Journal of Mathematical Physics, Analysis, Geometry
2015, vol. 11, No. 1, pp. 75–99

The Singular Limit of the Dissipative Zakharov System

A.S. Shcherbina

Department of Mechanics and Mathematics, Karazin Kharkiv National University
4 Svobody Sq., Kharkiv 61077, Ukraine
E-mail: shcherbina@mail.ru

Received November 7, 2013, revised September 9, 2014

The dissipative Zakharov system which models the propagation of Langmuir waves in plasmas is considered on the interval $[0, L]$. We are interested in the case of large ion acoustic speed λ. After the formal limiting transition $\lambda \to \infty$ this system turns into the coupling system of the parabolic and Schrödinger equations. We prove that this limit system has a solution and generates a dissipative dynamical system possessing a global compact attractor. Our main result is the upper semicontinuity of the attractor as $\lambda \to \infty$.

Key words: dissipative dynamical system, dissipative Zakharov system, global compact attractor.

Mathematics Subject Classification 2010: 35Q55; 35B40, 34G20.

Introduction

The description of the propagation of Langmuir waves in plasma by the system of coupled equations

$$\begin{cases} \dfrac{1}{\lambda^2} n_{tt} - \Delta\left(n + |E|^2\right) = 0, \\ iE_t + \Delta E - nE = 0 \end{cases} \tag{1}$$

was proposed by Zakharov in [12]. Here $E : \mathbb{R}_x \times \mathbb{R}_t^+ \to \mathbb{C}$ and $n : \mathbb{R}_x \times \mathbb{R}_t^+ \to \mathbb{R}$. The complex function E represents the slowly varying envelop of the highly oscillating electric field, and n is the fluctuation of the ion density about its equilibrium value. The parameter λ is proportional to the ion acoustic speed (see [12]).

In this paper we are interested in the one-dimensional dissipative case

$$\begin{cases} \varepsilon n_{tt} + n_t - \Delta\left(n + |E|^2\right) = f(x), & x \in (0, L), \\ iE_t + \Delta E - nE + i\gamma E = g(x), & x \in (0, L), \\ n_t(x,0) = m_0(x),\ n(x,0) = n_0(x),\ E(x,0) = E_0(x), \end{cases} \tag{2}$$

where a positive damping parameter γ, the external forces $f(x)$ and $g(x)$ are given. For simplicity, we denote $\varepsilon = \lambda^{-2}$ and consider the case $\varepsilon \to 0$. This limit corresponds to the assumption that the plasma responds instantaneously to variations in the electric field (see discussion in [9]).

Formally letting ε tend to 0, we obtain the system

$$\begin{cases} n_t - \Delta\left(n + |E|^2\right) = f(x), & x \in (0, L), \\ iE_t + \Delta E - nE + i\gamma E = g(x), & x \in (0, L), \\ n(x,0) = n_0(x),\ E(x,0) = E_0(x). \end{cases} \tag{3}$$

In the paper, we prove that the system (3) with Dirichlet boundary conditions has the unique strong solution for every initial data in the corresponding energy space. Moreover, this problem generates the dissipative dynamical system possessing the compact global attractor $\mathcal{A}$. Our main result is the proof of the convergence of the attractors $\mathcal{A}_\varepsilon$ for the system (2) to $\mathcal{A}$ as $\varepsilon \to 0$.

The limit $\lambda \to \infty$ for the Zakharov problem (1) without dissipation for $x \in \mathbb{R}^d$, $d \le 3$, was studied in [9]. But this result concerns sufficiently smooth solutions on the finite interval $[0, T]$.

A similar problem was studied in [1] for the system of Shrödinger and Klien–Gordon equations with Yukawa coupling

$$\begin{cases} \beta^2\varphi_{tt} + \beta\varphi_t - \Delta\varphi - |\psi|^2 = f, \\ i\psi_t + \Delta\psi + \varphi\psi + i\gamma\psi = g \end{cases}$$

in the bounded domain $\Omega \subset \mathbb{R}^n$, $n \le 3$. The case $\beta, \gamma \to 0$ was considered and the convergence (on each finite time interval) of the corresponding solutions of this system to those of the limit problem was proven. But this result was obtained for the case where the ratio γ/β belongs to a fixed interval $[1, M]$.

The system (2) with Dirichlet boundary conditions was studied by Flahaut in [4]. The author proved that this problem has a unique solution and generates a dynamical system in the energetic spaces $\mathcal{E}_1 \equiv L_2 \times H_0^1 \times \left(H_0^1 \bigcap H^2\right)$ and $\mathcal{E}_2 \equiv H_0^1 \times \left(H_0^1 \bigcap H^2\right) \times \left(H_0^1 \bigcap H^3\right)$. Moreover, it was shown that there exists a bounded absorbing set and a weak attractor for this system. This result was improved by O. Goubet and I. Moise [5]. They proved the existence of the (uniform) compact global attractor $\mathcal{A}_2 \subset \mathcal{E}_2$ for the Zakharov problem with Dirichlet boundary conditions. This attractor $\mathcal{A}_2$ is also a global attractor in the space $\mathcal{E}_1$. It means that the global attractor in $\mathcal{E}_1$ possesses additional spatial smoothness.

The case of periodic boundary conditions for the system (2) was studied in [7]. It was proven that the elements of the global attractor for the dissipative Zakharov system with periodic boundary conditions are the analytic functions of the spatial variable.

Zakharov's system with two spatial variables was studied in [2, 8]. The existence of the global attractor was shown under the conditions which hold in the case of large enough γ or for the thin domain. Another interesting example of the interaction of the wave and quantum dynamics is Schroedinger–Boussinesq equtions (see [3] and references therein).

The paper is organized as follows. In Sec. 1, we obtain some ε-uniform estimates for the solution of (2). In Sec. 2, we prove that the problem (3) has a unique solution and generates the dissipative dynamical system possessing a compact global attractor. In Sec. 3, we prove the convergence of the attractors $\mathcal{A}_\varepsilon$ to $\mathcal{A}$ as $\varepsilon \to 0$.

1. The ε-Uniform Estimates

In this section we consider the problem (2) with Dirichlet boundary conditions

$$n(0,t) = n(L,t) = 0, \quad E(0,t) = E(L,t) = 0. \tag{4}$$

We recall that it was proven for every $\varepsilon > 0$ that the problem (2), (4) is well-posed in the spaces $\mathcal{E}_1 \equiv L_2(\Omega) \times H_0^1(\Omega) \times \left(H_0^1(\Omega) \bigcap H^2(\Omega)\right)$ and $\mathcal{E}_2 \equiv H_0^1(\Omega) \times \left(H_0^1(\Omega) \bigcap H^2(\Omega)\right) \times \left(H_0^1(\Omega) \bigcap H^3(\Omega)\right)$ (see [4]). We also recall that this problem generates the dissipative dynamical system in $\mathcal{E}_1$ possessing the global compact attractor $\mathcal{A}_\varepsilon \subset \mathcal{E}_2$ (see [5]).

Our goal is to obtain the ε-uniform estimates. The proof is split into several steps presented as separate lemmas. All of these lemmas contain a common part. We consider a functional $W_k(U(t))$ which is equivalent to the square of the norm of $U(t) \equiv (n_t(t), n(t), E(t))$ in the corresponding phase space. Then we compute the derivative of this functional on the trajectories and obtain the inequality of the type

$$\frac{\mathrm{d}}{\mathrm{dt}} W_k(U(t)) + \eta_k W_k(U(t)) + C\|U(t)\|^2 \le R_k(U(t)),$$

where η_k is a suitable constant which does not depend on ε. Then we estimate the terms in $R_k(t)$ and get

$$\frac{\mathrm{d}}{\mathrm{dt}} W_k(U(t)) + \eta_k W_k(U(t)) \le C.$$

Using the Gronwall lemma, we have

$$W_k(U(t)) \le W_k(U(0)) e^{-\eta_k t} + C(1 - e^{-\eta_k t})/\eta_k,$$

which implies $W_k(U(t)) \le C$. Since $W_k(U(t))$ is equivalent to the $\|U(t)\|^2$, we obtain the necessary estimates.

To estimate the nonlinear terms in $R_k(t)$, we have to recall two well-known functional inequalities which will be useful for us. The first of them is the Agmon inequality

$$\|u\|_{L_\infty} \le C\|u\|^{1/2}\|\nabla u\|^{1/2},\ u \in H^1(\Omega), \tag{5}$$

where $\|\,.\,\|$ denotes the usual $L_2(\Omega)$ norm. And the second one is the Gagliardo–Nirenberg inequality

$$\|u\|_{L^4} \le C\|u\|^{3/4}\|\nabla u\|^{1/4},\ u \in H^1(\Omega). \tag{6}$$

We note also that formally we consider $W_k(U_N(t))$, where $U_N(t)$ is the Galerkin approximation of $U(t)$. Therefore, first we prove our estimates for $U_N(t)$ and then pass to the limit as $N \to \infty$. However, for simplicity, we omit this procedure.

We start from the following.

Lemma 1.1. *Suppose that ε belongs to $[0, \varepsilon_0]$, $f \in L_2(\Omega)$, $g \in H^1(\Omega)$ and let (n_t, n, E) be a solution of (2), (4) in $\mathcal{E}_1$. Then there exists κ_0 such that*

$$\varepsilon\|(-\Delta)^{-1/2}n_t(t)\|^2 + \|n(t)\|^2 + \|\nabla E(t)\|^2 \le C_1 + C_2 e^{-\kappa_0 t}, \tag{7}$$

where C_i does not depend on ε and C_1 does not depend on the initial data.

P r o o f. Testing (2) by E and taking the imaginary part of the result, we get

$$\frac{\mathrm{d}}{\mathrm{dt}}\|E(t)\|^2 + 2\gamma\|E(t)\|^2 = 2\Im(g, E).$$

Since

$$2\Im(g, E) \le 2\|E\|\|g\| \le \gamma\|E\|^2 + \frac{1}{\gamma}\|g\|^2,$$

then the Gronwall lemma implies

$$\|E(t)\|^2 \le \|E_0\|^2\, e^{-\gamma t} + \frac{1}{\gamma^2}\|g\|^2. \tag{8}$$

Let us define the functional

$$\begin{aligned} W_0(t) \;=\; & \varepsilon\|n_t\|_{-1}^2 + \|n\|^2 + 2\|\nabla E\|^2 + \eta_0\left\{2\varepsilon(n_t, n)_{-1} + \|n\|_{-1}^2\right\} \\ & -2(f, n)_{-1} + 4\Re(g, E) + 2(|E|^2, n), \end{aligned} \tag{9}$$

where $\|\,.\,\|_{-1}$ and $(\cdot,\cdot)_{-1}$ are $\|(-\Delta)^{-1/2}\cdot\|$ and $((-\Delta)^{-1/2}\cdot, (-\Delta)^{-1/2}\cdot)$, respectively, and the positive parameter η_0 will be chosen later.

