The KPZ Equation and Moments of Random Matrices

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Dedicated to V.A. Marchenko on his 95th birthday

The logarithm of the diagonal matrix element of a high power of a random matrix converges to the Cole–Hopf solution of the Kardar–Parisi–Zhang equation in the sense of one-point distributions.

Key words: KPZ equation, Cole–Hopf solution, Airy process, random matrices.

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1. Introduction and results

1.1. KPZ equation. Let $Z(t, x)$ be the solution of the stochastic heat equation

$$\frac{\partial Z}{\partial t} = \frac{1}{2}\frac{\partial^2 Z}{\partial x^2} - \dot{W}Z, \quad x \in \mathbb{R},\ t \geq 0; \quad Z(0, x) = \delta(x), \tag{1.1}$$

where $\dot{W}$ is the space-time white noise. The logarithm $H = -\log Z$ is known as the Cole–Hopf solution of the Kardar–Parisi–Zhang [22] equation with the narrow-wedge initial condition. Informally, it solves a singular stochastic PDE

$$\frac{\partial H}{\partial t} = \frac{1}{2}\frac{\partial^2 H}{\partial t^2} - \frac{1}{2}\left(\frac{\partial H}{\partial t}\right)^2 + \dot{W}; \tag{1.2}$$

the properties of the solutions of (1.2) are discussed in [8, 18, 30], and a rigorous regularisation — in [19–21].

We also need a half-line version of (1.1), which was recently studied in [4, 26]. Introduce $\tilde{Z}(t, x)$ as the solution to

$$\frac{\partial \tilde{Z}}{\partial t} = \frac{1}{2}\frac{\partial^2 \tilde{Z}}{\partial x^2} - \dot{W}\tilde{Z}, \qquad x \geq 0,\ t \geq 0;$$

$$\left.\left(\frac{\partial \tilde{Z}}{\partial x} + \frac{1}{2}\tilde{Z}\right)\right|_{x=0} = 0, \qquad t > 0; \qquad\qquad \tilde{Z}(0, x) = \delta(x).$$

$\tilde{H} = -\log\tilde{Z}$ is the Cole–Hopf solution for KPZ with a Neumann boundary condition at $x = 0$.

1.2. Random matrix edge. On the other side of the picture, let X_N be a $N \times N$ matrix of independent, identically distributed Gaussian random variables. In $\beta = 1$ case they are real $N(0, 2)$ and in $\beta = 2$ case they are complex with real and imaginary part independent $N(0, 1)$. The distribution of the Hermitian matrix $M_N = \frac{1}{2}(X_N + X_N^*)$ is known as GOE at $\beta = 1$ and GUE at $\beta = 2$. Let

$$\lambda_{1,N} > \lambda_{2,N} > \cdots > \lambda_{N,N}$$

be the eigenvalues of M_N. Then (see [15, 34, 35] and the textbooks [16, 28])

$$\left\{ N^{\frac{1}{6}} \left(\lambda_{j,N} - 2\sqrt{N} \right) \right\}_{j=1}^{\infty} \underset{N \to \infty}{\longrightarrow} \mathfrak{Ai}_\beta, \quad \beta = 1, 2, \tag{1.3}$$

where $\mathfrak{Ai}_\beta$ is the Airy$_\beta$ point process, a realisation of which is an ordered sequence $\lambda_1 > \lambda_2 > \cdots$ The relation (1.3) means that the right-most k rescaled eigenvalues converge in distribution to the k rightmost points of the Airy process, for any k.

For complex matrices, $\mathfrak{Ai}_2$ is a determinantal point process on the real line defined by the kernel

$$K_{\text{Airy}}(\lambda, \lambda') = \frac{\text{Ai}(\lambda) \, \text{Ai}'(\lambda') - \text{Ai}'(\lambda) \, \text{Ai}(\lambda')}{\lambda - \lambda'}.$$

For real matrices, $\mathfrak{Ai}_1$ is a Pfaffian point process whose kernel is also written in terms of the Airy function $\text{Ai}(x)$. For both $\beta = 1, 2$, the distribution of the first particle λ_1 is known as the Tracy–Widom$_\beta$ law, TW$_\beta$. See further [35] for the case $\beta = 4$ and [29] for arbitrary $\beta > 0$.

