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Ricci Solitons of Four-Dimensional Lorentzian Damek–Ricci Spaces

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In this paper, we show that the four-dimensional Lorentzian Damek–Ricci spaces are not a Ricci soliton. This is a generalization of the result of Tan and Deng (see [11]) who proved that these spaces are not a Ricci soliton only with respect to the left-invariant vector fields.

Key words: Damek–Ricci spaces, Ricci soliton, left-invariant metrics, Lorentzian metrics

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1. Introduction

A pseudo-Riemannian manifold (M, g) is called a Ricci soliton if there exists a smooth vector field X such that

$$L_X g + \varrho = \lambda g, \tag{1.1}$$

where L_X denotes the Lie derivative in the direction of X, ϱ is the Ricci tensor and λ is a real number. A Ricci soliton is said to be *shrinking*, *steady* or *expanding* according to whether $\lambda > 0, \lambda = 0$ or $\lambda < 0$, respectively.

Ricci solitons are natural generalizations of Einstein metrics, they also are self-similar solutions of Hamilton's Ricci flow [9], and they are important since they were applied by Grigori Perelman for solving the long standing Poincaré conjecture. In recent years, there have been seen much interest and increased activities in studying Ricci solitons, which are called quasi-Einstein metrics in physics literature (see [8]). For more details and further results we may refer to [4] and references therein.

Damek–Ricci spaces are semidirect products of Heisenberg groups with the real line. They were constructed by Damek and Ricci in [6], where the authors provided the examples of harmonic manifolds that were not symmetric and proved that the conjecture posed by Lichnerowicz fails in the non-compact case.

The geometry of these spaces has been studied by many authors. In [7], Degla and Todjihounde proved the nonexistence of a proper (nongeodesic) biharmonic curve in a four-dimensional Damek–Ricci space although such curves exist in three-dimensional Heisenberg groups. In [1], they studied the dispersive properties of the linear wave equation on Damek–Ricci spaces and their application to nonlinear Cauchy problems. In [3], it was constructed unaccountably many isoparametric families of hypersurfaces in Damek–Ricci spaces by characterizing those of them that have the constant principal curvatures. In [10], Koivogui and Todjihounde gave a setting for constructing the Weierstrass representation formulas for simply connected minimal surfaces in four-dimensional Riemannian Damek–Ricci spaces. This was extended to the case of spacelike and timelike minimal surfaces in the four-dimensional Damek–Ricci spaces equipped with the left-invariant Lorentzian metric [5].

In [11], Tan and Deng considered the four-dimensional Lorentzian Damek–Ricci spaces and studied some of their geometrical properties, including some problems related to Ricci solitons, harmonicity of invariant vector fields and curvature properties. In particular, they proved that these spaces did not even admit a left-invariant Ricci soliton (that is, the left-invaraint vector field satisfying equation (1.1)). In this paper, we consider the left-invariant Lorentzian metrics admitted by the four-dimensional Damek–Ricci spaces and prove the existence of the vector field for which the soliton equation (1.1) holds.

The paper is organized in the following way. In Section 2, we give some basic information about four-dimensional Damek–Ricci spaces and their left-invariant metrics in global coordinates, we also describe their Levi-Civita connection and Ricci tensor. In Section 3, Ricci solitons of four-dimensional Damek–Ricci spaces are characterized via a system of partial differential equations. In particular, we prove that some of Lorentzian Damek–Ricci spaces admit different vector fields resulting in expanding and shrinking Ricci solitons.

2. Geometry of four-dimensional Damek–Ricci spaces

We start with a short description of four-dimensional Damek–Ricci spaces by referring to [2] and [6] for more details and further results. First, we are to recall the so-called generalized Heisenberg group since Damek–Ricci spaces depend on it.

2.1. Generalized Heisenberg group. The generalized Heisenberg algebras are defined as follows. Let b and z be the finite-dimensional real vector spaces such that $\mathfrak{n}$ is the orthogonal sum $\mathfrak{n} = b \oplus z$. We define in $\mathfrak{n}$ the bracket

$$[U + X, V + Y] = \beta(U, V), \tag{2.1}$$

where $\beta : b \times b \to z$ is a skew-symmetric bilinear map. This product defines a Lie algebra structure on $\mathfrak{n}$.