Since

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}}\left\{\varepsilon\|n_t\|_{-1}^2 + \|n\|^2 - 2(f, n)_{-1}\right\} = -2\|n_t\|_{-1}^2 - 2(n_t, |E|^2), \\ &\frac{\mathrm{d}}{\mathrm{dt}}\left\{2\varepsilon(n_t, n)_{-1} + \|n\|_{-1}^2\right\} = 2\varepsilon\|n_t\|_{-1}^2 - 2\|n\|^2 - 2(n, |E|^2) + 2(n, f)_{-1}, \\ &\frac{\mathrm{d}}{\mathrm{dt}}\left\{2\|\nabla E\|^2 + 4\Re(g, E) + 2(|E|^2, n)\right\} = 2(n_t, |E|^2) - 4\gamma\|\nabla E\|^2 \\ &\qquad\qquad - 4\gamma(n, |E|^2) - 4\gamma\Re(g, E), \end{aligned}$$

by addition we get

$$\frac{\mathrm{d}}{\mathrm{dt}} W_0(t) + 2(1-\varepsilon\eta_0)\|n_t\|_{-1}^2 + 2\eta_0\|n\|^2 + 4\gamma\|\nabla E\|^2 = R_0(t), \tag{10}$$

where

$$R_0(t) = 2\eta_0(f,n)_{-1} - 2(\eta_0+2\gamma)(|E|^2,n) - 4\gamma\Re(g,E). \tag{11}$$

We note that it follows from (6) and (8) that

$$\|E\|_{L_4} \le C\|E\|^{3/4}\|\nabla E\|^{1/4} \le (C_1 + C_2e^{-\gamma t})\|\nabla E\|^{1/4},$$

where C_1 is independent of the initial data. Therefore,

$$|(n,|E|^2)| \le C\|n\|\|E\|_{L_4}^2 \le (C_1 + C_2e^{-\gamma t})\|n\|\|\nabla E\|^{1/2}.$$

Taking into account this inequality, it is easy to see that

$$\begin{aligned} R_0(t) &\le C(1+\|n\|) + (C_1 + C_2e^{-\gamma t})\|n\|\|\nabla E\|^{1/2} \\ &\le \eta_0\|n\|^2 + \gamma\|\nabla E\|^2 + (C_1 + C_2e^{-\gamma t}), \end{aligned} \tag{12}$$

and

$$\begin{aligned} W_0(t) &\ge \frac{\varepsilon}{2}\|n_t\|_{-1}^2 + \frac{1}{2}\|n\|^2 + \|\nabla E\|^2 - C_1 - C_2e^{-\gamma t}, \\ W_0(t) &\le \frac{3\varepsilon}{2}\|n_t\|_{-1}^2 + 2\|n\|^2 + 3\|\nabla E\|^2 + C_1 + C_2e^{-\gamma t}. \end{aligned} \tag{13}$$

Substituting (12) into (10), we get

$$\frac{\mathrm{d}}{\mathrm{dt}} W_0(t) + 2(1-\varepsilon\eta_0)\|n_t\|_{-1}^2 + \eta_0\|n\|^2 + 3\gamma\|\nabla E\|^2 \le C_1 + C_2e^{-\gamma t}.$$

Now we put $\eta_0 = 1/(2\varepsilon_0)$ and $\kappa_0 = \min\{\varepsilon_0/4, \gamma\}$. Then, for sufficiently small ε_0, from (13) we obtain

$$2(1-\varepsilon\eta_0)\|n_t\|_{-1}^2 + \eta_0\|n\|^2 + 3\gamma\|\nabla E\|^2 \ge \kappa_0 W_0(t) - C_1 - C_2e^{-\gamma t}.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{dt}} W_0(t) + \kappa_0 W_0(t) \le C_1 + C_2e^{-\gamma t}.$$

Taking into account (13), we deduce (7) from the Gronwall lemma.

Lemma 1.2. *Let ε belong to $[0,\varepsilon_0]$, $f \in L_2(\Omega)$, $g \in H^1(\Omega)$ and let (n_t, n, E) be a solution of (2), (4) in $\mathcal{E}_1$. Then there exists κ_1 such that*

$$\varepsilon\|n_t(t)\|^2 + \|\nabla n(t)\|^2 + \|\Delta E(t)\|^2 \le C_1 + C_2e^{-\kappa_1 t}, \tag{14}$$

where C_i does not depend on ε and C_1 is independent of the initial data.

P r o o f. Let us consider the functional

$$\begin{aligned} W_1(t) \quad = \quad & \frac{1}{2}\left(\varepsilon\|n_t\|^2 + \|\nabla n\|^2\right) + \|\Delta E\|^2 - 2\Re(nE, \Delta E) - 2\Re(g, \Delta E) \\ & -(n, f) + \eta_1\left\{\varepsilon(n, n_t) + \frac{1}{2}\|n\|^2\right\}. \end{aligned} \tag{15}$$

From the Agmon inequality (5) and (7) we obviously have $\|E(t)\|_{L_\infty} \leq C_1 + C_2 e^{-\kappa_0 t}$. Therefore,

$$\begin{aligned} & |(nE, \Delta E)| \leq \|n\|\|E\|_{L_\infty}\|\Delta E\| \leq (C_1 + C_2 e^{-\kappa_0 t})\|\Delta E\|, \\ & |(g, \Delta E)| \leq \|g\|\|\Delta E\| \leq C\|\Delta E\|, \\ & |(n, f)| \leq \|n\|\|f\| \leq C_1 + C_2 e^{-\kappa_0 t}, \\ & \varepsilon|(n, n_t)| \leq \varepsilon\|n_t\|\|n\| \leq \frac{\varepsilon}{4}\|n_t\|^2 + \varepsilon(C_1 + C_2 e^{-\kappa_0 t}). \end{aligned} \tag{16}$$

Using these inequalities, we get

$$\begin{aligned} W_1(t) \quad \geq \quad & \frac{1}{4}\left(\varepsilon\|n_t\|^2 + \|\nabla n\|^2\right) + \frac{1}{2}\|\Delta E\|^2 - C_1 - C_2 e^{-\kappa_0 t}, \\ W_1(t) \quad \leq \quad & \frac{3}{4}\left(\varepsilon\|n_t\|^2 + \|\nabla n\|^2\right) + \frac{3}{2}\|\Delta E\|^2 + C_1 + C_2 e^{-\kappa_0 t}. \end{aligned} \tag{17}$$

It is easy to prove that

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2}\left(\varepsilon\|n_t\|^2 + \|\nabla n\|^2\right) - (n, f)\right) = -\|n_t\|^2 + (n_t, \Delta|E|^2), \\ & \frac{\mathrm{d}}{\mathrm{dt}}\left(\varepsilon(n, n_t) + \frac{1}{2}\|n\|^2\right) = \varepsilon\|n_t\|^2 - \|\nabla n\|^2 + (n, \Delta|E|^2) + (n, f), \\ & \frac{\mathrm{d}}{\mathrm{dt}}\left(\|\Delta E\|^2 - 2\Re(nE, \Delta E) - 2\Re(g, \Delta E)\right) = -2\gamma\|\Delta E\|^2 - 2\Re(n_t E, \Delta E) + R_{1,1}(t), \end{aligned}$$

where

$$R_{1,1}(t) = 4\gamma\Re(nE, \Delta E) - 2\Im(n^2 E, \Delta E) + 2\gamma\Re(g, \Delta E) + 2\Im(ng, \Delta E). \tag{18}$$

By using these relations, we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}W_1(t) + (1 - \varepsilon\eta_1)\|n_t\|^2 + \eta_1\|\nabla n\|^2 + 2\gamma\|\Delta E\|^2 = R_1(t), \tag{19}$$

where

$$R_1(t) = 2(n_t, |\nabla E|^2) + \eta_1(n, \Delta|E|^2) + R_{1,1}(t). \tag{20}$$

Taking into account the Agmon inequality (5), the Gagliardo–Nirenberg inequality and the estimate (7), we get

$$\begin{aligned} & \|n\|_{L_\infty} \leq (C_1 + C_2 e^{-\kappa_0 t})\|\nabla n\|^{1/2}, \quad \|\nabla E\|_{L_\infty} \leq (C_1 + C_2 e^{-\kappa_0 t})\|\Delta E\|^{1/2}, \\ & \|n\|_{L_4} \leq (C_1 + C_2 e^{-\kappa_0 t})\|\nabla n\|^{1/4}, \quad \|\nabla E\|_{L_4} \leq (C_1 + C_2 e^{-\kappa_0 t})\|\Delta E\|^{1/4}. \end{aligned}$$

Using these relations, we estimate the terms in $R_1(t)$ as

$$
\begin{aligned}
&|(n_t, |\nabla E|^2)| \le \|n_t\| \|\nabla E\|_{L_4}^2 \le (C_1 + C_2 e^{-\kappa_0 t}) \|n_t\| \|\Delta E\|^{1/2},\\
&|(n, \Delta |E|^2)| \le 2\|\nabla n\| \|\nabla E\| \|E\|_{L_\infty} \le (C_1 + C_2 e^{-\kappa_0 t}) \|\nabla n\|,\\
&|(nE, \Delta E)| \le \|n\| \|E\|_{L_\infty} \|\Delta E\| \le (C_1 + C_2 e^{-\kappa_0 t}) \|\Delta E\|,\\
&|(n^2 E, \Delta E)| \le \|n\|_{L_4}^2 \|E\|_{L_\infty} \|\Delta E\| \le (C_1 + C_2 e^{-\kappa_0 t}) \|\nabla n\|^{1/2} \|\Delta E\|,\\
&|(g, \Delta E)| \le \|g\| \|\Delta E\| \le C \|\Delta E\|,\\
&|(ng, \Delta E)| \le \|n\| \|g\|_{L_\infty} \|\Delta E\| \le (C_1 + C_2 e^{-\kappa_0 t}) \|\Delta E\|.
\end{aligned}
$$

Hence,

$$
R_1(t) \le \frac{1}{8}\|n_t\|^2 + \frac{\eta_1}{4}\|\nabla n\|^2 + \frac{\gamma}{2}\|\Delta E\|^2 + C_1 + C_2 e^{-\kappa_0 t}.
$$

Substituting this inequality into (19), we get for $\eta_1 = 1/(8\varepsilon_0)$

$$
\frac{\mathrm{d}}{\mathrm{dt}} W_1(t) + \frac{3}{4}\|n_t\|^2 + \frac{3\eta_1}{4}\|\nabla n\|^2 + \frac{3\gamma}{2}\|\Delta E\|^2 \le C_1 + C_2 e^{-\kappa_0 t}. \tag{21}
$$

We can see from (17) that for $\kappa_1 = \min\{\varepsilon_0/8, \gamma/2\}$ the following relation is true:

$$
\frac{3}{4}\|n_t\|^2 + \frac{3\eta_1}{4}\|\nabla n\|^2 + \frac{3\gamma}{2}\|\Delta E\|^2 \ge \kappa_1 W_1(t) - C_1 - C_2 e^{-\kappa_0 t}.
$$

Thus, in view of (21), we have

$$
\frac{\mathrm{d}}{\mathrm{dt}} W_1(t) + \kappa_1 W_1(t) \le C_1 + C_2 e^{-\kappa_0 t}.
$$

From the Gronwall lemma and (17) we obtain the second uniform estimate (14).

