

Some Sharp Estimates for Convex Hypersurfaces of Pinched Normal Curvature

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For a convex domain D bounded by the hypersurface ∂D in a space of constant curvature we give sharp bounds on the width $R - r$ of a spherical shell with radii R and r that can enclose ∂D , provided that normal curvatures of ∂D are pinched by two positive constants. Furthermore, in the Euclidean case we also present sharp estimates for the quotient R/r . From the obtained estimates we derive stability results for almost umbilical hypersurfaces in the constant curvature spaces.

Key words: convex hypersurface, space of constant curvature, pinched normal curvature, λ -convexity, spherical shell, stability, almost umbilical hypersurface.

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1. Preliminaries and the Main Results

In [1], A. Borisenko and V. Miquel proved that a closed hypersurface with normal curvatures k_n satisfying the inequality $k_n \geq 1$ in the Lobachevsky space $\mathbb{H}^m(-1)$ can be put into a spherical shell between two concentric spheres of radii R and r such that the *width* $R - r$ of the shell satisfies $R - r \leq \ln 2$. A similar estimate holds in Hadamard manifolds (see [2]). In [3], these results were extended to Riemannian manifolds of constant-signed sectional curvatures and hypersurfaces with normal curvatures bounded below.

In the present paper, we refine some results from [3]. For this purpose we consider hypersurfaces with normal curvatures at any point and in any direction pinched by two positive constants. Such a restriction allows us to obtain sharper estimates for the width $R - r$ than in [3] (see [4] for some related results).

Furthermore, for surfaces of pinched normal curvature we are able to derive an upper bound on the quotient R/r , which, in contrast, can be arbitrarily large for a hypersurface with normal curvatures just bounded below.

Besides deriving sharp estimates for $R - r$ and R/r , the normal curvature pinching condition enables us to obtain accurate stability results for the so-called *almost umbilical* hypersurfaces, that is, the hypersurfaces whose normal curvatures are pinched between κ and $(1 + \varepsilon)\kappa$, in the constant curvature spaces. These results are the best possible for such a uniform pinching (see [5] for a weaker Euclidean version of the result; for the results on stability of almost umbilical hypersurfaces with pointwise principal curvatures pinching see [6, p. 493], [7]; for stability with some integral pinching conditions see [8, 9]).

Let us denote by $\mathbb{M}^m(c)$ with $m \geq 2$ a complete simply connected m -dimensional Riemannian manifold of constant sectional curvature equal to c . In order to state the main results, we need the following definition.

Definition 1.1. *A hypersurface $F \subset \mathbb{M}^m(c)$ is said to be κ_1, κ_2 -convex (with $\kappa_2 \geq \kappa_1$, and for $c = 0$ we assume that $\kappa_1 > 0$, for $c > 0$ we assume that $\kappa_1 \geq 0$, and for $c < 0$ we assume that $\kappa_1 > \sqrt{-c}$) if for any point $P \in F$ there exist two nested geodesic spheres $S_2 \subset S_1 \subset \mathbb{M}^m(c)$ of constant normal curvatures equal to, respectively, κ_1 and κ_2 , passing through P such that locally near P the hypersurface F lies inside S_1 and outside S_2 .*

Observe that κ_1, κ_2 -convex hypersurfaces are, in particular, κ_1 -convex (see [2] and [3]). The usual notion of convexity can be viewed as $0, \infty$ -convexity assuming that S_2 in this case is a point and S_1 is a totally geodesic hyperplane.

We should note that for C^r -smooth hypersurfaces with $r \geq 2$ the property of being κ_1, κ_2 -convex is equivalent to that all its normal curvatures k_n (or, equivalently, second fundamental forms) with respect to the inner normal vector field are κ_1, κ_2 -pinched, that is, $\kappa_1 \leq k_n \leq \kappa_2$. In general, since some small neighborhood of any point P on a κ_1, κ_2 -convex hypersurface F lies between two tangent at P geodesic spheres, we have that F is $C^{1,1}$ -smooth (see [10]). Therefore, by generalized Rademacher's theorem (see again [10]), at almost all its points a κ_1, κ_2 -convex hypersurface has well-defined normal curvatures (second fundamental forms) satisfying the inequality shown above.

