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On Compact Super Quasi-Einstein Warped Product with Nonpositive Scalar Curvature

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This note deals with super quasi-Einstein warped product spaces. Here we establish that if M is a super quasi-Einstein warped product space with nonpositive scalar curvature and compact base, then M is simply a Riemannian product space. Next we give an example of super quasi-Einstein space-time. In the last section a warped product is defined on it.

Key words: Einstein manifold, super quasi-Einstein manifold, Ricci tensor, Hessian tensor, warped product, warping function.

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1. Introduction

An n-dimensional $(n > 2)$ Riemannian manifold is Einstein if its Ricci tensor S of type (0,2) is of the form $S = \alpha g$, where α is a smooth function, which turns into $S = \frac{r}{n}g$, r being the scalar curvature of the manifold. The above equation is also called the Einstein metric condition [1]. Let (M^n, g), $n > 2$, be a Riemannian manifold and $U_S = \{x \in M : S \neq \frac{r}{n}g \text{ at } x\}$, then the manifold (M^n, g) is said to be quasi-Einstein manifold [5, 7] if on $U_S \subset M$ we have

$$S - \alpha g = \beta A \otimes A, \tag{1.1}$$

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where A is a 1-form on U_S, and α and β are some functions on U_S. It is clear that the 1-form A, as well as the function β, is nonzero at every point on U_S. From the above definition, it follows that every Einstein manifold is quasi-Einstein. In particular, every Ricci-flat manifold (e.g., Schwarzchild space-time) is quasi-Einstein. The scalars α, β are known as the associated scalars of the manifold. Also, the 1-form A is called the associated 1-form of the manifold defined by $g(X,\rho) = A(X)$ for any vector field X, ρ being a unit vector field, called the generator of the manifold. Such an n-dimensional quasi-Einstein manifold is denoted by $(QE)_n$.

M.C. Chaki introduced the super quasi-Einstein manifold in [4], denoted by $S(QE)_n$, where the Ricci tensor S of type $(0,2)$, which is not identically zero, satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] + \delta D(X,Y), \tag{1.2}$$

where $\alpha, \beta, \gamma, \delta$ are scalar functions such that β, γ, δ are nonzero and A, B are two nonzero 1-forms such that $g(X,U) = A(X)$ and $g(X,V) = B(X)$, U, V being unit vectors which are orthogonal, i. e., $g(U,V) = 0$ and D is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition $D(X,U) = 0$, $\forall X \in \chi(M)$.

Here $\alpha, \beta, \gamma, \delta$ are called the associated scalars, and A, B are called the associated main and auxiliary 1-forms, respectively, U, V are the main and auxiliary generators, and D is called the associated tensor of the manifold.

The notion of a warped product generalizes that of a surface of revolution. It was introduced in [3] for studying manifolds of negative curvature. Let (B, g_B) and (F, g_F) be two Riemannian manifolds with $\dim B = m > 0$, $\dim F = k > 0$ and $f : B \to (0,\infty)$, $f \in C^\infty(B)$. Consider the product manifold $B \times F$ with its projections $\pi : B \times F \to B$ and $\sigma : B \times F \to F$. The warped product $B \times_f F$ is the manifold $B \times F$ with the Riemannian structure such that $\|X\|^2 = \|\pi^*(X)\|^2 + f^2(\pi(p))\|\sigma^*(X)\|^2$ for any vector field X on M. Thus we have $g_M = g_B + f^2 g_F$ holds on M. Here B is called the base of M and F the fiber. The function f is called the warping function of the warped product [10]. We will denote by Ric_M, Ric_B, Ric_F, and H^f the Ricci curvature of M, the lifts to M of the Ricci curvatures of B and F, and the Hessian of f, respectively. A Riemannian manifold is said to be super quasi-Einstein if its Ricci tensor is proportional to the metric, that is,

$$\mathrm{Ric}_M = \alpha g_M(X,Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] + \delta D(X,Y). \tag{1.3}$$