Lemma 1.3. *Let $U(t) = (n_t(t), n(t), E(t))$ be a solution of the system* (2) *and $U(t)$ belong to the global attractor $\mathcal{A}_\varepsilon$. Then for $f \in L_2(\Omega)$, $g \in H^1(\Omega)$*

$$
\varepsilon \|\nabla n_t(t)\|^2 + \|\Delta n(t)\|^2 + \|\nabla \Delta E(t)\|^2 \le C_3, \tag{22}
$$

where C_3 does not depend on ε.

P r o o f. Let us consider the functional

$$
\begin{aligned}
W_2(t) \;=\; & \frac{1}{2}\left(\varepsilon\|\nabla n_t(t)\|^2 + \|\Delta n(t)\|^2\right) + \|\nabla\Delta E(t)\|^2 + (\Delta n, f)\\
& -2\Re(\nabla g, \nabla \Delta E) + 2\Re(\Delta n \Delta E, E) - 2(\Delta n, |\nabla E|^2)\\
& +\eta_2\left(\varepsilon(\nabla n_t, \nabla n) + \frac{1}{2}\|\nabla n\|^2\right),
\end{aligned} \tag{23}
$$

where η_2 is a small parameter, which will be chosen later. Let us recall now that from (14) we have that $\|\nabla n\| \le C$ and $\|\Delta E\| \le C$. From the definition of the functional $W_2(t)$ and from the inequalities

$$\begin{array}{l} |(\Delta n, f)| \le \|\nabla n\| \|\nabla f\| \le C, \\ |(\nabla g, \nabla \Delta E)| \le \|\nabla g\| \|\nabla \Delta E\| \le C\|\nabla \Delta E\|, \\ |(\Delta n \Delta E, E)| \le \|\Delta n\| \|\Delta E\| \|E\|_{L_\infty} \le C\|\Delta n\|, \\ |(\Delta n, |\nabla E|^2)| \le \|\Delta n\| \|\nabla E\|^2_{L_4} \le C\|\Delta n\|, \\ |(\nabla n_t, \nabla n)| \le \|\nabla n_t\| \|\nabla n\| \le C\|\nabla n_t\|, \end{array}$$

we obtain that the following estimates for $W_2(t)$ remain true:

$$\begin{aligned} W_2(t) &\ge \frac{1}{4}\|\nabla n_t\|^2 + \frac{1}{2}\|\Delta n\|^2 + \|\nabla \Delta E\|^2 - C \\ W_2(t) &\le \frac{3}{4}\|\nabla n_t\|^2 + \frac{3}{2}\|\Delta n\|^2 + 3\|\nabla \Delta E\|^2 + C. \end{aligned} \tag{24}$$

By straightforward computation we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\varepsilon}{2}\|\nabla n_t(t)\|^2 + \frac{1}{2}\|\Delta n(t)\|^2 + (\Delta n, f)\right) &= -\|\nabla n_t\|^2 - 2\Re(\Delta E \Delta n_t, E) \\ &\quad - 2(|\nabla E|^2, \Delta n_t), \\ \frac{\mathrm{d}}{\mathrm{dt}}\left(\varepsilon(\nabla n_t, \nabla n) + \frac{1}{2}\|\nabla n\|^2\right) &= \varepsilon\|\nabla n_t\|^2 - \|\Delta n\|^2 - (\Delta n, \Delta|E|^2) + (\Delta n, f). \end{aligned}$$

Taking into account that the second equation of (2) implies that $E_t = i\Delta E - inE - \gamma E - ig$, one can obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}}\left(\|\nabla \Delta E(t)\|^2 - 2\Re(\nabla g, \nabla \Delta E) + 2\Re(\Delta n \Delta E, E) - 2(\Delta n, |\nabla E|^2)\right) \\ &= -2\gamma\|\nabla \Delta E\|^2 + 2\Re(\Delta n_t \Delta E, E) + R_2(t), \end{aligned}$$

where

$$\begin{aligned} R_{2,1}(t) = {} & 4\Re(\nabla n_t \Delta E, \nabla E) - 2\gamma\Re(E\nabla n + n\nabla E, \nabla \Delta E) + 2\gamma\Re(\nabla g, \nabla \Delta E) \\ & -2\Im(n\nabla \Delta E, E\nabla n + n\nabla E) - 2\gamma\Re(n\nabla \Delta E, \nabla E) - 2\Im(n\nabla \Delta E, g) \\ & +6\Im(\nabla n \Delta E, \nabla \Delta E) - 6\gamma\Re(\nabla E \nabla n, \Delta E) - 6\Im(\nabla n \Delta E, \nabla g) \\ & +2\Re(\Delta n \Delta E, E_t) - 6\Im(\nabla n \Delta E, E\nabla n + n\nabla E). \end{aligned} \tag{25}$$

From these equalities we can conclude that

$$\frac{\mathrm{d}}{\mathrm{dt}}W_2(t) + (1 - \eta_2\varepsilon)\|\nabla n_t\|^2 + \eta_2\|\Delta n\|^2 + 2\gamma\|\nabla \Delta E\|^2 = R_2(t), \tag{26}$$

where
$$R_2(t) = R_{2,1} - 2(|\nabla E|^2, \Delta n_t) - \eta_2(\Delta|E|^2 - f, \Delta n).$$
Put $\eta_2 = 1/(4\varepsilon_0)$ and $\kappa_2 = \min\{\varepsilon_0/2, \gamma\}$. Then (24) and (26) imply that

$$\frac{\mathrm{d}}{\mathrm{dt}}W_2(t) + \kappa_2 W_2(t) + \frac{1}{4}\|\nabla n_t\|^2 + \frac{\eta_2}{2}\|\Delta n\|^2 + \gamma\|\nabla\Delta E\|^2 \le C + R_2(t). \tag{27}$$

Now we estimate the terms in the r.h.s. of (27). In these estimates we use the Agmon inequality (5), (14) and the inequality $\|E_t\| \le \|\Delta E\| + \|n\|\|E\|_{L_\infty} + \gamma\|E\| + \|g\| \le C$,

$$\begin{aligned}
&|(\nabla n_t \Delta E, \nabla E)| \le \|\nabla n_t\|\|\Delta E\|\|\nabla E\|_{L_\infty} \le C\|\nabla n_t\|,\\
&|(E\nabla n + n\nabla E, \nabla\Delta E)| \le \|\nabla\Delta E\|(\|E\|_{L_\infty}\|\nabla n\| + \|\nabla E\|\|n\|_{L_\infty}) \le C\|\nabla\Delta E\|,\\
&|(\nabla g, \nabla\Delta E)| \le \|\nabla g\|\|\nabla\Delta E\| \le C\|\nabla\Delta E\|,\\
&|(n\nabla\Delta E, E\nabla n + n\nabla E)| \le \|\nabla\Delta E\|\|n\|_{L_\infty}(\|E\|_{L_\infty}\|\nabla n\| + \|\nabla E\|\|n\|_{L_\infty})\\
&\qquad\qquad \le C\|\nabla\Delta E\|,\\
&|(n\nabla\Delta E, \nabla E)| \le \|\nabla\Delta E\|\|n\|_{L_\infty}\|\nabla E\| \le C\|\nabla\Delta E\|,\\
&|(n\nabla\Delta E, g)| \le \|\nabla\Delta E\|\|n\|_{L_\infty}\|g\| \le C\|\nabla\Delta E\|,\\
&|(\nabla n\Delta E, E\nabla n + n\nabla E)| \le \|\nabla n\|_{L_\infty}\|\Delta E\|(\|E\|_{L_\infty}\|\nabla n\| + \|\nabla E\|\|n\|_{L_\infty})\\
&\qquad\qquad \le C\|\Delta n\|^{1/2},\\
&|(\nabla E\nabla n, \Delta E)| \le \|\nabla E\|_{L_\infty}\|\nabla n\|\|\Delta E\| \le C,\\
&|(\nabla n\Delta E, \nabla g)| \le \|\Delta E\|\|\nabla n\|_{L_\infty}\|\nabla g\| \le C\|\Delta n\|^{1/2},\\
&|(\Delta n\Delta E, E_t)| \le \|\Delta n\|\|\Delta E\|_{L_\infty}\|E_t\| \le C\|\Delta n\|\|\nabla\Delta E\|^{1/2}.
\end{aligned}$$

Substituting these inequalities in (27), we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}W_2(t) + \kappa_2 W_2(t) + \frac{1}{4}\|\nabla n_t\|^2 + \frac{\eta_2}{2}\|\Delta n\|^2 + \gamma\|\nabla\Delta E\|^2$$
$$\le C\left(1 + \|\nabla n_t\| + \|\Delta n\|^{1/2} + \|\Delta n\| + \|\nabla\Delta E\| + \|\Delta n\|\|\nabla\Delta E\|^{1/2}\right). \tag{28}$$

Hence,
$$\frac{\mathrm{d}}{\mathrm{dt}}W_2(t) + \kappa_2 W_2(t) \le C.$$
Thus we get (22) from the Gronwall lemma and (24).

Lemma 1.4. *Let $U(t) = (n_t(t), n(t), E(t))$ be a solution of the system* (2) *and $U(t)$ belong to the global attractor $\mathcal{A}_\varepsilon$. Then*

$$\varepsilon\|n_{tt}(t)\|^2 + \|\nabla n_t(t)\|^2 + \|\Delta E_t(t)\|^2 \le C_4, \tag{29}$$

where C_4 does not depend on ε.

