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On the Correlation Functions of the Characteristic Polynomials of Random Matrices with Independent Entries: Interpolation Between Complex and Real Cases

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The paper is concerned with the correlation functions of the characteristic polynomials of random matrices with independent complex entries. We investigate how the asymptotic behavior of the correlation functions depends on the second moment of the common probability law of the matrix entries, wherein the second moment can be treated as a sort of "reality measure" of the entries. It is shown that the correlation functions behave like those for the Complex Ginibre Ensemble up to a factor depending only on the second moment and the fourth absolute moment of the common probability law of the matrix entries.

Key words: random matrix theory, Ginibre ensemble, correlation functions of characteristic polynomials, moments of characteristic polynomials, SUSY

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1. Introduction

The ensemble of random matrices with independent entries was introduced by Ginibre in 1965 [33]. To be exact, he introduced a partial case when entries of the matrices have Gaussian distribution. Anyway, the ensemble appeared to be significant and has been attracting scientists' attention since that time.

Random matrices with independent entries are usually considered over three fields: complex numbers, real numbers and quaternions. An asymptotic behavior of the correlation functions of the characteristic polynomials was recently computed in the complex case [2] and in the real case [3]. The goal of the current paper is to obtain a similar result in the intermediate case between the complex and the real ones.

Let us proceed to precise definitions. We consider the matrices of the form

$$M_n = \frac{1}{\sqrt{n}} X = \frac{1}{\sqrt{n}} (x_{jk})_{j,k=1}^n, \tag{1.1}$$

where x_{jk} are i.i.d. complex random variables such that

$$\mathbf{E}\{x_{jk}\} = 0, \quad \mathbf{E}\{|x_{jk}|^2\} = 1, \quad \mathbf{E}\{x_{jk}^2\} =: \kappa_{2,0}. \tag{1.2}$$

Here and everywhere below $\mathbf{E}$ denotes an expectation with respect to all random variables. In the particular case, if the entries x_{jk} are complex or real Gaussian this ensemble is known as Complex or Real Ginibre Ensemble, respectively (Gin($\mathbb{C}$) or Gin($\mathbb{R}$)). The parameter $\kappa_{2,0}$ plays a role of a “reality measure”. Indeed, on the one hand, $\kappa_{2,0} = 0$ in the complex case. On the other hand, $\kappa_{2,0} = 1$ in the real case.

Notice that the ensemble (1.1) has various applications in physics, neuroscience, economics, etc. For detailed information, see [4] and references therein.

Define the Normalized Counting Measure (NCM) of eigenvalues as

$$N_n(\Delta) = \#\{\lambda_j^{(n)} \in \Delta,\ j = 1, \ldots, n\}/n,$$

where Δ is an arbitrary Borel set in the complex plane, $\left\{\lambda_j^{(n)}\right\}_{j=1}^{n}$ are the eigenvalues of M_n. The NCM is known to converge to the uniform distribution on the unit disc. This distribution is called the circular law. This result has a long and rich history. Mehta was the first who obtained it for x_{jk} being a complex Gaussian in 1967 [43]. The proof strongly relied on the explicit formula for the common probability density of eigenvalues due to Ginibre [33]. Unfortunately, there is no such a formula in the general case. That is why other methods should be used. The Hermitization approach introduced by Girko [34] appeared to be an effective method. The main idea is to reduce the study of matrices (1.1) to the study of Hermitian matrices using the logarithmic potential of a measure

$$P_\mu(z) = \int_{\mathbb{C}} \log|z - \zeta|\, d\mu(\zeta).$$

This approach was successfully developed by Girko in his works [35–38]. The final result in the most general case was established by Tao and Vu [57]. Notice that there are a lot of partial results besides those listed above. The interested reader is referred to [8].

The Central Limit Theorem (CLT) for linear statistics of non-Hermitian random matrices of the form (1.1) was first proven for radial-invariant test functions in the complex case by Forrester [27]. The study was continued in the complex case by Rider and Silverstein [48], Rider and Virag [49], in the real case by O’Rourke and Renfrew [46], in both cases by Tao and Vu [58] and Kopel [41]. The best result for today was obtained by Cipolloni, Erdős and Schröder for the complex case in [15] and for the real case in [16]. They proved CLT for a bit more than twice differentiable test functions assuming that the common distribution of matrix entries has finite moments.