By τ_M, τ_B and τ_F, we will understand the scalar curvatures of M, B and F, that is, $\tau_M = \mathrm{Tr}(\mathrm{Ric}_M)$, $\tau_B = \mathrm{Tr}(\mathrm{Ric}_B)$ and $\tau_F = \mathrm{Tr}(\mathrm{Ric}_F)$. Therefore we have the followings [10]:

Proposition 1.1. *The Ricci curvature* Ric *of the warped product* $M = B \times_f F$ *with* $k = \dim F$ *satisfies*

(1) $\mathrm{Ric}(X,Y) = \mathrm{Ric}_B(X,Y) - \frac{k}{f} H^f(X,Y)$,

(2) $\mathrm{Ric}(X,V) = 0$,

(3) $\mathrm{Ric}(V,W) = \mathrm{Ric}_F(V,W) - g(V,W) f^{\#},\ f^{\#} = \frac{-\Delta f}{f} + \frac{k-1}{f^2}|\nabla f|^2$

for any horizontal vectors X, Y *(that is* $X, Y \in \tau(TB)$*) and any vertical vectors* V, W *(that is* $V, W \in \tau(TF)$*), where* H^f *and* Δf *denote the Hessian of* f *and the Laplacian of* f *given by* $\Delta f = -\operatorname{tr}(H^f)$*, respectively.*

Proposition 1.2. *Let* $M = B \times_f F$ *be a warped product manifold. Then the scalar curvature of* M *is given by*

$$\tau_M = \tau_B + \frac{\tau_F}{f^2} + 2k\frac{\Delta f}{f} - k(k-1)\frac{|\nabla f|^2}{f^2}.$$

From the above Proposition 1.1 we get the following theorem.

Theorem 1.1. *Let* $M = B \times_f F$ *be a warped product manifold which is also a super quasi-Einstein manifold. Then the following conditions hold.*

i) *When* U, V *are orthogonal and tangent to the base* B*, then the Ricci tensors of* B *and* F *satisfy the following conditions:*

a) $$\begin{aligned}\mathrm{Ric}_B(X,Y) = \alpha g_B(X,Y) &+ \beta g_B(X,U) g_B(Y,U) \\ &+ \gamma[g_B(X,U)g_B(Y,V) + g_B(Y,U)g_B(X,V)] \\ &+ \delta D_B(X,Y) + \frac{k}{f}H^f(X,Y),\end{aligned}$$

b) $$\mathrm{Ric}_F(X,Y) = g_F(X,Y)\left[\alpha f^2 - f\Delta f + (k-1)|\nabla f|^2\right] + \delta D_F(X,Y);$$

ii) *When* U, V *are orthogonal and tangent to the fibre* F*, then the Ricci tensors of* B *and* F *satisfy the following conditions:*

a) $$\mathrm{Ric}_B(X,Y) = \alpha g_B(X,Y) + \frac{k}{f}H^f(X,Y) + \delta D_B(X,Y),$$

b) $$\begin{aligned}\mathrm{Ric}_F(X,Y) = g_F(X,Y)&\left[\alpha f^2 - f\Delta f + (k-1)|\nabla f|^2\right] \\ &+ \beta f^4 g_F(X,U)g_F(Y,U) + \gamma f^4[g_F(X,U)g_F(Y,V) \\ &+ g_F(Y,U)g_F(X,V)] + \delta D_F(X,Y).\end{aligned}$$

Corollary 1.1. *Taking the traces of Theorem* 1.1*, we get the scalar curvature of* M, B *and* F *of two different cases.*

i) $\tau_M = \alpha(m+k)+\beta$, $\tau_B = \alpha m - k\frac{\Delta f}{f} + \beta$, $\tau_F = k\left[\alpha f^2 - f\Delta f + (k-1)|\nabla f|^2\right]$.

ii) $\tau_M = \alpha(m+k)+\beta$, $\tau_B = \alpha m - k\frac{\Delta f}{f}$, $\tau_F = k\left[\alpha f^2 - f\Delta f + (k-1)|\nabla f|^2\right] + \beta f^4$.