P r o o f. Let us denote $m = n_t$ and $u = E_t$. Now we differentiate the system (2) with respect to t:

$$\begin{cases} m_t - \Delta m - 2\Re(\Delta(u\overline{E})) = 0, \\ iu_t + \Delta u - nu - mE + i\gamma u = 0. \end{cases} \tag{30}$$

Consider the functional

$$\begin{aligned} W_3(t) &= \frac{1}{2}\left(\varepsilon\|m_t\|^2 + \|\nabla m\|^2\right) + \|\Delta u\|^2 - 2\Re(mE, \Delta u) \\ &\quad + \eta_3\left(\varepsilon(m_t, m) + \frac{1}{2}\|m\|^2\right), \end{aligned} \tag{31}$$

where $\eta_3 = 1/(4\varepsilon_0)$. We note that it follows from (22) and the first equation of (2) that $\|\nabla u\| \le C$. Therefore,

$$|(mE, \Delta u)| \le \|\nabla u\|(\|E\|_{L_\infty}\|\nabla m\| + \|m\|_{L_\infty}\|\nabla E\|) \le C\|\nabla m\|.$$

Then we conclude that

$$\frac{1}{4}\left(\varepsilon\|m_t\|^2 + \|\nabla m\|^2\right) + \frac{1}{2}\|\Delta u\|^2 - C \le W_3(t) \le \varepsilon\|m_t\|^2 + \|\nabla m\|^2 + 2\|\Delta u\|^2 + C. \tag{32}$$

By straightforward computation it is easy to see that

$$\frac{\mathrm{d}}{\mathrm{dt}}W_2(t) + (1 - \varepsilon\eta_3)\|m_t\|^2 + \eta_3\|\nabla m\|^2 + 2\gamma\|\Delta u\|^2 = R_3(t), \tag{33}$$

where

$$\begin{aligned} R_3(t) &= 2\Re(m_t u, \Delta E) + 4\Re(m_t\nabla u, \nabla E) - 2\Re(mE_t, \Delta u) \\ &\quad + 2\eta_3\Re(mu, \Delta E) + 4\eta_3\Re(m\nabla u, \nabla E) + 2(\eta_3 + \gamma)\Re(mE, \Delta u) \\ &\quad - 2\Im(\Delta(nu), mE) + 2\Im(u\Delta n + 2\nabla n\nabla u, \Delta u). \end{aligned} \tag{34}$$

For $\kappa_3 = \min\{1/2, \gamma\}$, from (32) and (33) we deduce that

$$\frac{\mathrm{d}}{\mathrm{dt}}W_3(t) + \kappa_3 W_3(t) + \frac{1}{4}\|m_t\|^2 + \frac{\eta_3}{2}\|\nabla m\|^2 + \gamma\|\Delta u\|^2 \le C + R_2(t). \tag{35}$$

Now we estimate the terms in the r.h.s. of (33) as follows:

$$\begin{aligned} &|(m_t u, \Delta E)| \le \|m_t\|\|\Delta E\|\|u\|_{L_\infty} \le C\|m_t\|, \\ &|(m_t\nabla u, \nabla E)| \le \|m_t\|\|\nabla E\|_{L_\infty}\|\nabla u\| \le C\|m_t\|, \\ &|(mE_t, \Delta u)| \le \|\nabla u\|(\|\nabla m\|\|E_t\|_{L_\infty} + \|\nabla E_t\|\|m\|_{L_\infty}) \le C\|\nabla m\|, \\ &|(mu, \Delta E)| \le \|m\|_{L_\infty}\|u\|\|\Delta E\| \le C\|\nabla m\|, \\ &|\Re(m\nabla u, \nabla E)| \le \|m\|\|\nabla u\|\|\nabla E\|_{L_\infty} \le C\|\nabla m\|, \\ &|(\Delta(nu), mE)| \le \|\nabla u\|\|n\|_{L_\infty}(\|\nabla m\|\|E\|_{L_\infty} + \|\nabla E\|\|m\|_{L_\infty}) \\ &\qquad\qquad + \|\nabla n\|\|u\|_{L_\infty}(\|\nabla m\|\|E\|_{L_\infty} + \|\nabla E\|\|m\|_{L_\infty}) \le C\|\nabla m\|, \\ &|(u\Delta n + 2\nabla n\nabla u, \Delta u)| \le \|\Delta u\|(\|u\|_{L_\infty}\|\Delta n\| + 2\|\nabla n\|_{L_\infty}\|\nabla u\|) \le C\|\Delta u\|. \end{aligned}$$

Therefore (35) implies that

$$\frac{\mathrm{d}}{\mathrm{dt}} W_3(t) + \eta_3 W_3(t) \leq C.$$

This relation, the Gronwall lemma and (32) yield (29).

2. The Limit Problem

This section is devoted to the investigation of the long time behavior of the solution of the system (3), (4). We understand the solutions of this problem in the sense of the following definition.

Definition 2.1. A pair $(n; E)$ is said to be *semi-strong* to the problem (3), (4) on $[0, +\infty)$ iff

$$(n; E) \in L_\infty \left([0, +\infty);\ H_0^1(\Omega) \times H_0^1(\Omega) \bigcap H^2(\Omega) \equiv \mathcal{H} \right), \tag{36}$$

(i) the first two relations in (3) are fulfilled in the sense of distributions, (ii) the initial data hold.

We call this solution "semi-strong" because it is weak with respect to n and strong with respect to E.

2.1. Existence and uniqueness of the solution

Our first result is the following theorem.

Theorem 2.1. *Let the initial data* (n_0, E_0) *belong to* $\mathcal{H}$ *and the external forces* $f(x)$ *and* $g(x)$ *belong to* $L_2(\Omega)$ *and* $H_0^1(\Omega)$*. Then the system* (3), (4) *has a unique semi-strong solution on* R^+*.*

2.1.1. Existence. The proof of the existence is based on the compactness method. We define P_N as a projector on the first N eigenvectors of the operator $-\Delta$ with Dirichlet boundary conditions. Let us consider the approximation of the system (3):

$$\begin{cases} n_t^N - \Delta \left(n^N + P_N |E^N|^2 \right) = P_N f(x), & x \in (0, L), \\ iE_t^N + \Delta E^N - P_N(n^N E^N) + i\gamma E^N = P_N g(x), & x \in (0, L), \\ n^N(x, 0) = P_N n_0(x),\ E^N(x, 0) = P_N E_0(x). \end{cases} \tag{37}$$

This system is a system of ordinary differential equations. Hence, there exists the local (in time) solution $\left(n^N, E^N\right)$ on $[0, T_N]$. We note that Lemma 1.1 and Lemma 1.2 remain true for $\varepsilon = 0$. Then from (14) we get that $(n^N, E^N) \in L_\infty(R^+, \mathcal{H})$. Hence, for any $T > 0$, we get that $\left(n^N, E^N\right)$ belongs to a bounded set in $L_\infty((0,T),\mathcal{H})$. Then there exists a subsequence, still denoted by $\left(n^N, E^N\right)$, such that there exists $(n,E) \in L^\infty((0,T),\mathcal{H})$ such that $\left(n^N, E^N\right) \to (n,E)$, which is weak-star in $L^\infty((0,T),\mathcal{H})$, as $N \to \infty$.

Let us prove that (n,E) satisfies (3). Since n^N and E^N belong to the bounded sets in $L_\infty\left((0,T),H^1\right)$ and $L_\infty\left((0,T),H^2\right)$, respectively, $n_t^N = \Delta(n^N + P_N|E|^2) + P_N f$ and $E_t^N = i\Delta E^N - iP_N(n^N E^N) - \gamma E^N - iP_N g$ belong to $L_2\left((0,T),H^{-1}\right)$ and $L_2((0,T),L_2)$. It follows from Aubin's imbedding theorem (see [10, Corollary 4]) that there exists a new subsequence $\left(n^N, E^N\right)$ such that $\left(n^N, E^N\right) \to (n^*, E^*)$ strongly in $C\left(0,T; L_2 \times H_0^1\right)$. It is easy to see that $n^* = n$ and $E^* = E$. Thus we can pass to the limit in the nonlinear terms.

2.1.2. Uniqueness. The uniqueness of the solution of (3) follows from the next proposition.

Proposition 2.1. *Suppose $(n^{(1)}, E^{(1)})$ and $(n^{(2)}, E^{(2)})$ are two solutions of (3) in $\mathcal{H}$ with the initial data $(n_0^{(1)}, E_0^{(1)})$ and $(n_0^{(2)}, E_0^{(2)})$, respectively, which belong to the ball of radius R in $\mathcal{H}$. Then*

$$\|\nabla n(t)\|^2 + \|\Delta E(t)\|^2 \le C_R\left(\|\nabla n_0\|^2 + \|\Delta E_0\|^2\right) e^{tC_R}, \tag{38}$$

where $n = n^{(1)} - n^{(2)}$, $E = E^{(1)} - E^{(2)}$, $n_0 = n_0^{(1)} - n_0^{(2)}$ and $E_0 = E_0^{(1)} - E_0^{(2)}$.

P r o o f. First, we prove the estimate (38) for Galerkin's approximations. Then, passing to the limit, we obtain (38) for the solutions of (3).

For simplicity, we omit index N of the number of Galerkin's approximations.

Since the initial data belong to the ball of radius R in $\mathcal{H}$, then there exists a constant C_R such that

$$\|\nabla n(t)\|^2 + \|\Delta E(t)\|^2 \le C_R. \tag{39}$$

It is obvious that (n,E) is a solution of

$$\begin{cases} n_t - \Delta n = \Delta(E^{(1)}\overline{E} + E\overline{E^{(2)}}), \\ iE_t + \Delta E + i\gamma E = n^{(1)}E + nE^{(2)}. \end{cases} \tag{40}$$

We note that the second equation from (40) implies

$$\begin{array}{ll} \exists c_{1,R} > 0, & c_{1,R}\|\Delta E\| \le \|E_t\| + \|n\| + \|E\|, \\ \exists c_{2,R} > 0, & \|E_t\| \le c_{2,R}(\|E\| + \|n\| + \|\Delta E\|). \end{array} \tag{41}$$

Let us rewrite now the second equation from (40) in the form

$$iE_t + \Delta E = -i\gamma E + n^{(1)}E + nE^{(2)}.$$

Taking into account that

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\|E\|^2+\|\nabla E\|^2\right)=2\Im(iE_t+\Delta E, E-\Delta E),$$

we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\left(\|E\|^2+\|\nabla E\|^2\right)+2\gamma\left(\|E\|^2+\|\nabla E\|^2\right)=2\Im(n^{(1)}E+nE^{(2)}, E-\Delta E)\\&\le(\|E^{(2)}\|_{L_\infty}\|n\|+\|E\|_{L_\infty}\|\nabla n^{(1)}\|+\|\nabla n\|\|E^{(2)}\|_{L_\infty}+\|n\|_{L_\infty}\|\nabla E^{(2)}\|)\|\nabla E\|\\&\le C_R\left\{\|\nabla E\|^2+\|\nabla n\|^2\right\}.\end{aligned}\tag{42}$$

Now we differentiate the second equation in (40) with respect to t:

$$iE_{tt}+\Delta E_t=-i\gamma E_t+n_t^{(1)}E^{(1)}+n^{(1)}E_t^{(1)}-n_t^{(2)}E^{(2)}-n^{(2)}E_t^{(2)}.$$

Then

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\|E_t(t)\|^2+2\gamma\|E_t(t)\|^2=2\Im(n_t^{(1)}E+n_tE^{(2)}+nE_t^{(2)}, E_t)\\&\le\left(\|n_t^{(1)}\|\|E\|_{L_\infty}+\|n_t\|\|E^{(2)}\|_{L_\infty}+\|n\|_{L_\infty}\|E_t^{(2)}\|\right)\|E_t\|\\&\le C_R\left(\|n_t\|\|E_t\|+\|\nabla n\|^2+\|E_t\|^2+\|\nabla E\|^2\right).\end{aligned}\tag{43}$$