We equip b with a positive inner product and z with a positive or Lorentzian inner product denoting the product metric by $\langle\cdot,\cdot\rangle_{\mathfrak{n}}$. Define a linear map $J : Z \in z \to J_Z \in \mathrm{End}(b)$ by

$$\langle J_Z U, V\rangle_{\mathfrak{n}} = \langle\beta(U, V), Z\rangle_{\mathfrak{n}} \quad \text{for all } U, V \in b \text{ and } Z \in z. \tag{2.2}$$

Then $\mathfrak{n}$ is a two-step nilpotent Lie algebra with center z.

If the inner product in z is positive and $J_Z^2 = -\langle Z, Z\rangle_{\mathfrak{n}}\, id_b$ for all $Z \in z$, then the Lie algebra $\mathfrak{n}$ is called a generalized Riemannian Heisenberg algebra, and the associated simply connected nilpotent Lie group, endowed with the induced left-invariant Riemannian metric, is called a generalized Riemannian Heisenberg group.

If the inner product in z is Lorentzian and

$$J_Z^2 = \begin{cases} -\langle Z, Z\rangle_{\mathfrak{n}}\, id_b, & \text{when } Z \text{ is spacelike,} \\ \langle Z, Z\rangle_{\mathfrak{n}}\, id_b, & \text{when } Z \text{ is timelike,} \end{cases} \tag{2.3}$$

then the Lie algebra $\mathfrak{n}$ is called a generalized Lorentzian Heisenberg algebra, and the associated simply connected nilpotent Lie group, endowed with the induced left-invariant Lorentzian metric, is called a generalized Lorentzian Heisenberg group.

2.2. Damek–Ricci spaces. Now, let $\varepsilon = \pm 1$ and $\mathfrak{a}_{-\varepsilon}$ be a one-dimensional pseudo-Riemannian real vector space, which is Lorentzian for $\varepsilon = 1$ and Riemannian for $\varepsilon = -1$, and let $\mathfrak{n}_\varepsilon = b \oplus z$ be a generalized Heisenberg algebra which is Lorentzian if $\varepsilon = -1$ and Riemannian if $\varepsilon = 1$.

Consider a new vector space $\mathfrak{a}_{-\varepsilon} \oplus \mathfrak{n}_\varepsilon$ as the vector space direct sum of $\mathfrak{a}_{-\varepsilon}$ and $\mathfrak{n}_\varepsilon$. Let $s, r \in \mathbb{R}$, $U, V \in b$ and $X, Y \in z$. We define the Lorentzian product $\langle \cdot, \cdot \rangle$ and a Lie bracket $[\cdot, \cdot]$ on $\mathfrak{a}_{-\varepsilon} \oplus \mathfrak{n}_\varepsilon$ by

$$\begin{aligned} \langle rA + U + X, sA + V + Y\rangle &= \langle U + X, V + Y\rangle_{\mathfrak{n}_\varepsilon} - \varepsilon rs, \\ [rA + U + X, sA + V + Y] &= [U, V]_{\mathfrak{n}_\varepsilon} + \frac{1}{2} rV - \frac{1}{2} sU + rY - sX \end{aligned} \tag{2.4}$$

for a non zero vector A in $\mathfrak{a}_{-\varepsilon}$. Therefore $\mathfrak{a}_{-\varepsilon} \oplus \mathfrak{n}_\varepsilon$ becomes a solvable Lie algebra. The corresponding simply connected Lie group, equipped with the induced left-invariant Lorentzian metric, is called a Lorentzian Damek–Ricci space and will be denoted by $\mathbb{S}_\varepsilon$.

2.3. Four-dimensional Lorentzian Damek–Ricci spaces. Consider the four-dimensional Lorentzian Damek–Ricci spaces $(\mathbb{S}_\varepsilon^4, g_\varepsilon)$ equipped with the left-invariant Lorentzian metric g_ε. Note that Damek–Ricci spaces of dimension four are diffeomorphic to $\mathbb{R}^4$ because they are simply connected solvable Lie groups. So we can use global coordinates (x, y, z, t), and throughout the paper we will denote the coordinate basis $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right\}$ by $\{\partial_x, \partial_y, \partial_z, \partial_t\}$.

