On Integration of One Class of Systems of Lax-Type Equations

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A nonlinear system of Lax-type equations is studied. The system is the basis of the construction of triangular models for commutative systems of linear non-selfadjoint bounded operators. Some of its solutions for $n = 4$ are described. In one of the cases, the general solution is explicitly expressed in terms of special (elliptic) functions.

Key words: triangular models, nonlinear differential equations, commutative systems of linear non-selfadjoint operators.

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1. Introduction

As is well known from [1, 5], an approach based on Lax's idea to write the initial nonlinear equation in the form $L'(x) = i[L(x), A(x)]$, where L, A are some differential operators, is the main method of integration of nonlinear equations. It is established that if the Lax pair $\{L, A\}$ is found for a nonlinear equation, then this equation can be 'integrated'.

In this paper, we study the system of equations

$$\frac{d}{dx}\left(\sigma_2 + \lambda\gamma(x)\right) = i\left[a(x), \sigma_2 + \lambda\gamma(x)\right], \quad \gamma(0) = \gamma^+, \quad x \in [0, l], \quad \lambda \in \mathbb{C}, \quad (1.1)$$

which is equivalent to the special system of the Lax-type equations

$$\begin{cases} [a(x), \gamma(x)] = 0, & x \in [0, l], \\ \gamma'(x) = i[a(x), \sigma_2], & x \in [0, l], \\ \gamma(0) = \gamma^+ \end{cases} \qquad (1.2)$$

where $a(x)$ is a spectral matrix measure, $\gamma(x), \sigma_2, \gamma^+$ are selfadjoint $n \times n$ matrices, and

$$a(x) \geqslant 0, \qquad tr\, a(x) \equiv 1, \qquad x \in [0, l]. \qquad (1.3)$$

System (1.2) appears in the construction of triangular models for commutative systems of non-selfadjoint bounded operators.

The purpose of this paper is to describe and study all the pairs of matrix functions $\{a(x), \gamma(x)\}$, the solutions of system (1.2) (when $n = 4$), for the given selfadjoint $n \times n$ matrices γ^+ and σ_2 such that

$$\gamma(x) \in AC([0, l]; C^{n \times n}), \qquad a(x) \in L^1([0, l]; C^{n \times n}), \qquad (1.4)$$

and (1.3) takes place.

In [7], general qualities of the solutions of system (1.2) for the case $n = 3$ are obtained and the descriptions of all solutions of this system for different cases are given. The idea of paper [7] is used in this paper. This idea lies in the fact that in the case when γ^+ (and so $\gamma(x)$ also) has a simple spectrum, $a(x)$ is a polynomial of $\gamma(x)$ of no higher degree than $n - 1$ (with scalar coefficients depending on x).

When $n = 4$ and the matrices σ_2, γ^+ have a simple spectrum, the explicit form of the solution in terms of elliptic functions is obtained (see Theorem 2.2 and Corollary 2.3). In Example 2.4, the solutions expressed by trigonometric functions are found.

When studying cubic dependency of $a(x)$ from $\gamma(x)$ ($n = 4$), the explicit form of the solution is also expressed in terms of special (elliptic) functions (see Theorem 3.2 and Corollary 3.3).

2. Description of the Solutions of System (1.2)

Proposition 2.1. *Let*

$$\sigma_2 = diag(b_1, \ldots, b_n), \quad \gamma^+ = \alpha_1 \sigma_2 + \alpha_0 I + iC, \qquad (2.1)$$

where $\alpha_1, \alpha_0 \in R$, the matrix $C = (c_{jk})_{j,k=1}^n = -C^*$ and $c_{jj} = 0$ when $j \in \{1, \ldots, n\}$.

Further, let κ_0, κ_1, $\kappa_2 \in L^1[0, l]$ be real functions. Then the pair $\{a(\cdot), \gamma(\cdot)\}$, where $a(x) = \kappa_2(x)\gamma(x)^2 + \kappa_1(x)\gamma(x) + \kappa_0(x)$, $x \in [0, l]$ and $\gamma(\cdot) = (\gamma_{jk}(\cdot))_{j,k=1}^n$,

is the solution of system (1.2) *if and only if the equalities*

$$\gamma_{jj}(x) = \gamma_{jj}^+, \quad j \in \{1, \ldots, n\}, \tag{2.2}$$

$$\gamma_{jk}(x) = i e^{i(b_j - b_k)\left(K_1(x) + (\gamma_{jj}^+ + \gamma_{kk}^+)K_2(x)\right)} y_{jk}(x), \quad j \neq k, \tag{2.3}$$

hold as $x \in [0, l]$, *where*

$$K_j(x) := \int_0^x \kappa_j(t)dt, \quad j \in \{1, 2\}, \tag{2.4}$$

and the functions $y_{jk}(\cdot)$, $j \neq k$, *satisfy the system*

$$
\begin{cases}
y'_{jk}(x) = (b_k - b_j)\kappa_2(x) \sum_{\substack{s=1 \\ s \neq j,k}}^{n} y_{js}(x)y_{sk}(x), & x \in [0, l], \quad j \neq k, \\
y_{kj}(x) = \overline{-y_{jk}(x)}, & x \in [0, l], \quad j \neq k, \\
y_{jk}(0) = c_{jk}, & j \neq k.
\end{cases}
\tag{2.5}
$$

Besides, if $c_{jk} \in R$, $j \neq k$, *every solution of system* (2.5) *is real.*

P r o o f. Since $a(x)$ commutes with $\gamma(x)$, then system (1.2) has the form

$$
\begin{cases}
\gamma'(x) = i[\kappa_2(x)\gamma(x)^2 + \kappa_1(x)\gamma(x), \sigma_2], & x \in [0, l], \\
\gamma(0) = \gamma^+.
\end{cases}
\tag{2.6}
$$