The first equation in (40) and the first inequality in (41) imply that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\|\nabla n(t)\|^2+2\|n_t(t)\|^2=2\left(\Delta(E^{(1)}\overline{E}+E\overline{E^{(2)}}), n_t\right)\\&\le C_R\left(\|\nabla E\|+\|E_t\|+\|\nabla n\|\right)\|n_t\|.\end{aligned}\tag{44}$$

Adding (42), (43) and (44), we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{dt}}\left(\|\nabla n(t)\|^2+\|E(t)\|^2+\|\nabla E(t)\|^2+\|E_t\|^2\right)+2\|n_t(t)\|^2\\&\le C_R\left(\|\nabla E\|+\|E_t\|+\|\nabla n\|\right)\|n_t\|+C_R\left(\|\nabla n\|^2+\|E_t\|^2+\|\nabla E\|^2\right)\\&\le 2\|n_t(t)\|^2+C_R\left(\|\nabla n\|^2+\|E\|^2+\|\nabla E\|^2+\|E_t\|^2\right).\end{aligned}$$

From the Gronwall lemma we have

$$\begin{aligned}&\|\nabla n(t)\|^2+\|E(t)\|^2+\|\nabla E(t)\|^2+\|E_t(t)\|^2\\&\le\left(\|\nabla n(0)\|^2+\|E(0)\|^2+\|\nabla E(0)\|^2+\|E_t(0)\|^2\right)e^{tC_R}.\end{aligned}\tag{45}$$

Taking into account (41), we can deduce that

$$\begin{aligned}&\|\nabla n(t)\|^2+\|E(t)\|^2+\|\nabla E(t)\|^2+\|E_t(t)\|^2\ge\frac{1}{2}\|\nabla n(t)\|^2+C_R\|\Delta E(t)\|^2,\\&\|\nabla n(0)\|^2+\|E(0)\|^2+\|\nabla E(0)\|^2+\|E_t(0)\|^2\le C_R\left(\|\nabla n_0\|^2+\|\Delta E_0\|^2\right).\end{aligned}$$

Thus, substituting these inequalities in (45), we obtain (38).

2.1.3. Construction of the evolution operator for the limit system. Let us rewrite the first equation from (3) as

$$n_t - \Delta n = f + \Delta |E|^2.$$

Taking into account that the r.h.s. of the equation belongs to $L^\infty(\mathbb{R}^+, L_2)$, we can derive that $n \in C\left(\mathbb{R}^+, H_0^1\right)$. In the same way, by rewriting the second equation from (3) as the linear Schrödinger equation and using the result of Lions and Magenes [6], we obtain that $E \in C(\mathbb{R}^+, H_0^1 \bigcap H^2)$.

We note that since Lemma 1.2 remains true for $\varepsilon = 0$, then the following proposition holds.

Proposition 2.2. *The problem* (3) *generates the dissipative dynamical system* $S(t) : (n_0, E_0) \in \mathcal{H} \to (n(t), E(t)) \in \mathcal{H}$.

2.2. Existence of the Global Attractor for the limit system

Our goal is the proof the following result concerning the existence of the compact global attractor in $\mathcal{H}$.

Theorem 2.2. *Let* $f(x)$ *and* $g(x)$ *be the functions from* $L_2(\Omega)$ *and* $H_0^1(\Omega)$. *Then the problem* (3) *generates the dynamical system possessing a compact global attractor* $\mathcal{A} \in \mathcal{H}$.

The proof of the existence of the global attractor is based on the well-known theorem of the general theory of dynamical systems (see, for instance, [11, Theorem I.1.1]). We construct a decomposition of the evolution operator $S(t)$ as $S_1(t) + S_2(t)$ with the properties required in Theorem, i.e.,

(1) $S_1(t)$ is uniformly compact in $\mathcal{H}$ for t large, i.e., for any bounded set B there exists t_0 such that the closure of $\bigcup_{t \geq t_0} S_1(t)B$ is a compact set in $\mathcal{H}$.

(2) $S_2(t) : \mathcal{H} \to \mathcal{H}$ is continuous for any $t \geq 0$, and for any bounded set $B_1 \subset \mathcal{H}$, the following relation holds:

$$\sup_{\phi \in B_1} \|S_2(t)\phi\|_{\mathcal{H}} \to 0 \text{ as } t \to \infty.$$

Decomposition. Consider the initial data (n_0, E_0) from the ball $B_R \subset \mathcal{H}$ centered at the origin. Proposition 2.2 states that there exists a bounded absorbing set in $\mathcal{H}$, denoted by $\mathcal{B}$. Then there exists the time of dissipation $t_0 = t_0(\mathcal{B}, R)$ such that $S(t)(n_0, E_0) \in \mathcal{B}$ for $\forall t \geq t_0$. Thus,

$$\|\nabla n(t)\|^2 + \|\Delta E(t)\|^2 \leq C, \qquad \forall t \geq t_0. \tag{46}$$

We set

$$E = P_N E + Q_N E \tag{47}$$

and we write $y = P_N E$. Now we split the high frequency part of E as

$$Q_N E = Z + \chi, \tag{48}$$

and n:

$$n = p + q, \tag{49}$$

where (p, Z) is a solution of the problem

$$\begin{cases} p_t - \Delta(p + |y + Z|^2) = f, \\ iZ_t + \Delta Z - Q_N(p(y + Z)) + i\gamma Z = Q_N g, \\ p(t, x) = p(t, x + L), Z(t, x) = Z(t, x + L), \\ p(t_0, x) = Z(t_0, x) = 0. \end{cases} \tag{50}$$

Let us prove now the lemma below.

Lemma 2.1. *There exists $N_0 \in \mathbb{N}$ such that for any $N > N_0$ the system* (50) *admits a unique solution belonging to $Q_N \mathcal{V}_s = Q_N \mathcal{H}$. Moreover, for any $N > N_0$ and $t > t_0$,*

$$\|\nabla p(t)\|^2 + \|\Delta Z\|^2 \le R_1^2. \tag{51}$$

P r o o f. Let $M > N$ be a positive number. If we substitute Q_N by $Q_{M,N} \equiv P_M - P_N$ in the system (50), then it is easy to see that (50) has a unique solution in some interval $(t_0, t_0 + T_{M,N})$. If we prove the uniform bound (51) in the interval $(t_0, t_0 + T_{M,N})$, then we can conclude that this solution can be extended to the half-axes $(t_0, +\infty)$. Then we send $M \to \infty$ and obtain the existence of the solution of (50) with property (51). It is easy to show (in the same way as for the system (3)) that this solution is unique. Thus we need only to prove the uniform estimate (51) for the solution on any interval of the existence. To simplify notations below, we omit the subscript M.

We prove (51) in several steps. First, we prove that

$$\|p(t)\|^2 + \|\nabla Z\|^2 \le R_1^2. \tag{52}$$

Then, using (52), we can start to prove (51).

Step 1. We consider the functional

$$J_0(t) = \|p\|^2 + 2\|\nabla Z\|^2 - 2(f, p)_{-1} + 4\Re(g, Z) + 2(p, |y + Z|^2) + \mu_0 \|p\|_{-1}^2. \tag{53}$$

Taking into account (50), by straightforward computation we get

$$\frac{\mathrm{d}}{\mathrm{dt}}(\|p\|^2 - 2(f,p)_{-1}) = -2\|p_t\|_{-1}^2 - 2(p_t, |y+Z|^2),$$
$$\frac{\mathrm{d}}{\mathrm{dt}}\|p\|_{-1}^2 = -2\|p\|^2 + 2(f,p)_{-1} - 2(p, |y+Z|^2),$$
$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\|\nabla Z\|^2 + 2\Re(g,Z) + (p,|y+Z|^2)\right) = -2\gamma\|\nabla Z\|^2 + (p_t, |y+Z|^2) + 2\Re(p(y+Z), y_t) - 2\gamma\Re(p(y+Z), Z) - 2\gamma\Re(g,Z).$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{dt}}J_0(t) + 2\|p_t\|_{-1}^2 + 2\mu_1\|p\|^2 + 4\gamma\|\nabla Z\|^2 = R_4(t), \tag{54}$$

where

$$R_4(t) = 2\mu_0(f,p)_{-1} - 2\mu_0(p,|y+Z|^2) + 4\Re(p(y+Z), y_t) - 4\gamma\Re(p(y+Z) - g, Z). \tag{55}$$

Now we remark that for fixed $N \in \mathbb{N}$ and for every $S_1 > S_2 \geq 0$,

$$\begin{cases} \|P_N\varphi\|_{H^{S_1}} \leq \lambda_N^{(S_1-S_2)/2}\|P_N\varphi\|_{H^{S_2}}, & \forall\varphi \in H^{S_2}, \\ \|Q_N\varphi\|_{H^{S_2}} \leq \lambda_{N+1}^{-(S_1-S_2)/2}\|Q_N\varphi\|_{H^{S_1}}, & \forall\varphi \in H^{S_1}. \end{cases} \tag{56}$$

Then, from the Agmon inequality (5) and the Gagliardo–Nirenberg inequality (6) for $Z = Q_N Z$, we get

$$\|Z\|_{L_\infty} \leq C\lambda_{N+1}^{-1/4}\|\nabla Z\|, \quad \|Z\|_{L_4} \leq C\lambda_{N+1}^{-3/8}\|\nabla Z\|. \tag{57}$$

Now, taking into account that y and y_t belong to the bounded subset in $H^2(\Omega)$ and $L_2(\Omega)$, respectively, and using (57), we get

$$|(f,p)_{-1}| \leq C\|p\|,$$
$$|(p,|y+Z|^2)| \leq \|p\|\|y+Z\|_{L_4}^2 \leq C\|p\|(1 + \lambda_{N+1}^{-3/8}\|\nabla Z\| + \lambda_{N+1}^{-3/4}\|\nabla Z\|^2),$$
$$|(p(y+Z), y_t)| \leq \|y_t\|\|p\|\|y+Z\|_{L_\infty} \leq C\|p\|(1 + \lambda_{N+1}^{-1/4}\|\nabla Z\|),$$
$$|(p(y+Z) - g, Z)| \leq C\lambda_{N+1}^{-1/2}\|\nabla Z\| + \|p\|\|y+Z\|_{L_\infty}\|Z\| \leq C\lambda_{N+1}^{-1/2}\|\nabla Z\| + C\lambda_{N+1}^{-1/2}\|p\|\|\nabla Z\|(1 + \|\nabla Z\|).$$