The proves of Theorem 1.1 and Corollary 1.1 follow similarly to Theorem 2.1 from the paper [12]. We also have the following propositions from [2, 10], where the expression of Ricci curvature of a warped product space was obtained.

Many authors, like M.C. Chaki [4], C. Özgür [11], etc., have studied super quasi-Einstein manifolds. In [6], D. Dumitru gave a characterization of the warped product on quasi-Einstein manifold and B. Pal, A. Bhattacharyya studied a characterization of the warped product on mixed super quasi-Einstein manifold in [12]. In [9], D. Kim discussed about a compact Einstein warped space with nonpositive scalar curvature. Motivated by the above papers, in this work we study super quasi-Einstein warped product spaces with nonpositive scalar curvature. Also, we establish the four-dimensional example of super quasi-Einstein space-time, and in the last section we give the example of a warped product on it.

2. Super Quasi-Einstein Warped Product Spaces with Nonpositive Scalar Curvature

From Proposition 1.1, we get the following result where equation (1.2) becomes

Result 2.1. *When U, V are orthogonal and tangent to the base B, the warped product $M = B \times_f F$ is a super quasi-Einstein manifold with*

$$\mathrm{Ric}_M(X,Y) = \alpha g_M(X,Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] + \delta D(X,Y),$$

where $D(X,Y) = g(lX,Y)$, l is a symmetric endomorphism if and only if

$$\text{(2.a)}\quad \begin{aligned}\mathrm{Ric}_B(X,Y) = \alpha g_B(X,Y) &+ \beta g_B(X,U)g_B(Y,U)\\ &+ \gamma[g_B(X,U)g_B(Y,V) + g_B(Y,U)g_B(X,V)]\\ &+ \delta D_B(X,Y) + \frac{k}{f}H^f(X,Y),\end{aligned}$$

$$\text{(2.b)}\quad \mathrm{Ric}_F(X,Y) = \mu g_F(X,Y) + \delta D_F(X,Y),$$

$$\text{(2.c)}\quad \mu = \left[\alpha f^2 - f\Delta f + (k-1)|\nabla f|^2\right].$$

Now, we state a lemma whose detailed proof is given in [9].

Lemma 2.1. *Let f be a smooth function on a Riemannian manifold B, then for any vector X, the divergence of the Hessian tensor H^f satisfies*

$$\operatorname{div}\left(H^f\right)(X) = \operatorname{Ric}(\nabla f, X) - \Delta(df)(X), \tag{2.1}$$

where $\Delta = d\delta + \delta d$ denotes the Laplacian on B acting on differential forms.

Now we prove the following proposition.

Proposition 2.1. *Let (B^m, g_B) be a compact Riemannian manifold of dimension $m \geq 2$. Suppose that f is a nonconstant smooth function on B satisfying (2.a) for a constant $\alpha \in R$ and a natural number $k \in N$, and if the condition*

$$\beta g_B(X,U)g_B(\nabla f,U) + \gamma[g_B(X,U)g_B(\nabla f,V) + g_B(\nabla f,U)g_B(X,V)] + g_B(lX,\nabla f) = 0$$

holds, then f satisfies (2.c) for a constant $\mu \in R$. Hence, for a compact Riemannian manifold F with $\operatorname{Ric}_F(X,Y) = \mu g_F(X,Y) + \delta D_F(X,Y)$, we can make a compact super quasi-Einstein warped product space $M = B \times_f F$ with

$$\operatorname{Ric}_M(X,Y) = \alpha g_M(X,Y) + \beta A(X)A(Y) + \gamma[A(X)B(Y) + A(Y)B(X)] + \delta D(X,Y),$$

where $D(X,Y) = g(lX,Y)$, l is a symmetric endomorphism when U, V are orthogonal and tangent to the base B.