1.3. Main results. The key notion of our article is the decorated Airy process $\mathfrak{Ai}^{\text{dec}}$.

Definition 1.1. Let u_i^j, v_i^j, $i, j = 1, 2, \ldots$ be i.i.d. standard real Gaussian random variables. For $n = 1, 2, \ldots$, let W_n be a $\mathbb{N} \times \mathbb{N}$ Hermitian rank 1 matrix

$$[W_n]_{a,b} = \begin{cases} u_a^n u_b^n, & \beta = 1, \\ \frac{1}{2}(u_a^n + \mathbf{i} v_a^n)(u_b^n - \mathbf{i} v_b^n), & \beta = 2. \end{cases}$$

$\mathfrak{Ai}_\beta^{\text{dec}}$ is (the distribution of) the random matrix-valued measure

$$\sum_{n=1}^{\infty} W_n \delta_{\lambda_n}, \quad \{\lambda_n\} \sim \mathfrak{Ai}_\beta.$$

For each continuous function $f : \mathbb{R} \to \mathbb{C}$ such that $\int_{-\infty}^{0} |f(\lambda)| \sqrt{|\lambda|} \, d\lambda < \infty$, the integral $\int f(\lambda) \mathfrak{Ai}_\beta^{\text{dec}}(d\lambda)$ is a Hermitian random matrix of size $\mathbb{N} \times \mathbb{N}$ with matrix elements

$$\left[\int_{\mathbb{R}} f(\lambda) \mathfrak{Ai}_\beta^{\text{dec}}(d\lambda) \right]_{a,b} = \sum_{n=1}^{\infty} [W_n]_{a,b} \, f(\lambda_n), \qquad a, b = 1, 2, \ldots. \tag{1.4}$$

Remark 1.2. The formula (1.4) defines a law of an infinite Hermitian matrix invariant under conjugations by matrices from the infinite-dimensional orthogonal/unitary group ($O(\infty) = \bigcup_N O(N)$, $U(\infty) = \bigcup_N U(N)$, where we embed the group of dimension N into the group of dimension $N + 1$ by fixing the last basis vector). In the notation of [25], the ergodic decomposition of this matrix is given by the point process $f(\mathfrak{Ai}_\beta^{\text{dec}})$ on the α-parameters of the decomposition. It would be interesting to compute the joint law of several elements of the matrix (1.4) explicitly (at least for specific choices of f).

The first result asserts that the one-point distributions for the stochastic heat equation at $x = 0$ are given by the integrals of exponents against a diagonal element of $\mathfrak{Ai}^{\text{dec}}$.

Theorem 1.3. *For each $\alpha > 0$ (but not jointly for several α's), in distribution*

$$\tilde{Z}(2\alpha^3, 0) \exp(\alpha^3/12) \overset{d}{=} 2 \left[\int e^{\alpha\lambda} \mathfrak{Ai}_1^{\text{dec}}(d\lambda) \right]_{1,1}$$

$$= 2 \sum_{n=1}^{\infty} (u_1^n)^2 \exp(\alpha\lambda_n), \quad \{\lambda_n\} \sim \mathfrak{Ai}_1, \qquad (1.5)$$

$$Z(2\alpha^3, 0) \exp(\alpha^3/12) \overset{d}{=} \left[\int e^{\alpha\lambda} \mathfrak{Ai}_2^{\text{dec}}(d\lambda) \right]_{1,1}$$

$$= \sum_{n=1}^{\infty} \frac{(u_1^n)^2 + (v_1^n)^2}{2} \exp(\alpha\lambda_n), \quad \{\lambda_n\} \sim \mathfrak{Ai}_2. \qquad (1.6)$$

The $\beta = 2$ part of Theorem 1.3 is a close relative of the determinantal formula for the Laplace transform of the one-point distribution of SHE computed in [1, 7, 13, 31] and linked to the Airy point process in [5]. The $\beta = 1$ part is related to Laplace transforms in [4, 26]. The proof of Theorem 1.3 (based on all these results) is given in Section 2.