A local regime for matrices (1.1) has been less studied. The asymptotic behavior of the k-point correlation function for Ginibre ensembles is well known, see [33, 43] for Gin($\mathbb{C}$) and [9, 23, 26] for Gin($\mathbb{R}$). A general distribution case was considered in [58]. It was established in both cases that the k-point correlation function converges in vague topology to that for Ginibre ensemble if x_{jk} has the first four moments as in the Gaussian case. The condition of matching moments was recently overcome at the edge of the spectrum (i.e., at $|z| = 1$) in [17]. The last result strongly relies on an estimate for the least singular value obtained in [14] using the supersymmetry technique (SUSY).

One can observe that non-Hermitian random matrices are more complicated than their Hermitian counterparts. Indeed, the Hermitian case was successfully dealt with using the Stieltjes transform or the moments method. However, a measure in the plane can not be recovered from its Stieltjes transform or its moments. Thus these approaches to the analysis fail in the non-Hermitian case.

The present paper suggests using the SUSY. It is a rather powerful method which is widely applied at the physical level of rigor (for instance, [12, 13, 28, 30, 32, 45, 61]). There have been a lot of rigorous results obtained by using SUSY in the recent years [6, 11, 14, 18–22, 50–53]. The supersymmetry technique is usually used in order to obtain an integral representation for ratios of determinants. Since the main spectral characteristics such as density of states, spectral correlation functions, etc. often can be expressed via ratios of determinants, SUSY allows one to get the integral representation for these characteristics too. For detailed discussion on connection between spectral characteristics and ratios of determinants, see [10, 39, 56]. See also [32, 47].

Let us consider the second spectral correlation function R_2 defined by the equality

$$\mathbf{E}\left\{2\sum_{1\le j_1<j_2\le n}\eta\left(\lambda_{j_1}^{(n)},\lambda_{j_2}^{(n)}\right)\right\}=\int_{\mathbb{C}^2}\eta(\lambda_1,\lambda_2)R_2(\lambda_1,\lambda_2)d\bar{\lambda}_1 d\lambda_1 d\bar{\lambda}_2 d\lambda_2,$$

where the function $\eta\colon \mathbb{C}^2 \to \mathbb{C}$ is bounded, continuous and symmetric in its arguments. Using the logarithmic potential, R_2 can be represented via ratios of the determinants of M_n with the most singular term of the form

$$\int_0^{\varepsilon_0}\int_0^{\varepsilon_0}\frac{\partial^2}{\partial\delta_1\partial\delta_2}\mathbf{E}\left\{\prod_{j=1}^{2}\frac{\det\left((M_n-z_j)(M_n-z_j)^*+\delta_j\right)}{\det\left((M_n-z_j)(M_n-z_j)^*+\varepsilon_j\right)}\right\}\Bigg|_{\delta=\varepsilon}d\varepsilon_1 d\varepsilon_2 \tag{1.3}$$

The integral representation for (1.3) obtained by SUSY will contain both commuting and anti-commuting variables. Integrals of this type are rather difficult to analyze. That is why one should investigate a similar but simpler integral to shed light on the situation. This integral arises from the study of the correlation functions of the characteristic polynomials. Moreover, the correlation functions of the characteristic polynomials are of independent interest. They were studied for many ensembles of Hermitian and real symmetric matrices, see, for instance, [1, 12, 13, 52, 54, 55], etc. The other result on the asymptotic behavior of the correlation functions of the characteristic polynomials of non-Hermitian matrices of the form $H + i\Gamma$, where H is from Gaussian Unitary Ensemble (GUE) and Γ is a fixed matrix of rank M, was obtained in [29]. The kernel computed there, in the limit of rank $M \to \infty$ of the perturbation Γ (taken after matrix size $n \to \infty$) after appropriate rescaling approaches the form (1.8). It was shown in [31, Sec. 2.2].

Let us introduce the m^{th} correlation function of the characteristic polynomials

$$\mathsf{f}_m(Z) = \mathbf{E}\left\{ \prod_{j=1}^{m} \det\left(M_n - z_j\right)\left(M_n - z_j\right)^* \right\}, \tag{1.4}$$

where

$$Z = \operatorname{diag}\{z_1, \dots, z_m\} \tag{1.5}$$

and $z_1, \dots, z_m$ are complex parameters which may depend on n. We are interested in the asymptotic behavior of (1.4), as $n \to \infty$, for

$$z_j = z_0 + \frac{\zeta_j}{\sqrt{n}}, \quad j = 1, 2, \dots, m, \tag{1.6}$$

where z_0 is either in the bulk ($|z_0| < 1$) or at the edge ($|z_0| = 1$) of the spectrum and $\zeta_1, \dots, \zeta_m$ are n-independent complex numbers. The functions (1.4) are well studied for the Complex Ginibre Ensemble, see [5, 60]. A general distribution case was considered in [2, 3]. It was shown that in the complex case for any z_0 in the unit disk,