Proof. By taking the trace of both sides of (2.a), we have

$$S = \alpha m - k\frac{\Delta f}{f} + \beta, \tag{2.2}$$

where S denotes the scalar curvature of B given by tr(Ric). Note that the second Bianchi identity implies (see [10])

$$dS = 2\operatorname{div}(\operatorname{Ric}). \tag{2.3}$$

From equations (2.2) and (2.3), we obtain

$$\operatorname{div}\operatorname{Ric}(X) = \frac{k}{2f^2}\{\Delta f df - f d(\Delta f)(X)\}. \tag{2.4}$$

On the other hand, by the definition, we have

$$\operatorname{div}\left(\frac{1}{f}H^f\right)(X) = \sum_i \left(D_{E_i}\left(\frac{1}{f}H^f\right)\right)(E_i, X)$$

$$= -\frac{1}{f^2} H^f(\nabla f, X) + \frac{1}{f} \operatorname{div} H^f(X)$$

for any vector field X and an orthonormal frame $E_1, E_2, \ldots, E_m$ of B. Since $H^f(\nabla f, X) = (D_X\, df)(\nabla f) = \frac{1}{2}\, d\left(|\nabla f|^2\right)(X)$, the last equation becomes

$$\operatorname{div}\left(\frac{1}{f} H^f\right)(X) = -\frac{1}{2f^2}\, d\left(|\nabla f|^2\right)(X) + \frac{1}{f} \operatorname{div} H^f(X)$$

for a vector field X on B. Hence, from (2.a) and (2.1), it follows that

$$\begin{aligned} \operatorname{div}(\frac{1}{f} H^f)(X) = {} & \frac{1}{2f^2}\left\{(k-1) d\left(|\nabla f|^2\right) - 2f\, d(\Delta f) + 2\alpha f\, df\right\} \\ & + \frac{1}{f} \beta g_B(X, U) g_B(\nabla f, U) \\ & + \frac{1}{f} \gamma [g_B(X, U) g_B(\nabla f, V) + g_B(\nabla f, U) g_B(X, V)] \\ & + \frac{1}{f} \delta D_B(X, \nabla f). \end{aligned} \tag{2.5}$$

But, (2.a) gives $\operatorname{div} \mathrm{Ric}_B = \operatorname{div}(\frac{k}{f} H^f) + \operatorname{div} D_B$. Therefore, (2.4) and (2.5) imply that $d\left(-f\Delta f + (k-1)|\nabla f|^2 + \alpha f^2\right) = 0$, that is, $-f\Delta f + (k-1)|\nabla f|^2 + \alpha f^2 = \mu$ for some constant μ. Thus the first part of the proposition is proved. For a compact Riemannian manifold (F, g_F) of dimension k with $\mathrm{Ric}_F = \mu g_F + \delta D_F$, we can construct a compact super quasi-Einstein warped product $M = B \times_f F$ by the sufficiencies of Result 2.1. □

In a similar way, we get the following result and proposition when U, V are orthogonal and tangent to the fiber F.

Result 2.2. *When U, V are orthogonal and tangent to the fiber F, the warped product $M = B \times_f F$ is a super quasi-Einstein manifold with* $\mathrm{Ric}_M(X, Y) = \alpha g_M(X, Y) + \beta A(X) A(Y) + \gamma[A(X) B(Y) + A(Y) B(X)] + \delta D(X, Y)$, *where $D(X, Y) = g(lX, Y)$, l is a symmetric endomorphism. if and only if*

(2.d) $\mathrm{Ric}_B(X, Y) = \alpha g_B(X, Y) + \dfrac{k}{f} H^f(X, Y) + \delta D_B(X, Y)$,

(2.e) $\mathrm{Ric}_F(X, Y) = g_F(X, Y)\left[\alpha f^2 - f\Delta f + (k-1)|\nabla f|^2\right]$
$+ \beta f^4 g_F(X, U) g_F(X, U)$
$+ \gamma f^4 [g_F(X, U) g_F(Y, V) + g_F(Y, U) g_F(X, V)] + \delta D_F(X, Y)$,

(2.f) $\mu = \left[\alpha f^2 - f\Delta f + (k-1)|\nabla f|^2\right]$.