A closed domain $D \subset \mathbb{M}^m(c)$ is called κ_1, κ_2 -convex if its boundary ∂D is a κ_1, κ_2 -convex hypersurface. Such domains are homeomorphic to geodesic balls of the corresponding spaces.

We recall that for κ_1, κ_2 -convex domains, well-known *Blaschke's rolling theorem* holds (see [11–13] for a smooth case, and [14, 15] for a general case). More precisely, it states the following. Suppose $D \subset \mathbb{M}^m(c)$ is a κ_1, κ_2 -convex domain; for any point $P \in \partial D$ let S_1 and S_2 be two nested spheres of normal curvature equal to, respectively, κ_1 and κ_2 , and that are tangent to ∂D at P ; then

$B_2 \subseteq D \subseteq B_1$, where B_i is the closed geodesic ball bounded by S_i , $i \in \{1, 2\}$.

We are now ready to state the main results of the paper.

Theorem 1. *If $D \subset \mathbb{M}^m(c)$ is a κ_1, κ_2 -convex domain, then the hypersurface ∂D can be put into a spherical shell between two concentric spheres of radii R and r (with $R \geq r$) such that*

1. for $c = 0$,

$$R - r \leq (\sqrt{2} - 1)(R_1 - R_2); \tag{1.1}$$

2. for $c = k^2$ with $k > 0$,

$$R - r \leq \frac{2}{k} \arccos \sqrt{\cos(k(R_1 - R_2))} - (R_1 - R_2); \tag{1.2}$$

3. for $c = -k^2$ with $k > 0$,

$$R - r \leq \frac{2}{k} \operatorname{arccosh} \sqrt{\cosh(k(R_1 - R_2))} - (R_1 - R_2), \tag{1.3}$$

where R_1 and R_2 are the radii of circles with geodesic curvatures equal to, respectively, κ_1 and κ_2 , and lying in the corresponding 2-planes $\mathbb{M}^2(c)$.

Moreover, these estimates are sharp.

R e m a r k 1.1. It is known that $R_i = 1/\kappa_i$ for $c = 0$, $R_i = 1/k \operatorname{arccot}(\kappa_i/k)$ for $c = k^2$, and $R_i = 1/k \operatorname{arccoth}(\kappa_i/k)$ for $c = -k^2$, $i \in \{1, 2\}$.

R e m a r k 1.2. As $\kappa_2 \rightarrow \infty$, estimates (1.1)–(1.3) tend to the corresponding estimates in the spaces of constant curvature from [3].

R e m a r k 1.3. By sharpness of the inequalities above and below we mean that the shown bounds are attained by the so-called *rounded κ_1, κ_2 -convex spindle-shaped surfaces* (see Fig. 1), which we describe in details in the next section.

In the Euclidean case, we can give even more interesting estimate for the quotient R/r .

Theorem 2. *If $D \subset \mathbb{E}^m$ is a κ_1, κ_2 -convex domain in the Euclidean space, then the hypersurface ∂D can be put into a spherical shell between two concentric spheres of radii R and r (with $R \geq r$) such that*

$$\frac{R}{r} \leq \frac{\sqrt{\frac{\kappa_2}{\kappa_1} + \sqrt{2}}}{\sqrt{\frac{\kappa_1}{\kappa_2} + \sqrt{2}}}. \tag{1.4}$$

Moreover, this estimate is sharp.

Corollary 1.1. *In the condition of Theorem 2, it is true that*

$$\frac{R}{r} \leq \frac{\kappa_2}{\kappa_1}.$$

R e m a r k 1.4. In the theorems and the corollary above, as we will see from the proofs, the center of the shell can be chosen to coincide with the center of the inscribed ball for the domain D . Compare this to [4], where Corollary 1.1 is proved for the center of the shell being one of the curvature centroids of ∂D .

R e m a r k 1.5. We note that if in the above $\kappa_1 = \kappa_2$, then from all estimates (1.1)–(1.4) it follows that $R = r$, and thus the domain D is a geodesic ball of the corresponding space.