$$\lim_{n\to\infty} n^{-\frac{m^2-m}{2}} \frac{\mathsf{f}_m(Z)}{\mathsf{f}_1(z_1)\cdots\mathsf{f}_1(z_m)} = e^{\frac{m^2-m}{2}(1-|z_0|^2)^2\kappa_{2,2}} \frac{\det(K_{\mathbb{C}}(\zeta_j,\zeta_k))_{j,k=1}^{m}}{|\triangle(\mathcal{Z})|^2}, \tag{1.7}$$

where $\kappa_{2,2} = \mathbf{E}\{|x_{11}|^4\} - \left|\mathbf{E}\{x_{11}^2\}\right|^2 - 2$ and

$$K_{\mathbb{C}}(z,w) = e^{-|z|^2/2 - |w|^2/2 + z\overline{w}}, \tag{1.8}$$

$$\mathcal{Z} = \operatorname{diag}\{\zeta_1, \dots, \zeta_m\}, \tag{1.9}$$

and $\triangle(\mathcal{Z})$ is a Vandermonde determinant of $\zeta_1, \dots, \zeta_m$. Whereas in the real case for any $z_0 \in [-1, 1]$,

$$\lim_{n\to\infty} n^{-2} \frac{\mathsf{f}_2(Z)}{\mathsf{f}_1(z_1)\mathsf{f}_1(z_2)} = Ce^{(1-|z_0|^2)^2\kappa_{2,2}} \frac{\operatorname{Pf}(K_{\mathbb{R}}(\zeta_j,\zeta_k))_{j,k=1}^{2}}{\triangle(\zeta_1,\zeta_2,\bar{\zeta}_1,\bar{\zeta}_2)},$$

where

$$K_{\mathbb{R}}(\zeta_j,\zeta_k) = e^{-\frac{|\zeta_j|^2}{2} - \frac{|\zeta_k|^2}{2}} \begin{pmatrix} (\zeta_j - \zeta_k)e^{\zeta_j\zeta_k} & (\zeta_j - \bar{\zeta}_k)e^{\zeta_j\bar{\zeta}_k} \\ (\bar{\zeta}_j - \zeta_k)e^{\bar{\zeta}_j\zeta_k} & (\bar{\zeta}_j - \bar{\zeta}_k)e^{\bar{\zeta}_j\bar{\zeta}_k} \end{pmatrix}.$$

In the current paper we extend the results of [2, 3] to the case of arbitrary $\kappa_{2,0}$, $|\kappa_{2,0}| \le 1$. The main result is

Theorem 1.1. *Let an ensemble of real random matrices M_n be defined by (1.1) and (1.2). Let the first $2m$ moments of the common distribution of entries of M_n be finite and z_j, $j = 1, \dots, m$, have the form (1.6). Let also z_0 and $\kappa_{2,0}$ satisfy at least one of the two following conditions:*

(i) $|\kappa_{2,0}| < 1$ *and* $|z_0| < 1$*;*

(ii) $|\kappa_{2,0}| = 1$ *and* $|z_0| < 1$*,* $z_0 \notin \mathbb{R}$*.*

Then the m^{th} *correlation function of the characteristic polynomials* (1.4) *satisfies the asymptotic relation*

$$\lim_{n\to\infty} n^{-\frac{m^2-m}{2}} \frac{\mathsf{f}_m(Z)}{\mathsf{f}_1(z_1)\cdots\mathsf{f}_1(z_m)} = C_{m,z_0} e^{d(\kappa_{2,0},\kappa_{2,2})} \frac{\det(K_{\mathbb{C}}(\zeta_j,\zeta_k))_{j,k=1}^m}{|\triangle(\mathcal{Z})|^2}, \qquad (1.10)$$

where C_{m,z_0} *is some constant, which does not depend on the common distribution of entries and on* $\zeta_1, \ldots, \zeta_m$*;* $\kappa_{2,2} = \mathbf{E}\{|x_{11}|^4\} - \left|\mathbf{E}\{x_{11}^2\}\right|^2 - 2$*,*

$$d(\kappa_{2,0},\kappa_{2,2}) = -m\log\Big\{\left|1-|\kappa_{2,0}|\,z_0^2\right|^2 - |\kappa_{2,0}|^2\left(1-|z_0|^2\right)^2\Big\} + \frac{m^2-m}{2}\left(1-|z_0|^2\right)^2\kappa_{2,2}, \qquad (1.11)$$

$\triangle(\mathcal{Z})$ *is a Vandermonde determinant of* $\zeta_1, \ldots, \zeta_m$ *and* $K_{\mathbb{C}}(z,w)$ *is defined in* (1.8)*.*

Notice that (1.10) has an additional factor compared with (1.7). This factor shows the dependence of the asymptotics of f_m (here and below we omit Z only if $Z = \operatorname{diag}\{z_1, \ldots, z_m\}$) on $\kappa_{2,0}$.