Proposition 2.2. *Let (B^m, g_B) be a compact Riemannian manifold of dimension $m \geq 2$. Suppose that f is a nonconstant smooth function on B satisfying (2.d) for a constant $\alpha \in R$ and a natural number $k \in N$, and if the condition $\delta g_B(lX, \nabla f) = 0$ holds, then f satisfies (2.f) for a constant $\mu \in R$. Hence, for a compact super quasi-Einstein manifold F with*

$$\begin{aligned}\mathrm{Ric}_F(X,Y) = g_F(X,Y)[\alpha f^2 - f\Delta f + (k-1)|\nabla f|^2 + \beta f^4 g_F(X,U) g_F(Y,U) \\ + \gamma f^4 [g_F(X,U) g_F(Y,V) + g_F(Y,U) g_F(X,V)] + \delta D_F(X,Y),\end{aligned}$$

we can make a compact super quasi-Einstein warped product space $M = B \times_f F$ with

$$\begin{aligned}\mathrm{Ric}_M(X,Y) = \alpha g_M(X,Y) + \beta A(X) A(Y) \\ + \gamma[A(X)B(Y) + A(Y)B(X)] + \delta D(X,Y),\end{aligned}$$

where $D(X,Y) = g(lX,Y)$, l is a symmetric endomorphism when U, V are orthogonal and tangent to the fiber F.

Proof. By taking the trace of both sides of (2.d), we have

$$S = \alpha m - k\frac{\Delta f}{f}, \tag{2.6}$$

where S denotes the scalar curvature of B given by tr(Ric). From equations (2.6) and (2.3), we obtain

$$\operatorname{div}\mathrm{Ric}(X) = \frac{k}{2f^2}\{\Delta f\, df - f\, d(\Delta f)(X)\}. \tag{2.7}$$

Hence, from (2.d) and (2.1), it follows that

$$\begin{aligned}\operatorname{div}\left(\frac{1}{f}H^f\right)(X) = \frac{1}{2f^2}\{(k-1)\, d\left(|\nabla f|^2\right) \\ - 2f\, d(\Delta f) + 2\lambda f\, df\} + \frac{1}{f}\delta D_B(X, \nabla f).\end{aligned} \tag{2.8}$$

But, (2.d) gives $\operatorname{div}\mathrm{Ric}_B = \operatorname{div}\left(\frac{k}{f}H^f\right) + \operatorname{div} D_B$. Therefore, (2.7) and (2.8) imply that $d(-f\Delta f + (k-1)|\nabla f|^2 + \lambda f^2) = 0$, that is, $-f\Delta f + (k-1)|\nabla f|^2 + \alpha f^2 = \mu$ for some constant μ. Thus the first part of Proposition 2.2 is proved. For a compact Riemannian manifold (F, g_F) of dimension k with

$$\begin{aligned}\mathrm{Ric}_F(X,Y) = g_F(X,Y)\left[\alpha f^2 - f\Delta f + (k-1)|\nabla f|^2\right] + \beta f^4 g_F(X,U) g_F(X,U) \\ + \gamma f^4 [g_F(X,U) g_F(Y,V) + g_F(Y,U) g_F(X,V)] + \delta D_F(X,Y),\end{aligned}$$

we can construct a compact super quasi-Einstein warped product $M = B \times_f F$ by the sufficiencies of Result 2.2. □

Now we prove the following theorem.

Theorem 2.1. *Let $M = B \times_f F$ be a compact super quasi-Einstein warped space. If M has nonpositive scalar curvature, then the warped product becomes a Riemannian product.*

Proof. Equations (2.c) and (2.f) become

$$\operatorname{div}(f\Delta f) + (k-2)|\nabla f|^2 + \alpha f^2 = \mu. \tag{2.9}$$

By integrating (2.9) over B, we get

$$\mu = \frac{k-2}{V(B)} \int_B |\nabla f|^2 + \frac{\alpha}{V(B)} \int_B f^2, \tag{2.10}$$

where $V(B)$ denotes the volume of B.