As it was pointed in [5], the left-invariant Lorentzian metric g_ε on the four-dimensional space $\mathbb{S}_\varepsilon^4$ is given by

$$g_\varepsilon = e^{-t} \mathrm{d}x^2 + e^{-t} \mathrm{d}y^2 + \varepsilon e^{-2t} \left(\mathrm{d}z + \frac{c}{2} y \mathrm{d}x - \frac{c}{2} x \mathrm{d}y\right)^2 \tag{2.5}$$

where $c \in \mathbb{R}$.

Following [5], let us denote

$$e_1 = e^{\frac{t}{2}}\left(\frac{\partial}{\partial x} - \frac{cy}{2}\frac{\partial}{\partial z}\right),\ e_2 = e^{\frac{t}{2}}\left(\frac{\partial}{\partial y} + \frac{cx}{2}\frac{\partial}{\partial z}\right),\ e_3 = e^t\frac{\partial}{\partial z},\ e_4 = \frac{\partial}{\partial t}. \tag{2.6}$$

Then $\{e_1, e_2, e_3, e_4\}$ form an orthonormal basis of the Lie algebra $\mathfrak{s}^4$ of $\mathbb{S}^4_\varepsilon$ for which

$$g_\varepsilon(e_1, e_1) = g_\varepsilon(e_2, e_2) = 1, \quad g_\varepsilon(e_3, e_3) = -g_\varepsilon(e_4, e_4) = \varepsilon. \tag{2.7}$$

The bracket operation in $\mathfrak{s}^4$ is given by the formulas:

$$\begin{aligned} &[e_1, e_2] = ce_3, && [e_1, e_3] = 0, && [e_1, e_4] = -\frac{1}{2}e_1, \\ &[e_2, e_3] = 0, && [e_2, e_4] = -\frac{1}{2}e_2, && [e_3, e_4] = -e_3. \end{aligned} \tag{2.8}$$

As stated in [11], it is easy to see that for these metrics $J^2_{e_3} = -c^2 id_b$, and from the definition of the map J_Z it follows that $J^2_{e_3} = -id_b$, so $c^2 = 1$.

Let ϱ denote the Ricci tensor of $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$. The non-zero components $\varrho_{ij} = \varrho(e_i, e_j)$ of the Ricci tensor are [11]:

$$\varrho_{11} = \varrho_{22} = \frac{\varepsilon}{2}, \quad \varrho_{33} = \frac{5}{2}, \quad \varrho_{44} = -\frac{3}{2}. \tag{2.9}$$

3. Ricci solitons of four-dimensional Lorentzian Damek–Ricci spaces

In this section, we analyze the existence of Ricci solitons on four-dimensional Lorentzian Damek–Ricci spaces $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$.

Let $X = f_1e_1 + f_2e_2 + f_3e_3 + f_4e_4$ be a vector field on $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$, where $f_1, .., f_4$ are smooth functions of the variables x, y, z, t. Then the Lie derivative of the metric (2.5) with respect to X is given by :

$$\begin{aligned}
(L_X g_\varepsilon)(e_1, e_1) &= 2e_1(f_1) - f_4, \\
(L_X g_\varepsilon)(e_1, e_2) &= e_1(f_2) + e_2(f_1), \\
(L_X g_\varepsilon)(e_1, e_3) &= \varepsilon(cf_2 + e_1(f_3)) + e_3(f_1), \\
(L_X g_\varepsilon)(e_1, e_4) &= \frac{1}{2}f_1 - \varepsilon e_1(f_4) + e_4(f_1), \\
(L_X g_\varepsilon)(e_2, e_2) &= -f_4 + 2e_2(f_2), \\
(L_X g_\varepsilon)(e_2, e_3) &= \varepsilon(e_2(f_3) - cf_1) + e_3(f_2), \\
(L_X g_\varepsilon)(e_2, e_4) &= \frac{1}{2}f_2 - \varepsilon e_2(f_4) + e_4(f_2), \\
(L_X g_\varepsilon)(e_3, e_3) &= 2\varepsilon(e_3(f_3) - f_4), \\
(L_X g_\varepsilon)(e_3, e_4) &= \varepsilon(f_3 + e_4(f_3) - e_3(f_4)), \\
(L_X g_\varepsilon)(e_4, e_4) &= -2\varepsilon e_4(f_4).
\end{aligned} \tag{3.1}$$