The paper is organized as follows. In Section 2, a suitable integral representation for f_m is discussed. In Section 3, we apply the steepest descent method to the suitable integral representation and find out the asymptotic behavior of f_m. In order to compute it, the Harish–Chandra/Itsykson–Zuber formula is used. For the reader convenience the latter section is divided into two parts. The first part deals with a simpler partial case and the second one treats a general case.

1.1. Notation. Throughout the paper lower-case letters denote scalars, bold lower-case letters denote vectors, upper-case letters denote matrices and bold upper-case letters denote sets of matrices. We use the same letter for a matrix, for its columns and for its entries. Table 1.1 shows the exact correspondence. Besides, for any matrix A we denote by $(A)_j$ its j-th column and by $(A)_{kj}$ its entry in the k-th row and in the j-th column.

Set of matrices	Matrix	Vector	Entry
$\boldsymbol{Q}$	$Q_{p,s}$		$q_{\alpha\beta}^{(p,s)}$
		$\boldsymbol{\phi}$	ϕ_j
		$\boldsymbol{\theta}$	θ_j
	$Y_{p,s}$		$y_{\alpha\beta}^{(p,s)}$
	U		u_{kj}
	V		v_{kj}

Table 1.1: Notation correspondence

The term “Grassmann variable” is a synonym for “anti-commuting variable”. The variables of integration ϕ, θ and ρ are Grassmann variables, all the other variables of integration unspecified by an integration domain are either complex or real. We split all the generators of Grassmann algebra into two equal sets and consider the generators from the second set as the “conjugates” of those from the first set, i.e., for the Grassmann variable υ we use υ^* to denote its “conjugate”. Furthermore, if $\Upsilon = (\upsilon_{jk})$ means a matrix of Grassmann variables, then Υ^+ is a matrix (υ^*_{kj}). d-dimensional vectors are identified with $d \times 1$ matrices.

Integrals without limits denote either integration over Grassmann variables or integration over the whole space $\mathbb{C}^d$ or $\mathbb{R}^d$. Let also $d\boldsymbol{t}^* d\boldsymbol{t}$ ($\boldsymbol{t} = (t_1, \ldots, t_d)^T \in \mathbb{C}^d$) denote the measure $\prod_{j=1}^d d\bar{t}_j dt_j$ on the space $\mathbb{C}^d$. Similarly, for vectors with anti-commuting entries $d\boldsymbol{\tau}^+ d\boldsymbol{\tau} = \prod_{j=1}^d d\tau_j^* d\tau_j$. Note that the space of matrices is a linear space over $\mathbb{C}$. The same notations are also used for matrices.

By $\langle \cdot, \cdot \rangle$, denote a standard scalar product on $\mathbb{C}^d$. For matrices, $\langle A, B \rangle = \operatorname{tr} B^* A$. For sets of matrices, $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \sum_j \langle A_j, B_j \rangle$. The norm we use is defined by $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$.

Note that $\binom{m}{p} \times \binom{m}{s}$ matrices appear in the statement of Proposition 2.1. It is natural to number rows and columns of such matrices by subsets of an m-element set. To this end, set

$$\mathcal{I}_{m,p'} = \{\alpha \in \mathbb{Z}^{p'} \mid 1 \le \alpha_1 < \ldots < \alpha_{p'} \le m\}. \tag{1.12}$$

If $p' = 0$, we define $\mathcal{I}_{m,p'}$ as $\{\varnothing\}$.

The cumulants $\kappa_{p,s}$ are defined as follows. Consider the function

$$\psi(t_1, t_2) := \mathbf{E}\left\{ e^{t_1 x_{11} + t_2 \overline{x}_{11}} \right\}.$$

Then

$$\kappa_{p,s} = \frac{\partial^{p+s}}{\partial^p t_1 \partial^s t_2} \log \psi(t_1, t_2) \bigg|_{t_1 = t_2 = 0}. \tag{1.13}$$

In particular, $\kappa_{2,2} = \mathbf{E}\{|x_{11}|^4\} - \left|\mathbf{E}\{x_{11}^2\}\right|^2 - 2$.

Throughout the paper, $U(m)$ denotes a group of unitary $m \times m$ matrices. μ denotes a corresponding Haar measure. In addition, C, C_1 denote various n-independent constants which can be different in different formulas.