1. Suppose $k \geq 3$. Let p be a maximum point of f on B. Then we have $f(p) > 0, \nabla f(p) = 0$ and $\Delta f(p) \geq 0$. Hence, from (2.c), (2.f) and (2.10), we obtain the following:

$$\begin{aligned} 0 \leq f(p)\Delta f(p) &= \alpha f^2(p) - \mu \\ &= \frac{2-k}{V(B)} \int_B |\nabla f|^2 + \frac{\alpha}{V(B)} \int_B \left(f^2(p) - f^2\right) \leq 0. \end{aligned} \tag{2.11}$$

If $\alpha < 0$, then f is constant.

2. Suppose $k = 1, 2$. Let p be a minimum point of f on B. Then we have $f(q) > 0, \nabla f(q) = 0$ and $\Delta f(p) \leq 0$. Hence, from (2.c), (2.f) and (2.10), we obtain the following:

$$\begin{aligned} 0 \geq f(q)\Delta f(q) &= \alpha f^2(q) - \mu \\ &= \frac{2-k}{V(B)} \int_B |\nabla f|^2 + \frac{\alpha}{V(B)} \int_B \left(f^2(q) - f^2\right) \geq 0. \end{aligned} \tag{2.12}$$

If $k = 1$ and $\alpha < 0$, then from (2.12), f is constant. If $k = 2$ and $\alpha = 0$, (2.9) and (2.10) imply that f is harmonic on B, then f is constant. This completes the proof of the theorem.

□

3. Example of 4-Dimensional Super Quasi-Einstein Space-Time

Here we construct a nontrivial concrete example of a super quasi-Einstein space-time. Let us consider a Lorentzian metric g on M^4 by

$$ds^2 = g_{ij}dx^i dx^j = -\frac{k}{r}(dt)^2 + \frac{1}{\frac{c}{r} - 4}(dr)^2 + r^2(d\theta)^2 + (r\sin\theta)^2(d\phi)^2,$$

where $i, j = 1, 2, 3, 4$ and k, c are constant. Then the only nonvanishing components of Christofell symbols, the curvature tensors, and the Ricci tensors are:

$$\Gamma^2_{33} = 4r - c, \quad \Gamma^1_{12} = -\frac{1}{2r}, \quad \Gamma^2_{22} = \frac{c}{2r(c-4r)}, \quad \Gamma^3_{32} = \Gamma^4_{42} = \frac{1}{r},$$
$$\Gamma^2_{33} = 4r - c, \quad \Gamma^4_{43} = \cot\theta, \quad \Gamma^2_{44} = (4r-c)(\sin\theta)^2, \quad \Gamma^3_{44} = -\frac{\sin 2\theta}{2} \tag{3.1}$$

$$R_{1221} = -\frac{k(c-3r)}{r^3(c-4r)}, \quad R_{1331} = \frac{k(c-4r)}{2r^2}, \quad R_{1441} = \frac{k(c-4r)(\sin\theta)^2}{2r^2},$$
$$R_{2332} = \frac{c}{2(4r-c)}, \quad R_{2442} = \frac{c(\sin\theta)^2}{2(4r-c)}, \quad R_{3443} = r(c-5r)(\sin\theta)^2,$$
$$R_{11} = -\frac{k}{r^3}, \quad R_{22} = -\frac{3}{r(c-4r)}, \quad R_{33} = -3, \quad R_{44} = -3(\sin\theta)^2. \tag{3.2}$$

From the above, it can be said that M^4 is a Lorentzian manifold of the nonvanishing scalar curvature and the scalar curvature $r_1 = -\frac{8}{r^2}$. We shall now show that this manifold is $S(QE)_4$.

Let us consider the associated scalars α, β, γ and δ and the associated tensor D as follows:

$$\alpha = -\frac{3}{r^2}, \quad \beta = -\frac{1}{r}, \quad \gamma = \frac{1}{r}, \quad \delta = \frac{1}{r^2}, \tag{3.3}$$

