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Gap Control by Singular Schrödinger Operators in a Periodically Structured Metamaterial

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This paper is dedicated to Volodymyr Olexandrovych Marchenko on the occasion of his jubilee

We consider a family $\{\mathcal{H}^\varepsilon\}_{\varepsilon>0}$ of $\varepsilon\mathbb{Z}^n$-periodic Schrödinger operators with δ'-interactions supported on a lattice of closed compact surfaces; within a minimum period cell one has $m \in \mathbb{N}$ surfaces. We show that in the limit when $\varepsilon \to 0$ and the interactions strengths are appropriately scaled, $\mathcal{H}^\varepsilon$ has at most m gaps within finite intervals, and moreover, the limiting behavior of the first m gaps can be completely controlled through a suitable choice of those surfaces and of the interactions strengths.

Key words: periodic Schrödinger operators, δ' interaction, spectral gaps, eigenvalue asymptotics.

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1. Introduction

Spectral analysis of operators with periodic coefficients is a frequent topic in mathematical physics. Recent advances in investigation of various sorts of *metamaterials* motivate the study of operators. One of the central questions concerns the structure of spectral gaps in view of their importance for conductivity properties of such substances, in particular, the possibility of engineering the gap structure by choosing an appropriate material. In the present paper, we investigate this problem for a class of such operators; we are going to show that using a suitable lattice of "traps" arranged periodically in combination with a scaling transformation that makes these traps smaller and weaker one can approximate any prescribed finite family of spectral gaps. Let us recall in this connection that similar ideas can also appear in a different context, for instance, concerning the gap creation by "decoration" of quantum graphs [2, 5, Sec. 5.1].

The idea to employ δ' traps was first used in our recent paper [7] where we demonstrated that it can provide an approximation to the first spectral gap in the particular case of operators used to model *nanowires* regarding them as electron waveguides. In the said paper, we focused our attention at guides with Neumann boundary characteristic for metallic nanowires, and we also supposed that the scaling made the duct thin. Here we extend this result in two directions. First of all, we suppose that the family of traps is periodic in more than one direction, and secondly, we manage to get an approximation with any finite number of prescribed gaps. What is equally important, however, not only the present result is more general, but also the method we employ differs from that used in [7] where the argument was based on eigenvalue convergence for the elements of the fiber decomposition by constructing approximations for the eigenfunctions.

In contrast, in the current paper, we identify the limiting operators using Simon's results for a monotonic sequence of forms [13]. The convergence of the eigenvalues is then proven using a (slightly modified) lemma from [8]. This allows us not only to prove the said convergence of eigenvalues, but also to estimate its rate. Location of spectral gaps can be then controlled by a suitable choice of the interaction 'strength' and the trap shapes, that is, surfaces supporting these interactions, following a result from [10]. In the next section, we describe the problem properly and state the main result. Section 3 is then devoted to its proof; in the Appendix we recall the lemma mentioned above.

2. Setting of the problem and main result

Let $m \in \mathbb{N}$ and let $\{\Omega_j\}_{j=1}^m$ be a family of simply connected Lipschitz domains in $\mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$. We assume that

$$\overline{\Omega_j} \cap \overline{\Omega_{j'}} = \emptyset \text{ as } j \neq j' \quad \text{and} \quad \overline{\cup_{j=1}^m \Omega_j} \subset Y := (0,1)^n.$$

Also, we set

$$\Omega_0 := Y \setminus \bigcup_{j=1}^m \overline{\Omega_j}.$$

In what follows, $\varepsilon > 0$ will be a small parameter. For $i \in \mathbb{Z}^n$ and $j \in \{1, \dots, m\}$, we set

$$\Gamma_{ij}^\varepsilon := \varepsilon(\partial\Omega_j + i).$$

Next we describe the family of operators $\mathcal{H}^\varepsilon$ which will be the main object of our interest in this paper. We denote

$$\Gamma^\varepsilon = \bigcup_{i\in\mathbb{Z}^n} \bigcup_{j=1}^m \Gamma_{ij}^\varepsilon$$

and introduce the sesquilinear form $\mathfrak{h}^\varepsilon$ in the Hilbert space $\mathsf{L}^2(\mathbb{R}^n)$ via

$$\begin{aligned} \mathfrak{h}^\varepsilon[u,v] := &\int_{\mathbb{R}^n\setminus\Gamma^\varepsilon} \nabla u \cdot \nabla \bar{v} \, \mathrm{d}x \\ &+ \varepsilon \sum_{i\in\mathbb{Z}} \sum_{j=1}^m q_j \int_{\Gamma_{ij}^\varepsilon} (u \restriction_{\Gamma_{ij}^\varepsilon}^{\mathrm{ext}} - u \restriction_{\Gamma_{ij}^\varepsilon}^{\mathrm{int}}) \overline{(v \restriction_{\Gamma_{ij}^\varepsilon}^{\mathrm{ext}} - v \restriction_{\Gamma_{ij}^\varepsilon}^{\mathrm{int}})} \, \mathrm{d}s, \quad q_j > 0, \end{aligned} \tag{2.1}$$

with the domain of form $\mathrm{dom}(\mathfrak{h}^\varepsilon) = \mathsf{H}^1(\mathbb{R}^n \setminus \Gamma^\varepsilon)$. Here $f \upharpoonright^{\rm ext}_{\Gamma^\varepsilon_{ij}}$ (respectively, $f \upharpoonright^{\rm int}_{\Gamma^\varepsilon_{ij}}$) stands for the trace of the function f taken from the exterior (respectively, interior) side of Γ^ε_{ij}; $\mathrm{d}s$ is the 'area' measure on Γ^ε_{ij}.

Remark 2.1. From the viewpoint of physical motivation mentioned in the introduction, the cases $n = 2, 3$ are important. However, there is no problem in stating and proving the result for any dimension; what matters is that the codimension of the interaction support is one. In general, a trap lattice may have different periods in different dimensions, but using suitable scaling transformations one can reduce such situations to the case considered here.

