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Ricci Solitons and Gradient Ricci Solitons on $N(k)$-Paracontact Manifolds

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An η-Einstein paracontact manifold M admits a Ricci soliton (g, ξ) if and only if M is a K-paracontact Einstein manifold provided one of the associated scalars α or β is constant. Also we prove the non-existence of Ricci soliton in an $N(k)$-paracontact metric manifold M whose potential vector field is the Reeb vector field ξ. Moreover, if the metric g of an $N(k)$-paracontact metric manifold M^{2n+1} is a gradient Ricci soliton, then either the manifold is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4, or M^{2n+1} is an Einstein manifold. Finally, an illustrative example is given.

Key words: paracontact manifold, $N(k)$-paracontact manifold, Ricci soliton, gradient Ricci soliton, Einstein manifold.

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1. Introduction

A natural generalization of an Einstein metric is a Ricci soliton [4]. In a pseudo-Riemannian manifold (M, g) a Ricci soliton is a triplet (g, V, λ), with g, a pseudo-Riemannian metric, V, a smooth vector field (called the potential vector field) and λ, a constant such that

$$\pounds_V g + 2S - 2\lambda g = 0, \tag{1.1}$$

where $\pounds_V g$ is the Lie derivative of g along a vector field V and S is the Ricci tensor of type $(0, 2)$. Obviously, a Ricci soliton with V zero or Killing is an Einstein metric. The Ricci soliton is said to be shrinking, steady or expanding depending on λ being positive, zero or negative, respectively. Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t} g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings. They often arise as blow-up limits for the Ricci flow on compact manifolds. Metrics satisfying (1.1) are interesting and useful in physics. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. For some aspects in this direction we refer to Friedan [14]. A Ricci soliton on a compact manifold has a constant curvature in dimension 2 (Hamilton [16]) and also in dimension 3 (Ivey [17]). On the other hand, a Ricci soliton on a compact manifold is a gradient Ricci soliton [22]. If the complete vector field V is the gradient of a potential function $-f$, then g is said to be a gradient Ricci soliton and equation (1.1) takes the form

$$\text{Hess}\, f = S - \lambda g, \tag{1.2}$$

where $\text{Hess}\, f$ denotes the Hessian of a smooth function f on M and is defined by $\text{Hess} f = \nabla\nabla f$. For details on Ricci solitons and gradient Ricci solitons, we refer to Chow and Knopf [10], Bejan and Crasmareanu [1].

Sharma [24] started to study Ricci solitons in contact geometry as a K-contact metric. In a K-contact manifold the structure vector field ξ is Killing, that is, $\pounds_\xi g = 0$, generally it is not true in an $N(k)$-paracontact metric manifold. Recently, Ricci solitons and gradient Ricci solitons on several types of (almost) contact metric manifolds were studied by many authors. Cho [8,9] obtained some results about Ricci solitons in almost contact and contact geometry. Instantly, Yildiz et al. [28] and Turan et al. [25] also studied Ricci solitons in 3-dimensional f-Kenmotsu manifolds and 3-dimensional trans-Sasakian manifolds, respectively. In [11], De and Matsuyama studied Ricci solitons and gradient Ricci solitons in a Kenmotsu manifold. Ricci solitons were also studied by Deshmukh et al. [12,13], Ghosh [15], Wang et al. [26] and others.

Motivated by these circumstances, in this paper, we study Ricci solitons and gradient Ricci solitons in $N(k)$-paracontact metric manifolds. The present paper is organized as follows: Section 2 contains some preliminary results of $N(k)$-paracontact metric manifolds. In Section 3, we prove that an η-Einstein paracontact manifold M admits a Ricci soliton (g, ξ) if and only if M is a K-paracontact Einstein manifold provided one of the associated scalars α or β is constant. In the next section we prove the non-existence of Ricci soliton in an $N(k)$-paracontact metric manifold M whose potential vector field is the Reeb vector field ξ. Finally, we study a gradient Ricci soliton in an $N(k)$-paracontact metric manifold M^{2n+1} and prove that if the metric g of M^{2n+1} is a gradient Ricci soliton, then either the manifold is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4, or, M^{2n+1} is an Einstein manifold. Finally, an illustrative example is given.