One application of Theorem 1.3 is that it makes the $T \to +\infty$ limit for the solution to the KPZ equation immediate, leading to TW_β distribution after proper centering and rescaling.

On the random matrix side, the decorated Airy process governs the edge asymptotic behavior of large random matrices.

Theorem 1.4. *Let M_N be a Hermitian (Wigner) matrix with independent (up to symmetry) real/complex elements, so that the moments of the matrix elements $\mathbb{E}[M_N]_{i,j}^k \overline{[M_N]_{i,j}^\ell}$ with $k + \ell \leq 4$ match those of GOE/GUE, respectively, and suppose that*

$$\sup_N \max_{1 \leq i,j \leq N} \mathbb{E}|[M_N]_{i,j}|^{C_0} < \infty , \qquad (1.7)$$

where C_0 is the absolute constant from [33]. Then for all $\alpha > 0$

$$\lim_{N\to\infty} \frac{N}{2} \left[\left(\frac{M_N}{2\sqrt{N}} \right)^{2\lfloor \alpha N^{2/3} \rfloor} + \left(\frac{M_N}{2\sqrt{N}} \right)^{2\lfloor \alpha N^{2/3} \rfloor + 1} \right]_{a,b=1}^{\infty}$$

$$= \int_{\mathbb{R}} \exp(\alpha\lambda) \mathfrak{Ai}_\beta^{\mathrm{dec}}(d\lambda), \quad (1.8)$$

in the sense of the convergence of joint distributions for finitely many a, b, and α's.

If instead of the matrix elements we deal with the trace of the matrix in the LHS of (1.8), then Theorem 1.4 (with different technical conditions) turns into the statement of [32] describing the universality of the eigenvalue distribution at the spectral edges. The matrix elements, which we now consider, contain the information on the eigenvectors in addition to the eigenvalues. If we deal with GOE/GUE, then eigenvectors are uniformly distributed on the unit sphere (of $\mathbb{R}^N$) for $\beta = 1$ and of $\mathbb{C}^N$ for $\beta = 2$, and it immediately follows that as $N \to \infty$ their components become independent Gaussians. This is the reason for the appearance of $\{u_i^j\}$, $\{v_i^j\}$ in Definition 1.1. The extension to more general Wigner matrices in Theorem 1.4 uses the four moments theorem of [33], which essentially says that the asymptotic in the general case is the same as for GOE/GUE. A proof of Theorem 1.4 along these lines is given in Section 2.

Combining Theorems 1.3, 1.4 we arrive at an intriguing corollary.

Corollary 1.5. *Let $Z^\beta = Z$ for $\beta = 2$ and $\tilde{Z}$ for $\beta = 1$. The one-point distribution of Z^β satisfies*

$$\lim_{N\to\infty} \frac{N}{\beta} \left[\left(\frac{M_N}{2\sqrt{N}} \right)^{2\lfloor \alpha N^{2/3} \rfloor} + \left(\frac{M_N}{2\sqrt{N}} \right)^{2\lfloor \alpha N^{2/3} \rfloor + 1} \right]_{1,1} \stackrel{d}{=} Z^\beta(2\alpha^3, 0) \exp(\alpha^3/12),$$

$$(1.9)$$

for any M_N as in Theorem 1.4 and $\alpha > 0$.

Remark 1.6. The conditions on the matrix M_N may be relaxed: The matching of the fourth moment could probably be dropped using the methods of [23]. As to the tail decay, the maximal-generality condition for the eigenvalue convergence (1.3) was found in [24]; we expect it to be sufficient also for the problem considered here.

In a different direction, passing to a slightly different matrix ensemble (e.g., sample covariance matrices or shifted Wigner matrices), one can obtain a statement similar to Corollary 1.5 with one, rather than two, terms in the left-hand side.

In yet another direction, Theorem 1.3 and Corollary 1.5 should generalize to unitary (or orthogonally) invariant random matrix ensembles, the local eigenvalue statistics of which are discussed in [12,27] and the monographs [11,28].