By Lemma 3.12 from [7], in view of the diagonal form of σ_2, (2.23) is true for every solution of system (2.6). Taking this into account, system (2.6) takes the form

$$
\begin{cases}
\gamma'_{jk}(x) = i(b_j - b_k)(\kappa_1(x) + \kappa_2(x)(\gamma_{jj} + \gamma_{kk})) \\
\qquad\quad + i(b_j - b_k)\kappa_2(x) \sum_{\substack{s=1 \\ s \neq j,k}}^{n} \gamma_{js}(x)\gamma_{sk}(x), & x \in [0, l], \quad j \neq k, \\
\gamma_{kj}(x) = \overline{\gamma_{jk}(x)}, & x \in [0, l], \quad j \neq k, \\
\gamma_{jk}(0) = \gamma_{jk}, & j \neq k.
\end{cases}
\tag{2.7}
$$

We search the solution of this system in the form of (2.24). In view of (2.1), $\gamma_{jj} = \alpha_1 b_j + \alpha_0$, $j \in \{1, \ldots, n\}$, holds. Therefore formula (2.24) becomes

$$\gamma_{jk}(x) = iE_j(x)/E_k(x)y_{jk}(x), \quad E_j(x) := e^{ib_j(K_1(x) + (\alpha_1 b_j + 2\alpha_0)K_2(x))}. \tag{2.8}$$

After substituting formula (2.8) into system (2.7), it is easy to check that it is equivalent to (2.5).

Now let the matrix C be real. We are to prove that every solution of system (2.5) is real. Let $\{y_{jk}(\cdot)\}_{j \neq k}$ be some (complex-valued) solution of system (2.5). Let

$$u_{jk}(\cdot) := \operatorname{Re} y_{jk}(\cdot), \quad v_{jk}(\cdot) := \operatorname{Im} y_{jk}(\cdot), \quad j \neq k. \tag{2.9}$$

By separating the imaginary part from the equations of problem (2.5) and taking into account that $c_{jk} \in R$, $j \neq k$, we obtain the following system on the functions $v_{jk}(\cdot)$:

$$\begin{cases} v'_{jk} = (b_k - b_j)\kappa_2(x) \sum\limits_{\substack{s=1 \\ s \neq j,k}}^{n} (u_{sk}v_{js} + u_{js}v_{sk}), \quad x \in [0,l], \quad j \neq k, \\ v_{jk}(0) = 0, \quad j \neq k. \end{cases} \tag{2.10}$$

System (2.10) is a Cauchy problem for the system of linear ordinary differential equations with zero initial data. Therefore, by the uniqueness theorem, $v_{jk}(\cdot) = 0$, $j \neq k$, which signifies the reality of the solution $\{y_{jk}(\cdot)\}_{j \neq k}$.

Theorem 2.2. *Let $n = 4$,*

$$b_4 < b_1 < b_2 < b_3, \quad b_1 + b_2 = b_3 + b_4, \tag{2.11}$$

$$\alpha_3 := \frac{b_3 - b_2}{b_3 - b_1}, \quad \alpha_4 := \frac{b_4 - b_2}{b_4 - b_1}. \tag{2.12}$$

Further, let

$$c_{jk} \in R, \quad c_{jk} = -c_{kj}, \quad j, k \in \{1, 2, 3, 4\}, \tag{2.13}$$

$$c_{1k} > 0, \quad k \in \{2, 3, 4\}, \tag{2.14}$$

$$c_{23} = \sqrt{\alpha_3} \cdot c_{13}, \quad c_{24} = -\sqrt{\alpha_4} \cdot c_{14}, \quad c_{34} = 0. \tag{2.15}$$

Next suppose

$$\beta_3 := \frac{b_2 - b_1}{b_3 - b_1}, \quad \beta_4 := \frac{b_2 - b_1}{b_4 - b_1}, \tag{2.16}$$

$$\alpha := \frac{c_{14}}{c_{13}}, \quad \beta := \beta_3 + \beta_4\alpha^2, \tag{2.17}$$

$$F(y) := \int\limits_{c_{13}}^{y} \frac{du}{u\sqrt{c_{12}^2 + \beta(u^2 - c_{13}^2)}}, \tag{2.18}$$

$$\rho := \sqrt{(b_3 - b_1)(b_3 - b_2)}, \tag{2.19}$$

$$v(x) := F^{-1}\left(\rho K_2(x)\right), \tag{2.20}$$

where $F^{-1}(\cdot)$ is the function inverse to the function $F(\cdot)$. Let $(y_0^-, y_0^+) \subset \overline{R}$ be the largest by inclusion interval containing the number c_{13}, and the inequality

$$c_{12}^2 + \beta(y^2 - c_{13}^2) > 0, \quad y_0^- < y < y_0^+, \tag{2.21}$$

be true. Further, let $\kappa_0, \kappa_1\kappa_2 \in L^1[0, l]$ be real functions and the functions $K_1(x)$, $K_2(x)$ be given by the equalities $K_j(x) := \int_0^x \kappa_j(t)dt$, $j = 1, 2$, and

$$F(y_0^-) < \rho K_2(x) < F(y_0^+), \quad x \in [0, l]. \tag{2.22}$$

Then the pair $\{a(\cdot), \gamma(\cdot)\}$, where $a(x) = \kappa_2(x)\gamma(x)^2 + \kappa_1(x)\gamma(x) + \kappa_0(x)$, $x \in [0, l]$, is the solution of system (1.2) if and only if the equalities

$$\gamma_{jj}(x) = \gamma_{jj}^+, \quad j \in \{1, \ldots, n\}, \tag{2.23}$$

$$\gamma_{jk}(x) = i\mathrm{e}^{i(b_j - b_k)\left(K_1(x) + (\gamma_{jj}^+ + \gamma_{kk}^+)K_2(x)\right)} y_{jk}(x), \quad j \neq k, \tag{2.24}$$