2. Preliminaries on $N(k)$-paracontact metric manifolds

By an almost paracontact manifold we mean a $(2n+1)$-dimensional smooth manifold M which admits a tensor field ϕ of type $(1,1)$, a vector field ξ (called the Reeb vector field), a 1-form η and for any $X \in \chi(M)$ satisfying [18]:

(i) $\phi^2 X = X - \eta(X)\xi$,

(ii) $\phi(\xi) = 0$, $\eta \circ \phi = 0$, $\eta(\xi) = 1$,

(iii) the tensor field ϕ induces an almost paracomplex structure on each fibre of $\mathcal{D} = \ker(\eta)$, that is, the eigendistributions $\mathcal{D}^+_\phi$ and $\mathcal{D}^-_\phi$ of ϕ corresponding to the eigenvalues 1 and -1, respectively, have the same dimension n.

An almost paracontact manifold equipped with a pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.1}$$

for all $X, Y \in \chi(M)$, is said to be an almost paracontact metric manifold, where the signature of g is $(n+1, n)$. An almost paracontact structure is said to be normal [29] if the $(1,2)$-type torsion tensor $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi$ vanishes identically where $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$. In an almost paracontact structure $g(X, \phi Y) = d\eta(X, Y)$ implies the structure is a paracontact structure [29]. The manifold M will be called a paracontact metric manifold [2] if it is endowed with a paracontact metric structure (ϕ, ξ, η, g). In an almost paracontact metric manifold, there always exists a special type of basis, the so-called pseudo-orthonormal ϕ basis $\{X_i, \phi X_i, \xi\}$, where X_i's and ξ are space-like vector fields and ϕX_i's are time-like vector fields. For this reason an almost paracontact metric manifold is an odd dimensional manifold. A normal paracontact metric manifold is a para-Sasakian manifold and satisfies

$$R(X, Y)\xi = -(\eta(Y)X - \eta(X)Y), \tag{2.2}$$

for any $X, Y \in \chi(M)$, but unlike contact metric geometry the relation (2.2) does not imply that the paracontact manifold is a para-Sasakian manifold. It is clear that every para-Sasakian manifold is a K-paracontact manifold, but the converse is not always true as it is shown in the three dimensional case [5]. In a paracontact metric manifold M, we define a $(1,1)$-tensor field h by $2h = \pounds_\xi \phi$. Then we observe that h is symmetric and anticommutes with ϕ. Also, h satisfies the following [29]:

$$h\xi = tr(h) = tr(\phi h) = 0, \tag{2.3}$$

$$\nabla_X \xi = -\phi X + \phi h X \tag{2.4}$$

for all $X \in \chi(M)$. Clearly, the tensor $h = 0$ holds if and only if ξ is a Killing vector field and consequently M is said to be a K-paracontact manifold [21].

The (k, μ)-nullity distribution $N(k, \mu)$ [7] of a paracontact metric manifold M is defined by

$$N(k, \mu) : p \to N_p(k, \mu) = \{W \in T_pM | R(X, Y)W = (kI + \mu h)(g(Y, W)X - g(X, W)Y)\}$$

for all $X, Y \in T_pM$ and $k, \mu \in \mathbb{R}$. A paracontact metric manifold M with $\xi \in N(k, \mu)$ is called a (k, μ)-paracontact metric manifold. Then we must have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY] \tag{2.5}$$

for all $X, Y \in \chi(M)$. In [19, 20], Martin-Molina studied (k, μ)-paracontact metric spaces and constructed some examples.

In particular, if $\mu = 0$, then the (k, μ)-nullity distribution will be called an $N(k)$-nullity distribution. Thus (2.5) reduces to

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y]. \tag{2.6}$$

For an $N(k)$-paracontact metric manifold $M^{2n+1}(n>1)$, the following relations hold [6, 23]:

$$h^2 = (k+1)\phi^2, \tag{2.7}$$
$$(\nabla_X \phi)Y = -g(X-hX,Y)\xi + \eta(Y)(X-hX) \quad \text{for } k\neq -1, \tag{2.8}$$
$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X], \tag{2.9}$$
$$QY = 2(1-n)Y + 2(n-1)hY + [2(n-1)+2nk]\eta(Y)\xi \quad \text{for } k\neq -1, \tag{2.10}$$
$$S(X,\xi) = 2nk\eta(X), \tag{2.11}$$
$$(\nabla_X h)Y = -[(1+k)g(X,\phi Y) + g(X,\phi hY)]\xi + \eta(Y)\phi h(hX-X) \quad \text{for } k\neq -1, \tag{2.12}$$
$$(\nabla_X \eta)Y = g(X,\phi Y) + g(\phi hX, Y), \tag{2.13}$$
$$(\nabla_X h)Y - (\nabla_Y h)X = -(1+k)[2g(X,\phi Y)\xi + \eta(X)\phi Y - \eta(Y)\phi X] + \eta(X)\phi hY - \eta(Y)\phi hX \quad \text{for } k\neq -1, \tag{2.14}$$

where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and for any vector field $X, Y \in \chi(M)$. In fact, in a (k,μ)-paracontact metric manifold there is no restriction for k [6], whereas in a (k,μ)-contact metric manifold, $k \leq 1$ [3]. Also, in the contact case, $k=1$ implies the manifold is a Sasakian manifold, but in the paracontact case, $k=-1$ (equivalently, $h^2=0$ and $h\neq 0$) does not imply the manifold is a para-Sasakian manifold.

An $N(k)$-paracontact metric manifold M is said to be η-Einstein manifold if the Ricci tensor S satisfies the condition

$$S = \alpha g + \beta \eta \otimes \eta, \tag{2.15}$$

where α, β are smooth functions on M. Moreover, if $\beta = 0$, then the manifold is an Einstein manifold. We recall some results.

Lemma 2.1 ([30, Theorem 3.3]). *Let $M^{2n+1}, n>1$, be a paracontact metric manifold which satisfies $R(X,Y)\xi = 0$ for all $X, Y \in \chi(M)$. Then M^{2n+1} is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4.*

Lemma 2.2 ([29, Corollary 3.2]). *On a paracontact metric manifold M^{2n+1} the Ricci curvature in the direction of ξ is given by*

$$S(\xi,\xi) = -2n + |h|^2. \tag{2.16}$$

On a K-paracontact metric manifold M^{2n+1} we have

$$S(\xi,\xi) = -2n. \tag{2.17}$$

3. η-Einstein paracontact metric as a Ricci soliton

The following lemma is very crucial for this section:

Lemma 3.1 ([8]). *If (g,V) is a Ricci soliton of a Riemannian manifold, then we have*

$$\frac{1}{2}\|\pounds_V g\|^2 = dr(V) + 2\,\mathrm{div}(\lambda V - QV), \tag{3.1}$$

where r denotes the scalar curvature of g and Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$.

The above Lemma 3.1 also holds for a pseudo-Riemannian manifold. Suppose M^{2n+1} is an η-Einstein paracontact metric manifold which admits a Ricci soliton (g,ξ). Substituting (2.15) in (1.1), we get

$$(\pounds_V g)(Y,Z) = -2(\alpha-\lambda)g(Y,Z) - 2\beta\eta(Y)\eta(Z) \tag{3.2}$$

for any vector fields Y, Z. Taking covariant differentiation of (3.2) along an arbitrary vector field X and making use of (2.13), we obtain

$$\begin{aligned}(\nabla_X \pounds_V g)(Y,Z) = &-2X(\alpha)g(Y,Z) - 2X(\beta)\eta(Y)\eta(Z)\\ &-2\beta g(X,\phi Y)\eta(Z) - 2\beta g(\phi hX,Y)\eta(Z)\\ &-2\beta g(X,\phi Z)\eta(Y) - 2\beta g(\phi hX,Z)\eta(Y).\end{aligned} \tag{3.3}$$

The following formula follows from Yano [27]:

$$(\pounds_V\nabla_X g - \nabla_X\pounds_V g - \nabla_{[V,X]}g)(Y,Z) = -g((\pounds_V\nabla)(X,Y),Z) - g((\pounds_V\nabla)(X,Z),Y)$$

for all X,Y,Z on M^{2n+1}. Since $\nabla g = 0$, then it follows from the above equation that

$$(\nabla_X\pounds_V g)(Y,Z) = g((\pounds_V\nabla)(X,Y),Z) + g((\pounds_V\nabla)(X,Z),Y) \tag{3.4}$$

for all X,Y,Z on M^{2n+1}. As $\pounds_V\nabla$ is a symmetric $(1,2)$-tensor, that is, $(\pounds_V\nabla)(X,Y) = (\pounds_V\nabla)(Y,X)$, then we have from (3.4) that

$$\begin{aligned}g((\pounds_V\nabla)(X,Y),Z) = &\frac{1}{2}(\nabla_X\pounds_V g)(Y,Z) + \frac{1}{2}(\nabla_Y\pounds_V g)(Z,X)\\ &-\frac{1}{2}(\nabla_Z\pounds_V g)(X,Y).\end{aligned} \tag{3.5}$$