The matrix element $[M_N^k]_{1,1}$ admits the following combinatorial interpretation:

$$[M_N^k]_{1,1} = \sum_{j_1,\cdots,j_{k-1}} M_N(1, j_1) M_N(j_1, j_2) \cdots M_N(j_{k-1}, 1). \qquad (1.10)$$

The tuple $1, j_1, \ldots, j_k, 1$ can be thought of as a path in the complete graph, the product in the right-hand side of (1.10) collects the random weights along the path.

On the other hand, the Feynman–Kac formula for the Stochastic Heat Equation identifies $Z(t, x)$ with a partition function of a Brownian directed polymer, see [2] for a detailed treatment. This continuous polymer is a universal limit for discrete directed polymers in the intermediate disorder regime [3]. Such identification bears similarities with (1.10), however, note that there is no clear spatial structure in the latter. It would be interesting to find a more direct relation between these models.

A $1d$ spatial structure can be introduced to GOE/GUE by passing to tridiagonal models of [14]. Consider the matrix

$$
\widetilde{M}_N = \begin{pmatrix}
\mathfrak{a}(N) & \mathfrak{b}(N-1) & 0 & \cdots & 0 \\
\mathfrak{b}(N-1) & \mathfrak{a}(N-1) & \mathfrak{b}(N-2) & \ddots & \vdots \\
0 & \mathfrak{b}(N-2) & \mathfrak{a}(N-2) & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \mathfrak{b}(1) \\
0 & \cdots & 0 & \mathfrak{b}(1) & \mathfrak{a}(1)
\end{pmatrix},
$$

where all $\mathfrak{a}(m)$, $m = 1, 2, \ldots$, have the normal distribution $N(0, 2/\beta)$, and the $\mathfrak{b}(m)$, $m = 1, 2, \ldots$, are $\beta^{-1/2}$ multiples of χ-distributed random variables with parameters βm. Here, the density of the χ distribution with parameter a on $\mathbb{R}_{\geq 0}$ is

$$
\frac{2^{1-k/2}}{\Gamma(a/2)} \, x^{a-1} \, e^{-x^2/2}, \quad x > 0.
$$

For $\beta = 1, 2$, $\widetilde{M}_N$ are obtained from GOE/GUE by the tridiagonalization procedure, which keeps the spectral measure of the $(1, 1)$ matrix element (i.e. the functional $f \mapsto \langle f(M_N) e_1, e_1 \rangle$, where e_1 is the first basis vector) unchanged. This implies that the laws of $[M_N^k]_{1,1}$ and $[\widetilde{M}_N^k]_{1,1}$ coincide (even jointly for several k), and therefore, M_N can be replaced with $\widetilde{M}_N$ in Corollary 1.5. The large moments of $\widetilde{M}_N$ were studied in detail in [17]. A slight generalization of the results therein leads to an alternative form of (1.9).

Proposition 1.7. *The random variables of formula* (1.9) *and Theorem 1.3 coincide with the distributional limit*

$$
\lim_{N \to \infty} \frac{N}{\beta} \left[\left(\frac{\widetilde{M}_N}{2\sqrt{N}} \right)^{2\lfloor \alpha N^{2/3} \rfloor} + \left(\frac{\widetilde{M}_N}{2\sqrt{N}} \right)^{2\lfloor \alpha N^{2/3} \rfloor + 1} \right]_{1,1}, \tag{1.11}
$$

and is also given by

$$
\frac{1}{\beta \alpha \sqrt{\pi \alpha}} \mathbb{E}_{\mathfrak{e}} \left[\exp \left(-\frac{1}{2} \int_0^{2\alpha} \mathfrak{e}(t) \, \mathrm{d}t + \frac{1}{\sqrt{\beta}} \int_0^{\infty} L_a(\mathfrak{e}) \, \mathrm{d}W(a) \right) \right], \tag{1.12}
$$

where $\mathfrak{e}(t)$ *is a standard Brownian excursion going from* 0 *to* 0 *in time* 2α, L_a *is its local time, and* $W(a)$ *is a standard Brownian motion.*

Remark 1.8. The limit (1.11) exists and is given by (1.12) for quite general choice of distributions for $\mathfrak{a}(n)$, $\mathfrak{b}(n)$ in the definition of $\widetilde{M}_N$, see [17, Assumption 2.1].