hold as $x \in [0, l]$, where the functions y_{jk}, $j \neq k$ are given by

$$y_{12}(x) = \sqrt{c_{12}^2 + \beta(v^2(x) - c_{13}^2)}, \tag{2.25}$$

$$y_{13}(x) = v(x), \tag{2.26}$$

$$y_{14}(x) = \alpha v(x), \tag{2.27}$$

$$y_{23}(x) = \sqrt{\alpha_3}\, v(x), \tag{2.28}$$

$$y_{24}(x) = -\sqrt{\alpha_4}\, \alpha v(x), \tag{2.29}$$

$$y_{34}(x) = 0, \tag{2.30}$$

$$y_{kj}(x) = y_{jk}(x), \quad 1 \leqslant j < k \leqslant 4. \tag{2.31}$$

P r o o f. First check that the set $\{y_{12}(\cdot), y_{13}(\cdot), y_{14}(\cdot), y_{23}(\cdot), y_{24}(\cdot), y_{34}(\cdot)\}$ is the solution of system (2.5). Inequality (2.22) implies that the function $v(\cdot)$, given by formula (2.20), is correctly defined on the segment $[0, l]$, besides, $v(x) \in (y_0^-, y_0^+)$, $x \in [0, l]$. Therefore the equality

$$F(v(x)) = \rho K_2(x), \quad x \in [0, l],$$

is true. Differentiating it, we obtain

$$v'(x)F'(v(x)) = \rho\kappa_2(x), \quad x \in [0, l].$$

Hence, taking into account (2.18), (2.25), (2.26), (2.28), we obtain

$$\begin{aligned}
y_{13}'(x) = v'(x) &= \rho\kappa_2(x)v(x)\sqrt{c_{12}^2 + \beta(v^2(x) - c_{13}^2)} \\
&= \rho\kappa_2(x)v(x)y_{12}(x) \tag{2.32} \\
&= \frac{\rho}{\sqrt{\alpha_3}}\,\kappa_2(x)y_{12}(x)y_{23}(x) \\
&= (b_3 - b_1)\kappa_2(x)y_{12}(x)y_{23}(x), \quad x \in [0, l]. \tag{2.33}
\end{aligned}$$

Note that accordingly to (2.11), the equality $\alpha_3\alpha_4 = 1$ is true. Therefore,

$$\frac{\sqrt{\alpha_3}(b_1 - b_3)}{\sqrt{\alpha_4}} = \alpha_3(b_1 - b_3) = b_2 - b_3 = b_4 - b_1,$$

and in virtue of (2.26), (2.27), (2.33),(2.28), (2.29), we have

$$\begin{aligned}
y'_{14}(x) &= \alpha y'_{13}(x) = \alpha(b_3 - b_1)\kappa_2(x)y_{12}(x)y_{23}(x) \\
&= \frac{\sqrt{\alpha_3}(b_1 - b_3)}{\sqrt{\alpha_4}}\kappa_2(x)y_{12}(x)y_{24}(x) \\
&= (b_4 - b_1)\kappa_2(x)y_{12}(x)y_{24}(x), \quad x \in [0, l].
\end{aligned} \tag{2.34}$$

Then we have

$$\begin{aligned}
y'_{23}(x) &= \sqrt{\alpha_3}\, y'_{13}(x) = \sqrt{\alpha_3}\,(b_3 - b_1)\kappa_2(x)y_{12}(x)y_{23}(x) \\
&= \alpha_3(b_3 - b_1)\kappa_2(x)y_{12}(x)y_{13}(x) \\
&= (b_3 - b_2)\kappa_2(x)y_{12}(x)y_{13}(x), \quad x \in [0, l],
\end{aligned} \tag{2.35}$$

$$\begin{aligned}
y'_{24}(x) &= -\sqrt{\alpha_4}\, y'_{14}(x) = -\sqrt{\alpha_4}\,(b_4 - b_1)\kappa_2(x)y_{12}(x)y_{24}(x) \\
&= \alpha_4(b_4 - b_1)\kappa_2(x)y_{12}(x)y_{14}(x) \\
&= (b_4 - b_2)\kappa_2(x)y_{12}(x)y_{14}(x), \quad x \in [0, l].
\end{aligned} \tag{2.36}$$

From (2.25) and (2.32), we obtain

$$y'_{12}(x) = \frac{\beta v(x)v'(x)}{y_{12}(x)} = \rho\beta\kappa_2(x)v^2(x), \quad x \in [0, l]. \tag{2.37}$$

Further, taking into account (2.26)–(2.29), we have

$$y_{13}(x)y_{23}(x) + y_{14}(x)y_{24}(x) = (\sqrt{\alpha_3} - \sqrt{\alpha_4}\alpha^2)v^2(x), \quad x \in [0, l]. \tag{2.38}$$

Note that in virtue of (2.11) the equality

$$\rho = \sqrt{(b_3 - b_2)(b_3 - b_1)} = \sqrt{(b_4 - b_2)(b_4 - b_1)} \tag{2.39}$$

is true. Therefore, since $b_4 < b_1 < b_3$, then

$$\begin{aligned}
\rho\beta &= \rho(\beta_3 + \beta_4\alpha^2) = \sqrt{(b_3 - b_2)(b_3 - b_1)}\,\frac{b_2 - b_1}{b_3 - b_1} \\
&\quad + \sqrt{(b_4 - b_2)(b_4 - b_1)}\,\frac{b_2 - b_1}{b_4 - b_1}\,\alpha^2 \\
&= (b_2 - b_1)\left(\sqrt{\frac{b_3 - b_2}{b_3 - b_1}} - \sqrt{\frac{b_4 - b_2}{b_4 - b_1}}\,\alpha^2\right) \\
&= (b_2 - b_1)\left(\sqrt{\alpha_3} - \sqrt{\alpha_4}\,\alpha^2\right).
\end{aligned} \tag{2.40}$$

Now the relations (2.37)–(2.40) yield

$$y'_{12}(x) = (b_2 - b_1)\kappa_2(x)(y_{13}(x)y_{23}(x) + y_{14}(x)y_{24}(x)), \ x \in [0, l]. \qquad (2.41)$$

Since $\alpha_3\alpha_4 = 1$, then in virtue of (2.26)–(2.30),

$$(b_4 - b_3)\kappa_2(x)(y_{13}(x)y_{14}(x) + y_{23}(x)y_{24}(x))$$
$$= (b_4 - b_3)\kappa_2(x)(y_{13}(x)y_{14}(x) - \sqrt{\alpha_3}\, y_{13}(x)\sqrt{\alpha_4}\, y_{14}(x))$$
$$= 0 = y'_{34}(x). \quad (2.42)$$