The definition of $\mathfrak{h}^\varepsilon[u, v]$ makes sense: the second sum in (2.1) is finite as one can check applying the standard trace inequalities within each period cell. Furthermore, it is straightforward to check that the form $\mathfrak{h}^\varepsilon[u, v]$ is densely defined, closed, and positive. Then, by the first representation theorem [9, Chapter 6, Theorem 2.1], there is a unique self-adjoint and positive operator associated with the form $\mathfrak{h}^\varepsilon$, which we denote as $\mathcal{H}^\varepsilon$,

$$(\mathcal{H}^\varepsilon u, v)_{\mathsf{L}^2(\mathbb{R}^n)} = \mathfrak{h}^\varepsilon[u, v], \quad \forall u \in \mathrm{dom}(\mathcal{H}^\varepsilon),\ \forall v \in \mathrm{dom}(\mathfrak{h}^\varepsilon).$$

Let $u \in \mathrm{dom}(\mathcal{H}^\varepsilon) \cap \mathsf{C}^2(\mathbb{R}^n \setminus \Gamma^\varepsilon)$. Integrating by parts, one can easily show that

$$(\mathcal{H}^\varepsilon u)(x) = -\Delta u(x) \quad \text{at} \quad x \in \mathbb{R}^n \setminus \Gamma^\varepsilon,$$

while on Γ^ε_{ij} one has the following interface matching conditions:

$$(\partial_{\mathbf{n}} u) \upharpoonright^{\rm ext}_{\Gamma^\varepsilon_{ij}} = (\partial_{\mathbf{n}} u) \upharpoonright^{\rm int}_{\Gamma^\varepsilon_{ij}} = \varepsilon q_j (u \upharpoonright^{\rm ext}_{\Gamma^\varepsilon_{ij}} - u \upharpoonright^{\rm int}_{\Gamma^\varepsilon_{ij}}),$$

where $\partial_{\mathbf{n}}$ is the derivative along the outward-pointing unit normal to Γ^ε_{ij}. This supports our interpretation of $\mathcal{H}^\varepsilon$ as the Hamiltonians describing a lattice of periodically spaced obstacles, or "traps" in the form of given by δ' interaction supported by Γ^ε_{ij}; the interaction becomes "weak" as $\varepsilon \to 0$. For more details on Schrödinger operators in $\mathbb{R}^n$ with δ' interactions supported by hypersurfaces we refer to [3, 4].

We denote by $\sigma(\mathcal{H}^\varepsilon)$ the spectrum of $\mathcal{H}^\varepsilon$. Due to the Floquet–Bloch theory, $\sigma(\mathcal{H}^\varepsilon)$ is a locally finite union of compact intervals called *bands*. In general, the bands may touch each other or even overlap. The non-empty bounded open interval $(A, B) \subset \mathbb{R}$ is called a *gap* in the spectrum of $\mathcal{H}^\varepsilon$ if

$$(A, B) \cap \sigma(\mathcal{H}^\varepsilon) = \emptyset, \quad A, B \in \sigma(\mathcal{H}^\varepsilon).$$

First we give a simple estimate from above to the number of gaps.

Proposition 2.2. *The spectrum $\sigma(\mathcal{H}^\varepsilon)$ has at most m gaps within the interval $[0, \Lambda\varepsilon^{-2}]$ with some constant $\Lambda > 0$ depending on the set Ω_0 only.*

The proof of this proposition is simple, but we postpone it to Section 3, cf. Corollary 3.2 since we need to do some preliminary work first. The constant Λ is given by (3.7).

Our main goal is to detect gaps in the spectrum of $\mathcal{H}^\varepsilon$ within the interval $[0, \Lambda\varepsilon^{-2}]$ and to describe their asymptotic behavior as $\varepsilon \to 0$. To state the result, we have to introduce some notations.

In what follows, we denote by C, C_1, etc. generic constants independent of ε and of functions appearing in the estimates and equalities where these constants occur, however, they may depend on n, Ω_j and q_j.

For $j \in \{1, \ldots, m\}$, we set

$$A_j := q_j \frac{|\partial\Omega_j|}{|\Omega_j|},$$

where the symbol $|\cdot|$ serves both for the volume of domain in $\mathbb{R}^n$ and for the "area" of $(n-1)$-dimensional surface in $\mathbb{R}^n$. We assume that the domains Ω_j and the numbers q_j are chosen in such a way that

$$A_j < A_{j+1},\ j \in \{1, \ldots, m-1\}. \tag{2.2}$$

Furthermore, we consider the rational function

$$F(\lambda) := 1 + \sum_{j=1}^{m} \frac{A_j |\Omega_j|}{|\Omega_0|(A_j - \lambda)}. \tag{2.3}$$

It is easy to show that $F(\lambda)$ has exactly m roots, those are real and interlace with A_j provided (2.2) holds. We denote them B_j, $j \in \{1, \ldots, m\}$ assuming them to be renumbered in the ascending order,

$$A_j < B_j < A_{j+1}, \quad j \in \{1, \ldots, m-1\}, \quad A_m < B_m < \infty. \tag{2.4}$$

Now we are in position to formulate the main results of this work.

Theorem 2.3. *The spectrum of $\mathcal{H}^\varepsilon$ has the following form within the interval $[0, \Lambda\varepsilon^{-2}]$:*

$$\sigma(\mathcal{H}^\varepsilon) \cap [0, \Lambda\varepsilon^{-2}] = [0, \Lambda\varepsilon^{-2}] \setminus \left(\bigcup_{j=1}^{m} (A_j^\varepsilon, B_j^\varepsilon) \right).$$

The endpoints of the intervals $(A_j^\varepsilon, B_j^\varepsilon)$ satisfy

$$A_j^\varepsilon \in [A_j - C\varepsilon,\, A_j], \quad B_j^\varepsilon \in [B_j - C\varepsilon,\, B_j],$$

provided ε is small enough.

Remark 2.4. In the above theorem, "provided ε is small enough" means $\varepsilon < \varepsilon_0$ for some ε_0 which depends in general on q_j and Ω_j. It will be apparent from the proof, cf. Lemma 3.4, that ε_0 can be given explicitly, but the formula looks rather cumbersome, in particular, it depends on the constants appearing in the Poincaré and trace inequalities for Ω_j.

Using a lemma from [10], *one can choose the domains* Ω_j *and the numbers* q_j *in such a way that the limiting intervals* (A_j, B_j) *coincide with predefined segments.* Indeed, let us define the map

$$\mathcal{L} : \mathrm{dom}(\mathcal{L}) \subset \mathbb{R}^{2m} \to \mathbb{R}^{2m}, \quad (a_1, \dots, a_m, b_1, \dots, b_m) \overset{\mathcal{L}}{\mapsto} (A_1, \dots, A_m, B_1, \dots, B_m)$$

with the domain

$$\mathrm{dom}(\mathcal{L}) = \Big\{(a_1, \dots, a_m, b_1, \dots, b_m) \in \mathbb{R}^{2m} : a_j > 0,\ b_j > 0, \\ \sum_{j=1}^m b_j < 1,\ \frac{a_j}{b_j} < \frac{a_{j+1}}{b_{j+1}}\Big\}$$

acting as follows: $A_j = \dfrac{a_j}{b_j}$, B_j are the roots of the function

$$1 + \sum_{j=1}^m \frac{A_j b_j}{b_0(A_j - \lambda)}, \quad \text{where } b_0 := 1 - \sum_{j=1}^m b_j,$$

renumbered according to (2.4). The indicated result [10, Lemma 2.1] then reads as follows:

Lemma 2.5. *$\mathcal{L}$ maps* $\mathrm{dom}(\mathcal{L})$ *onto the set of* $(A_1, \dots, A_m, B_1, \dots B_m) \in \mathbb{R}^{2m}$ *satisfying* (2.4)*. Moreover, $\mathcal{L}$ is a one-to-one map, and the inverse map $\mathcal{L}^{-1}$ is given by the formulæ*

$$a_j = A_j \frac{\rho_j}{1 + \sum\limits_{i=1}^m \rho_i}, \quad b_j = \frac{\rho_j}{1 + \sum\limits_{i=1}^m \rho_i}, \tag{2.5}$$

where

$$\rho_j = \frac{B_j - A_j}{A_j} \prod_{i=1,\dots,m|i\neq j} \left(\frac{B_i - A_j}{A_i - A_j}\right).$$

Now it is clear how to choose the sought Ω_j and q_j, cf. the statement following Remark 2.4. Specifically, assume that the intervals (A_j, B_j) satisfying (2.4) are given. We define for them the numbers a_j, b_j by formulæ (2.5) and then we choose the domains Ω_j, $j \in \{1, \dots, m\}$, in such a way that $|\Omega_j| = b_j$. Obviously, this can be always done since $b_j > 0$ and $\sum_{j=1}^m b_j < 1$; recall that the closures of Ω_j must be pairwise disjoint by assumption and belong to the unit cube. Needless to say, such a choice is not unique. Finally, with these Ω_j we define the numbers q_j by $q_j = A_j \frac{|\Omega_j|}{|\partial\Omega_j|}$.

3. Proof of the results

3.1. Preliminaries. We introduce the sets

- $\Gamma_j = \partial\Omega_j$, where $j \in \{1, \dots, m\}$,
- $\Gamma_{ij} = \partial\Omega_j + i$, where $i \in \mathbb{Z}^n$, $j \in \{1, \dots, m\}$,
- $\Gamma = \cup_{i\in\mathbb{Z}^n} \cup_{j\in\{1,\dots,m\}} \Gamma_{ij}$.

The operator $\mathcal{H}^\varepsilon$ is by construction $\mathbb{Z}^n$-periodic with the period cell εY. It is convenient to perform a change of coordinates $x = \varepsilon y$ (from the old coordinates x to the new coordinates y) that would allow us to work with an ε-independent period cell. More precisely, we introduce the sesquilinear form $\widehat{\mathfrak{h}}^\varepsilon$ in the Hilbert space $\mathsf{L}^2(\mathbb{R}^n)$ via

$$\widehat{\mathfrak{h}}^\varepsilon[u,v] := \frac{1}{\varepsilon^2}\int_{\mathbb{R}^n\setminus\Gamma} \nabla u\cdot\nabla\bar v\,\mathrm{d}x + \sum_{i\in\mathbb{Z}}\sum_{j=1}^m q_j \int_{\Gamma_{ij}} (u\restriction_{\Gamma_{ij}}^{\mathrm{ext}} - u\restriction_{\Gamma_{ij}}^{\mathrm{int}})\overline{(v\restriction_{\Gamma_{ij}}^{\mathrm{ext}} - v\restriction_{\Gamma_{ij}}^{\mathrm{int}})}\,\mathrm{d}s, \quad q_j > 0,$$

with the form domain $\mathrm{dom}(\widehat{\mathfrak{h}}^\varepsilon) = \mathsf{H}^1(\mathbb{R}^n\setminus\Gamma)$. Finally, by $\widehat{\mathcal{H}}^\varepsilon$ we denote the unique self-adjoint and positive operator associated with the form $\widehat{\mathfrak{h}}^\varepsilon$. It is easy to see that

$$\sigma(\widehat{\mathcal{H}}^\varepsilon) = \sigma(\mathcal{H}^\varepsilon).$$

Moreover, the operator $\widehat{\mathcal{H}}^\varepsilon$ is periodic with respect to the ε-independent period cell Y.

The Floquet–Bloch theory — see, e.g., [6,11,12] — establishes a relationship between $\sigma(\widehat{\mathcal{H}}^\varepsilon)$ and the spectra of certain operators on Y. Let $\phi = (\phi_1, \dots, \phi_n) \in [0, 2\pi)^n$, the dual cell to Y. We introduce the space $\mathsf{H}^1_\phi(Y \setminus \cup_{j=1}^m \Gamma_j)$, which consists of functions from $\mathsf{H}^1(Y\setminus\cup_{j=1}^m\Gamma_j)$ satisfying the following conditions at the opposite faces of ∂Y, usually referred to as *quasi-periodic boundary conditions*,

$$\forall k\in\{1,\dots,n\}\quad u(x+e_k) = \exp(i\phi_k)u(x) \quad \text{for } x = (x_1, x_2, \dots, \underset{\substack{\uparrow\\ k\text{th place}}}{0}, \dots, x_n), \tag{3.1}$$

where $e_k = (0, 0, \dots, 1, \dots, 0)$.

In the space $\mathsf{L}^2(Y)$, we introduce the sesquilinear form $\widehat{\mathfrak{h}}^\varepsilon_\phi$ defined by

$$\widehat{\mathfrak{h}}^\varepsilon_\phi[u,v] := \frac{1}{\varepsilon^2}\int_{Y\setminus\cup_{j=1}^m\Gamma_j} \nabla u\cdot\nabla\bar v\,\mathrm{d}x + \sum_{j=1}^m q_j \int_{\Gamma_j} (u\restriction_{\Gamma_j}^{\mathrm{ext}} - u\restriction_{\Gamma_j}^{\mathrm{int}})\overline{(v\restriction_{\Gamma_j}^{\mathrm{ext}} - v\restriction_{\Gamma_j}^{\mathrm{int}})}\,\mathrm{d}s \tag{3.2}$$

with the domain $\mathsf{H}^1_\phi(Y \setminus \cup_{j=1}^m\Gamma_j)$. We denote by $\widehat{\mathcal{H}}^\varepsilon_\phi$ the associated self-adjoint and positive operator. Its domain consists of the functions $u \in \mathsf{H}^2(Y\setminus\cup_{j=1}^m\Gamma_j)\cap \mathsf{H}^1_\phi(Y\setminus\cup_{j=1}^m\Gamma_j)$ satisfying also

$$\forall k\in\{1,\dots,n\}\quad \frac{\partial u}{\partial x_k}(x+e_k) = \exp(i\phi_k)\frac{\partial u}{\partial x_k}(x)$$