Continuing the discussion started after Remark 1.6, we see that the expression (1.12) can be treated as a partition function of a certain polymer. However, an important difference to the KPZ is that the white noise $dW(a)$ in (1.12) is one-dimensional, while $\dot{W}$ in the Stochastic Heat Equation is 2-dimensional space-time white noise. It would be interesting to find a direct proof of the distributional identity between (1.12) and the expressions of Theorem 1.3.

2. Proofs

Proof of Theorem 1.3. For $\beta = 2$ our starting point is the following exact relation between the distribution of $Z(t, 0)$ and the Airy process which was recently found in [5, Theorem 2.1] as a reformulation of the results of [1, 7, 13, 31]: for each $\alpha > 0$ and $u \geq 0$,

$$\mathbb{E}\exp\left[-uZ(2\alpha^3, 0)\exp(\alpha^3/12)\right] = \mathbb{E}\left[\prod_{k=1}^{\infty}\frac{1}{1 + u\exp(\alpha\lambda_k)}\right], \quad \{\lambda_i\} \sim \mathfrak{Ai}_2. \quad (2.1)$$

An analogue for $\beta = 1$ is [4, Theorem B], [26, Corollary 1.3]: for each $\alpha > 0$ and $u \geq 0$,

$$\mathbb{E}\exp\left[-u\tilde{Z}(2\alpha^3, 0)\exp(\alpha^3/12)\right] = \mathbb{E}\left[\prod_{k=1}^{\infty}\frac{1}{\sqrt{1 + 4u\exp(\alpha\lambda_k)}}\right], \quad \{\lambda_i\} \sim \mathfrak{Ai}_1. \quad (2.2)$$

By a uniqueness theorem for the Laplace transform, the result of Theorem 1.3 would follow from

$$\mathbb{E}\exp\left[-u \times \text{LHS of (1.5) or (1.6)}\right] = \mathbb{E}\exp\left[-u \times \text{RHS of (1.5) or (1.6)}\right] \quad (2.3)$$

for all $u \geq 0$. For the left-hand side, the answer is given by (2.1), (2.2). For the right-hand side, the squared standard Gaussian and sum of the squares of two independent Gaussians are particular cases of χ_β^2 distribution at $\beta = 1, 2$. The Laplace transform of χ_β^2 is given by

$$\mathbb{E}\exp\left(-v\chi_\beta^2\right) = \frac{1}{(1 + 2v)^{\beta/2}}, \quad v > -1/2.$$

Therefore, using the factorisation of the Laplace transform of the sum of independent random variables factorizes, we have for $\beta = 1, 2$

$$\mathbb{E}\exp\left(-v\beta\left[\int e^{\alpha\lambda}\mathfrak{Ai}_\beta^{\text{dec}}(d\lambda)\right]_{1,1}\right) = \mathbb{E}\left[\prod_{k=1}^{\infty}\frac{1}{(1 + 2v\exp(\alpha\lambda_k))^{\beta/2}}\right], \quad v > 0,$$

where $\{\lambda_i\} \sim \mathfrak{Ai}_\beta$. Setting $v = 2u/\beta^2$, we arrive at (2.1) and (2.2). $\qquad\square$

Proof of Theorem 1.4. First consider the case of GOE/GUE. We first observe that

$$\lim_{N\to\infty} \frac{1}{2} \operatorname{tr}\left[\left(\frac{M_N}{2\sqrt{N}}\right)^{2\lfloor \alpha N^{2/3}\rfloor} + \left(\frac{M_N}{2\sqrt{N}}\right)^{2\lfloor \alpha N^{2/3}\rfloor+1}\right] = \int_{\mathbb{R}} \exp(\alpha\lambda)\mathfrak{Ai}_\beta(d\lambda) \quad (2.4)$$

in distribution. Although (2.4) is not a formal consequence of (1.3), it can be deduced from the latter using a simple estimate on the mean eigenvalue density of the GOE/GUE. (The relation (2.4) is also proved for more general Wigner matrices in [32].)