These estimates and (55) yield

$$|R_4(t)| \leq C\|p\|(1 + \lambda_{N+1}^{-3/8}\|\nabla Z\| + \lambda_{N+1}^{-3/4}\|\nabla Z\|^2) + C\lambda_{N+1}^{-1/2}\|\nabla Z\|, \tag{58}$$

and (53) implies that

$$\frac{1}{2}(\|p\|^2 + \|\nabla Z\|^2) - C - C\lambda_{N+1}^{-3/8}(\|p\|^2 + \|\nabla Z\|^2)^2 \leq J_0(t) \leq \frac{3}{2}(\|p\|^2 + \|\nabla Z\|^2) + C + C\lambda_{N+1}^{-3/8}(\|p\|^2 + \|\nabla Z\|^2)^2. \tag{59}$$

Substituting (58) and (59) into (54), for some sufficiently small δ we get

$$\frac{\mathrm{d}}{\mathrm{dt}} J_0(t) + \delta J_0(t) \le C + C\lambda_{N+1}^{-3/8} \left(\|p(t)\|^2 + \|\nabla Z(t)\|^2\right)^2 . \tag{60}$$

Now, after integrating (60) on $[t_0, t]$, and using that $J_0(0) = 0$, we have

$$\begin{aligned} \|p(t)\|^2 + \|\nabla Z(t)\|^2 \le C_1 \Bigg\{ & 1 + \lambda_{N+1}^{-3/8} \left(\|p(t)\|^2 + \|\nabla Z(t)\|^2\right)^2 \\ & + \lambda_{N+1}^{-3/8} \int_{t_0}^{t} \left(\|p(\tau)\|^2 + \|\nabla Z(\tau)\|^2\right)^2 e^{\delta(\tau - t)} d\tau \Bigg\} . \end{aligned} \tag{61}$$

Set $\varphi(t) = \sup\limits_{t_0 \le \tau \le t} \left\{\|p(\tau)\|^2 + \|\nabla Z(\tau)\|^2\right\}$. Hence, from (61) we get

$$\varphi(t) \le C_2 \lambda_{N+1}^{-3/8} \varphi(t)^2 + C_1 .$$

Then the inequality $F(\varphi(t)) \ge 0$ is true for the function $F(\varphi(t)) = C_2 \lambda_{N+1}^{-1/4} \varphi(t)^2 + C_1 - \varphi(t)$. Let us notice that $\varphi(t)$ is a continuous function, and $\varphi(0) = 0$. Choosing again N large enough to provide $4C_1C_2\lambda_{N+1}^{-3/8} < 1$, we obtain that $\varphi(t) \le \alpha_1$, where α_1 is the first root of F. Thus (52) is obtained.

Step 2. Let us take now the real part of the inner product in L_2 of the second equation from (50) with $4\Delta Z_t + 4i\gamma\Delta Z$:

$$\frac{\mathrm{d}}{\mathrm{dt}} \left(2\|\Delta Z\|^2 - 4\Re(g, \Delta Z)\right) + 4\gamma\|\Delta Z\|^2 - 4\Re(p(y+Z), \Delta Z_t + \gamma\Delta Z) = 4\gamma\Re(g, \Delta Z). \tag{62}$$

Taking into account that $Z_t = i\Delta Z - iQ_N(p(y+Z)) - \gamma Z - iQ_N g$, we can transform the term $\Re(p(y+Z), \Delta Z_t)$ as

$$\begin{aligned} \Re(p(y+Z)), \Delta Z_t) = & \frac{\mathrm{d}}{\mathrm{dt}} \Re(p(y+Z), \Delta Z) - \Re(p_t(y+Z), \Delta Z) - \Re(py_t, \Delta Z) \\ & - \Im(pQ_N(p(y+Z)), \Delta Z) + \gamma\Re(pZ, \Delta Z) - \Im(pQ_N g, \Delta Z). \end{aligned}$$

Substituting this relation in (62), we obtain

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{dt}} \left\{2\|\Delta Z\|^2 - 4\Re(p(y+Z), \Delta Z) - 4\Re(g, \Delta Z)\right\} \\ & + 4\gamma\|\Delta Z\|^2 + 4\Re(p_t(y+Z), \Delta Z) = R_3(t), \end{aligned} \tag{63}$$

where

$$\begin{aligned} R_3(t) = & 4\gamma\Re(g, \Delta Z) - 4\Im(pQ_N(p(y+Z)), \Delta Z) - 4\Re(py_t, \Delta Z) \\ & + 8\gamma\Re(pZ, \Delta Z) - 4\Im(pQ_N g, \Delta Z). \end{aligned} \tag{64}$$

Now we multiply the first equation from (50) by $2p_t + 2p$. It is straightforward to get

$$\frac{\mathrm{d}}{\mathrm{dt}}\left\{\|p\|^2 + \|\nabla p\|^2 - 2(f,p)\right\} + 2\|p_t\|^2 + 2\|\nabla p\|^2 - 4\Re(p_t(y+Z),\Delta Z) = R_4(t), \tag{65}$$

where

$$\begin{aligned} R_4(t) \quad = \quad & 2(f,p) + 2(\Delta|y+Z|^2,p) + 2(\Delta|y|^2,p_t) + 4(|\nabla Z|^2,p_t) \\ & +4\Re(p_t\Delta y, Z) + 8\Re(p_t\nabla y,\nabla Z). \end{aligned} \tag{66}$$

We set

$$J_1(t) = \|p\|^2 + \|\nabla p\|^2 + 2\|\Delta Z\|^2 - 4\Re(p(y+Z),\Delta Z) - 2(f,p) - 4\Re(g,\Delta Z), \tag{67}$$

and let $\mu_1 > 0$ be a positive parameter, small enough. Then we rewrite the sum of (63) and (65) in the form

$$\frac{\mathrm{d}}{\mathrm{dt}}J_1(t) + \mu_1 J_1(t) + 2\|p_t\|^2 + 2\|\nabla p\|^2 + 4\gamma\|\Delta Z\|^2 = R_3(t) + R_4(t) + \mu_1 J_1(t). \tag{68}$$

From (56), the Agmon inequality (5) and the Gagliardo–Nirenberg inequality (6), for $Z = Q_N Z$ we get

$$\begin{cases} \|Z\|_{L_\infty} \le C\lambda_{N+1}^{-3/4}\|\Delta Z\|, & \|Z\|_{L_4} \le C\lambda_{N+1}^{-7/8}\|\Delta Z\|, \\ \|\nabla Z\|_{L_\infty} \le C\lambda_{N+1}^{-1/4}\|\Delta Z\|, & \|\nabla Z\|_{L_4} \le C\lambda_{N+1}^{-3/8}\|\Delta Z\|. \end{cases} \tag{69}$$

Now we can start to estimate the terms in the r.h.s. of (68). We estimate the terms from $R_3(t)$ as follows:

$$\begin{aligned} &|(g,\Delta Z)| \le \|g\|\|\Delta Z\| \le C\|\Delta Z\|, \\ &|(pQ_N(p(y+Z)),\Delta Z)| \le \|p\|_{L_\infty}\|p\|_{L_4}\|y+Z\|_{L_4}\|\Delta Z\| \le C\|\nabla p\|^{3/4}\|\Delta Z\|, \\ &|(py_t,\Delta Z)| \le \|p\|_{L_\infty}\|y_t\|\|\Delta Z\| \le C\|\Delta Z\|\|\nabla p\|^{1/2}, \\ &|(pZ,\Delta Z)| \le \|p\|\|Z\|_{L_\infty}\|\Delta Z\| \le C\|\Delta Z\|, \\ &|(pQ_N g,\Delta Z)| \le \|p\|_{L_\infty}\|g\|\|\Delta Z\| \le C\|\Delta Z\|\|\nabla p\|^{1/2}. \end{aligned}$$

Then we estimate the terms from $R_4(t)$:

$$\begin{aligned} &|(f,p)| \le \|f\|\|p\| \le C, \\ &|(\Delta|y+Z|^2,p)| \le 2(\|\Delta y + \Delta Z\|\|y+Z\|_{L_\infty} + \|\nabla y + \nabla Z\|_{L_4}^2)\|p\| \le C\|\Delta Z\|, \\ &|(\Delta|y|^2,p_t)| \le 2(\|\Delta y\|\|y\|_{L_\infty} + \|\nabla y\|_{L_4}^2)\|p_t\| \le C\|p_t\|, \\ &|(|\nabla Z|^2,p_t)| \le \|\nabla Z\|_{L_4}^2\|p_t\| \le C\|p_t\|\|\Delta Z\|^{1/2}, \\ &|(p_t\Delta y, Z)| \le \|p_t\|\|\Delta y\|\|Z\|_{L_\infty} \le C\|p_t\|, \\ &|(p_t\nabla y,\nabla Z)| \le \|p_t\|\|\nabla y\|_{L_4}\|\nabla Z\|_{L_4} \le C\|p_t\|\|\Delta Z\|^{1/4}. \end{aligned}$$

These inequalities and (67) yield

$$\frac{1}{2}\left(\|\nabla p\|^2+\|\Delta Z\|^2\right)-C\le J_1(t)\le\frac{3}{2}\left(\|\nabla p\|^2+\|\Delta Z\|^2\right)+C, \tag{70}$$

and (68) implies

$$\frac{\mathrm{d}}{\mathrm{dt}}J_1(t)+\mu_1 J_1(t)\le C.$$

Using the Gronwall lemma and (70), we obtain (51). This completes the proof of Lemma 2.1.

Now we prove the additional estimate for $\|p_t\|$, which will be useful later.

Lemma 2.2. *There exists $N_1\in\mathbb{N}$ such that for every fixed $N>N_1$,*

$$\|p_t(t)\|\le C(1+\lambda_N^{1/2}),\ t\ge t_0, \tag{71}$$

where the constant C is independent of N.