Thus, by using (2.5) and (3.1) in (1.1), a standard calculation gives that the four-dimensional Lorentzian Damek–Ricci space $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$ is a Ricci soliton if and only if the following system holds:

$$\begin{aligned}
&2e^{\frac{t}{2}}(\partial_x f_1 - \frac{cy}{2}\partial_z f_1) - f_4 = \lambda - \varrho_{11},\\
&\partial_x f_2 - \frac{cy}{2}\partial_z f_2 + \partial_y f_1 + \frac{cx}{2}\partial_z f_1 = 0,\\
&e^t\partial_z f_1 + \varepsilon\left[cf_2 + e^{\frac{t}{2}}(\partial_x f_3 - \frac{cy}{2}\partial_z f_3)\right] = 0,\\
&\partial_t f_1 - \varepsilon e^{\frac{t}{2}}(\partial_x f_4 - \frac{cy}{2}\partial_z f_4) + \frac{f_1}{2} = 0,\\
&2e^{\frac{t}{2}}(\partial_y f_2 + \frac{cx}{2}\partial_z f_2) - f_4 = \lambda - \varrho_{22},\\
&\varepsilon e^{\frac{t}{2}}(\partial_y f_3 + \frac{cx}{2}\partial_z f_3) + e^t\partial_z f_2 - \varepsilon c f_1 = 0,\\
&\partial_t f_2 - \varepsilon e^{\frac{t}{2}}(\partial_y f_4 + \frac{cx}{2}\partial_z f_4) + \frac{f_2}{2} = 0,\\
&2\varepsilon(e^t\partial_z f_3 - f_4) = \varepsilon\lambda - \varrho_{33},\\
&\partial_t f_3 - e^t\partial_z f_4 + f_3 = 0,\\
&2\varepsilon\partial_t f_4 = \varepsilon\lambda + \varrho_{44}.
\end{aligned} \tag{3.2}$$

(3.2) yields

$$f_4 = \left(\frac{\lambda + \varepsilon\varrho_{44}}{2}\right)t + A, \tag{3.3}$$

where $A = A(x, y, z)$ is a real-valued smooth function on $\mathbb{S}^4_\varepsilon$.

Taking the derivative of the fourth equation of (3.2) with respect to t and using (3.3), we obtain

$$\partial_t f_1 + 2\partial_t^2 f_1 + \varepsilon e^{\frac{t}{2}}\left(\frac{cy}{2}\partial_z A - \partial_x A\right) = 0, \tag{3.4}$$

which, together with the fourth equation of (3.2), gives

$$f_1 = He^{\frac{t}{2}} + \overline{H}e^{-\frac{t}{2}} \tag{3.5}$$

for some smooth functions $H = H(x, y, z)$ and $\overline{H} = \overline{H}(x, y, z)$.

Next, taking the derivative of the seventh equation of (3.2) with respect to t and using (3.3), we have

$$\partial_t f_2 + 2\partial_t^2 f_2 - \varepsilon e^{\frac{t}{2}}\left(\frac{cx}{2}\partial_z A + \partial_y A\right) = 0. \tag{3.6}$$

Thus, from the seventh equation of (3.2) we deduce that

$$f_2 = Ge^{\frac{t}{2}} + \overline{G}e^{-\frac{t}{2}}, \tag{3.7}$$

where $G = G(x, y, z)$ and $\overline{G} = \overline{G}(x, y, z)$ are smooth functions.

Again, we derive the ninth equation of (3.2) with respect to t and use (3.3), to get

$$\partial_t f_3 + \partial_t^2 f_3 - e^t \partial_z A = 0. \tag{3.8}$$

Hence, by the ninth equation of (3.2), we prove that

$$f_3 = Ke^t + \overline{K}e^{-t}. \tag{3.9}$$

where $K = K(x, y, z)$ and $\overline{K} = \overline{K}(x, y, z)$ are smooth functions.

Now, replacing (3.5) and (3.3) in the fourth equation of (3.2), we obtain

$$H = \varepsilon \left(\partial_x A - \frac{cy}{2} \partial_z A \right). \tag{3.10}$$

Then, replacing (3.7) and (3.3) in the seventh equation of (3.2), we have

$$G = \varepsilon \left(\partial_y A + \frac{cx}{2} \partial_z A \right). \tag{3.11}$$