Further, the eigenbasis of the GOE/GUE is independent of the spectrum (as follows from the orthogonal/unitary invariance of the GOE/GUE probability density). The matrix of coordinates of eigenvectors is a uniformly-random (i.e. Haar-distributed) element O/U of the N-dimensional orthogonal/unitary group. As $N \to \infty$, the matrix elements of O/U multiplied by $\sqrt{N}$ become i.i.d. standard real/complex Gaussians. This is a folklore fact which can be proved, for example, by sampling O/U through the Gram–Schmidt orthogonalization applied to GOE/GUE. We refer to [10] for more details and quantitative estimates, and to [9] for a historical discussion.

At this point, we write the (a, b)-th matrix element in the LHS of (1.8) through the eigenvalue-eigenvector expansion as:

$$\frac{N}{2} \sum_{j=1}^{N} z_a^j \overline{z_b^j}\left(\left(1 + \frac{\lambda_j - 2\sqrt{N}}{2\sqrt{N}}\right)^{2\lfloor \alpha N^{2/3}\rfloor} + \left(1 + \frac{\lambda_j - 2\sqrt{N}}{2\sqrt{N}}\right)^{2\lfloor \alpha N^{2/3}\rfloor+1}\right), \quad (2.5)$$

where λ_j, $j = 1, \ldots, N$, are ordered eigenvalues of M_N and z_i^j are (real or complex) coordinates of corresponding eigenvectors. The convergence of $N^{1/6}(\lambda_j - 2\sqrt{N})$ and $N^{1/6}(\lambda_{N-j} + 2\sqrt{N})$ to points of two (independent) $\mathfrak{Ai}_\beta$ point processes together with convergence of $N z_a^j \overline{z_b^j}$ to products of independent Gaussian random variables, implies that terms of (2.5) converge to (1.4) with $f(\lambda) = \exp(\alpha\lambda)$. To justify the exchange of summation with the limit, we take conditional expectation (conditioned on the eigenvalues) and use (2.4). This implies (1.8).

Now we pass to the case of general M_N. Let us consider the $(1, 1)$ matrix element with a fixed α (the proof for the joint distribution of several matrix elements with different values of α only adds indices to the notation). We rely on [33, Theorem 8], which we use in the following form. For a Hermitian matrix M, denote by $p_j(M)$ the squared absolute value of the first coordinate of the eigenvector corresponding to the j-the eigenvalue (ordered from the largest to the smallest one).

Theorem (Tao-Vu). *For any two Wigner matrices M, M' satisfying the assumptions of Theorem 1.4 with the same β, there exists a small $\delta > 0$ such that the following holds. Let $k \leq N^\delta$, and let $G \in C^5(\mathbb{R}^k \times \mathbb{R}^k)$ be such that*

$$\sup_{x\in\mathbb{R}^k\times\mathbb{C}^k} \max_{0\leq i\leq 5} \|\nabla^i G(x)\| \leq N^\delta. \quad (2.6)$$

Then for all sufficiently large N, any $1 \le j_1, \ldots, j_k \le N$

$$\left| \mathbb{E} G((\sqrt{N}\lambda_{j_\alpha}(M))^k_{\alpha=1}, (Np_{j_\alpha}(M))^k_{\alpha=1}) \right.$$
$$\left. - \mathbb{E} G((\sqrt{N}\lambda_{j_\alpha}(M'))^k_{\alpha=1}, (Np_{j_\alpha}(M'))^k_{\alpha=1}) \right| \le N^{-\delta}.$$

The application of the theorem closely follows the strategy of the proof of [33, Theorem 7]. Let M' be sampled from the GOE or GUE for $\beta = 1$ or 2. Set $k = \lfloor N^{\frac{\delta}{100}} \rfloor$. Pick a smooth bump function $\chi \in C^\infty(\mathbb{C})$ which is identically one for $[-\frac{1}{2}, \frac{1}{2}]$ and vanishes outside $[-1, 1]$. Then let

$$G_1(\vec{x}, \vec{y}) = \prod_{j=1}^k \chi\left(\frac{y_j}{k}\right), \qquad G_2(\vec{x}, \vec{y}) = \chi\left(\left(\frac{x_1}{2N} - 1\right)^2 N^{\frac{4}{3}} (\log\log N)^{-2}\right),$$

$$G_3(\vec{x}, \vec{y}) = 1 - \chi\left(\left(\frac{x_k}{2N} - 1\right)^2 N^{\frac{4}{3}} k^{-\frac{1}{2}}\right).$$