Now the equalities (2.41), (2.33), (2.34), (2.35), (2.36), (2.42) imply that the functions $y_{12}, y_{13}, y_{14}, y_{23}, y_{24}, y_{34}$ satisfy the equations of system (2.5). Further, since $c_{12} > 0$, then

$$y_{13}(0) = v(0) = F^{-1}(\rho K_2(0)) = F^{-1}(0) = c_{13}, \qquad (2.43)$$

$$y_{12}(0) = \sqrt{c_{12}^2 + \beta(v^2(0) - c_{13}^2)} = c_{12}, \qquad (2.44)$$

$$y_{14}(0) = \alpha v(0) = \frac{c_{14}}{c_{13}} c_{13} = c_{14}, \qquad (2.45)$$

$$y_{23}(0) = \sqrt{\alpha_3}\, y_{13}(0) = \sqrt{\alpha_3}\, c_{13} = c_{23}, \qquad (2.46)$$

$$y_{24}(0) = -\sqrt{\alpha_4}\, y_{14}(0) = -\sqrt{\alpha_4}\, c_{14} = c_{24}, \qquad (2.47)$$

$$y_{34}(0) = 0 = c_{34}. \qquad (2.48)$$

Thus the functions $y_{12}(\cdot), y_{13}(\cdot), y_{14}(\cdot), y_{23}(\cdot), y_{24}(\cdot), y_{34}(\cdot)$ satisfy also the initial data of system (2.5). General uniqueness theorems for the initial problem imply that this system has the unique solution on $[0, l]$.

We will show the way of finding expressions for $y_{ks}(\cdot), \ k, s \in 1, \ldots, n$ (2.25) – (2.31) as solutions of the corresponding system of differential equations (2.5). Let now the functions $y_{12}(\cdot), \ y_{13}(\cdot), \ y_{14}(\cdot) \ y_{23}(\cdot), \ y_{24}(\cdot), \ y_{34}(\cdot) \in AC[0, l]$ satisfy problem (2.5). Multiply equation (2.33) by $(b_3 - b_2)y_{13}(x)$ and equation (2.35) by $(b_3 - b_1)y_{23}(x)$. After summarizing the results, we arrive at the relation $y'_{23}(x)y_{23}(x) - \alpha_3 y'_{13}(x)y_{13}(x) = 0$, integrating which we obtain

$$y_{23}(x) = \sqrt{c_{23}^2 + \alpha_3(y_{13}^2(x) - c_{13}^2)}. \qquad (2.49)$$

Taking into account (2.15), we have (2.28). Following the same procedure with (2.34) and (2.36), we get $y_{24}(x) = \sqrt{c_{24}^2 + \alpha_4(y_{14}^2(x) - c_{14}^2)}$. After using (2.15), we obtain

$$y_{24}(x) = -\sqrt{\alpha_4}\, y_{14}(x). \qquad (2.50)$$

Use (2.49) and (2.50) for the equations (2.33) and (2.34) to get

$$y'_{13}(x) = \kappa_2(x)(b_3 - b_1)y_{12}(x)\sqrt{\alpha_3}\, y_{13}(x), \qquad (2.51)$$

$$y'_{14}(x) = \kappa_2(x)(b_4 - b_1)y_{12}(x)\sqrt{\alpha_4}\, y_{14}(x). \qquad (2.52)$$

Express $\kappa_2(x)y_{12}(x)$ from (2.51) and taking into account (2.39) equate the obtained relations

$$\frac{y'_{13}(x)}{y_{13}(x)} = \frac{y'_{14}(x)}{y_{14}(x)}. \tag{2.53}$$

Thus (2.53) yields

$$y_{14}(x) = \frac{c_{14}}{c_{13}}\, y_{13}(x) = \alpha y_{13}(x). \tag{2.54}$$

Thus the correctness of equality (2.27) is confirmed. After substituting it into (2.50), the correctness of (2.29) is also confirmed. By using the obtained results, we transform the equations (2.41) and (2.33), subject to (2.40),

$$
\begin{aligned}
y'_{12}(x) &= \kappa_2(x)(b_2 - b_1)y_{13}^2(x)\left(\sqrt{\alpha_3} - \alpha^2\sqrt{\alpha_4}\right) \\
&= \kappa_2(x)\rho\beta y_{13}^2(x); \tag{2.55} \\
y'_{13}(x) &= \kappa_2(x)\rho y_{12}(x)y_{13}. \tag{2.56}
\end{aligned}
$$

Whence we obtain that

$$\frac{y'_{12}(x)}{y_{13}(x)\beta} = \frac{y'_{13}(x)}{y_{12}(x)} \tag{2.57}$$

or

$$y_{12}(x) = \sqrt{c_{12}^2 + \beta(y_{13}^2(x) - c_{13}^2)}. \tag{2.58}$$

Thus (2.25) takes place. Substituting the obtained expressions for $y_{12}(x)$ (2.25) and $y_{23}(x)$ (2.28) into the equation for $y'_{13}(x)$ (2.33), we obtain

$$y'_{13}(x) = \kappa_2(x)\rho y_{13}(x)\sqrt{c_{12}^2 + \beta(y_{13}^2(x) - c_{13}^2)}, \tag{2.59}$$

or, using the notation (2.19),

$$\frac{y'_{13}(x)}{y_{13}(x)\sqrt{c_{12}^2 + \beta(y_{13}^2(x) - c_{13}^2)}} = \rho\kappa_2(x). \tag{2.60}$$

It is easy to see that (2.60), subject to notation (2.18), becomes (2.26).