Using (3.3) in (3.5), we get

$$\begin{aligned}g((\pounds_V\nabla)(X,Y),Z) = &-X(\alpha)g(Y,Z) - X(\beta)\eta(Y)\eta(Z) - Y(\alpha)g(X,Z)\\ &-Y(\beta)\eta(X)\eta(Z) + Z(\alpha)g(X,Y) + Z(\beta)\eta(X)\eta(Y)\\ &-\beta\{g(X,\phi Y)\eta(Z) + g(\phi hX,Y)\eta(Z)\\ &+g(X,\phi Z)\eta(Y) + g(\phi hX,Z)\eta(Y)\\ &+g(Y,\phi Z)\eta(X) + g(\phi hY,Z)\eta(X)\\ &+g(Y,\phi X)\eta(Z) + g(\phi hY,X)\eta(Z)\\ &-g(Z,\phi X)\eta(Y) - g(\phi hZ,X)\eta(Y)\\ &-g(Z,\phi Y)\eta(X) - g(\phi hZ,Y)\eta(X)\}.\end{aligned} \tag{3.6}$$

Now putting $X = Y = e_i$ in (3.6), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over $i = 1, 2, \dots, 2n+1$, we get

$$\sum_{i=1}^{2n+1} g((\pounds_V \nabla)(e_i, e_i), Z) = (2n-1)Z(\alpha) + Z(\beta) - 2\xi(\beta)\eta(Z). \tag{3.7}$$

Also taking the covariant differentiation of (1.1) along an arbitrary vector field X, we have

$$(\nabla_X \pounds_V g)(Y, Z) = -2(\nabla_X S)(Y, Z) \tag{3.8}$$

for any vector fields X, Y, Z. Substituting (3.8) in (3.5) yields

$$g((\pounds_V \nabla)(X, Y), Z) = -(\nabla_X S)(Y, Z) - (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y). \tag{3.9}$$

By setting $X = Y = e_i$ in (3.9), we have

$$\sum_{i=1}^{2n+1} g((\pounds_V \nabla)(e_i, e_i), Z) = 0. \tag{3.10}$$

The combining of equations (3.7) and (3.10) yields

$$(2n-1)Z(\alpha) + Z(\beta) - 2\xi(\beta)\eta(Z) = 0. \tag{3.11}$$

Let us consider that α is constant. Then equation (3.11) gives us

$$Z(\beta) - 2\xi(\beta)\eta(Z) = 0. \tag{3.12}$$

Putting $Z = \xi$ in the above equation, we get $\xi(\beta) = 0$. Using this relation and (3.12), one gets $Z(\beta) = 0$ from which it follows that $\beta = \text{const}$.

On the other hand, if we consider that β is constant, then from (3.11) we obtain $Z(\alpha) = 0$, that is, α is constant. Thus, we see that if one of the associated scalars α or β is constant, then the other is constant.

From (1.1), we have

$$g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) + 2S(X, Y) - 2\lambda g(X, Y) = 0. \tag{3.13}$$

Applying (2.4) in (3.13) gives

$$S(X, Y) + g(\phi h X, Y) - \lambda g(X, Y) = 0. \tag{3.14}$$

Substitute $X = \xi$ in the above equation. Since $h\xi = 0$, we have $Q\xi = \lambda\xi$. Also contracting (2.15) we get $r = \alpha(2n+1) + \beta$, which is a constant. Now we suppose that $V = \xi$ in (3.1). Then we obtain that ξ is a Killing vector field, and hence M^{2n+1} is a K-paracontact metric manifold. Moreover, from (1.1) we have that the manifold M^{2n+1} is an Einstein manifold. This leads to the following:

Theorem 3.2. *An η-Einstein paracontact manifold M admits a Ricci soliton (g, ξ) if and only if M is a K-paracontact Einstein manifold provided one of the associated scalars α or β is constant.*

