathfrak{T}_s^-(\mathfrak{T}_s^-)^* = \mathfrak{P}_s^-, \quad \mathfrak{S}\mathfrak{T}_s^- = -\mathfrak{T}_s^- \tag{47b}$$

hold, where $\mathfrak{P}_c^+$, $\mathfrak{P}_c^-$, $\mathfrak{P}_s^+$ and $\mathfrak{P}_s^-$ are orthogonal projectors from $L^2(\mathbb{R})_+$ onto the eigensubspaces $\mathcal{C}_{+1}, \mathcal{C}_{-1}, \mathcal{S}_{+1}$ and $\mathcal{S}_{-1}$, respectively, and $(\mathfrak{T}_c^+)^*$, $(\mathfrak{T}_c^-)^*$, $(\mathfrak{T}_s^+)^*$, $(\mathfrak{T}_s^-)^*$ are the operators Hermitian-conjugated to the operators $(\mathfrak{T}_c^+)$, $(\mathfrak{T}_c^-)$, $\mathfrak{T}_s^+)$, $(\mathfrak{T}_s^-)$ with respect to the standard scalar product in the Hilbert space $L^2(\mathbb{R}_+)$.

In particular, the operators $(\mathfrak{T}_c^+)^*$, $(\mathfrak{T}_c^-)^*$, $(\mathfrak{T}_s^+)^*$ and $(\mathfrak{T}_s^-)^*$ are generalized inverses* of the operators $\mathfrak{T}_c^+$, $\mathfrak{T}_c^-$, $\mathfrak{T}_s^+$ and $\mathfrak{T}_s^-$, respectively.

* In the sense of Moore–Penrose, for example.

It is worth mentioning that

$$\big((\mathfrak{T}_c^{+})^{*}x\big)(\tau)=\int\limits_{\mathbb{R}_+}e_c^{+}(t,\tau)x(t)\,dt,\quad \big((\mathfrak{T}_c^{-})^{*}x\big)(\tau)=\int\limits_{\mathbb{R}_+}e_c^{-}(t,\tau)x(t)\,dt, \tag{48a}$$

$$\big((\mathfrak{T}_s^{+})^{*}x\big)(\tau)=\int\limits_{\mathbb{R}_+}e_s^{+}(t,\tau)\,x(t)\,dt,\quad \big((\mathfrak{T}_s^{-})^{*}x\big)(\tau)=\int\limits_{\mathbb{R}_+}e_s^{-}(t,\tau)\,x(t)\,dt. \tag{48b}$$

Theorem 3 is a consequence of the following

Lemma 6. *Let a function $x(t)$ belong to $L^2(\mathbb{R}_+)$, and $\hat{x}_c(t)$ and $\hat{x}_s(t)$ be the cosine and sine Fourier transforms of the function x:*

$$\hat{x}_c(t)=\sqrt{\frac{2}{\pi}}\int\limits_0^\infty x(s)\,\cos(ts)\,ds, \tag{49a}$$

$$\hat{x}_s(t)=\sqrt{\frac{2}{\pi}}\int\limits_0^\infty x(s)\,\sin(ts)\,ds. \tag{49b}$$

Let $\Phi_x(\zeta)$, $\Phi_{\hat{x}_c}(\zeta)$ and $\Phi_{\hat{x}_s}(\zeta)$ be the Melline transforms of the functions x, $\hat{x}_c$ and $\hat{x}_s$, respectively. (All three functions x, $\hat{x}_c$, $\hat{x}_s$ belong to $L^2(0,\infty)$, so their Melline transforms exist and they are the L^2 functions on the vertical line $\operatorname{Re}\zeta=\frac{1}{2}$.)