Corollary 2.3. *In the conditions of Theorem 2 suppose $c_{12} = 0$ which determines the form of the matrix C in (2.1) in the following way:*

$$iC = \begin{pmatrix} 0 & 0 & c_{13} & c_{14} \\ 0 & 0 & c_{23} & c_{24} \\ \overline{c_{13}} & \overline{c_{23}} & 0 & 0 \\ \overline{c_{14}} & \overline{c_{24}} & 0 & 0 \end{pmatrix}.$$

Then for the functions $y_{12}(\cdot), y_{13}(\cdot)$ the following representations take place:

$$y_{12}(x) = c_{13}\sqrt{\beta}\tan(z(x)), \ \beta > 0, \tag{2.61}$$

$$y_{13}(x) = \frac{c_{13}}{\cos(z(x))}, \ \beta > 0, \tag{2.62}$$

or

$$y_{12}(x) = c_{13}\sqrt{\beta}, \ \beta < 0, \tag{2.63}$$

$$y_{13}(x) = 0, \tag{2.64}$$

or

$$y_{12}(x) = c_{13}\sqrt{\beta}\,\mathrm{th}(z(x)), \ \beta < 0, \tag{2.65}$$

$$y_{13}(x) = \frac{c_{13}}{\mathrm{ch}(z(x))}, \ \beta < 0, \tag{2.66}$$

where

$$z(x) = c_{13}\rho\sqrt{\beta}\,K_2(x). \tag{2.67}$$

P r o o f. Consider the case $\beta > 0$. The definition of the function $F(\cdot)$ and formula (2.26) imply that

$$\rho K_2(x) = F(y_{13}(x)) = \frac{1}{\sqrt{\beta}} \int\limits_{c_{13}}^{y_{13}(x)} \frac{du}{u\sqrt{u^2 - c_{13}^2}}\,. \tag{2.68}$$

Then (2.68) has the form

$$-c_{13}\sqrt{\beta}\,rhoK_2(x) = \arcsin\left|\frac{c_{13}}{y_{13}(x)}\right| - \frac{\pi}{2}.$$

Using notation (2.67), we obtain

$$-\sin(z(x)) = \sin\left(\arcsin\left|\frac{c_{13}}{y_{13}(x)}\right| - \frac{\pi}{2}\right)$$

$$= \sin\left(\arcsin\left|\frac{c_{13}}{y_{13}(x)}\right|\right)\cos\frac{\pi}{2} - \cos\left(\arcsin\left|\frac{c_{13}}{y_{13}(x)}\right|\right)\sin\frac{\pi}{2}$$

$$= -\sqrt{1 - \frac{c_{13}^2}{y_{13}^2(x)}}\,.$$

Suppose $z(x)$ is such that $\sin(z(x))$ and $\cos(z(x))$ are positive, then we obtain (2.62). Substituting (2.62) into (2.25)–(2.30), we will come to the form of (2.61) for $y_{12}(\cdot)$ and to the corresponding expressions for the functions $y_{14}(\cdot)$, $y_{23}(\cdot)$, $y_{24}(\cdot)$, $y_{34}(\cdot)$.

In the case $\beta < 0$, in virtue of inequality (2.21), we obtain that $y_{13}^2(x) - c_{13}^2 < 0$. Thus,

$$\rho K_2(x) = F(y_{13}(x)) = \frac{1}{\sqrt{\beta}} \int\limits_{c_{13}}^{y_{13}(x)} \frac{du}{u\sqrt{c_{13}^2 - u^2}}. \tag{2.69}$$

Whence it follows that

$$-z(x) = \ln\left(\frac{\sqrt{c_{13}^2 - u^2} + c_{13}}{y_{13}(x)}\right)$$

or

$$\sqrt{c_{13}^2 - y_{13}^2(x)} = y_{13}(x)\exp^{-z(x)} - c_{13}.$$

As a result of transformations, we have

$$y_{13}(x)\left(y_{13}(x)(\exp^{-2z(x)} + 1) - 2\exp^{-z(x)} c_{13}\right) = 0,$$

i.e., either $y_{13}(x) = 0$, which results in the solutions of (2.63)–(2.64), or $y_{13}(x) = \dfrac{2c_{13}}{\exp^{z(x)} + \exp^{-z(x)}}$, which corresponds to the solutions in the forms of (2.65) and (2.66).

E x a m p l e 2.4. Let

$$\gamma^+ = \begin{pmatrix} 0 & 0 & 2i & i \\ 0 & 0 & \sqrt{2}i & -\sqrt{2}i \\ -2i & -\sqrt{2}i & 0 & 0 \\ -i & \sqrt{2}i & 0 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 2b & 0 \\ 0 & 0 & 0 & -b \end{pmatrix}, \quad b > 0. \tag{2.70}$$

Then the pair $\{a(x), \gamma(x)\}$, where $a(x) = \gamma^2(x)$, is the solution of system (1.2) if and only if

$$\gamma(x) = \begin{pmatrix} 0 & \sqrt{2}\,\mathrm{tg}(2bx) & \dfrac{2}{\cos(2bx)} & \dfrac{1}{\cos(2bx)} \\ -\sqrt{2}\,\mathrm{tg}(2bx) & 0 & \dfrac{\sqrt{2}}{\cos(2bx)} & -\dfrac{\sqrt{2}}{\cos(2bx)} \\ -\dfrac{2}{\cos(2bx)} & -\dfrac{\sqrt{2}}{\cos(2bx)} & 0 & 0 \\ -\dfrac{1}{\cos(2bx)} & \dfrac{\sqrt{2}}{\cos(2bx)} & 0 & 0 \end{pmatrix}. \tag{2.71}$$

3. Case of Cubic Dependency of $a(x)$ from $\gamma(x)$

Remind the statement proved in [7].

Theorem 3.1. *Let*

$$\gamma^+ = diag(l_1 I_{n_1}, \ldots, l_r I_{n_r}), \quad n_1 + \ldots n_r = n, \tag{3.1}$$

where $l_1, \ldots, l_r$ are different real numbers.