P r o o f. As in the previous lemmas, we consider a functional $J_2(t)$ and compute the derivative of this functional with respect to t. Since these calculations are straightforward and similar to the previous one, we omit them. We get

$$\frac{\mathrm{d}}{\mathrm{dt}}J_2(t)=-2\|\nabla p_t\|^2-2\|\Delta p\|^2-4\gamma\|\nabla\Delta Z\|^2+R_5(t), \tag{72}$$

where

$$J_2(t)=\|\Delta p\|^2+2\|\nabla\Delta Z\|^2+2(f,\Delta p)+4\Re(g,\Delta^2 Z)+4\Re(p(y+Z),\Delta^2 Z), \tag{73}$$

and

$$\begin{aligned}R_5(t)=&-2(f,\Delta p)-2(\Delta|y+Z|^2,\Delta p)+2(\nabla\Delta|y|^2,\nabla p_t)+4\Re(\nabla\Delta y\nabla p_t,Z)\\&+12\Re(\Delta y\nabla p_t,\nabla Z)+12\Re(\nabla y\nabla p_t,\Delta Z)+4\Re(\nabla\Delta|Z|^2,\nabla p_t)\\&-4\Re(p_t\nabla y,\nabla\Delta Z)+4\Re(p_tZ,\Delta^2 Z)+4\Re(py_t,\Delta^2 Z)-4\gamma\Re(g,\Delta^2 Z)\\&-4\gamma\Re(p(y+Z),\Delta^2 Z)+4\Im(\nabla p\Delta Z,\nabla\Delta Z)-4\gamma\Re(pZ,\Delta^2 Z)\\&+4\Im(pQ_N(p(y+Z)),\Delta^2 Z)+4\Im(pQ_Ng,\Delta^2 Z).\end{aligned} \tag{74}$$

Note that (56) implies that for $y=P_Ny$ and for $Z=Q_NZ$

$$\|\nabla\Delta y\|\le C\lambda_N^{1/2}\|\Delta y\|\text{ and }\|\Delta Z\|\le C\lambda_{N+1}^{-1/2}\|\nabla\Delta Z\|.$$

Taking into account this relation, (51), (69) and the Agmon inequality (5), it is easy to prove that

$$\begin{aligned}|R_5(t)|\le&\,C\|\Delta p\|+C(1+\lambda_N^{1/2})(\|\nabla p_t\|+\|\nabla\Delta Z\|)\\&+C\lambda_{N+1}^{-1/2}\|\nabla p_t\|\|\nabla\Delta Z\|+C\|\Delta p\|^{1/2}\|\nabla\Delta Z\|,\end{aligned} \tag{75}$$

and

$$\frac{1}{2}\|\Delta p\|^2 + \|\nabla\Delta Z\|^2 - C \le J_2(t) \le \frac{3}{2}\|\Delta p\|^2 + 3\|\nabla\Delta Z\|^2 + C. \tag{76}$$

Now, choosing sufficiently small μ_2, from (72), (75) and (76) we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}J_2(t) + \mu_2 J_2(t) \le C(1+\lambda_N).$$

From the Gronwall lemma and (76) we have

$$\frac{1}{2}\|\Delta p\|^2 + \|\nabla\Delta Z\|^2 \le C(\lambda_N + 1).$$

Thus, taking into account this result, from the first equation of (50) we get (71).

Lemma 2.3. *There exists $N_1 \in \mathbb{N}$ such that for every fixed $N > N_1$*

$$\|\nabla q(t)\|^2 + \|\Delta\chi(t)\|^2 \le C_4 e^{-\mu_2 t},\ t \ge t_0, \tag{77}$$

where C_4 is a constant depending on the initial data uniformly for $(n_0, E_0) \in \mathcal{B}_R$.

P r o o f. As in the previous case, we split the proof into two steps. In the first step we prove that there exists some N_0 such that for $N \ge N_0$

$$\|q(t)\|^2 + \|\nabla\chi(t)\|^2 \le C_3 e^{-\mu_2 t},\ t \ge t_0. \tag{78}$$

Then, taking into account (78), we prove (77).

It follows from (48) that $q = n - p$ and $\chi = Q_N E - Z$. Therefore, from (3) and (50) we get that (q, χ) is a solution of

$$\begin{cases} q_t - \Delta q = \Delta(|\chi|^2 + 2\Re((y+Z)\overline{\chi})), \\ i\chi_t + \Delta\chi - Q_N(p\chi + qE) + i\gamma\chi = 0, \\ q(t,x) = q(t,x+L),\ \chi(t,x) = \chi(t,x+L), \\ q(t_0,x) = Q_N n(t_0,x),\ Z(t_0,x) = Q_N E(t_0,x). \end{cases} \tag{79}$$

Step I. Consider the functional

$$J_2(t) = \|q\|_{-1}^2 + \|q\|^2 + 2\|\nabla\chi\|^2 + 4\Re(qE,\chi) - 2(q,|\chi|^2). \tag{80}$$

Since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\left(\|q\|_{-1}^2 + \|q\|^2\right) &= -2\|q_t\|_{-1}^2 - 2\|q\|^2 - 2(|\chi|^2, q_t + q) \\ &\quad -4\Re((q+q_t)(y+Z),\chi),\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\left(\|\nabla\chi\|^2 + 2\Re(qE,\chi) - (q,|\chi|^2)\right) &= -2\gamma\|\nabla\chi\|^2 + 2\gamma(n,|\chi|^2) - (q_t,|\chi|^2) \\ &\quad +2\Re(q_tE + qE_t,\chi) + 2\Im(\nabla n\chi, \nabla\chi) \\ &\quad -2\Re(\gamma\chi + in\chi, Q_N(p\chi + qE)),\end{aligned}$$

we have

$$\frac{\mathrm{d}}{\mathrm{dt}}J_2(t) + 2\|q\|_{-1}^2 + 2\|q\|^2 + 4\gamma\|\nabla\chi\|^2 = R_5(t), \tag{81}$$

where

$$\begin{aligned} R_5(t) \quad = \quad & -2(q, |\chi|^2) - 4\Re(q(y+Z), \chi) + 4\gamma(n, |\chi|^2) + 4\Re(qE_t, \chi) \\ & +4\Im(\nabla n\chi, \nabla\chi) - 4\Re(\gamma\chi + in\chi, Q_N(p\chi + qE)). \end{aligned} \tag{82}$$

We notice that from the Agmon inequality (5) and the Gagliardo–Nirenberg inequality (6) for $\chi = Q_N\chi$ and from (56) it follows that

$$\|\chi\|_{L_\infty} \le C\lambda_{N+1}^{-1/4}\|\nabla\chi\|, \quad \|\chi\|_{L_4} \le C\lambda_{N+1}^{-3/8}\|\nabla\chi\|. \tag{83}$$

Hence, taking into account (14), (51) and (83), we estimate the terms in (82) as

$$\begin{aligned} &|(q, |\chi|^2)| \le \|q\|\|\chi\|\|\chi\|_{L_\infty} \le C\lambda_{N+1}^{-1/4}\|q\|\|\nabla\chi\|, \\ &|(q(y+Z), \chi)| \le \|q\|\|y+Z\|_{L_\infty}\|\chi\| \le C\lambda_{N+1}^{-1/2}\|q\|\|\nabla\chi\|, \\ &|(n, |\chi|^2)| \le \|n\|_{L_\infty}\|\chi\|^2 \le C\lambda_{N+1}^{-1}\|\nabla\chi\|^2, \\ &|(qE_t, \chi)| \le \|q\|\|E_t\|\|\chi\|_{L_\infty} \le C\lambda_{N+1}^{-1/4}\|q\|\|\nabla\chi\|, \\ &|(\nabla n\chi, \nabla\chi)| \le \|\nabla n\|\|\chi\|_{L_\infty}\|\nabla\chi\| \le C\lambda_{N+1}^{-1/4}\|\nabla\chi\|^2, \\ &|(\chi + in\chi, Q_N(p\chi + qE))| \le \|p\|_{L_\infty}\|\chi\|^2 + \|n\|_{L_\infty}\|\chi\|(\|p\|\|\chi\|_{L_\infty} + \|q\|\|E\|_{L_\infty}) \\ &\qquad\qquad \le C\lambda_{N+1}^{-1/4}\|\nabla\chi\|(\|\nabla\chi\| + \|q\|). \end{aligned}$$

These estimates imply that there exists a sufficiently large number N_0 such that for all $N \ge N_0$ the functional $J_2(t)$ can be estimated as

$$\frac{1}{2}(\|q\|^2 + \|\nabla\chi\|^2) \le J_2(t) \le \frac{3}{2}(\|q\|^2 + \|\nabla\chi\|^2), \tag{84}$$

and $R_5(t)$ can be estimated as

$$|R_5(t)| \le \|q\|^2 + \gamma\|\nabla\chi\|.$$

Substituting it into (80), for sufficiently small μ_2 we get

$$\frac{\mathrm{d}}{\mathrm{dt}}J_2(t) + \mu_2 J_2(t) \le 0.$$

Thus, from the Gronwall lemma we obtain (78).

Step II. We set

$$J_3(t) = \|q\|^2 + \|\nabla q\|^2 + 2\|\Delta\chi\|^2 - 4\Re(p\chi + qE, \Delta\chi). \tag{85}$$

Taking into account that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\left(\|q\|^2 + \|\nabla q\|^2\right) \quad = \quad & -2\|q_t\|^2 - 2\|\nabla q\|^2 + 4\Re(yq_t, \Delta\chi) + 2(\Delta|\chi|^2, q_t + q) \\ & +4\Re(\Delta(Z\overline{\chi}) + \Delta y\overline{\chi} + 2\nabla y \cdot \nabla\overline{\chi}, q_t + q), \end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\|\Delta\chi\|^2 - 2\Re(p\chi + qE, \Delta\chi) = -2\gamma\|\chi\|^2 + 2\gamma\Re(2p\chi + qE, \Delta\chi) - 2\Re(yq_t, \Delta\chi) - 2\Re(qE_t, \Delta\chi) - 2\Re(q_t(Z+\chi), \Delta\chi) - 2\gamma\Re(p_t\chi, \Delta\chi) - 2\Im(pQ_N(p\chi + qE), \Delta\chi),$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}J_3(t) + 2\|q_t\|^2 + 2\|\nabla q\|^2 + 4\gamma\|\Delta\chi\|^2 = R_6(t), \tag{86}$$

where

$$R_6(t) = 2(\Delta|\chi|^2, q_t + q) + 4\Re(\Delta(Z\overline{\chi}), q_t + q) + 4\Re(\Delta y\overline{\chi} + 2\nabla y \cdot \nabla\overline{\chi}, q_t + q) + 4\gamma\Re(2p\chi + qE, \Delta\chi) - 4\Re(q_t(Z+\chi) + qE_t, \Delta\chi) - 4\gamma\Re(p_t\chi, \Delta\chi) - 4\Im(pQ_N(p\chi + qE), \Delta\chi). \tag{87}$$

We notice that from the Agmon inequality (5) and the Gagliardo–Nirenberg inequality (6) for $\chi = Q_N\chi$ and from (56) it follows that

$$\begin{cases} \|\chi\|_{L_\infty} \le C\lambda_{N+1}^{-3/4}\|\Delta\chi\|, & \|\chi\|_{L_4} \le C\lambda_{N+1}^{-7/8}\|\Delta\chi\|, \\ \|\nabla\chi\|_{L_\infty} \le C\lambda_{N+1}^{-1/4}\|\Delta\chi\|, & \|\nabla\chi\|_{L_4} \le C\lambda_{N+1}^{-3/8}\|\Delta\chi\|. \end{cases} \tag{88}$$

Using these relations, we have

$$|(p\chi + qE, \Delta\chi)| \le \|\nabla\chi\|(\|\nabla\chi\|\|p\|_{L_\infty} + \|\chi\|_{L_\infty}\|\nabla p\| + \|\nabla q\|\|E\|_{L_\infty} + \|q\|_{L_\infty}\|\nabla E\|) \le C\lambda_{N+1}^{-1/2}\|\Delta\chi\|(\|\Delta\chi\| + \|\nabla q\|).$$