$$\text{for } x = (x_1, x_2, \ldots, \underset{\substack{\uparrow \\ k\text{th place}}}{0}, \ldots, x_n) \tag{3.3}$$

and the following δ' interface matching conditions on Γ_j:

$$(\partial_{\mathbf{n}} u) \restriction_{\Gamma_j}^{\text{ext}} = (\partial_{\mathbf{n}} u) \restriction_{\Gamma_j}^{\text{int}} = \varepsilon^2 q_j (u \restriction_{\Gamma_j}^{\text{ext}} - u \restriction_{\Gamma_j}^{\text{int}}),$$

where $\partial_{\mathbf{n}}$ is the derivative along the outward-pointing unit normal to Γ_j. The operator $\widehat{\mathcal{H}}_\phi^\varepsilon$ acts as

$$(\widehat{\mathcal{H}}_\phi^\varepsilon u) \restriction_{\Omega_j} = -\frac{1}{\varepsilon^2} (\Delta u) \restriction_{\Omega_j}, \quad j \in \{0, \ldots, m\}.$$

The spectrum of $\widehat{\mathcal{H}}_\phi^\varepsilon$ is purely discrete. We denote by $\{\lambda_{k,\phi}^\varepsilon\}_{k\in\mathbb{N}}$ the sequence of its eigenvalues arranged in the ascending order and repeated according to their multiplicity.

By the Floquet–Bloch theory, we have

$$\sigma(\widehat{\mathcal{H}}^\varepsilon) = \bigcup_{k=1}^{\infty} \bigcup_{\phi\in[0,2\pi)^n} \{\lambda_{k,\phi}^\varepsilon\}, \tag{3.4}$$

and moreover, for any fixed $k \in \mathbb{N}$ the set $\cup_{\phi\in[0,2\pi)^n}\{\lambda_{k,\phi}^\varepsilon\}$ is a compact interval, conventionally referred to as the kth spectral band.

Along with the operators $\widehat{\mathcal{H}}_\phi^\varepsilon$ we also introduce the operators $\widehat{\mathcal{H}}_N^\varepsilon$ and $\widehat{\mathcal{H}}_D^\varepsilon$, which differ from $\widehat{\mathcal{H}}_\phi^\varepsilon$ only by the boundary conditions at ∂Y: instead of the quasi-periodic conditions one imposes here the Neumann and the Dirichlet ones, respectively. More precisely, we introduce in $\mathsf{L}^2(Y)$ the sesquilinear forms $\widehat{\mathfrak{h}}_N^\varepsilon$ and $\widehat{\mathfrak{h}}_D^\varepsilon$ with the domains

$$\operatorname{dom}(\widehat{\mathfrak{h}}_N^\varepsilon) = \mathsf{H}^1(Y \setminus \cup_{j=1}^m \Gamma_j) \quad \text{and} \quad \operatorname{dom}(\widehat{\mathfrak{h}}_D^\varepsilon) = \{u \in \mathsf{H}^1(Y \setminus \cup_{j=1}^m \Gamma_j) :\ u \restriction_Y = 0\}$$

and the action specified by (3.2); then $\widehat{\mathcal{H}}_N^\varepsilon$ and $\widehat{\mathcal{H}}_D^\varepsilon$ are the operators associated with these forms. The spectra of these operators are purely discrete. We denote by $\{\lambda_{k,N}^\varepsilon\}_{k\in\mathbb{N}}$ (respectively, $\{\lambda_{k,D}^\varepsilon\}_{k\in\mathbb{N}}$) the sequence of eigenvalues of $\widehat{\mathcal{H}}_N^\varepsilon$ (respectively, of $\widehat{\mathcal{H}}_D^\varepsilon$) arranged in the ascending order and repeated according to their multiplicity. Since

$$\forall \phi \in [0, 2\pi)^n : \quad \operatorname{dom}(\widehat{\mathfrak{h}}_N^\varepsilon) \supset \operatorname{dom}(\widehat{\mathfrak{h}}_\phi^\varepsilon) \supset \operatorname{dom}(\widehat{\mathfrak{h}}_D^\varepsilon),$$

using the min-max principle [12, Sec. XIII.1], we obtain

$$\forall k \in \mathbb{N},\ \forall \phi \in [0, 2\pi)^n : \quad \lambda_{k,N}^\varepsilon \le \lambda_{k,\phi}^\varepsilon \le \lambda_{k,D}^\varepsilon. \tag{3.5}$$

For a fixed $\phi \in [0, 2\pi)^n$, we denote by $\Delta_{N,\phi}(\Omega_0)$ the Laplace operator on Ω_0 subject to the Neumann conditions on $\cup_{j=1}^m \partial\Omega_j$ and conditions (3.1), (3.3) on ∂Y.

Lemma 3.1. *For each $\phi \in [0, 2\pi)^n$, one has*

$$\frac{1}{\varepsilon^2}\Lambda_\phi \le \lambda^\varepsilon_{m+1,\phi},$$

where Λ_ϕ is the smallest eigenvalue of the operator $-\Delta_{N,\phi}(\Omega_0)$.

Proof. We consider the *decoupled* operator

$$\widehat{\mathcal{H}}^\varepsilon_{\phi,\mathrm{dec}} = \left(-\frac{1}{\varepsilon^2}\Delta_{N,\phi}(\Omega_0)\right) \oplus \left(\oplus_{j=1}^m \left(-\frac{1}{\varepsilon^2}\Delta_N(\Omega_j)\right)\right),$$

where $\Delta_N(\Omega_j)$ is the Neumann Laplacian on Ω_j, $j = 1, \dots, m$. Since $q_j > 0$, we get

$$\widehat{\mathfrak{h}}^\varepsilon_{\phi,\mathrm{dec}} \le \widehat{\mathfrak{h}}^\varepsilon_\phi, \tag{3.6}$$

where $\widehat{\mathfrak{h}}^\varepsilon_{\phi,\mathrm{dec}}$ is the form associated with $\widehat{\mathcal{H}}^\varepsilon_{\phi,\mathrm{dec}}$. Using the min-max principle, we conclude from (3.6) that the kth eigenvalue of $\widehat{\mathcal{H}}^\varepsilon_{\phi,\mathrm{dec}}$ is smaller than or equal to the kth eigenvalue of $\widehat{\mathcal{H}}^\varepsilon_\phi$ for any $k \in \mathbb{N}$. It is clear that the first m eigenvalues of $\widehat{\mathcal{H}}^\varepsilon_{\phi,\mathrm{dec}}$ are equal to zero, while the $(m+1)$th one equals $\varepsilon^{-2}\Lambda_\phi$, whence we obtain the desired result. □

Now we set

$$\Lambda := \max_{\phi \in [0,2\pi)^n} \Lambda_\phi. \tag{3.7}$$

It is easy to see that $\Lambda < \infty$. Indeed, due to the min-max principle, $\Lambda \le \Lambda_D$, where Λ_D is the smallest eigenvalue of the Laplace operator in Ω_0 subject to the Neumann conditions at $\cup_{j=1}^m \partial\Omega_j$ and the Dirichlet conditions at ∂Y. Note that $\Lambda \ne \Lambda_D$ in general.