Then for $\zeta:\ \operatorname{Re}\zeta=\frac{1}{2}$, the equalities

$$\Phi_{\hat{x}_c}(\zeta)=\Phi_x(1-\zeta)\cdot 2^{\zeta-\frac{1}{2}}\frac{\Gamma\big(\frac{\zeta}{2}\big)}{\Gamma\big(\frac{1}{2}-\frac{\zeta}{2}\big)}\,, \tag{50a}$$

$$\Phi_{\hat{x}_s}(\zeta)=\Phi_x(1-\zeta)\cdot 2^{\zeta-\frac{1}{2}}\frac{\Gamma\big(\frac{1}{2}+\frac{\zeta}{2}\big)}{\Gamma\big(1-\frac{\zeta}{2}\big)} \tag{50b}$$

hold.

P r o o f. It is enough to prove Eqs. (50) assuming that the functions $x(t),\hat{x}_c(t),\hat{x}_s(t)$ are continuous and belong to $L^2(\mathbb{R}_+)\cap L^1(\mathbb{R}_+)$: the set of these functions x is dense in $L^2(\mathbb{R})$ and all three transforms, cosine, sine and Melline transforms, act continuously from L^2 to L^2. Under these extra assumptions on the functions $x(t),\hat{x}_c(t),\hat{x}_s(t)$, the Melline transforms $\Phi_x(\zeta)$, $\Phi_{\hat{x}_c}(\zeta)$, $\Phi_{\hat{x}_c}(\zeta)$ are defined everywhere on the vertical line $\operatorname{Re}\zeta=\frac{1}{2}$ and are continuous functions. For such x, Eqs. (50) will be established for every $\zeta:\ \operatorname{Re}\zeta=\frac{1}{2}$.

We fix ζ: $\operatorname{Re}\zeta = \frac{1}{2}$. The Melline transform $\Phi_{\hat{x}_c}(\zeta)$ is

$$\Phi_{\hat{x}_c}(\zeta) = \lim_{R\to\infty} \int\limits_0^R \hat{x}_c(t) t^{\zeta-1} dt.$$

Substituting (49a) for $\hat{x}_c(t)$ into the last formula, we obtain

$$\Phi_{\hat{x}_c}(\zeta) = \lim_{R\to\infty} \int\limits_0^R \Big(\sqrt{\frac{2}{\pi}} \int\limits_0^\infty x(s)\, \cos(ts)\, ds\Big) t^{\zeta-1}\, dt. \tag{51}$$

For fixed finite R, we change the order of integration

$$\int\limits_0^R \Big(\int\limits_0^\infty x(s)\, \cos(ts)\, ds\Big) t^{\zeta-1}\, dt = \int\limits_0^\infty x(s) \Big(\int\limits_0^R \cos(ts)\, t^{\zeta-1}\, dt\Big)\, ds\,.$$

The change of order of integration is justified by Fubini's theorem. Changing the variable $ts = \tau$, we get

$$\int\limits_0^R \cos(ts)\, t^{\zeta-1}\, dt = s^{-\zeta} \int\limits_0^{Rs} \cos(\tau)\, \tau^{\zeta-1}\, d\tau\,.$$

Thus

$$\int\limits_0^R \Big(\sqrt{\frac{2}{\pi}} \int\limits_0^\infty x(s)\, \cos(ts)\, ds\Big) t^{\zeta-1}\, dt$$
$$= \int\limits_0^\infty x(s) s^{-\zeta} \Big(\sqrt{\frac{2}{\pi}} \int\limits_0^{Rs} \cos(\tau)\, \tau^{\zeta-1}\, d\tau\Big)\, ds\,. \tag{52}$$

According to Lemma 1, for every $s > 0$,

$$\lim_{R\to\infty} \int\limits_0^{Rs} \cos(\tau)\, \tau^{\zeta-1}\, d\tau = \Big(\cos\, \frac{\pi}{2}\zeta\Big)\, \Gamma(\zeta)\,.$$