Theorems 1 and 2 are based on the following result, which is useful by itself. It gives the sharp upper bound on the outer radius R of the spherical shell in terms of the inner radius r of the shell.

Theorem 3. *If $D \subset \mathbb{M}^m(c)$ is a κ_1, κ_2 -convex domain, then the hypersurface ∂D can be put into a spherical shell between two concentric spheres of radii R and r (with $R \geq r$) such that*

1. for $c = 0$,

$$R \leq \sqrt{(R_1 - R_2)^2 - (R_1 - r)^2} + R_2; \tag{1.5}$$

2. for $c = k^2$ with $k > 0$,

$$R \leq \frac{1}{k} \arccos \frac{\cos(k(R_1 - R_2))}{\cos(k(R_1 - r))} + R_2; \tag{1.6}$$

3. for $c = -k^2$ with $k > 0$,

$$R \leq \frac{1}{k} \operatorname{arccosh} \frac{\cosh(k(R_1 - R_2))}{\cosh(k(R_1 - r))} + R_2, \tag{1.7}$$

where R_1 and R_2 are the radii of circles with geodesic curvature equal to, respectively, κ_1 and κ_2 , and lying in the corresponding 2-planes $\mathbb{M}^2(c)$.

Moreover, estimates (1.5)–(1.7) are sharp.

As it was said earlier, the results of Theorems 1 and 2 allow us to derive accurate stability results for almost umbilical hypersurfaces. In particular, we prove the following

Theorem 4. *If $D \subset \mathbb{M}^m(c)$ is a $\kappa, (1 + \varepsilon)\kappa$ -convex domain for some $\varepsilon \geq 0$, then the hypersurface ∂D can be put into a spherical shell between two concentric spheres with radii R and r (with $R \geq r$) such that*

$$R - r < C(\kappa, c)\varepsilon$$

with $C(\kappa, c) = \frac{\kappa}{\kappa^2 + c} (\sqrt{2} - 1)$, and this constant is the best possible.

Moreover, for $c = 0$, the hypersurface ∂D can be put into a spherical shell between two concentric spheres with radii R and r ($R \geq r$) such that

$$\frac{R}{r} - 1 < C\varepsilon$$

with $C = \sqrt{2} - 1$, and this constant is the best possible.

2. Proofs of the Main Results

We are going to prove the theorems above by using a comparison argument. Let us introduce an object to compare with.

In $\mathbb{M}^m(c)$, let us consider a spindle-shaped κ_1 -convex hypersurface, that is a hypersurface obtained by rotating a smaller circular arc PQ of geodesic curvature equal to κ_1 (see [3]). This surface has two vertexes P and Q where its normal curvatures blow up. After smoothing these vertexes by using two spherical caps of normal curvature equal to κ_2 , whose centers lie on the geodesic line PQ , we obtain a convex $C^{1,1}$ -smooth hypersurface (see Fig. 1). We will call these surfaces the *rounded κ_1, κ_2 -convex spindle-shaped hypersurfaces*.

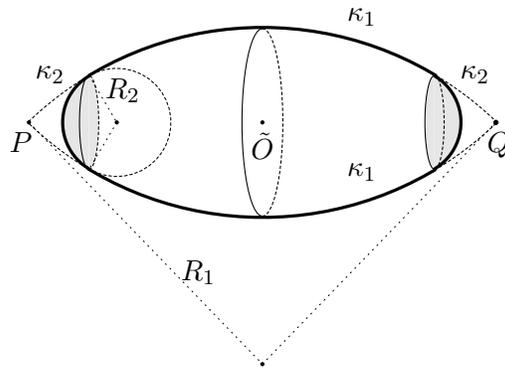


Fig. 1. Rounded κ_1, κ_2 -convex spindle-shaped hypersurface

Since the constructed above rounded κ_1, κ_2 -convex spindle-shaped hypersurface is centrally symmetric with respect to the midpoint \tilde{O} of the geodesic segment PQ (see Fig. 1), the center of its inscribed ball coincides with \tilde{O} , and the radius

\tilde{r} of this ball is equal to the distance from \tilde{O} to the hypersurface. Note that \tilde{r} varies from R_2 to R_1 , and for each $\tilde{r} \in [R_2, R_1]$ there exists a unique rounded κ_1, κ_2 -convex spindle-shaped hypersurface with \tilde{r} being the radius of its inscribed ball. Thus we get a one-parametric family of such surfaces.