From the above lemma and (3.4), we immediately obtain the following corollary justifying the claim of Proposition 2.2:

Corollary 3.2. $\sigma(\widehat{\mathcal{H}}^\varepsilon)$ *(hence also $\sigma(\mathcal{H}^\varepsilon)$) has at most m gaps on the interval* $[0, \Lambda\varepsilon^2]$.

Now we are able to proceed to the proof of our main result. First we sketch our strategy.

3.2. Sketch of the proof.

We distinguish two points of the dual lattice cell, usually referred to as *Brillouin zone*, denoting

$$\phi_0 = (0, 0, \dots, 0), \quad \phi_\pi = (\pi, \pi, \dots, \pi).$$

In view of (3.4)–(3.5), the left edge of the kth spectral band of $\widehat{\mathcal{H}}^\varepsilon$ is located between $\lambda^\varepsilon_{k,N}$ and λ_{k,ϕ_0}, while the right one is located between $\lambda^\varepsilon_{k,\phi_\pi}$ and $\lambda_{k,D}$. Clearly, $\lambda^\varepsilon_{1,N} = \lambda^\varepsilon_{1,\phi_0} = 0$ holds. Our goal is to prove that

$$\lim\nolimits_{\varepsilon\to 0} \lambda^\varepsilon_{k,N} = \lim\nolimits_{\varepsilon\to 0} \lambda^\varepsilon_{k,\phi_0} = B_{k-1}, \quad k = 2, \dots, m+1,$$
$$\lim\nolimits_{\varepsilon\to 0} \lambda^\varepsilon_{k,D} = \lim\nolimits_{\varepsilon\to 0} \lambda^\varepsilon_{k,\phi_\pi} = A_k, \quad k = 1, \dots, m,$$

and, moreover, that the rate of this convergence is of order $C\varepsilon$. These results taken together give the claim of Theorem 2.3.

Let us start from the Neumann eigenvalues. The idea is to find a limit operator $\widehat{\mathcal{H}}_N$ the eigenvalues of which will approach $\lambda_{k,N}^\varepsilon$ as $\varepsilon \to 0$. It is not difficult to guess — using, e.g., Simon's results [13] about monotonic sequences of forms — how the "limit" operator should look like: it is associated with the form

$$\begin{aligned} \operatorname{dom}(\widehat{\mathfrak{h}}_N) &= \left\{ u \in \cap_{\varepsilon>0} \operatorname{dom}(\widehat{\mathfrak{h}}_N^\varepsilon) :\ \sup_{\varepsilon>0} \widehat{\mathfrak{h}}_N^\varepsilon[u,u] < \infty \right\}, \\ \widehat{\mathfrak{h}}_N[u,v] &= \lim_{\varepsilon\to 0} \widehat{\mathfrak{h}}_N^\varepsilon[u,v]. \end{aligned} \tag{3.8}$$

Evidently, $\operatorname{dom}(\widehat{\mathfrak{h}}_N)$ consists of the functions constant on each Ω_j, and the value of the form on functions is given by $\sum\limits_{j=1}^m q_j|\Gamma_j||u_j-u_0|^2$, with the abuse of notation written as $u = (u_0,\dots,u_m) \in \mathbb{C}^{m+1}$. Moreover, it turns out that the eigenvalues of $\widehat{\mathcal{H}}_N$ are $0, B_1, \dots, B_m$, with reference to the result obtained in [2].

The limit operator for $\widehat{\mathcal{H}}_{\phi_0}^\varepsilon$ is again $\widehat{\mathcal{H}}_N$ since a function constant on Ω satisfies ϕ_0-periodic boundary conditions and hence (3.8) leads to the same operator.

The limit operator for $\widehat{\mathcal{H}}_D^\varepsilon$ is associated with the form $\widehat{\mathfrak{h}}_D$ defined by (3.8) except that $\widehat{\mathfrak{h}}_N^\varepsilon$ is replaced by $\widehat{\mathfrak{h}}_D^\varepsilon$. Since the only constant satisfying the Dirichlet boundary conditions is zero, we conclude that $\operatorname{dom}(\widehat{\mathfrak{h}}_D) = \mathbb{C}^m$ and the action of this form on $u = (u_1,\dots,u_m) \in \mathbb{C}^m$ is $\sum\limits_{j=1}^m q_j|\Gamma_j||u_j|^2$. The eigenvalues of $\widehat{\mathcal{H}}_D$ are thus $A_1,\dots,A_m$.

Finally, the limit operator for $\widehat{\mathcal{H}}_{\phi_\pi}^\varepsilon$ is $\widehat{\mathcal{H}}_D$ since the functions constant on Ω can satisfy ϕ_π-periodic boundary conditions *iff* the functions are zero.

In the subsequent sections we will implement this strategy. Our asymptotic analysis will be based on a (slightly modified) result from [8] which for the reader's convenience is presented in Appendix.

3.3. Asymptotic behavior of $\lambda_{k,N}^\varepsilon$ and $\lambda_{k,\phi_0}^\varepsilon$. In the following, we will work with the space $\mathbb{C}^{m+1}$ denoting its elements by bold letters, $\mathbf{u}$, $\mathbf{v}, \dots$. Their entries will be enumerated starting from zero,

$$\mathbf{u} \in \mathbb{C}^{m+1} \Rightarrow \mathbf{u} = (u_0,\dots,u_m) \text{ with } u_j \in \mathbb{C}.$$

Let $\mathbb{C}_\Omega^{m+1}$ be the same space $\mathbb{C}^{m+1}$, but equipped with the weighted scalar product,

$$(\mathbf{u},\mathbf{v})_{\mathbb{C}_\Omega^{m+1}} = \sum_{j=0}^m u_j\overline{v_j}|\Omega_j|. \tag{3.9}$$