Then for every solution $\{a(\cdot), \gamma(\cdot)\}$ of system (1.2) there exists a unique unitary matrix function $U \in AC([0, l]; C^{n \times n})$ such that

$$\gamma(x) = U(x)\gamma^+ U^*(x), \quad x \in [0, l], \qquad U(0) = I_n, \tag{3.2}$$

and the matrix function $C(\cdot) := -iU^{-1}(\cdot)U'(\cdot)$ is self-adjoint and has the zero block diagonal relative to the decomposition $C^n = C^{n_1} \oplus \ldots \oplus C^{n_r}$. Besides,

$$a(x) = U(x)A(x)U^*(x), \qquad x \in [0, l], \tag{3.3}$$

where

$$A(\cdot) = diag(A_1(\cdot), \ldots, A_r(\cdot)) = A^*(\cdot), \tag{3.4}$$

$$A_j \in L^1([0, l]; C^{n_j \times n_j}), \quad j \in \{1, \ldots, r\}. \tag{3.5}$$

Moreover, for $B(x) := U^*(x)\sigma_2 U(x)$,

$$[C(x), \gamma^+] = [A(x), B(x)], \quad x \in [0, l], \tag{3.6}$$

$$B'(x) = i[B(x), C(x)], \quad x \in [0, l], \quad B(0) = \sigma_2 \tag{3.7}$$

take place.

Conversely, if for the self-adjoint matrix functions

$$A, C \in L^1([0, l]; C^{n \times n}), \quad B \in AC([0, l]; C^{n \times n}), \tag{3.8}$$

(3.4), (3.5), (3.6), (3.7) take place, and $U \in AC([0, l]; C^{n \times n})$ is the solution of the initial problem

$$U'(x) = iU(x)C(x), \quad x \in [0, l], \qquad U(0) = I_n, \tag{3.9}$$

then $U(x)$ is unitary for every $x \in [0, l]$,

$$B(x) = U^*(x)\sigma_2 U(x), \quad x \in [0, l], \tag{3.10}$$

and the pair $\{a(\cdot), \gamma(\cdot)\}$, given by (3.2), (3.3), is the solution of system (1.2).

Theorem provides the following step-by-step procedure [7] for finding all the solutions of system (1.2):

1. Choose an orthonormal basis in C^n, in which the matrix γ^+ has the diagonal form (3.1).

2. Choose an arbitrary matrix function $A(\cdot)$ satisfying conditions (3.4)–(3.5).

3. Solve the Cauchy problem for the nonlinear system of ordinary differential equations on the matrix B obtained from (3.6), (3.7) in the previous remark.

4. If it has the global solution on the segment $[0, l]$, then we calculate the matrix $C(x)$ by the formula

$$C_{jk}(x) = (l_k - l_j)^{-1}(A_j(x)B_{jk}(x) - B_{jk}(x)A_k(x)),$$
$$x \in [0, l], \quad j, k \in \{1, \ldots, r\}, \quad j \neq k, \quad (3.11)$$

supposing that its diagonal blocks are equal to zero.

5. Find $U(\cdot)$ as a unique solution of the Cauchy problem (3.9) for the system of linear ordinary differential equations.

6. Finally, we obtain the solution $\{a(\cdot), \gamma(\cdot)\}$ of system (1.2) by the formulas (3.2), (3.3).

Using Theorem 3.1, we will show how to find the explicit form of the matrix $B(x)$ for the case $n = 4$ and $a(x) = \kappa(x)\gamma^3(x)$, where $\kappa(x)$ is a real function such that $\kappa \in L^1[0, l]$.

Theorem 3.2. *Let $n = 4$, $\lambda_1, \ldots, \lambda_4$ be different real numbers such that λ_3 and λ_4 are between λ_1 and λ_2,*

$$\gamma^+ = diag(\lambda_1, \ldots, \lambda_4),$$
$$\sigma_2 = (b_{jk}^{(0)})_{j,k=1}^4 = i\beta_{jk}, \ \beta_{jk} > 0, \ \beta_{jk} \in R \ as \quad j \neq k,$$
$$b_{jj}^{(0)} = \beta_1\lambda_j + \beta_0, \ j \in \{\overline{1, 4}\},$$
$$C(x) = (c_{jk}(x))_{j,k=1}^4, \ besides \quad c_{jj} = 0,$$
$$a(x) = \kappa(x)\gamma^3(x),$$

where $a(x)$ is a matrix function and $\kappa(x) \in L^1[0, l]$ is a real function, $x \in [0, l]$, and the matrix B is such that $B(x) = (b_{jk}(x))_{j,k=1}^4 = B^*(x)$ as $j \neq k$.
Let further

$$\alpha_j = \sqrt{\frac{\lambda_2 - \lambda_j}{\lambda_j - \lambda_1}}, \ \beta_j = \frac{\lambda_2 - \lambda_1}{\lambda_j - \lambda_1}, \ j = 3, 4, \quad (3.12)$$

$$\alpha = \frac{\sqrt{(\lambda_4 - \lambda_1)(\lambda_2 - \lambda_4)}\,(\lambda_1 + \lambda_2 + \lambda_4)}{\sqrt{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)}\,(\lambda_1 + \lambda_2 + \lambda_3)}, \quad (3.13)$$

$$\psi(y) = \sqrt{\beta_{12}^2 - \beta_3(y^2 - \beta_{13}^2) - \beta_4\beta_{14}^2 \left[\left(\frac{y}{\beta_{13}}\right)^{2\alpha} - 1\right]}, \qquad (3.14)$$

$$F(y) = \int\limits_{\beta_{13}}^{y} \frac{dt}{t\psi(t)} . \qquad (3.15)$$

Then the elements of the matrix $B(x)$ from (3.10) are given by

$$b_{jk}(x) = i \exp^{-i\beta_1 \cdot (\lambda_j^3 - \lambda_k^3) \int\limits_0^x \kappa(t)dt} y_{jk}(x), \ j \neq k, \qquad (3.16)$$

where the real functions $y_{jk}(\cdot)$ $(j \neq k)$ are given by the equalities

$$y_{12}(x) = \psi(y_{13}(x)), \qquad (3.17)$$

$$y_{13}(x) = F^{-1}\left(\sqrt{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)} \int\limits_0^x \kappa(t)dt\right), \qquad (3.18)$$

$$y_{14}(x) = \beta_{14}\left(\frac{y_{13}(x)}{\beta_{13}}\right)^\alpha, \qquad (3.19)$$

$$y_{2j}(x) = \alpha_j y_{1j}(x), \quad (j = 3,4), \qquad (3.20)$$

$$y_{34}(x) = 0, \qquad (3.21)$$

$$y_{kj}(\cdot) = y_{jk}(\cdot), \quad (j \neq k). \qquad (3.22)$$

Here $F^{-1}(\cdot)$ is a function inverse to the monotonously increasing function $F(\cdot)$.