Therefore, from (85), for sufficiently large N we get

$$\frac{1}{2}\left(\|\nabla q\|^2 + \|\Delta\chi\|^2\right) \le J_3(t) \le \frac{3}{2}\left(\|\nabla q\|^2 + \|\Delta\chi\|^2\right). \tag{89}$$

Then we can choose a small constant δ_1 such that

$$\delta_1 J_1(t) \le \|\nabla q\|^2 + 2\gamma\|\Delta\chi\|^2.$$

Substituting this relation into (86), we obtain

$$\frac{\mathrm{d}}{\mathrm{dt}}J_1(t) + \delta_1 J_1(t) + 2\|q_t\|^2 + \|\nabla q\|^2 + 2\gamma\|\Delta\chi\|^2 \le R_5(t). \tag{90}$$

Now we estimate the terms of $R_5(t)$. In these estimations we use the Agmon inequality (5), the Gagliardo–Nirenberg inequality (6), and the inequalities (88)

$$\|\nabla q\|^2 + \|\Delta\chi\|^2 \le C,$$

which follows from $q = n - p$ and $\chi = Q_N E - Z$ and estimates (14) for (n, E) and (51) for (p, Z), respectively,

$$\begin{aligned}
&|(\Delta|\chi|^2, q_t + q)| \le 2(\|q_t\| + \|q\|)(\|\Delta\chi\|\|\chi\|_{L_\infty} + \|\nabla\chi\|^2_{L_4}) \le C\lambda_{N+1}^{-1/4}\|\Delta\chi\|\|q_t\|,\\
&|(\Delta(Z\overline{\chi}), q_t + q)| \le (\|q_t\| + \|q\|)(\|\Delta\chi\|\|Z\|_{L_\infty} + \|\Delta Z\|\|\chi\|_{L_\infty} + 2\|\nabla\chi\|_{L_4}\|\nabla Z\|_{L_4})\\
&\qquad\qquad \le C\lambda_{N+1}^{-3/4}\|\Delta\chi\|(\|q_t\| + \|\nabla q\|),\\
&|(\Delta y\overline{\chi}, q_t + q)| \le \|\Delta y\|\|\chi\|_{L_\infty}(\|q_t\| + \|q\|) \le C\lambda_{N+1}^{-3/4}\|\Delta\chi\|(\|q_t\| + \|\nabla q\|),\\
&|(\nabla y \cdot \nabla\overline{\chi}, q_t + q)| \le \|\nabla y\|_{L_4}\|\nabla\chi\|_{L_4}(\|q_t\| + \|q\|) \le C\lambda_{N+1}^{-3/8}\|\Delta\chi\|(\|q_t\| + \|\nabla q\|),\\
&|(2p\chi + qE, \Delta\chi)| \le 2\|p\|_{L_\infty}\|\chi\|\|\Delta\chi\| + (\|\nabla q\|\|E\|_{L_\infty} + \|q\|_{L_\infty}\|\nabla E\|)\|\nabla\chi\|\\
&\qquad\qquad \le C\lambda_{N+1}^{-1/2}(\|\Delta\chi\| + \|\nabla q\|)\|\Delta\chi\|,\\
&|(q_t(Z+\chi) + qE_t, \Delta\chi)| \le (\|q_t\|\|Z+\chi\|_{L_\infty} + \|q\|_{L_\infty}\|E_t\|)\|\Delta\chi\|\\
&\qquad\qquad \le C\lambda_{N+1}^{-1/4}(\|q_t\| + \|\nabla q\|)\|\Delta\chi\| + C\|q\|^{1/2}\|\Delta\chi\|,\\
&|(pQ_N(p\chi + qE), \Delta\chi)| \le \|p\|^2_{L_\infty}\|\chi\|\|\Delta\chi\| + (\|\nabla q\|\|E\|_{L_\infty} + \|q\|_{L_\infty}\|\nabla E\|)\|\nabla\chi\|\\
&\qquad\qquad \le C\lambda_{N+1}^{-1/2}(\|\Delta\chi\| + \|\nabla q\|)\|\Delta\chi\|.
\end{aligned}$$

Taking into account (71), the term $(p_t\chi, \Delta\chi)$ can be estimated as

$$|(p_t\chi, \Delta\chi)| \le \|p_t\|\|\chi\|_{L_\infty}\|\Delta\chi\| \le C(1 + \lambda_N^{1/2})\lambda_{N+1}^{-3/2}\|\Delta\chi\|^2.$$

These inequalities, (78), and (90), for sufficiently large N imply

$$\frac{\mathrm{d}}{\mathrm{dt}}J_1(t) + 2\mu_1 J_1(t) \le C(\|q\| + \|\nabla\chi\|) \le Ce^{-t\mu_1}. \tag{91}$$

Thus the Gronwall lemma and (89) conclude the proof of Lemma 2.3.

P r o o f of Theorem 2.2. Let N be fixed large enough as above (see Lemma 2.1 and Lemma 2.3). Let

$$S(t)(n_0, E_0) = (n(t), E(t))$$

and t_0 be defined as in Proposition 2.2. We now define

$$S_1(t)(n_0, E_0) = (p(t), y(t) + Z(t)) \text{ and } S_2(t)(n_0, E_0) = (q(t), \chi(t)).$$

At this stage, Lemma 2.1 and Lemma 2.3 allow us to apply Theorem I.1.1 from [11]. Hence, it is proven that $S(t)$ possesses a compact global attractor $\mathcal{A}$ in $\mathcal{H}$.

3. Convergence of the Attractors

Theorem 3.1. *Suppose that the conditions of Theorem 2.2 are fulfilled. Then*

$$\lim_{\varepsilon\to 0} \sup\{dist_{\mathcal{E}_1}(y, \mathcal{A}^*) : y \in A_\varepsilon\} = 0, \tag{92}$$

where $\mathcal{A}_\varepsilon$ is the global attractor for the problem (2), and

$$\mathcal{A}^* = \left\{(z_0, z_1, z_2) : (z_1, z_2) \in \mathcal{A},\ z_0 = \Delta(z_1 + |z_2|^2) + f\right\}$$

with $\mathcal{A}$ being the global attractor for the problem (3), $dist_{\mathcal{E}_1}(y, A)$ being a distance from the element y to the set A in the space $\mathcal{E}_1$.

P r o o f. It follows from (22) and (29) that there exists some constant R_1 such that for an any bounded set B of the initial data there exists the moment $t_0(B)$ such that

$$\varepsilon\|n_{tt}(t)\|^2+\|\nabla n_t(t)\|^2+\|\Delta n(t)\|^2+\|\nabla\Delta E(t)\|^2+\|\Delta E_t(t)\|^2 \le R_1^2,\ t \ge t_0,\ \varepsilon \le \varepsilon_0.$$

Since $\mathcal{A}_\varepsilon$ is an invariant set, then for all complete trajectories in $\mathcal{A}_\varepsilon$ this relation implies

$$\varepsilon\|n_{tt}(t)\|^2 + \|\nabla n_t(t)\|^2 + \|\Delta n(t)\|^2 + \|\nabla\Delta E(t)\|^2 + \|\Delta E_t(t)\|^2 \le R_1^2. \qquad (93)$$

It is evident that there exists an element $y_\varepsilon = (m_0^\varepsilon, n_0^\varepsilon, E_0^\varepsilon)$ such that

$$\mathrm{dist}_{\mathcal{E}_1}(y_\varepsilon, \mathcal{A}^*) = \sup\{\mathrm{dist}_{\mathcal{E}_1}(y, \mathcal{A}^*),\ y \in \mathcal{A}\}.$$

Let $y_\varepsilon(t) = (n_t^\varepsilon(t), n^\varepsilon(t), E^\varepsilon(t))$ be a complete trajectory such that $y_\varepsilon(0) = y_\varepsilon$. It follows from (93) that there exists the subsequence ε_k and the element $y(t) = (n_t(t), n(t), E(t)) \in L_\infty(\mathbb{R}, \mathcal{E}_1)$ such that $y_{\varepsilon_k}(t)$ tends to $y(t)$ as $\varepsilon_k \to 0$ on any interval $[a, b]$ in the weakly* topology in $L_\infty([a, b], \mathcal{E}_2)$. From Aubin's imbedding theorem (see [10, Corollary 4]) it follows that $y_{\varepsilon_k}(t)$ tends to $y(t)$ strongly in $C([a, b], \mathcal{E}_1)$. Taking to the limit in (2) as $\varepsilon \to 0$ and using that $\varepsilon\|n_{tt}^\varepsilon\| \to 0$, we get that $y(t)$ is a bounded solution of the problem (3). Hence, $y(t)$ belongs to $\mathcal{A}^*$.

Acknowledgement. The author is grateful to professor I.D. Chueshov for the statement of the problem and the fruitful discussion.

References

[1] *P. Biler*, Attractors for the System of Schrödinger and Klein–Gordon Equations with Yukawa Coupling. — *SIAM J. Math. Anal.* **21** (1990), No. 5, 1190–1212.

[2] *I.D. Chueshov and A.S. Shcherbina*, On 2D Zakharov System in a Bounded Domain. — *Differential and Integral Equations* **18** (2005), No. 7, 781–812.

[3] *I. Chueshov and A. Shcherbina*, Semi-Weak Well-Posedness and Attractors for 2D Schroedinger–Boussinesq Equtions. — *Evolution Equations and Control Theory* **1** (2012), 57–80.

[4] *I. Flahaut*, Attractors for the Dissipative Zakharov System. — *Nonlinear Analysis* **16** (1991), 599–633.

[5] *O. Goubet and I. Moise*, Attractor for Dissipative Zakharov System. — *Nonlinear Analysis* **7** (1998), 823–847.

[6] *J.L. Lions and E. Magenes*, Problemes aux Limites Non Homogenes et Applications. Paris, Dunod, 1968.

[7] *A.S. Shcherbina*, Gevrey Regularity of the Global Attractor for the Dissipative Zakharov System. — *Dynamical Systems* **18** (2003), No. 3, 201–225.

[8] *A.S. Shcherbina*, Dissipative Zakharov System in Two-Dimensional Thin Domain. — *Mat. Fiz., Anal., Geom.* **12** (2005), No. 2, 230–245.

[9] *S.H. Schochet and M.I. Weinstein*, The Nonlinear Schrödinger Limit of the Zakharov Equations Governing Langmuir Turbulence. — *Communs Math. Phys.* **106** (1986), 569–580.

[10] *J. Simon*, Compact Sets in the Space $L^p(0, T; B)$. — *Annali di Matematica Pura ed Applicata* **148** (1987), 65–96.

[11] *R. Temam*, Infinite-dimensional Dynamical Systems in Mechanics and Physics. Springer-Verlag, New York, 1988.

[12] *V.E. Zakharov*, Collapse of Langmuir Waves. — *Sov. Phys. JETP* **35** (1972), 908–912.