In this space we introduce the sesquilinear form $\widehat{\mathfrak{h}}_N$

$$\widehat{\mathfrak{h}}_N[\mathbf{u},\mathbf{v}] = \sum_{j=1}^m q_j|\Gamma_j|(u_j-u_0)\overline{(v_j-v_0)}$$

with $\mathrm{dom}(\widehat{\mathfrak{h}}_N) = \mathbb{C}_\Omega^{m+1}$. Let $\widehat{\mathcal{H}}_N$ be an operator in $\mathbb{C}_\Omega^{m+1}$ associated with this form. It is obvious that $\widehat{\mathcal{H}}_N$ can be represented by the $(n+1)\times(n+1)$ matrix, symmetric with respect to the scalar product (3.9),

$$\widehat{\mathcal{H}}_N = \begin{pmatrix} \sum_{j=1}^{m} q_j|\Gamma_j||\Omega_0|^{-1} & -q_1|\Gamma_1||\Omega_0|^{-1} & -q_2|\Gamma_2||\Omega_0|^{-1} & \dots & -q_m|\Gamma_m||\Omega_0|^{-1} \\ -q_1|\Gamma_1||\Omega_1|^{-1} & q_1|\Gamma_1||\Omega_1|^{-1} & 0 & \dots & 0 \\ -q_2|\Gamma_2||\Omega_2|^{-1} & 0 & q_2|\Gamma_2||\Omega_2|^{-1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q_m|\Gamma_m||\Omega_m|^{-1} & 0 & 0 & \dots & q_m|\Gamma_m||\Omega_m|^{-1} \end{pmatrix}. \tag{3.10}$$

We denote by $\lambda_{1,N} \le \lambda_{2,N} \le \dots \le \lambda_{m+1,N}$ the eigenvalues of $\widehat{\mathcal{H}}_N$.

Lemma 3.3. *For any $k \in \{1,\dots,m+1\}$, one has*

$$\lambda_{k,N}^{\varepsilon} \le \lambda_{k,N}.$$

Proof. By the min-max principle, we have

$$\lambda_{k,N}^{\varepsilon} = \min_{V\in\mathfrak{V}[k]} \max_{u\in V\setminus\{0\}} \frac{\widehat{\mathfrak{h}}_N^{\varepsilon}[u,u]}{\|u\|^2_{\mathsf{L}^2(Y)}}, \tag{3.11}$$

where $\mathfrak{V}[k]$ is the family of all k-dimensional subspaces in $\mathrm{dom}(\widehat{\mathfrak{h}}_N^{\varepsilon})$. We introduce the operator $P : \mathbb{C}_\Omega^{m+1} \to \mathsf{L}^2(Y)$ by

$$P\mathbf{u} = \sum_{j=0}^{m} u_j \chi_{\Omega_j},$$

where χ_{Ω_j} is the indicator function of Ω_j. Since the Ω_j's are disjoint by assumption, we have

$$\|P\mathbf{u}\|_{\mathsf{L}^2(Y)} = \|\mathbf{u}\|_{\mathbb{C}_\Omega^{m+1}}, \quad \widehat{\mathfrak{h}}_N^{\varepsilon}[P\mathbf{u},P\mathbf{u}] = \widehat{\mathfrak{h}}_N[\mathbf{u},\mathbf{u}]. \tag{3.12}$$

Let $\mathbf{u}_{0,N},\dots,\mathbf{u}_{m,N}$ be an orthonormal system of eigenvectors of $\widehat{\mathcal{H}}_N$ such that $\widehat{\mathcal{H}}_N u_{j,N} = \lambda_{j,N} u_{j,N}$. We denote $W_k := \mathrm{span}(u_{0,N},\dots,u_{m,N})$, then it is easy to check that

$$\forall \mathbf{u} \in W_k \quad \frac{\widehat{\mathfrak{h}}_N[\mathbf{u},\mathbf{u}]}{\|\mathbf{u}\|^2_{\mathbb{C}_\Omega^{m+1}}} \le \lambda_{k,N}, \tag{3.13}$$

the equality in (3.13) being attained for $\mathbf{u} = \mathbf{u}_{k,N}$.

Finally, we set $V_k := PW_k$. It is obvious that $V_k \in \mathfrak{V}[k]$, and using (3.11)–(3.13), we obtain

$$\lambda_{k,N}^{\varepsilon} \le \max_{u\in V_k\setminus\{0\}} \frac{\widehat{\mathfrak{h}}_N^{\varepsilon}[u,u]}{\|u\|^2_{\mathsf{L}^2(Y)}} = \max_{\mathbf{u}\in W_k\setminus\{0\}} \frac{\widehat{\mathfrak{h}}_N[\mathbf{u},\mathbf{u}]}{\|\mathbf{u}\|^2_{\mathbb{C}_\Omega^{m+1}}} = \lambda_{k,N},$$

which concludes the proof. ∎

Lemma 3.4. *For any $k \in \{1,\dots,m+1\}$, one has*

$$\lambda_{k,N} \le \lambda^{\varepsilon}_{k,N} + C\varepsilon, \tag{3.14}$$

provided ε is small enough.

Proof. For $u \in \mathrm{dom}(\widehat{\mathfrak{h}}^{\varepsilon}_N)$, we introduce the norm

$$\|u\|_{1,\varepsilon} := \left(\widehat{\mathfrak{h}}^{\varepsilon}_N[u,u] + \|u\|^2_{\mathsf{L}^2(Y)}\right)^{1/2}.$$

Furthermore, we define the operator $\Phi : \mathrm{dom}(\widehat{\mathfrak{h}}^{\varepsilon}_N) \to \mathrm{dom}(\widehat{\mathfrak{h}}_N)$ by

$$(\Phi u)_j = \frac{1}{|\Omega_j|}\int_{\Omega_j} u(x)\,\mathrm{d}x, \quad j = 0,\dots,m.$$

Our goal is to prove that the following estimates hold for each $u \in \mathrm{dom}(\widehat{\mathfrak{h}}^{\varepsilon}_N)$:

$$\|u\|^2_{\mathsf{L}^2(Y)} \le \|\Phi u\|^2_{\mathbb{C}^{m+1}_{\Omega}} + C_1\varepsilon^2\|u\|^2_{1,\varepsilon}, \tag{3.15}$$

$$\widehat{\mathfrak{h}}_N[\Phi u,\Phi u] \le \widehat{\mathfrak{h}}^{\varepsilon}_N[u,u] + C_2\varepsilon\|u\|^2_{1,\varepsilon}. \tag{3.16}$$

Then, by means of Lemma 3.9, from Appendix we will get

$$\lambda^{\varepsilon}_{k,N} \le \lambda_{k,N} + \frac{\lambda^{\varepsilon}_{k,N}(1+\lambda^{\varepsilon}_{k,N})C_1\varepsilon^2 + (1+\lambda^{\varepsilon}_{k,N})C_2\varepsilon}{1-(1+\lambda^{\varepsilon}_{k,N})C_1\varepsilon^2}, \tag{3.17}$$

and since $\lambda^{\varepsilon}_{k,N} \le \lambda_{k,N}$ holds by Lemma 3.3, the sought estimate (3.14) will follow from (3.17).