2. Integral representation for f_m

The following integral representation is true

Proposition 2.1. *Let an ensemble M_n be defined by (1.1) and (1.2). Then the m^{th} correlation function of the characteristic polynomials f_m defined by (1.4) can be represented in the form*

$$\mathsf{f}_m = \left(\frac{n}{\pi}\right)^{c_m} \int g(\boldsymbol{Q}) e^{(n - c_m) f(\boldsymbol{Q})} d\boldsymbol{Q}, \tag{2.1}$$

where $c_m = 2^{2m-1}$, $\boldsymbol{Q} = (\boldsymbol{Q}_j)_{j=0}^m$, $\boldsymbol{Q}_j = \{Q_{p,s} \mid p+s = 2j,\ 0 \le p,s \le m\}$, $Q_{p,s}$ *is a complex* $\binom{m}{p} \times \binom{m}{s}$ *matrix,* $d\boldsymbol{Q} = \prod_{\substack{p+s \text{ is even} \\ 0\le p,s\le m}} dQ_{p,s}^* dQ_{p,s}$ *and*

$$f(\boldsymbol{Q}) = -\langle \boldsymbol{Q}, \boldsymbol{Q}\rangle + \log h(\boldsymbol{Q}); \tag{2.2}$$

$$g(\boldsymbol{Q}) = (h(\boldsymbol{Q})^{c_m} + n^{-1/2}\mathtt{p}_a(\boldsymbol{Q}))\exp\{-c_m\langle \boldsymbol{Q}, \boldsymbol{Q}\rangle\};$$

$$h(\boldsymbol{Q}) = \operatorname{Pf} F(\boldsymbol{Q}_1) + n^{-1/2}\tilde{h}(\boldsymbol{Q}_1, \boldsymbol{Q}_2) + n^{-1}\mathtt{p}_c(\boldsymbol{Q}_1, \boldsymbol{Q}_{>1}); \tag{2.3}$$

$$F(\boldsymbol{Q}_1) = \begin{pmatrix} \sqrt{\kappa_{2,0}}B_{2,0} & 0 & -Z & Q_1 \\ 0 & \sqrt{\kappa_{2,0}}B_{0,2}^* & -Q_1^* & -Z^* \\ Z & \overline{Q}_1 & \sqrt{\kappa_{2,0}}B_{2,0}^* & 0 \\ -Q_1^T & Z^* & 0 & \sqrt{\kappa_{2,0}}B_{0,2} \end{pmatrix}; \tag{2.4}$$

$B_{2,0}$ *and* $B_{0,2}$ *are skew-symmetric matrices such that*

$$(B_{2,0})_{\alpha_1\alpha_2} = -q_{\alpha\varnothing}^{(2,0)}, \quad (B_{0,2})_{\alpha_1\alpha_2} = -q_{\varnothing\alpha}^{(0,2)}, \quad \alpha \in \mathcal{I}_{m,2}$$

and $\mathcal{I}_{m,2}$ *is defined in* (1.12). *Moreover,*

$$\tilde{h}(\boldsymbol{Q}_1, \boldsymbol{Q}_2) = -\int \sum_{p+s=4} \left(\operatorname{tr} \tilde{Y}_{p,s}Q_{p,s} + \operatorname{tr} Q_{p,s}^* Y_{p,s}\right) e^{-\frac{1}{2}\boldsymbol{\rho}^T F \boldsymbol{\rho}} d\boldsymbol{\phi}^+ d\boldsymbol{\phi} d\boldsymbol{\theta}^+ d\boldsymbol{\theta}, \tag{2.5}$$

$$\boldsymbol{\rho} = \begin{pmatrix}\boldsymbol{\phi}^+ & \boldsymbol{\theta}^+ & \boldsymbol{\phi}^T & \boldsymbol{\theta}^T\end{pmatrix}^T, \tag{2.6}$$

$$\tilde{y}_{\beta\alpha}^{(p,s)} = \sqrt{\kappa_{p,s}}(-1)^p \prod_{r=s}^{1} \theta_{\beta_r} \prod_{q=p}^{1} \phi_{\alpha_q}^*, \tag{2.7}$$

$$y_{\alpha\beta}^{(p,s)} = \sqrt{\kappa_{p,s}} \prod_{q=1}^{p} \phi_{\alpha_q} \prod_{r=1}^{s} \theta_{\beta_r}^*, \tag{2.8}$$

$\kappa_{p,s}$ *are defined in* (1.13), $\mathtt{p}_a(\boldsymbol{Q})$ *and* $\mathtt{p}_c(\boldsymbol{Q}_1, \boldsymbol{Q}_{>1})$ *are certain polynomials such that* $\mathtt{p}_c(\boldsymbol{Q}_1, 0) = 0$, *and* $\boldsymbol{Q}_{>1}$ *contains all* $\boldsymbol{Q}_j$ *except* $\boldsymbol{Q}_1$.