Putting $X = Y = \xi$ in (3.14), we obtain

$$S(\xi, \xi) = \lambda. \tag{3.15}$$

Making use of (2.17) and (3.15), we get $\lambda = -2n$, a negative number. Therefore we can state the following:

Corollary 3.3. *If an η-Einstein paracontact manifold M admits a Ricci soliton (g, ξ), then the soliton is expanding.*

4. Non-existence of Ricci soliton in $N(k)$-paracontact metric manifolds

We consider a Ricci soliton whose potential vector field is the Reeb vector field. Then, from (1.1), we have

$$\frac{1}{2}\pounds_\xi g + S - \lambda g = 0. \tag{4.1}$$

Suppose an $N(k)$-paracontact metric manifold admits a Ricci soliton (g, ξ). Then (4.1) reduces to

$$\frac{1}{2}(g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)) + S(X, Y) - \lambda g(X, Y) = 0. \tag{4.2}$$

Making use of (2.4) and the above equation yields

$$S(X, Y) + g(\phi h X, Y) - \lambda g(X, Y) = 0. \tag{4.3}$$

Substituting $Y = \xi$ and using (2.11), we have

$$\lambda = 2nk. \tag{4.4}$$

Thus, (4.3) can be written as

$$S(X, Y) = 2nkg(X, Y) - g(\phi h X, Y). \tag{4.5}$$

Putting $X = \phi X$ in (4.5) gives

$$S(\phi X, Y) = 2nkg(\phi X, Y) + g(hX, Y). \tag{4.6}$$

Also, in an $N(k)$-paracontact metric manifold the following relation holds:

$$\begin{aligned} S(X, Y) = {} & 2(1-n)g(X, Y) + 2(n-1)g(X, hY) \\ & + [2(n-1) + 2nk]\eta(X)\eta(Y). \end{aligned} \tag{4.7}$$

Replacing X by ϕX in (4.7) implies

$$S(\phi X, Y) = 2(1-n)g(\phi X, Y) + 2(n-1)g(h\phi X, Y). \tag{4.8}$$

Comparing the right-hand sides of (4.6) and (4.8), we get

$$2(1-n-nk)g(\phi X,Y)+2(n-1)g(h\phi X,Y)=g(hX,Y). \tag{4.9}$$

Interchanging X and Y in (4.9) yields

$$2(1-n-nk)g(\phi Y,X)+2(n-1)g(h\phi Y,X)=g(hY,X). \tag{4.10}$$

By adding (4.9) and (4.10), one can easily get

$$2(n-1)g(h\phi X,Y)=g(hX,Y). \tag{4.11}$$

Once again substituting $X=\phi X$ in the above equation, we get

$$g(h\phi X,Y)=2(n-1)g(hX,Y). \tag{4.12}$$

Making use of (4.11) in (4.12) implies

$$\{4(n-1)^2-1\}g(hX,Y)=0. \tag{4.13}$$

But the equation $4(n-1)^2-1=0$ has no positive integer root. Thus, it follows from (4.13) that $g(hX,Y)=0$, that is, $h=0$. Applying $h=0$ in (2.7) gives $k=-1$, which is a contradiction as we consider $k\neq -1$. By the above discussions we can state the following:

Theorem 4.1. *There does not exist a Ricci soliton in an $N(k)$-paracontact manifold M^{2n+1}, $n>1$, whose potential vector field is the Reeb vector field ξ and $k\neq -1$.*

5. Gradient Ricci soliton in $N(k)$-paracontact metric manifolds

Let (M,g) be a $(2n+1)$-dimensional paracontact metric $N(k)$-manifold and g be a gradient Ricci soliton. Then equation (1.2) becomes

$$\nabla_Y Df=QY-\lambda Y \tag{5.1}$$

for any $Y\in\chi(M)$, where D denotes the gradient operator of g. From (5.1) it follows that

$$R(X,Y)Df=(\nabla_X Q)Y-(\nabla_Y Q)X \tag{5.2}$$

for any $X,Y\in\chi(M)$. Replacing X by ξ in (5.2) yields

$$g(R(\xi,Y)Df,\xi)=g((\nabla_\xi Q)Y-(\nabla_Y Q)\xi,\xi). \tag{5.3}$$

With the help of (2.10) we have

$$(\nabla_Y Q)X=\{2(n-1)+2nk\}[(\nabla_Y\eta)X\xi+\eta(X)\nabla_Y\xi]+2(n-1)(\nabla_Y h)X. \tag{5.4}$$