Now we can prove the key comparison lemma.

Lemma 2.1. *Let $D \subset \mathbb{M}^m(c)$ be a closed κ_1, κ_2 -convex domain, r be the radius of the inscribed sphere for D with center at a point O . Let $\tilde{F} \subset \mathbb{M}^m(c)$ be a rounded κ_1, κ_2 -convex spindle-shaped hypersurface, and let \tilde{r}, \tilde{R} be the radii of its inscribed and circumscribed spheres. If*

$$\tilde{r} = r,$$

then

$$\max \text{dist}(O, \partial D) \leq \tilde{R}. \tag{2.1}$$

Moreover, this bound is sharp.

P r o o f. We will argue by contradiction. Suppose that (2.1) is not true, and the inverse inequality

$$\max \text{dist}(O, \partial D) > \tilde{R} \tag{2.2}$$

holds. For simplicity, we denote the hypersurface ∂D by F .

Let $M \in F$ be a point such that $\max \text{dist}(O, F) = |OM|$ (here and below $|\cdot|$ denotes the distance between two points); then from (2.2) it follows that on the geodesic segment OM there exists a point A such that

$$|OA| = \tilde{R} < |OM|. \tag{2.3}$$

Assume that the rounded κ_1, κ_2 -convex spindle-shaped hypersurface \tilde{F} is centered at O , and its rotational axis coincides with the geodesic line OA . Then $A \in \tilde{F}$.

Since the point M is a point for which the maximal distance from O is attained, we have that a totally geodesic hyperplane which touches F at M is perpendicular to the geodesic line OM . Therefore, if $\omega_2 \subseteq D$ is a sphere with the center at a point O_2 , and with normal curvature equal to κ_2 that touches from inside the hypersurface F at M (such a sphere exists by Blaschke's rolling theorem), then O_2 lies on the segment OM . We note that from (2.3) it follows

$$|OO_2| > \tilde{R} - R_2, \tag{2.4}$$

where R_2 is the radius of ω_2 .

Let us denote the inscribed sphere for F by ω . Since the hypersurface F is κ_1, κ_2 -convex, then by Blaschke's rolling theorem, F lies in a ball of radius R_1 (we remind that R_1 is the radius of a sphere of normal curvature equal to κ_1). Hence, both ω and ω_2 lie in this ball too. Now we show that there is a sphere of radius R_1 that touches externally both ω and ω_2 simultaneously.

Denote $|OO_2|$ by d . It is clear that the sphere mentioned above exists if and only if the three numbers d , $R_1 - r$, and $R_1 - R_2$ satisfy the triangle inequality. Let us check this:

1. $(R_1 - r) + (R_1 - R_2) = 2R_1 - (r + R_2) > d$, since ω and ω_2 lie in a ball of radius R_1 ;
2. $(R_1 - R_2) + d > R_1 - r$, because ω is the inscribed sphere and thus either $\omega \equiv \omega_2$ and $r = R_1 = R_2$ that is a trivial case when F is a sphere, or ω_2 touches ω from inside, which is again a trivial case when F is a sphere, or ω_2 cannot lie entirely inside ω , hence $r + d < R_2$.
3. $(R_1 - r) + d > R_1 - R_2$, which is obviously true.

By rotational symmetry, along with a single sphere of radius R_1 , there exists a family of spheres of the same radius that touches simultaneously ω and ω_2 along the small $(m - 2)$ -dimensional spheres σ and σ_2 . To proceed we need the following lemma.