and

$$D_{11} = 0, \quad D_{22} = \frac{1}{r}, \quad D_{33} = \frac{1}{r}, \quad D_{44} = -\frac{2}{r},$$
$$D_{12} = \frac{2\sqrt{k}}{r}, \quad D_{21} = \frac{2\sqrt{k}}{r}, \quad D_{13} = \frac{2\sqrt{k}}{r}, \quad D_{31} = \frac{2\sqrt{k}}{r},$$
$$D_{14} = \frac{\sqrt{k}}{r}, \quad D_{41} = \frac{\sqrt{k}}{r}, \quad D_{23} = \frac{\sqrt{k}}{r}, \quad D_{32} = \frac{\sqrt{k}}{r},$$
$$D_{24} = \frac{1}{2r}, \quad D_{42} = \frac{1}{2r}, \quad D_{34} = \frac{1}{2r}, \quad D_{43} = \frac{1}{2r}, \tag{3.4}$$

and the 1-forms are given by

$$A_i(x) = \begin{cases} \frac{2\sqrt{k}}{r} & \text{for} \quad i = 1 \\ \frac{1}{r} & \text{for} \quad i = 2,3 \\ -\frac{1}{r} & \text{for} \quad i = 4 \end{cases} \quad \text{and} \quad B_i(x) = \begin{cases} -\frac{3}{2r} & \text{for} \quad i = 4 \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

i) $R_{11} = \alpha g_{11} + \beta A_1 A_1 + \gamma[A_1 B_1 + A_1 B_1] + \delta D_{11}$,

ii) $R_{22} = \alpha g_{22} + \beta A_2 A_2 + \gamma[A_2 B_2 + A_2 B_2] + \delta D_{22}$,

iii) $R_{33} = \alpha g_{33} + \beta A_3 A_3 + \gamma[A_3 B_3 + A_3 B_3] + \delta D_{44}$,

iv) $R_{44} = \alpha g_{44} + \beta A_4 A_4 + \gamma[A_4 B_4 + A_4 B_4] + \delta D_{44}$.

Since all the cases other than (i)–(iv) are trivial, we can say that

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma[A_i B_j + A_j B_i] + \delta D_{ij}, \quad i, j = 1, 2, 3, 4.$$

Example 3.1. Let (M^4, g) be a Lorentzian manifold endowed with the metric given by

$$ds^2 = g_{ij} dx^i dx^j = -\frac{k}{r}(dt)^2 + \frac{1}{\frac{c}{r} - 4}(dr)^2 + r^2(d\theta)^2 + (r\sin\theta)^2(d\phi)^2,$$

where $i, j = 1, 2, 3, 4$ and k, c are constant. Then (M^4, g) is an $S(QE)_4$ space-time with nonvanishing and nonconstant scalar curvature.

4. Example of Warped Product on Super Quasi-Einstein Space-Time

Here we consider the example (3.1), a 4-dimensional example of super quasi-Einstein space-time endowed with the Lorentzian metric given by

$$ds^2 = g_{ij} dx^i dx^j = -\frac{k}{r}(dt)^2 + \frac{1}{\frac{c}{r} - 4}(dr)^2 + r^2(d\theta)^2 + (r\sin\theta)^2(d\phi)^2,$$

where $i, j = 1, 2, 3, 4$ and k, c are constant. Now we have already proved that it is a super quasi-Einstein space-time with nonzero and constant scalar curvature.

Therefore the above space-time of the form $\mathbf{R} \times_f (\frac{c}{4}, \infty) \times \mathbf{S^2}$, where S^2 is the 2-dimensional Euclidean sphere, the warping function $f : \mathbf{R} \to (0, \infty)$ is given by $f(t) = \frac{1}{\sqrt{\frac{c}{r} - 4}}$, $r < \frac{c}{4}$. Here $\mathbf{R}$ is the base B, and $F = (\frac{c}{4}, \infty) \times \mathbf{S^2}$ is the fiber. Therefore the metric $ds_M^2 = ds_B^2 + f^2 ds_F^2$, that is,

$$ds^2 = g_{ij} dx^i dx^j = \frac{-k}{r}(dt)^2 + \frac{1}{\frac{c}{r} - 4}\left[(dr)^2 + (cr - 4r^2)((d\theta)^2 + \sin^2\theta(d\phi)^2)\right],$$

is the example of a warped product on $S(QE)_4$ space-time.

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