Estimate (3.15) is an easy consequence of the Poincaré inequality

$$\forall j \in \{0,\dots,m\} \quad \|u-(\Phi u)_j\|_{\mathsf{L}^2(\Omega_j)} \le C\|\nabla u\|_{\mathsf{L}^2(\Omega_j)}.$$

Indeed, we have

$$\begin{aligned}\|u\|^2_{\mathsf{L}^2(Y)} &= \sum_{j=0}^{m}\|u\|^2_{\mathsf{L}^2(\Omega_j)} = \|\Phi u\|^2_{\mathbb{C}^{m+1}_{\Omega}} + \sum_{j=0}^{m}\|u-(\Phi u)_j\|^2_{\mathsf{L}^2(\Omega_j)}\\ &\le \|\Phi u\|^2_{\mathbb{C}^{m+1}_{\Omega}} + C_1\sum_{j=0}^{m}\|\nabla u\|^2_{\mathsf{L}^2(\Omega_j)} \le \|\Phi u\|^2_{\mathbb{C}^{m+1}_{\Omega}} + C_1\varepsilon^2\|u\|^2_{1,\varepsilon}.\end{aligned}$$

Let us next prove (3.16). One has

$$\widehat{\mathfrak{h}}_N[\Phi u,\Phi u] \le \widehat{\mathfrak{h}}^{\varepsilon}_N[u,u] + \sum_{j=1}^{m} q_j R_j[u,u],$$

where

$$R_j[u,u] := \|(\Phi u)_0 - (\Phi u)_j\|^2_{\mathsf{L}^2(\Gamma_j)} - \|u\restriction^{\mathrm{ext}}_{\Gamma_j} - u\restriction^{\mathrm{int}}_{\Gamma_j}\|^2_{\mathsf{L}^2(\Gamma_j)},$$

and these expressions can be estimated in the following way:

$$\begin{aligned}|R_j[u,u]| &\le \left| \|(\Phi u)_0-(\Phi u)_j\|_{\mathsf{L}^2(\Gamma_j)} - \left\| u\restriction_{\Gamma_j}^{\mathrm{ext}} - u\restriction_{\Gamma_j}^{\mathrm{int}}\right\|_{\mathsf{L}^2(\Gamma_j)}\right| \\ &\quad\times\left(\|(\Phi u)_0-(\Phi u)_j\|_{\mathsf{L}^2(\Gamma_j)} + \left\| u\restriction_{\Gamma_j}^{\mathrm{ext}} - u\restriction_{\Gamma_j}^{\mathrm{int}}\right\|_{\mathsf{L}^2(\Gamma_j)}\right) \\ &\le \left(\left\|(\Phi u)_0 - u\restriction_{\Gamma_j}^{\mathrm{ext}}\right\|_{\mathsf{L}^2(\Gamma_j)} + \left\|(\Phi u)_j - u\restriction_{\Gamma_j}^{\mathrm{int}}\right\|_{\mathsf{L}^2(\Gamma_j)}\right) \\ &\quad\times\left(\|(\Phi u)_0\|_{\mathsf{L}^2(\Gamma_j)} + \|(\Phi u)_j\|_{\mathsf{L}^2(\Gamma_j)} + \|u\restriction_{\Gamma_j}^{\mathrm{ext}}\|_{\mathsf{L}^2(\Gamma_j)} + \|u\restriction_{\Gamma_j}^{\mathrm{int}}\|_{\mathsf{L}^2(\Gamma_j)}\right).\end{aligned}$$

Using the trace and the Poincaré inequalities, we get

$$\begin{aligned} j\in\{1,\dots,m\}:\quad \left\|(\Phi u)_j - u\restriction_{\Gamma_j}^{\mathrm{int}}\right\|_{\mathsf{L}^2(\Gamma_j)} &\le C\sqrt{\|(\Phi u)_j - u\|^2_{\mathsf{L}^2(\Omega_j)} + \|\nabla u\|^2_{\mathsf{L}^2(\Omega_j)}} \\ &\le C_1\|\nabla u\|_{\mathsf{L}^2(\Omega_j)} \le C_1\varepsilon\|u\|_{1,\varepsilon}, \end{aligned} \tag{3.18}$$

and similarly,

$$\left\|(\Phi u)_0 - u\restriction_{\Gamma_j}^{\mathrm{ext}}\right\|_{\mathsf{L}^2(\Gamma_j)} \le C\|\nabla u\|_{\mathsf{L}^2(\Omega_0)} \le C\varepsilon\|u\|_{1,\varepsilon}. \tag{3.19}$$

Using further the trace and the Cauchy–Schwarz inequalities, one finds

$$\begin{aligned}&\|(\Phi u)_0\|_{\mathsf{L}^2(\Gamma_j)} + \|(\Phi u)_j\|_{\mathsf{L}^2(\Gamma_j)} + \|u\restriction_{\Gamma_j}^{\mathrm{ext}}\|_{\mathsf{L}^2(\Gamma_j)} + \|u\restriction_{\Gamma_j}^{\mathrm{int}}\|_{\mathsf{L}^2(\Gamma_j)} \\ &\qquad\le C\|u\|_{\mathsf{H}^1(Y\setminus\cup_{j=1}^m\Gamma_j)} \le C\|u\|_{1,\varepsilon}.\end{aligned} \tag{3.20}$$

Combining now (3.18)–(3.20), we obtain the needed estimate

$$|R_j[u,u]| \le C\varepsilon\|u\|_{1,\varepsilon},$$

which implies the validity of (3.16) concluding thus the proof. □

We notice that the matrix of the form (3.10) has already been studied in [2] (using different notations). It is shown there that its eigenvalues are the roots of the function $\lambda F(\lambda)$, where $F(\lambda)$ is defined by (2.3). Taking this into account, we immediately obtain the following corollary from the last two lemmata.

Corollary 3.5. *One has*

$$\lambda^\varepsilon_{1,N} = 0,\quad \lambda^\varepsilon_{k,N} \le B_{k-1}\quad \textit{for}\ \ k\in\{2,\dots,m+1\}.$$

Moreover, for small enough ε there is also a lower bound

$$B_{k-1} - C\varepsilon \le \lambda^\varepsilon_{k,N}\quad \textit{for}\ \ k\in\{2,\dots,m+1\}.$$

As we have already noticed above, the limit operator in the ϕ_0-periodic situation has the same eigenvalues as the Neumann one. We have the following claim the proof of which repeats *verbatim* the arguments of Lemmata 3.3 and 3.4.