Lemma 2.2. [3] *If $D \subset \mathbb{M}^m(c)$ is a closed κ_1 -convex domain (where for $c = 0$ we assume that $\kappa_1 > 0$, for $c > 0$ we assume $\kappa_1 \geq 0$ and for $c < 0$ we assume that $\kappa_1 > \sqrt{-c}$), then for any two points A, B from D every smaller circular arc of geodesic curvature equal to κ_1 that joins A and B lies in the domain D .*

Since D is, in particular, a κ_1 -convex domain, then by Lemma 2.2, the part Θ of the envelope of this family lying between two hyperplanes π and π_2 (corresponding to the spheres $\sigma = \pi \cap \omega$ and $\sigma_2 = \pi_2 \cap \omega_2$), lies inside D (see Fig. 2).

Let ω^- and ω_2^+ be the spherical caps cut from ω and ω_2 by the planes π and π_2 in a way that Θ and ω^- , Θ and ω_2^+ lie on the different sides with respect to π and π_2 , correspondingly.

Consider a $C^{1,1}$ -smooth hypersurface $\omega^- \cup \Theta \cup \omega_2^+$ denoted by Ω . By the arguments above, Ω lies in D .

Now, using (2.4), we are going to show that it is possible to inscribe inside Ω a sphere of radius r' strictly greater than the radius r of the inscribed sphere. This will give us a desired contradiction since, by definition, the inscribed sphere is a sphere of maximum radius lying in a domain.

By symmetry, the center C of the inscribed in Ω sphere lies on the geodesic OO_2 . Let us consider a section of Ω by a totally geodesic two-dimensional plane

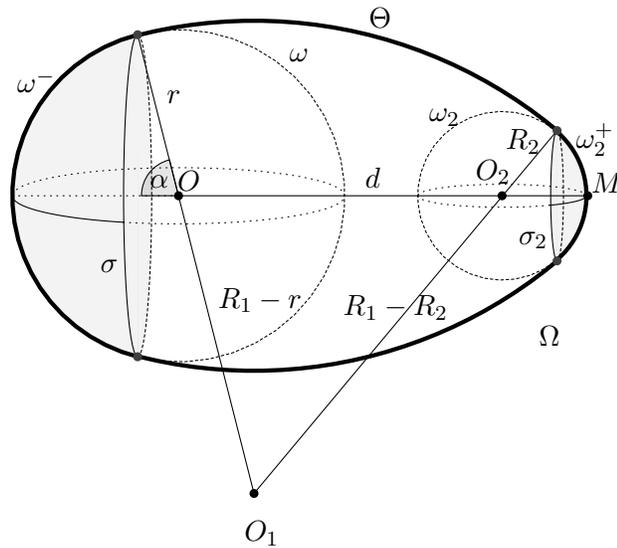


Fig. 2. The hypersurface $\Omega = \omega^- \cup \Theta \cup \omega_2^+$.

Π passing through OO_2 . Then the curve Ω' defined as $\Omega \cap \Pi$ consists of four parts: an arc ω' of a circle $\omega \cap \Pi$ of radius r , an arc ω_2' of a circle $\omega_2 \cap \Pi$ of radius R_2 , and two circular arcs of radius R_1 . Denote by O_1 the center of one of the corresponding circles (see Fig. 2). Then $|OO_1| = R_1 - r$, $|O_2O_1| = R_1 - R_2$. Let α be the angle between the geodesic lines OO_2 and OO_1 . Assume $\alpha \leq \pi/2$.

From the construction of \tilde{F} it follows that the triangle with the side lengths $R_1 - r$, $R_1 - R_2$, and $\tilde{R} - R_2$ is a right triangle. Denote $\tilde{R} - R_2$ by \tilde{d} . From (2.4) we have that $d > \tilde{d}$. By the law of cosines, from the geodesic triangle $\triangle OO_2O_1$ we deduce:

1. For $c = 0$,

$$\begin{aligned} (R_1 - R_2)^2 &= (R_1 - r)^2 + d^2 - 2d(R_1 - r)\cos\alpha \\ &> (R_1 - r)^2 + \tilde{d}^2 - 2d(R_1 - r)\cos\alpha. \end{aligned}$$

Since $(R_1 - R_2)^2 = (R_1 - r)^2 + \tilde{d}^2$, then from the computations above it follows that $\cos\alpha > 0$, thus $\alpha < \pi/2$.