Applying (2.4) and (2.13) in (5.4) gives

$$(\nabla_Y Q)X=\{2(n-1)+2nk\}[g(Y,\phi X)\xi+g(\phi hY,X)\xi-\eta(X)\phi Y+\eta(X)\phi hY]$$

$$+ 2(n-1)(\nabla_Y h)X. \tag{5.5}$$

Similarly, we obtain

$$(\nabla_X Q)Y = \{2(n-1) + 2nk\}[g(X, \phi Y)\xi + g(\phi hX, Y)\xi - \eta(Y)\phi X + \eta(Y)\phi hX] + 2(n-1)(\nabla_X h)Y. \tag{5.6}$$

Using (2.14), (5.5) and (5.6), we have

$$\begin{aligned}(\nabla_X Q)Y - (\nabla_Y Q)X = 2(n-1)[&-(k+1)\{2g(X, \phi Y)\xi + \eta(X)\phi Y \\ &- \eta(Y)\phi X\} + \eta(X)\phi hY - \eta(Y)\phi hX] \\ &+ \{2(n-1) + 2nk\}[2g(X, \phi Y)\xi + \eta(X)\phi Y \\ &- \eta(Y)\phi X + \eta(Y)\phi hX - \eta(X)\phi hY].\end{aligned} \tag{5.7}$$

Replacing $X = \xi$ in the above equation and then taking the inner product of (5.7) with ξ implies

$$g((\nabla_\xi Q)Y - (\nabla_Y Q)\xi, \xi) = 0. \tag{5.8}$$

In view of (5.3) and (5.8), we obtain

$$g(R(\xi, Y)Df, \xi) = 0. \tag{5.9}$$

Together with (2.9) it gives

$$k\{g(Y, Df) - g(Df, \xi)\eta(Y)\} = 0, \tag{5.10}$$

from which it follows that

$$k(Df - (\xi f)\xi) = 0. \tag{5.11}$$

Hence either $k = 0$, or

$$Df = (\xi f)\xi. \tag{5.12}$$

If $k = 0$, then equation (2.6) gives $R(X, Y)\xi = 0$. Thus, from Lemma 2.1 we can say that M^{2n+1}, $n > 1$ is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4.

Also, from (5.2) we have $R(X, Y)Df = 0$, from which we can say that the potential vector field Df is a nullity vector field.

On the other hand, if (5.12) holds, then we obtain from (5.1)

$$S(X, Y) - \lambda g(X, Y) = Y(\xi f)\eta(X) - (\xi f)g(\phi Y, X) + (\xi f)g(\phi hY, X). \tag{5.13}$$

Putting $X = \xi$ in the above equation and using (2.11) gives us

$$Y(\xi f) = (2nk - \lambda)\eta(Y). \tag{5.14}$$

By applying (5.14) in (5.13), we have

$$S(X, Y) - \lambda g(X, Y) = (2nk - \lambda)\eta(X)\eta(Y) - (\xi f)g(\phi Y, X)$$

$$+ (\xi f)g(\phi hY, X). \tag{5.15}$$

Interchanging X and Y in (5.15) yields

$$\begin{aligned} S(Y,X) - \lambda g(Y,X) = (2nk-\lambda)\eta(Y)\eta(X) - (\xi f)g(\phi X, Y) \\ + (\xi f)g(\phi hX, Y). \end{aligned} \tag{5.16}$$

Adding (5.15) and (5.16) implies

$$S(X,Y) - \lambda g(X,Y) = (2nk-\lambda)\eta(X)\eta(Y) + (\xi f)g(\phi hX, Y). \tag{5.17}$$

Making use of (5.1) and (5.17), we get

$$\nabla_Y Df = (2nk-\lambda)\eta(Y)\xi + (\xi f)\phi hY. \tag{5.18}$$

Using the above equation, we obtain

$$\begin{aligned} R(X,Y)Df = {} & (2nk-\lambda)\{2g(X,\phi Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y\} \\ & + (\xi f)\{-(k+1)(\eta(X)Y - \eta(Y)X) \\ & + \eta(X)hY - \eta(Y)hX\}. \end{aligned} \tag{5.19}$$