These three functions satisfy (2.6). For the GOE/GUE,

$$\mathbb{E} G_i\left((\sqrt{N}\lambda_j(M'))^k_{j=1}, (Np_j(M'))^k_{j=1}\right) = 1 - o(1), \quad i = 1, 2, 3,$$

hence according to the theorem also

$$\mathbb{E} G_i\left((\sqrt{N}\lambda_j(M))^k_{j=1}, (Np_j(M))^k_{j=1}\right) = 1 - o(1), \quad i = 1, 2, 3,$$

and similarly

$$\mathbb{E} G_i\left((-\sqrt{N}\lambda_{N+1-j}(M))^k_{j=1}, (Np_{N+1-j}(M))^k_{j=1}\right) = 1 - o(1), \quad i = 1, 2, 3.$$

In particular, the expressions inside the expectation are equal to 1 on an event of asymptotically full probability. For $i = 1$ it means that $Np_j(M)$, $Np_{N+1-j}(M)$ are not large for $j \le k$. For $i = 2$ we conclude that x_1 is close to $2N$ and x_N is close to $-2N$. For $i = 3$ we get that x_k is far enough from $2N$ and x_{N-k} is far enough from $-2N$.

In particular, from the relations corresponding to $i = 3$, we conclude that the contribution of eigenvalues λ_j, $k < j \le N - k$, in the expansion (2.5) is negligible.

Thus, it remains to prove that

$$\sum_{j=1}^k \left[\left(\frac{\lambda_j(M)}{2\sqrt{N}}\right)^{2\lfloor \alpha N^{2/3} \rfloor} \right.$$
$$\left. + \left(\frac{\lambda_j(M)}{2\sqrt{N}}\right)^{2\lfloor \alpha N^{2/3}+1 \rfloor} \right] \frac{Np_j(M)}{2} \to \int_{\mathbb{R}} \exp(\alpha\lambda) \mathfrak{Ai}^{\text{dec}}_\beta \, d\lambda,$$

$$\sum_{j=N-k+1}^N \left[\left(\frac{-\lambda_j(M)}{2\sqrt{N}}\right)^{2\lfloor \alpha N^{2/3} \rfloor} - \left(\frac{-\lambda_j(M)}{2\sqrt{N}}\right)^{2\lfloor \alpha N^{2/3}+1 \rfloor} \right] \frac{Np_j(M)}{2} \to 0, \quad (2.7)$$

in distribution as $N \to \infty$, and these relations are already established for M'. Take m satisfying $N^{\frac{2}{3}}(\log\log N)^{-1} \le m \le N^{\frac{2}{3}}\log\log N$, let ϕ be a smooth function with bounded derivatives up to order 5, and set

$$G(x_1, \ldots, x_{2k}, y_1, \ldots, y_{2k}) = \phi\left(\sum_{j=1}^{2k}\left(\frac{x_j}{2N}\right)^m y_j\right)$$

$$\times \prod_{i=1}^{3}\left[G_i(x_1, \ldots, x_k, y_1, \ldots, y_k)G_i(-x_{k+1}, \ldots, -x_{2k}, y_{k+1}, \ldots, y_{2k})\right].$$

It also satisfies (2.6), whence we can apply the Tao–Vu theorem with

$$j_\alpha = \begin{cases} \alpha\,, & 1 \le \alpha \le k, \\ N+1-\alpha\,, & k < \alpha \le 2k, \end{cases}$$

to reduce $\mathbb{E}G$ for M to that of M'. Since ϕ is arbitrary, we conclude that (2.7) holds. $\qquad\square$

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Рівняння КПЖ та моменти випадкових матриць

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Логарифм діагонального матричного елемента високого ступеня випадкової матриці збігається до розв'язку Коле–Гопфа рівняння Кардара–Парісі–Жанга в сенсі одноточкових розташувань.

Ключові слова: рівняння КПЖ, розв'язок Коле–Гопфа, процес Ейрі, випадкова матриця.