The value $\int\limits_0^\rho (\cos\tau)\tau^{\zeta-1}\, d\tau$, considered as a function of ρ, vanishes at $\rho = 0$, is a continuous function of ρ, and has a finite limit as $\rho \to \infty$. Therefore there exists a finite $M(\zeta) < \infty$ such that the estimate holds $\left|\int\limits_0^\rho (\cos\tau)\tau^{\zeta-1}\,d\tau\right| \leq M(\zeta)$, where the value $M(\zeta)$ does not depend on ρ. In other words,

$$\left|\int\limits_0^{Rs} \cos(\tau)\,\tau^{\zeta-1}\,d\tau\right| \leq M(\zeta) < \infty \quad \forall\, s, R:\ 0 \leq s < \infty,\ 0 \leq R < \infty\,.$$

By the Lebesgue dominated convergence theorem,

$$\lim_{R\to\infty}\int\limits_0^\infty x(s)s^{-\zeta}\biggl(\sqrt{\frac{2}{\pi}}\int\limits_0^{Rs}\cos(\tau)\,\tau^{\zeta-1}\,d\tau\biggr)\,ds$$
$$= \int\limits_0^\infty x(s)s^{-\zeta}\biggl(\sqrt{\frac{2}{\pi}}\int\limits_0^{\infty}\cos(\tau)\,\tau^{\zeta-1}\,d\tau\biggr)\,ds\,. \quad (53)$$

Taking into account equalities (51), (53) and using (16a) and (20a), we reduce the last equality to the form

$$\Phi_{\hat{x}_c}(\zeta) = \int\limits_0^\infty x(s)\,s^{-\zeta}\,ds\ \cdot\ 2^{\zeta-\frac{1}{2}}\,\frac{\Gamma\bigl(\frac{\zeta}{2}\bigr)}{\Gamma\bigl(\frac{1}{2}-\frac{\zeta}{2}\bigr)}\,.$$

To obtain (50a) from the previous equality, we need only to consider that

$$\int\limits_0^\infty x(s)\,s^{-\zeta}\,ds = \Phi_x(1-\zeta)\,.$$

Equality (50b) can be proved in a similar way. ■

R e m a r k 8. Equalities (50) can be presented in the form

$$\Phi_{\hat{x}_c}\bigl(\tfrac{1}{2}+i\tau\bigr) = \Phi_x\bigl(\tfrac{1}{2}-i\tau\bigr)\cdot c^2(\tau), \quad (54a)$$
$$\Phi_{\hat{x}_s}\bigl(\tfrac{1}{2}+i\tau\bigr) = \Phi_x\bigl(\tfrac{1}{2}-i\tau\bigr)\cdot s^2(\tau), \quad (54b)$$

where $c(\tau)$ and $s(\tau)$ were introduced in (30).

P r o o f. [Proof of Theorem 3] Let $x_c(t)$ be defined by (49a). The equality $\mathfrak{C}x = x$, i.e., the equality $x_c(t) = x(t)$ for the functions $x_c(t)$, $x(t)$, is equivalent to the equality

$$\Phi_{\hat{x}_c}\bigl(\tfrac{1}{2}+i\tau\bigr) = \Phi_x\bigl(\tfrac{1}{2}+i\tau\bigr)$$

for their Melline transforms. According to Lemma 6, (54a), the last equality can be reduced* to the form

$$\Phi_x\big(\tfrac{1}{2}-i\tau\big)\cdot c(\tau) = \quad \Phi_x\big(\tfrac{1}{2}+i\tau\big)\cdot c(-\tau), \quad -\infty<\tau<\infty. \tag{55a}$$

Analogously, the equalities $\mathfrak{C}x=-x$, $\mathfrak{S}x=x$ and $\mathfrak{S}x=-x$ for the functions $x(t)$ are equivalent to the equalities

$$\Phi_x\big(\tfrac{1}{2}-i\tau\big)\cdot c(\tau) = -\Phi_x\big(\tfrac{1}{2}+i\tau\big)\cdot c(-\tau), \quad -\infty<\tau<\infty, \tag{55b}$$

and

$$\Phi_x\big(\tfrac{1}{2}-i\tau\big)\cdot s(\tau) = \quad \Phi_x\big(\tfrac{1}{2}+i\tau\big)\cdot s(-\tau), \quad -\infty<\tau<\infty, \tag{56a}$$

$$\Phi_x\big(\tfrac{1}{2}-i\tau\big)\cdot s(\tau) = -\Phi_x\big(\tfrac{1}{2}+i\tau\big)\cdot s(-\tau), \quad -\infty<\tau<\infty. \tag{56b}$$