Since $g(R(X,Y)(\xi f)\xi, \xi) = 0$, we have from (5.19), $(2nk-\lambda)g(X,\phi Y) = 0$, from which it follows that

$$\lambda = 2nk. \tag{5.20}$$

In view of (5.20) and (5.13), we get

$$Y(\xi f) = 0, \tag{5.21}$$

which implies that $\xi f = c$, where c is a constant. Also, from (5.12) we have $df = (\xi f)\eta$. Its exterior derivative gives $0 = d^2 f = d(\xi f)\eta + (\xi f)d\eta$. Since $\xi f = c$ and $d\eta \neq 0$, we get $c = 0$. Consequently, f is constant. Applying this fact in (5.1) gives us $S(X,Y) = 2nkg(X,Y)$.

By the above discussions we have the following:

Theorem 5.1. *Let (M,g) be a $(2n+1)$-dimensional $(n > 1)$ $N(k)$-paracontact metric manifold. If g is a gradient Ricci soliton, then either the manifold is locally isometric to a product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4, or M^{2n+1} is an Einstein manifold, provided $k \neq -1$.*

6. Example of a 5-dimensional $N(k)$-paracontact metric manifold

In this section, we give an example of a 5-dimensional $N(k)$-paracontact metric manifold such that $k = -4$. Let $\mathfrak{g}$ be the Lie algebra of a Lie group G of basis $\{e_1, e_2, e_3, e_4, e_5\}$ such that

$$[e_1, e_5] = -2e_1 - 2e_2, \qquad [e_2, e_5] = 2e_2, \qquad [e_1, e_2] = 4e_5,$$

$$[e_3, e_4] = 4e_4 + 4e_5, \qquad [e_1, e_4] = 2e_1 + 2e_2, \qquad [e_2, e_4] = -2e_2.$$

We consider the metric such that

$$g(e_1, e_2) = g(e_5, e_5) = 1,$$
$$g(e_3, e_4) = -1 \text{ and } g(e_i, e_j) = 0, \text{ for all other values of } i, j.$$

Set $e_5 = \xi$ and denote by η its dual 1-form. We define a tensor ϕ by $\phi e_1 = e_1$, $\phi e_2 = -e_2$, $\phi e_3 = -e_3$, $\phi e_4 = e_4$, $\phi e_5 = 0$. Therefore, we have $\phi^2 X = X - \eta(X)\xi$ and $g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y)$. Thus (ϕ, ξ, η, g) makes G a paracontact metric manifold.

Using the well-known Koszul's folmula, we have:

$$\begin{array}{llll}
\nabla_{e_1} e_5 = -2e_1 - 2e_2, & \nabla_{e_2} e_5 = 2e_2, & \nabla_{e_3} e_5 = 2e_3, & \nabla_{e_4} e_5 = -2e_4, \\
\nabla_{e_5} e_1 = 0, & \nabla_{e_5} e_2 = 0, & \nabla_{e_5} e_3 = 2e_3, & \nabla_{e_5} e_4 = -2e_4, \\
\nabla_{e_1} e_1 = e_3, & \nabla_{e_1} e_2 = 2e_5, & \nabla_{e_1} e_3 = 0, & \nabla_{e_1} e_4 = 2e_2, \\
\nabla_{e_2} e_1 = -2e_5, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = 0, & \nabla_{e_2} e_4 = 0, \\
\nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = -e_3, & \nabla_{e_3} e_4 = 2e_5 + 4e_4, \\
\nabla_{e_4} e_1 = -2e_1, & \nabla_{e_4} e_2 = 2e_2, & \nabla_{e_4} e_3 = -2e_5, & \nabla_{e_4} e_4 = 0, \\
\nabla_{e_5} e_5 = 0. & & &
\end{array}$$

Comparing the above relations with (2.4), we get

$$he_1 = -e_1 + 2e_2, \; he_2 = -e_2, \; he_3 = -e_3, \; he_4 = -e_4, \; he_5 = 0.$$