Again, replacing (3.9) and (3.3) in the ninth equation of (3.2), we get

$$K = \frac{1}{2} \partial_z A. \tag{3.12}$$

Next, from the eighth equation of (3.2), (3.3), (3.9), and (3.12) we prove that

$$\left(\frac{\lambda + \varepsilon \varrho_{44}}{2} \right) t + A - (\partial_z K)\, e^{2t} - \partial_z \overline{K} = \frac{\varepsilon \varrho_{33} - \lambda}{2}, \tag{3.13}$$

which must hold for any value of t. Therefore, using (2.9), we have

$$\begin{cases} \lambda = \dfrac{3\varepsilon}{2}, \\ K = K(x, y), \\ \partial_z \overline{K} = A - \dfrac{\varepsilon}{2}. \end{cases} \tag{3.14}$$

By (3.3) and (3.7), the fifth equation in (3.2) implies, since $\lambda = \frac{3\varepsilon}{2}$,

$$(2\partial_y G + cx\partial_z G)\, e^t + 2\partial_y \overline{G} + cx\partial_z \overline{G} - A - \varepsilon = 0, \tag{3.15}$$

which holds for any value of t and thus is equivalent to

$$\begin{cases} 2\partial_y G + cx\partial_z G = 0, \\ 2\partial_y \overline{G} + cx\partial_z \overline{G} - A = \varepsilon. \end{cases} \tag{3.16}$$

Using the first equation in (3.2), (3.3), and (3.5), we obtain

$$(2\partial_x H - cy\partial_z H)\, e^t + 2\partial_x \overline{H} - cy\partial_z \overline{H} - A - \varepsilon = 0. \tag{3.17}$$

Therefore,

$$\begin{cases} 2\partial_x H - cy\partial_z H = 0, \\ 2\partial_x \overline{H} - cy\partial_z \overline{H} - A = \varepsilon. \end{cases} \tag{3.18}$$

By (3.5) and (3.7), the second equation in (3.2) yields

$$\begin{cases} \partial_x G + \partial_y H - \dfrac{cy}{2}\partial_z G + \dfrac{cx}{2}\partial_z H = 0, \\ \partial_x \overline{G} + \partial_y \overline{H} - \dfrac{cy}{2}\partial_z \overline{G} + \dfrac{cx}{2}\partial_z \overline{H} = 0. \end{cases} \tag{3.19}$$

From (3.5), (3.7), and (3.9), it follows that the third and the sixth equations of (3.2) are equivalent to (since $K = K(x, y)$)

$$\begin{cases} \varepsilon c G + \partial_z \overline{H} = 0, \\ c\overline{G} + \partial_x \overline{K} - \dfrac{cy}{2}\partial_z \overline{K} = 0, \\ \varepsilon \partial_x K + \partial_z H = 0 \end{cases} \tag{3.20}$$

and

$$\begin{cases} c\overline{H} - \partial_y \overline{K} - \dfrac{cx}{2}\partial_z \overline{K} = 0, \\ \partial_z G + \varepsilon \partial_y K = 0, \\ \varepsilon c H - \partial_z \overline{G} = 0, \end{cases} \tag{3.21}$$

respectively. We then replace (3.10) and (3.11) into the first equations of (3.16), (3.18) and (3.19), respectively, obtaining

$$\begin{cases} \partial_y^2 A + cx\partial_y \partial_z A = 0, \\ \partial_x^2 A - cy\partial_x \partial_z A = 0, \\ 2\partial_x \partial_y A + cx\partial_x \partial_z A - cy\partial_y \partial_z A = 0. \end{cases} \tag{3.22}$$

Next, replacing (3.11) and (3.12) into the second equation of (3.21), we deduce that

$$\partial_y \partial_z A = 0. \tag{3.23}$$

Therefore, using (3.22) and (3.23), we prove that A can be written as follows:

$$A = \overline{A}z + \tilde{A}y + \hat{A}, \tag{3.24}$$

where $\overline{A} = \overline{A}(x), \tilde{A} = \tilde{A}(x)$ and $\hat{A} = \hat{A}(x)$ are smooth functions such that

$$2\tilde{A}' + cx\overline{A}' = 0, \tag{3.25}$$

and

$$\overline{A}''z + \left(\tilde{A}'' - c\overline{A}'\right) y + \hat{A}'' = 0, \tag{3.26}$$

which must hold for any value of y and z. Consequently, A reduces to

$$A = ax + \delta y + \beta z + b \tag{3.27}$$

for some real constants $a, b, \delta, \beta \in \mathbb{R}$, and so (3.10), (3.11), and (3.12) become

$$\begin{cases} H = \varepsilon \left(a - \dfrac{\beta cy}{2}\right), \\ G = \varepsilon \left(\delta + \dfrac{\beta cx}{2}\right), \\ K = \dfrac{\beta}{2}. \end{cases} \tag{3.28}$$