Thus each of the equalities $\mathfrak{C}x=x$, $\mathfrak{C}x=-x$, $\mathfrak{S}x=x$, $\mathfrak{S}x=-x$ for the function $x(t)$, $0<t<\infty$, is equivalent to the symmetry condition for its Melline transform $\Phi_x\big(\tfrac{1}{2}+i\tau\big)$, $-\infty<\tau<\infty$. These symmetry conditions, which appear as conditions (55), (56), can be presented in the form

$$\Phi_x\big(\tfrac{1}{2}+i\tau\big) = \sqrt{\pi}\,c(\tau)\,\phi(|\tau|), \quad -\infty<\tau<\infty,$$

$$\Phi_x\big(\tfrac{1}{2}+i\tau\big) = \tfrac{1}{i}\,\mathrm{sign}\,(\tau)\,\sqrt{\pi}\,c(\tau)\,\phi(|\tau|), \quad -\infty<\tau<\infty,$$

and

$$\Phi_x\big(\tfrac{1}{2}+i\tau\big) = \sqrt{\pi}\,s(\tau)\,\phi(|\tau|), \quad -\infty<\tau<\infty,$$

$$\Phi_x\big(\tfrac{1}{2}+i\tau\big) = \tfrac{1}{i}\,\mathrm{sign}\,(\tau)\,\sqrt{\pi}\,s(\tau)\,\phi(|\tau|), \quad -\infty<\tau<\infty,$$

where $\phi(\tau)$ is the function defined for $0<\tau<\infty$. Comparing these expressions for the function $\Phi_x\big(\tfrac{1}{2}+i\tau\big)$ with the expressions (28), (29) for the eigenfunctions $e_c^+(t,\tau)$, $e_c^-(t,\tau)$, $e_s^+(t,\tau)$, $e_s^-(t,\tau)$, we can see that in each of the four cases the inversion formula

$$x(t) = \tfrac{1}{2\pi}\int\limits_{-\infty}^{\infty} t^{-\frac{1}{2}+i\tau}\,\Phi_x\big(\tfrac{1}{2}+i\tau\big)\,d\tau$$

for the Melline transform can be presented in terms of the function $\phi(\tau)$ as

$$x(t) = \int\limits_0^\infty e_c^+(t,\tau)\,\phi(\tau)\,d\tau, \quad x(t) = \int\limits_0^\infty e_c^-(t,\tau)\,\phi(\tau)\,d\tau, \tag{57a}$$

$$x(t) = \int\limits_0^\infty e_s^+(t,\tau)\,\phi(\tau)\,d\tau, \quad x(t) = \int\limits_0^\infty e_s^-(t,\tau)\,\phi(\tau)\,d\tau, \tag{57b}$$

* Remember that $c^{-1}(\tau)=c(-\tau)$.

respectively. Now the symmetries (55), (56) of the function $\Phi_x\left(\frac{1}{2}+i\tau\right)$ are hidden in the structure of the functions e_c^+, e_c^-, e_s^+, e_-^-.

Thus, the equalities $\mathfrak{C}x = x$, $\mathfrak{C}x = -x$ and $\mathfrak{S}x = x$, $\mathfrak{S}x = -x$ for the functions x are equivalent to the representability of x in one of the four forms (57), i.e., in the form $x = \mathfrak{T}_c^+\phi, x = \mathfrak{T}_c^-\phi$, $x = \mathfrak{T}_s^+\phi$ and $x = \mathfrak{T}_s^-\phi$, respectively (with $\phi \in L^2(\mathbb{R}_+)$). ■

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