Using the formula $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, we can calculate the following:

$$\begin{aligned}
& R(e_1, e_2)e_1 = 4e_1 + 4e_2, R(e_1, e_2)e_2 = -4e_2, R(e_1, e_2)e_3 = -8e_3, \\
& R(e_1, e_2)e_4 = 8e_4, R(e_1, e_3)e_1 = e_3, R(e_1, e_3)e_2 = -4e_3, \\
& R(e_1, e_3)e_3 = 0, R(e_1, e_3)e_4 = 4e_2 - 4e_1, R(e_1, e_4)e_1 = -4e_3 + 6e_5, \\
& R(e_1, e_4)e_2 = 4e_4, R(e_1, e_4)e_3 = 4e_1 + 4e_2, R(e_1, e_4)e_4 = -8e_2, \\
& R(e_1, e_5)e_1 = -4e_5, R(e_1, e_5)e_2 = 4e_5, R(e_1, e_5)e_5 = -4e_1, \\
& R(e_2, e_3)e_1 = 4e_3, R(e_2, e_3)e_2 = 0, R(e_2, e_3)e_3 = 0, \\
& R(e_2, e_3)e_4 = 4e_2, R(e_2, e_4)e_1 = -4e_4, R(e_2, e_4)e_2 = 0, \\
& R(e_2, e_4)e_3 = -4e_2, R(e_2, e_4)e_4 = 0, R(e_2, e_5)e_1 = 4e_5, \\
& R(e_2, e_5)e_4 = 0, R(e_2, e_5)e_5 = -4e_2, R(e_3, e_4)e_1 = 8e_1, \\
& R(e_3, e_4)e_2 = -8e_2, R(e_3, e_4)e_3 = -12e_3 + 6e_5, R(e_3, e_4)e_4 = 12e_4, \\
& R(e_3, e_5)e_3 = 2e_3, R(e_3, e_5)e_4 = -4e_5, R(e_3, e_5)e_5 = -4e_3, \\
& R(e_4, e_5)e_2 = 0, R(e_4, e_5)e_3 = -4e_5, R(e_4, e_5)e_5 = -4e_4.
\end{aligned}$$

With the help of the expressions of the curvature tensor we conclude that the manifold is an $N(k)$-paracontact metric manifold with $k = -4$. Also, from the above expressions we obtain

$$S(e_1,e_1)=S(e_1,e_2)=S(e_2,e_2)=-4,$$
$$S(e_3,e_4)=8,\quad S(e_3,e_3)=-12,\quad S(e_4,e_4)=-20.$$

For $X=e_1$, $Y=e_2$, by using the above results, we have from (4.2) that $\lambda=-4$. Substituting the value of λ in (4.2), we see that the relation (4.2) is not true for all values of X and Y. Thus Theorem 4.1 is verified.

Now, if we take the non-zero Lie brackets as

$$\begin{aligned}
&[e_1,e_5]=-(k_0+1)e_1-(k_0+1)e_2, && [e_2,e_5]=(k_0+1)e_2,\\
&[e_1,e_2]=2(k_0+1)e_5, && [e_3,e_4]=2(k_0+1)e_4+2(k_0+1)e_5,\\
&[e_1,e_4]=(k_0+1)e_1+(k_0+1)e_2, && [e_2,e_4]=-(k_0+1)e_2,
\end{aligned}$$

where k_0 is a real number such that $k_0\neq-1$, then it can be easily shown that the manifold under consideration is an $N(k)$-paracontact metric manifold with $k=-(k_0+1)^2$, $k_0\neq-1$.

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Солітони Річчі та градієнтні солітони Річчі на $N(k)$-параконтактних многовидах

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η-ейнштейнівський параконтактний многовид M допускає солітон Річчі (g, ξ) тоді і тільки тоді, коли M є K-параконтактним ейнштейнівським многовидом за умови, що одна з асоційованих скалярних величин α або β є постійною. Ми також доводимо неможливість існування солітона Річчі на $N(k)$-параконтактному метричному многовиді M, потенціальне векторне поле якого є рібовським векторним полем ξ. Більш того, якщо метрика g $N(k)$-параконтактного метричного многовиду M^{2n+1} є градієнтним солітоном Річчі, то або многовид локально ізометричний добутку плоского $(n+1)$-вимірного многовида і n-вимірного многовида з постійною негативною кривиною -4, або M^{2n+1} є ейнштейнівським многовидом. На додаток наведено ілюстративний приклад.

Ключові слова: параконтактний многовид, $N(k)$-параконтактний многовид, солітон Річчі, градієнтний солітон Річчі, ейнштейнівський многовид.