2. For $c = 1$,

$$\begin{aligned} \cos(R_1 - R_2) &= \cos(R_1 - r)\cos d + \sin(R_1 - r)\sin d\cos\alpha \\ &< \cos(R_1 - r)\cos\tilde{d} + \sin(R_1 - r)\sin d\cos\alpha. \end{aligned}$$

Recalling that $\cos(R_1 - R_2) = \cos(R_1 - r) \cos \tilde{d}$, we obtain $\cos \alpha > 0$, $\alpha < \pi/2$.

3. For $c = -1$,

$$\begin{aligned} \cosh(R_1 - R_2) &= \cosh(R_1 - r) \cosh d - \sinh(R_1 - r) \sinh d \cos \alpha \\ &> \cosh(R_1 - r) \cosh \tilde{d} + \sinh(R_1 - r) \sinh d \cos \alpha. \end{aligned}$$

And since $\cosh(R_1 - R_2) = \cosh(R_1 - r) \cosh \tilde{d}$, we get $\alpha < \pi/2$.

Therefore, in all the cases the angle α is strictly less than $\pi/2$. Hence, from the right triangle $\triangle OO_1C$ we have $|OO_1| > |O_1C|$. Thus, if r' is the radius of the inscribed in Ω' circle, then $r' = R_1 - |O_1C|$ and $r' > R_1 - |OO_1| = r$. And since it is true for any plane Π , we come to the contradiction which proves (2.1).

Inequality (2.1) is sharp since the equality is obviously attained for \tilde{F} . Lemma 2.1 is proved. ■

Theorem 3 is a direct consequence of Lemma 2.1. Indeed, by Blaschke's rolling theorem, if r is the radius of the inscribed sphere for D , then there exists a unique rounded κ_1, κ_2 -convex spindle-shaped hypersurface with $\tilde{r} = r$. As the straightforward computations show, the right-hand sides of inequalities (1.5) – (1.7) are the values of \tilde{R} in terms of the inscribed sphere's radius $\tilde{r} = r$, after which Lemma 2.1 implies Theorem 3 (the computations are omitted).

In order to prove Theorems 1 and 2, we should derive additional estimates for the spherical shell's width and the quotient of its radii in the case of rounded κ_1, κ_2 -convex spindle-shaped hypersurfaces. These estimates are summarized in the following lemma.

Lemma 2.3. *Suppose $\tilde{F} \subset \mathbb{M}^m(c)$ is a rounded κ_1, κ_2 -convex spindle-shaped hypersurface, \tilde{r} and \tilde{R} are the radii of the inscribed and circumscribed spheres for \tilde{F} ; then for the width $\tilde{R} - \tilde{r}$ estimates (1.1)–(1.3) hold. Moreover, when $c = 0$ for the quotient \tilde{R}/\tilde{r} estimate (1.4) holds.*

P r o o f. Estimates (1.1)–(1.3) are obtained similarly to those from [3]. Let us show (1.1). In the rest of the cases the computations are similar.

If $R_1 = 1/\kappa_1$ and $R_2 = 1/\kappa_2$ are, as usual, the radii of the spheres of the curvatures equal to κ_1 and κ_2 , then it is easy to see that

$$\tilde{R} = \sqrt{(R_1 - R_2)^2 - (R_1 - \tilde{r})^2} + R_2.$$

Let us introduce a function

$$w(\tilde{r}) := \tilde{R} - \tilde{r} = \sqrt{(R_1 - R_2)^2 - (R_1 - \tilde{r})^2} + R_2 - \tilde{r}$$

defined for $\tilde{r} \in [R_2, R_1]$. By construction, $w \geq 0$, and $w(R_1) = w(R_2) = 0$. Hence, this function attains the global maximum on (R_2, R_1) . Solving the equation for the derivative $dw/d\tilde{r} = 0$, which is a linear equation with respect to \tilde{r} , and substituting its solution in $w(\tilde{r})$, we will get (1.1).