Proof. Proposition 2.1 was proved for the case $\kappa_{2,0} = 1$ in [3, Proposition 2.1]. The most part of the provided proof goes in the frames of a general case, and only in the very end $\kappa_{2,0} = 1$ is substituted. Therefore it is easy to understand from [3] that the only distinction of the general case from the partial one is in the presence of $\kappa_{2,0}$ in (2.4). □

Remark 2.2. Let $Q_1 = U\Lambda V^*$ be the singular value decomposition of the matrix Q_1, i.e., $\Lambda = \operatorname{diag}\{\lambda_j\}_{j=1}^m$, $\lambda_j \ge 0$, $U, V \in U(m)$. In order to perform asymptotic analysis, let us change the variables $Q_1 = U\Lambda V^*$, $B_{2,0} \to UB_{2,0}U^T$, $B_{0,2} \to \overline{V}B_{0,2}V^*$ in (2.1). Since the Jacobian is $\frac{2^m \pi^{m^2}}{(\prod_{j=1}^{m-1} j!)^2}\triangle^2(\Lambda^2)\prod_{j=1}^m \lambda_j$ (see, e.g., [40]), we obtain

$$\mathsf{f}_m = Cn^{c_m}\int_{\mathcal{D}} \triangle^2(\Lambda^2)\prod_{j=1}^m \lambda_j \left[g_0(\Lambda, \hat{\boldsymbol{Q}}) + \frac{1}{\sqrt{n}} g_r(U\Lambda V^*, \hat{\boldsymbol{Q}})\right]$$

$$\times \exp\left\{(n-c_m)\left[f_0(\Lambda,\hat{\boldsymbol{Q}})+\frac{1}{\sqrt{n}}f_r(U\Lambda V^*,\hat{\boldsymbol{Q}})\right]\right\}$$
$$\times\, d\mu(U)d\mu(V)d\Lambda d\hat{\boldsymbol{Q}}, \tag{2.9}$$

where $\hat{\boldsymbol{Q}}$ contains all the matrices $Q_{p,s}$ except Q_1, $\mathcal{D}=\{(\Lambda,U,V,\hat{\boldsymbol{Q}}) \mid \lambda_j\geq 0,\ j=1,\dots,m,\ U,V\in U(m)\}$, μ is a Haar measure, $d\Lambda=\prod_{j=1}^m d\lambda_j$ and

$$f_0(\boldsymbol{Q})=-\langle \boldsymbol{Q},\boldsymbol{Q}\rangle+\log h_0(\boldsymbol{Q}_1), \tag{2.10}$$
$$g_0(\boldsymbol{Q})=h_0(\boldsymbol{Q}_1)^{c_m}\exp\{-c_m\langle \boldsymbol{Q},\boldsymbol{Q}\rangle\}=e^{c_m f_0(\boldsymbol{Q})},$$

$$h_0(\boldsymbol{Q}_1)=\operatorname{Pf}\tilde{F}(\boldsymbol{Q}_1),\ \ \tilde{F}(\boldsymbol{Q}_1):=\begin{pmatrix}\mathcal{B}_{2,0} & 0 & -z_0I_m & \Lambda\\ 0 & \mathcal{B}^*_{0,2} & -\Lambda & -\overline{z}_0 I_m\\ z_0I_m & \Lambda & \mathcal{B}^*_{2,0} & 0\\ -\Lambda & \overline{z}_0I_m & 0 & \mathcal{B}_{0,2}\end{pmatrix}, \tag{2.11}$$

$$f_r(\boldsymbol{Q})=\sqrt{n}(f(\boldsymbol{Q})-f_0(\boldsymbol{Q})), \tag{2.12}$$
$$g_r(\boldsymbol{Q})=\sqrt{n}(g(\boldsymbol{Q})-g_0(\boldsymbol{Q})),$$

$\mathcal{B}_{2,0}=\sqrt{\kappa_{2,0}}B_{2,0}$ and $\mathcal{B}_{0,2}=\sqrt{\kappa_{2,0}}B_{0,2}$. Notice that $f_0(U\Lambda V^*,\hat{\boldsymbol{Q}})=f_0(\Lambda,\hat{\boldsymbol{Q}})$ and the same for g_0.