P r o o f. (3.6) yields the relation

$$c_{jk}(x) = -\kappa(x)d_{jk}b_{jk}(x), \qquad (3.23)$$

where $d_{jk} = \frac{\lambda_j^3 - \lambda_k^3}{\lambda_j - \lambda_k} = \lambda_j^2 + \lambda_j\lambda_k + \lambda_k^2$. Substitute (3.23) into (3.7), then

$$b'_{jk}(x) = -i\kappa(x)d_{jk}(b_{jj}^{(0)} - b_{kk}^{(0)})b_{jk}(x) -$$

$$i\kappa(x) \sum_{l=1, l\neq k}^{4} (d_{kl} - d_{jl})b_{jl}(x)b_{lk}(x), \quad (3.24)$$

where $j \neq k$, and $d_{jk}(b_{jj}^{(0)} - b_{kk}^{(0)}) = \beta_1(\lambda_j^3 - \lambda_k^3)$. It is easy to see that

$$d_{kl} - d_{jl} = \lambda_k^2 + \lambda_k\lambda_l + \lambda_l^2 - \lambda_j^2 - \lambda_j\lambda_l - \lambda_l^2 = (\lambda_k - \lambda_j)(\lambda_j + \lambda_k + \lambda_l).$$

Let $b_{jk}(x)$ be given by (3.16). Then

$$
\begin{cases}
y'_{jk}(x) = \kappa(x) \displaystyle\sum_{l=1, l \neq j,k}^{4} (d_{kl} - d_{jl}) y_{jl}(x) y_{lk}(x), \\
y_{jk}(0) = \beta_{jk}.
\end{cases}
\tag{3.25}
$$

Use the fact that $y_{jk}(x)$ are real and $y_{34}(x) = 0$. Then $\beta_{34} = 0$, $y_{jk}(x) = y_{kj}(x)$, as $1 \leq j < k \leq 4$, and the system has the form

$$y'_{12} = \kappa(x)(d_{13} - d_{23})y_{13}y_{23} + \kappa(x)(d_{14} - d_{24})y_{14}y_{24}, \tag{3.26}$$
$$y'_{13} = \kappa(x)(d_{23} - d_{12})y_{12}y_{23}, \tag{3.27}$$
$$y'_{23} = \kappa(x)(d_{12} - d_{13})y_{12}y_{13}, \tag{3.28}$$
$$y'_{14} = \kappa(x)(d_{24} - d_{12})y_{12}y_{24}, \tag{3.29}$$
$$y'_{24} = \kappa(x)(d_{12} - d_{14})y_{12}y_{14}. \tag{3.30}$$

The condition $y_{34}(x) = 0$ implies

$$0 = \kappa(x)(d_{13} - d_{14})y_{13}y_{14} + (d_{23} - d_{24})y_{23}y_{24}. \tag{3.31}$$

It follows from (3.27) and (3.28) that

$$(d_{23} - d_{12})(y_{23}^2(x) - \beta_{23}^2) = (d_{12} - d_{13})(y_{13}^2(x) - \beta_{13}^2) \tag{3.32}$$

or

$$(\lambda_3 - \lambda_1)(y_{23}^2(x) - \beta_{23}^2) = (\lambda_2 - \lambda_3)(y_{13}^2(x) - \beta_{13}^2). \tag{3.33}$$

The restriction on λ_j, where $j = \overline{1,4}$, follows from the relation

$$(\lambda_3 - \lambda_1)\beta_{23}^2 = (\lambda_2 - \lambda_3)\beta_{13}^2, \tag{3.34}$$

i.e., $(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3) > 0$, thus λ_3 is between λ_1 and λ_2. Therefore,

$$y_{23}(x) = \sqrt{\frac{\lambda_2 - \lambda_3}{\lambda_3 - \lambda_1}}\, y_{13}(x). \tag{3.35}$$

Analogously, (3.29) and (3.30) imply that

$$y_{24}(x) = \sqrt{\frac{\lambda_2 - \lambda_4}{\lambda_4 - \lambda_1}}\, y_{14}(x) \tag{3.36}$$

if $(\lambda_4 - \lambda_1)\beta_{24}^2 = (\lambda_2 - \lambda_4)\beta_{14}^2$, and thus λ_4 is between λ_1 and λ_2. Taking into account (3.12), we write equality (3.31) in the form

$$(\lambda_1 + \lambda_3 + \lambda_4 + \alpha_3\alpha_4(\lambda_2 + \lambda_3 + \lambda_4))\, y_{13}(x)y_{14}(x) = 0, \tag{3.37}$$

besides, $\beta_{23} = \alpha_3\beta_{13}$, $\beta_{24} = \alpha_4\beta_{14}$. Thus $y_{2j}(x) = \alpha_j y_{1j}(x)$ for $j = 3, 4$. From (3.26), (3.27), (3.29), we find

$$y_{12}^2(x) - \beta_{12}^2 = -\beta_3(y_{13}^2(x) - \beta_{13}^2) - \beta_4(y_{14}^2(x) - \beta_{14}^2). \tag{3.38}$$

So the equations (3.27) and (3.29) become

$$y_{13}' = \kappa(x)(d_{23} - d_{12})\alpha_3 y_{12} y_{13}; \tag{3.39}$$
$$y_{14}' = \kappa(x)(d_{24} - d_{12})\alpha_4 y_{12} y_{14}. \tag{3.40}$$