Now let us prove estimate (1.4).

Similarly to the above, we introduce a function

$$q(\tilde{r}) := \frac{\tilde{R}}{\tilde{r}} = \frac{1}{\tilde{r}} \left(\sqrt{(R_1 - R_2)^2 - (R_1 - \tilde{r})^2} + R_2 \right) \quad (2.5)$$

defined for $\tilde{r} \in [R_2, R_1]$. Moreover, by construction, $q \geq 1$, and $q(R_1) = q(R_2) = 1$. Hence, this function attains the global maximum on (R_2, R_1) unless the trivial case $R_1 = R_2$. Let us find this maximal value.

Solving the equation $dq/d\tilde{r} = 0$ for \tilde{r} , which simplifies to a quadratic equation, we get the root

$$\tilde{r}_0 = \frac{1}{R_1^2 + R_2^2} \left(2R_1^2 R_2 - R_2 (R_1 - R_2) \sqrt{2R_1 R_2} \right).$$

The function q attains its maximum at \tilde{r}_0 . Therefore,

$$q(\tilde{r}) \leq q(\tilde{r}_0) \text{ for all } \tilde{r} \text{ from } [R_2, R_1]. \quad (2.6)$$

Now we proceed with computing $q(\tilde{r}_0)$. It is straightforward to check that

$$(R_1 - R_2)^2 - (R_1 - \tilde{r}_0)^2 = \frac{(R_1 - R_2)^2}{(R_1^2 + R_2^2)^2} \left(R_2 (R_1 - R_2) - R_1 \sqrt{2R_1 R_2} \right)^2.$$

Thus, since $R_1 \sqrt{2R_1 R_2} > R_2 (R_1 - R_2)$,

$$\begin{aligned} & \sqrt{(R_1 - R_2)^2 - (R_1 - \tilde{r}_0)^2} + R_2 = \\ & = \frac{\sqrt{2R_1 R_2}}{R_1^2 + R_2^2} \left(R_1 + \sqrt{2R_1 R_2} \right) \left(R_1 + R_2 - \sqrt{2R_1 R_2} \right). \end{aligned} \quad (2.7)$$

We can also rewrite \tilde{r}_0 in the same manner:

$$\tilde{r}_0 = \frac{\sqrt{2R_1 R_2}}{R_1^2 + R_2^2} \left(R_2 + \sqrt{2R_1 R_2} \right) \left(R_1 + R_2 - \sqrt{2R_1 R_2} \right). \quad (2.8)$$

Combining (2.7) and (2.8), and recalling that $R_i = 1/\kappa_i$, $i \in \{1, 2\}$, we get

$$q(\tilde{r}_0) = \frac{R_1 + \sqrt{2R_1 R_2}}{R_2 + \sqrt{2R_1 R_2}} = \frac{\sqrt{\frac{\kappa_2}{\kappa_1}} + \sqrt{2}}{\sqrt{\frac{\kappa_1}{\kappa_2}} + \sqrt{2}}. \quad (2.9)$$

Thereby, from (2.5), (2.6), and (2.9), we finally obtain

$$\frac{\tilde{R}}{\tilde{r}} \leq \frac{\sqrt{\frac{\kappa_2}{\kappa_1}} + \sqrt{2}}{\sqrt{\frac{\kappa_1}{\kappa_2}} + \sqrt{2}},$$

as desired.

The bound above is sharp and is attained for the rounded spindle-shaped κ_1, κ_2 -convex hypersurface with the radius of the inscribed sphere equal to \tilde{r}_0 . Lemma 2.3 is proved. ■

Now, Theorems 1 and 2 are the direct consequences of the comparison Lemma 2.1 and Lemma 2.3.

Theorem 4 is a corollary of Theorems 1 and 2, and is obtained by substituting $\kappa_1 = \kappa$, $\kappa_2 = (1 + \varepsilon)\kappa$ in the right-hand sides of (1.1)–(1.3) and (1.4) with the subsequent standard analysis of the Taylor expansion for ε of the obtained expression at $\varepsilon = 0$.

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