Remark 2.3. In the special case $m=1$, the matrices $B_{2,0}$ and $B_{0,2}$ are zeros and we have

$$\mathsf{f}_1(z)=\frac{n}{\pi}\int \exp\left\{n(-|q|^2+\log(|z|^2+|q|^2))\right\}d\bar{q}dq.$$

Changing variables to polar coordinates and performing a simple Laplace integration, we obtain

$$\mathsf{f}_1(z)=2n\int_0^{+\infty} r\exp\left\{n(-r^2+\log(|z|^2+r^2))\right\}dr$$
$$=\sqrt{2\pi n}\,e^{n(|z|^2-1)}(1+o(1)). \tag{2.13}$$

3. Asymptotic analysis

The goal of the section is to study the asymptotic behavior of the integral representation (2.9). To this end, the steepest descent method is applied. As usual, the hardest step is to choose stationary points of $f(\boldsymbol{Q})$ and a N-dimensional (real) manifold $M_*\subset\mathbb{C}^N$ such that for any chosen stationary point $\boldsymbol{Q}_*\in M_*$,

$$\Re f(\boldsymbol{Q})<\Re f(\boldsymbol{Q}_*),\quad \forall \boldsymbol{Q}\in M_*,\ \boldsymbol{Q}\text{ is not chosen.}$$

Note that N is equal to the number of real variables of the integration, i.e., in our case $N=2^{2m}$.

The present proof proceeds by a standard scheme for the case when the function $f(\boldsymbol{Q})$ has the form

$$f(\boldsymbol{Q})=f_0(\boldsymbol{Q})+n^{-1/2}f_r(\boldsymbol{Q}),$$

where $f_0(\boldsymbol{Q})$ does not depend on n, whereas $f_r(\boldsymbol{Q})$ may depend on n. We choose stationary points of $f_0(\boldsymbol{Q})$ of the form $Q_1 = U\Lambda_0 V^*$, $\hat{\boldsymbol{Q}} = 0$, where $\Lambda_0 = \lambda_0 I$, $\lambda_0 = \sqrt{1-|z_0|^2}$ and U, V vary in $U(m)$. The manifold M_* is $\mathbb{R}^N$. Then the steepest descent method is applied to the integral over Λ and $\hat{\boldsymbol{Q}}$. In the process U and V are considered as parameters and all the estimates are uniform in U and V. As soon as the domain of integration is restricted by a small neighborhood, we recall about the integration over U and V. After several changes of the variables, the integral is reduced to the form (1.10).

We start with the analysis of the function f_0.

Lemma 3.1. *Let the function $f_0\colon \mathbb{R}^{2^{2m}} \to \mathbb{C}$ be defined by (2.10). Then the function $\Re f_0(\Lambda, \hat{\boldsymbol{Q}})$ attains its global maximum value only at the point*

$$\lambda_1 = \cdots = \lambda_m = \lambda_0, \quad \hat{\boldsymbol{Q}} = 0, \tag{3.1}$$

where $\lambda_0 = \sqrt{1-|z_0|^2}$.

Proof. From (2.10) and (2.11), we get

$$\begin{aligned}\Re f_0(\Lambda, \hat{\boldsymbol{Q}}) &= -\sum_{j\neq 1}\langle \boldsymbol{Q}_j, \boldsymbol{Q}_j\rangle - \langle \boldsymbol{Q}_1, \boldsymbol{Q}_1\rangle + \frac{1}{2}\log\left|\det \tilde{F}\right| \\ &\leq -\langle \boldsymbol{Q}_1, \boldsymbol{Q}_1\rangle + \frac{1}{2}\log\left|\det \tilde{F}\right|. \end{aligned}\tag{3.2}$$

Hadamard's inequality yields

$$\begin{aligned}\frac{1}{2}\log\left|\det \tilde{F}\right| \leq \frac{1}{2}\log\Bigg\{\prod_{j=1}^{m}\Big(|z_0|^2+\lambda_j^2+|\kappa_{2,0}|\sum_{k=1}^{m}\Big|q^{(2,0)}_{(j,k)\varnothing}\Big|^2\Big)^{\frac{1}{2}} \\ \times\Big(|\overline{z}_0|^2+\lambda_j^2+|\kappa_{2,0}|\sum_{k=1}^{m}\Big|\overline{q^{(0,2)}_{\varnothing(j,k)}}\Big|^2\Big)^{\frac{1}{2}}\Big(|z_0|^2+\lambda_j^2+|\overline{\kappa_{2,0}}|\sum_{k=1}^{m}\Big|\overline{q^{(2,0)}_{(j,k)\varnothing}}\Big|^2\Big)^{\frac{1}{2}} \\ \times\Big(|\overline{z}_0|^2+\lambda_j^2+|\overline{\kappa_{2,0}}|\sum_{k=1}^{m}\Big|q^{(0,2)}_{\varnothing(j,k)}\Big|^2\Big)^{\frac{1}{2}}\Bigg\}.\end{aligned}\tag{3.3}$$