Lemma 3.6. *One has*

$$\lambda^\varepsilon_{1,\phi_0} = 0,\quad \lambda^\varepsilon_{k,\phi_0} \le B_{k-1}\quad \textit{for}\ \ k\in\{2,\dots,m+1\}.$$

Moreover, for small enough ε, there is also a lower bound

$$B_{k-1} - C\varepsilon \le \lambda^\varepsilon_{k,\phi_0}\quad \textit{for}\ \ k\in\{2,\dots,m+1\}.$$

3.4. Asymptotic behavior of $\lambda^{\varepsilon}_{k,D}$ and $\lambda^{\varepsilon}_{k,\phi_\pi}$. Keeping the boldface symbols from the previous section, we denote by $\mathbb{C}^m_\Omega$ the space of vectors $\mathbf{u} = (0, u_1, \ldots, u_m) \in \mathbb{C}^{m+1}$ equipped with the scalar product

$$(\mathbf{u}, \mathbf{v})_{\mathbb{C}^m_\Omega} = \sum_{j=1}^{m} u_j \overline{v_j} |\Omega_j|,$$

and introduce in this space the sesquilinear form $\widehat{\mathfrak{h}}_D$,

$$\widehat{\mathfrak{h}}_D[\mathbf{u}, \mathbf{v}] := \sum_{j=1}^{m} q_j |\Gamma_j| u_j \overline{v_j}$$

with $\mathrm{dom}(\widehat{\mathfrak{h}}_D) = \mathbb{C}^m_\Omega$. Let further $\widehat{\mathcal{H}}_D$ be an operator in $\mathbb{C}^m_\Omega$ associated with this form. It is clear that $\widehat{\mathcal{H}}_D$ acts as

$$\widehat{\mathcal{H}}_D \mathbf{u} = \sum_{j=1}^{m} q_1 |\Gamma_1| |\Omega_1|^{-1} u_j$$

and its eigenvalues are $A_1,\ A_2, \ldots,\ A_m$.

Lemma 3.7. *One has*

$$\lambda^{\varepsilon}_{k,D} \le A_k \quad \textit{for} \quad k \in \{1, \ldots, m\}.$$

Moreover, for small enough ε, there is a lower bound

$$A_k - C\varepsilon \le \lambda^{\varepsilon}_{k,D} \quad \textit{for} \quad k \in \{1, \ldots, m\}.$$

The proof of this lemma is again similar to those of Lemmata 3.3 and 3.4. The only essential difference here is that instead of the Poincaré inequality, in Ω_0 we use the Friedrichs inequality

$$\|u\|_{\mathsf{L}^2(\Omega_0)} \le \|\nabla u\|_{\mathsf{L}^2(\Omega_0)},$$

which is valid because the functions from $\mathrm{dom}(\widehat{\mathfrak{h}}^{\varepsilon}_D)$ have zero trace on ∂Y.

The analogous result is valid for eigenvalues in the ϕ_π-periodic situation.

Lemma 3.8. *One has*

$$\lambda^{\varepsilon}_{k,\phi_\pi} \le A_k \quad \textit{for} \quad k \in \{1, \ldots, m\}.$$

Moreover, for small enough ε, there is again a lower bound

$$A_k - C\varepsilon \le \lambda^{\varepsilon}_{k,\phi_\pi} \quad \textit{for} \quad k \in \{1, \ldots, m\}.$$

This brings us to the conclusion. Combining Corollary 3.5, Lemmata 3.6–3.8, and equations (3.4), (3.5), we arrive at the claim of Theorem 2.3.

Appendix

Here we recall a result from [8], which is a simple consequence of the min-max principle and serves to compare eigenvalues of two operators acting in different Hilbert spaces.

Let H and H' be two separable Hilbert spaces with the norms $\|\cdot\|$ and $\|\cdot\|'$. Let $\mathcal{H}$ and $\mathcal{H}'$ be non-negative self-adjoint operators in these spaces with purely discrete spectra, and $\mathfrak{h}$ and $\mathfrak{h}'$ be the corresponding forms. We denote by $\{\lambda_k\}_{k\in\mathbb{N}}$ and $\{\lambda'_k\}_{k\in\mathbb{N}}$ the corresponding sequences of eigenvalues, numbered in the ascending order and with account of their multiplicity. Finally, we set $\|u\|_n^2 := \|u\|^2 + \|\mathcal{H}^{n/2}u\|$.

Lemma 3.9 ([8]). *Suppose that* $\Phi : \mathrm{dom}(\mathfrak{h}) \to \mathrm{dom}(\mathfrak{h}')$ *is a linear map such that for all* $u \in \mathrm{dom}(\mathcal{H}^{\max\{n_1,n_2\}/2})$ *one has*

$$\|u\|^2 \le \|\Phi u\|'^2 + \delta_1\|u\|_{n_1}^2,$$
$$\mathfrak{h}'[\Phi u, \Phi u] \le \mathfrak{h}[u,u] + \delta_2\|u\|_{n_2}^2$$

with some constants $n_1, n_2 \ge 0$ *and* $\delta_1, \delta_2 \ge 0$*. Then for each* $k \in \mathbb{N}$*, we have*

$$\lambda'_k \le \lambda_k + \frac{\lambda_k(1+\lambda_k^{n_1})\delta_1 + (1+\lambda_k^{n_2})\delta_2}{1-(1+\lambda_k^{n_1})\delta_1}, \tag{3.21}$$

provided the denominator $1-(1+\lambda_k^{n_1})\delta_1$ *is positive.*

Remark 3.10. The above result was established in [8] under the assumption that $\dim H = \dim H' = \infty$, however, it is easy to see from its proof that the result remains valid for $\dim H < \infty$ as well. In that case (3.21) holds for $k \in \{1, \dots, \dim H\}$. This is the situation in the proof of Lemma 3.4, where we apply Lemma 3.9 to $H = \mathbb{C}_\Omega^{n+1}$.

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Лакунарний контроль сінгулярними операторами Шредінгера в періодично структурованому матеріалі

Pavel Exner and Andrii Khrabustovskyi

Ми розглядаємо сім'ю $\{\mathcal{H}^\varepsilon\}_{\varepsilon>0}$ $\varepsilon\mathbb{Z}^n$-періодичних операторів Шредінгера з δ'-взаємодіями, які локалізовані на сім'ї замкнених компактних поверхонь; мінімальна комірка періодичності містить $m \in \mathbb{N}$ таких поверхонь. Показано, що при $\varepsilon \to 0$ і при певному порядку сили взаємодії $\mathcal{H}^\varepsilon$ має на кінцевих інтервалах не більше m спектральних лакун. Крім того, гранична поведінка перших m лакун повністю контролюється за допомогою належного вибору цих поверхонь і сили взаємодії.

Ключові слова: періодичний оператор Шредінгера, δ'-взаємодія, спектральна лакуна, асимптотика власних значень.