Since $\dfrac{y_{14}'}{y_{14}} = \dfrac{\alpha_4}{\alpha_3}\dfrac{(d_{24} - d_{12})}{(d_{23} - d_{12})}\dfrac{y_{13}'}{y_{13}}$, then denoting $\alpha = \dfrac{(d_{24} - d_{12})\alpha_4}{(d_{23} - d_{12})\alpha_3}$ (which is equivalent to (3.13)), we search $y_{14}(x)$ (subject to the initial conditions) in the form $y_{14}(x) = \beta_{14}\left(\dfrac{y_{13}(x)}{\beta_{13}}\right)^\alpha$. Substitute

$$y_{12}(x) = \psi(y_{13}(x)) \tag{3.41}$$

into (3.38), where

$$\psi(y_{13}(x)) =$$
$$\sqrt{\beta_{12}^2 - \beta_3(y_{13}^2(x) - \beta_{13}^2) - \beta_4\beta_{14}^2\left(\left(\frac{y_{13}(x)}{\beta_{13}}\right)^{2\alpha} - 1\right)}. \tag{3.42}$$

By using (3.39), we can find

$$y_{13}'(x) = \kappa(x)\psi(y_{13}(x))y_{13}(x).$$

Since $F(y)$ is represented in the form (3.15), then

$$y_{13}(x) = F^{-1}\left(\sqrt{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)}\int_0^x \kappa(t)dt\right).$$

Thus, in the case $n = 4$ and cubic dependency of $a(x)$ from $\gamma(x)$, the elements of the matrix $B(x)$ are expressed in terms of elliptic functions.

Corollary 3.3. *In the conditions of Theorem 3 we suppose* $\lambda_1 = -a$, $\lambda_2 = a$, $\lambda_3 = -b$, $\lambda_4 = b$, *where* $a, b \in R$ *and the condition* $0 < a < 3b$ *is fulfilled, moreover,* $\beta_{13}^2 = \dfrac{1}{2}$, $\beta_{14}^2 = 2$, $\beta_{12}^2 = \dfrac{3ab - a^2}{a^2 - b^2}$, *then the solutions* $y_{jk}(x)$ *as* $j \neq k$,

(3.17)–(3.20), are given by

$$y_{12}(x) = -i\sqrt{\frac{2a}{a+b}}\, \frac{cn(z(x)+N,k)dn(z(x)+N,k)}{sn(z(x)+N)}, \tag{3.43}$$

$$y_{13}(x) = sn(z(x)+N,k), \tag{3.44}$$

$$y_{14}(x) = \frac{1}{sn(z(x)+N,k)}, \tag{3.45}$$

$$y_{23}(x) = k\, sn(z(x)+N,k), \tag{3.46}$$

$$y_{24}(x) = \frac{1}{k\, sn(z(x)+N,k)}, \tag{3.47}$$

where

$$k^2 = \frac{a+b}{a-b}, \quad z(x) = -i\sqrt{2a(a-b)}\int_0^x \kappa(t)dt, \tag{3.48}$$

$$N = \int_0^{\frac{1}{\sqrt{2}}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{3.49}$$

P r o o f. From (3.18), we obtain

$$\sqrt{(a-b)(a+b)}\int_0^x \kappa(t)dt = F(y_{13}(x)). \tag{3.50}$$

Applying (3.14), (3.15) and taking into account that under the given choice of λ_j, where $j = \overline{1,4}$, the value of $\alpha = -1$ (which is evident from (3.13)), we have

$$F(y_{13}(x)) = \int_{\beta_{13}}^{y_{13}(x)} \frac{dt}{\sqrt{\dfrac{2a}{b-a}t^4 + \dfrac{4a^2}{a^2-b^2}t^2 - \dfrac{2a}{b+a}}} =$$

$$i\sqrt{\frac{a+b}{2a}}\int_{\frac{1}{\sqrt{2}}}^{y_{13}(x)} \frac{dt}{\sqrt{\dfrac{a+b}{a-b}t^4 + \dfrac{2a}{a-b}t^2 + 1}} =$$

$$i\sqrt{\frac{a+b}{2a}}\int_{\frac{1}{\sqrt{2}}}^{y_{13}(x)} \frac{dt}{\sqrt{(1-t^2)\left(1-\dfrac{a+b}{a-b}t^2\right)}}. \tag{3.51}$$

Now relation (3.50) can be expressed as

$$-i\sqrt{2a(a-b)}\int_0^x \kappa(t)dt = \int_{\frac{1}{\sqrt{2}}}^{y_{13}(x)} \frac{dt}{\sqrt{(1-t^2)\left(1-\dfrac{a+b}{a-b}t^2\right)}}, \qquad (3.52)$$

i.e., taking into account (3.48),

$$z(x) = \int_{\frac{1}{\sqrt{2}}}^{y_{13}(x)} \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}} \qquad (3.53)$$

or

$$z(x) + \int_0^{\frac{1}{\sqrt{2}}} \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}} = \int_0^{y_{13}(x)} \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}}. \qquad (3.54)$$

In view of the notation and definition of elliptic functions, we have

$$z(x) + N = \int_0^{y_{13}(x)} \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}}, \qquad (3.55)$$

namely,

$$y_{13}(x) = sn(z(x) + N, k). \qquad (3.56)$$

From (3.17) and (3.14), we obtain

$$y_{12}(x) = \frac{1}{sn(z(x)+N)}\left(-i\sqrt{\frac{2a}{a+b}}\sqrt{(1-sn^2(z(x)+N)(1-k^2 sn^2(z(x)+N))}\right)$$

$$= -i\sqrt{\frac{2a}{a+b}}\,\frac{cn(z(x)+N,k)dn(z(x)+N,k)}{sn(z(x)+N)}. \qquad (3.57)$$

Substituting the values of λ_j, as $j = \overline{1,4}$, we have that $\alpha_3 = k$, and $\alpha_4 = \dfrac{1}{k}$, and then (3.20) implies that $y_{23}(x) = \alpha_3 y_{13}(x) = k\, sn(z(x)+N,k)$ and $y_{24}(x) = \alpha_4 y_{14}(x) = \dfrac{1}{k\, sn(z(x)+N,k)}$.

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