From the last equation of (3.21), the first equation of (3.20), by using (3.14) and (3.28), we get

$$\begin{cases} \overline{G} = c\left(a - \dfrac{\beta c}{2}y\right)z + \widetilde{G}, \\ \overline{H} = -c\left(\delta + \dfrac{\beta c}{2}x\right)z + \widetilde{H}, \\ \overline{K} = \left(ax + \delta y + b - \dfrac{\varepsilon}{2}\right)z + \dfrac{\beta}{2}z^2 + \widetilde{K} \end{cases} \tag{3.29}$$

for some smooth functions $\widetilde{G} = \widetilde{G}(x,y), \widetilde{H} = \widetilde{H}(x,y)$ and $\widetilde{K} = \widetilde{K}(x,y)$. Since $c^2 = 1$, then the second equation of (3.16) is equivalent to

$$\begin{cases} 2\partial_y \widetilde{G} - \delta y = \varepsilon + b, \\ \beta = 0. \end{cases} \tag{3.30}$$

Hence, (3.29) reduces to

$$\begin{cases} \overline{G} = caz + \widetilde{G}, \\ \overline{H} = -c\delta z + \widetilde{H}, \\ \overline{K} = \left(ax + \delta y + b - \dfrac{\varepsilon}{2}\right)z + \widetilde{K}. \end{cases} \tag{3.31}$$

Thus, the second equation of (3.20) is then equivalent to

$$a\left(c^2+1\right)z + \partial_x \widetilde{K} + c\widetilde{G} - \frac{c}{2}\left(ax + \delta y + b - \frac{\varepsilon}{2}\right)y = 0, \tag{3.32}$$

which must hold for any z, so $a(c^2+1) = 0$, that is $a = 0$. We have then

$$\partial_x \widetilde{K} + c\widetilde{G} - \frac{c}{2}\left(\delta y + b - \frac{\varepsilon}{2}\right)y = 0, \tag{3.33}$$

which, by derivation with respect to y, gives (since (3.30))

$$\partial_y \partial_x \widetilde{K} = \frac{c}{2}\left(\delta y - \frac{3\varepsilon}{2}\right). \tag{3.34}$$

Next, the second equation of (3.18) becomes

$$2\partial_x \widetilde{H} = \varepsilon + b. \tag{3.35}$$

By the first equation of (3.21) and using the second equation of (3.19), we prove that

$$\begin{cases} \partial_y \widetilde{K} - c\widetilde{H} + \dfrac{c}{2}\left(b - \dfrac{\varepsilon}{2}\right)x = 0, \\ \delta = 0, \\ \partial_x \widetilde{G} + \partial_y \widetilde{H} = 0. \end{cases} \tag{3.36}$$

Now, deriving the first equation of (3.36) with respect to x and using (3.35), we get

$$\partial_x \partial_y \widetilde{K} = \frac{3\varepsilon c}{4}. \tag{3.37}$$

Thus, $\partial_x\partial_y\widetilde{K} \neq \partial_y\partial_x\widetilde{K}$ for $c \neq 0$, which is a contradiction.

Summarizing, we have proved that the four-dimensional Lorentzian Damek–Ricci spaces $(\mathbb{S}^4_\varepsilon, g_\varepsilon)$ are not a Ricci soliton. This is a generalization of the result of Tan and Deng (see [11]) who proved that these spaces are not a Ricci soliton only with respect to the left-invariant vector fields. We have the following result:

Theorem 3.1. *All four-dimensional Lorentzian Damek–Ricci spaces are not a Ricci soliton.*

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Солітони Річчі у чотиривимірних лоренцевих просторах Дамек–Річчі

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У цій статті ми показуємо, що чотиривимірні лоренцеві простори Дамек–Річчі не є солітоном Річчі. Це є узагальненням результату Тана і Денга (див. [11]), які довели, що ці простори не є солітоном Річчі лише відносно ліво-інваріантних векторних полів.

Ключові слова: простори Дамек–Річчі, солітон Річчі, ліво-інваріантні метрики, лоренцеві метрики