where $q^{(2,0)}_{(j,k)\varnothing} = -q^{(2,0)}_{(k,j)\varnothing}$, $q^{(0,2)}_{(j,k)\varnothing} = -q^{(0,2)}_{(k,j)\varnothing}$ for $j > k$ and $q^{(2,0)}_{(j,j)\varnothing} = q^{(0,2)}_{(j,j)\varnothing} = 0$. Simplifying the right-hand side of (3.3) and taking into account (3.2), we obtain

$$\begin{aligned}\Re f_0(\Lambda, \hat{\boldsymbol{Q}}) \leq -\langle \boldsymbol{Q}_1, \boldsymbol{Q}_1\rangle + \frac{1}{2}\sum_{j=1}^{m}\log\Bigg\{\Big(|z_0|^2+\lambda_j^2+|\kappa_{2,0}|\sum_{k=1}^{m}\Big|q^{(2,0)}_{(j,k)\varnothing}\Big|^2\Big) \\ \times\Big(|z_0|^2+\lambda_j^2+|\kappa_{2,0}|\sum_{k=1}^{m}\Big|q^{(0,2)}_{\varnothing(j,k)}\Big|^2\Big)\Bigg\},\end{aligned}\tag{3.4}$$

The inequality $\log x \leq x-1$ and (3.4) imply

$$\Re f_0(\Lambda, \hat{\boldsymbol{Q}}) \leq -\langle \boldsymbol{Q}_1, \boldsymbol{Q}_1\rangle + \frac{1}{2}\sum_{j=1}^{m}\Bigg\{\Big(|z_0|^2+\lambda_j^2+|\kappa_{2,0}|\sum_{k=1}^{m}\Big|q^{(2,0)}_{(j,k)\varnothing}\Big|^2\Big)$$

$$+\left(|z_0|^2+\lambda_j^2+|\kappa_{2,0}|\sum_{k=1}^{m}\left|q^{(0,2)}_{\varnothing(j,k)}\right|^2\right)-2\Bigg\}$$

$$=-\langle \boldsymbol{Q}_1,\boldsymbol{Q}_1\rangle+m\,|z_0|^2-m+\sum_{j=1}^{m}\lambda_j^2+|\kappa_{2,0}|\sum_{\alpha\in\mathcal{I}_{m,2}}\left|q^{(2,0)}_{\alpha\varnothing}\right|^2$$

$$+|\kappa_{2,0}|\sum_{\alpha\in\mathcal{I}_{m,2}}\left|q^{(0,2)}_{\varnothing\alpha}\right|^2. \tag{3.5}$$

Finally, since $|\kappa_{2,0}|=\left|\mathbf{E}\{x_{11}^2\}\right|\le\mathbf{E}\{|x_{11}|^2\}=1$, we have

$$\Re f_0(\Lambda,\hat{\boldsymbol{Q}})\le-\langle \boldsymbol{Q}_1,\boldsymbol{Q}_1\rangle+\sum_{j=1}^{m}\lambda_j^2+\sum_{\alpha\in\mathcal{I}_{m,2}}\left\{\left|q^{(2,0)}_{\alpha\varnothing}\right|^2+\left|q^{(0,2)}_{\varnothing\alpha}\right|^2\right\}$$

$$+m\,|z_0|^2-m=-\langle \boldsymbol{Q}_1,\boldsymbol{Q}_1\rangle+\langle \boldsymbol{Q}_1,\boldsymbol{Q}_1\rangle+m\,|z_0|^2-m=m(|z_0|^2-1). \tag{3.6}$$

Therefore, the function $\Re f_0(\Lambda,\hat{\boldsymbol{Q}})$ attains its global maximum value at the point (3.1). It remains to show that there is no other point for which $\Re f_0(\Lambda,\hat{\boldsymbol{Q}})=m(|z_0|^2-1)$. Indeed, equality in (3.2) is attained if and only if $\boldsymbol{Q}_{>1}=0$. Moreover, the right-hand side of (3.6) and (3.5) are equal if and only if $|\kappa_{2,0}|=1$ or $q^{(2,0)}_{\alpha\varnothing}=q^{(0,2)}_{\varnothing\alpha}=0$. Let us consider the following two cases.

1. $|\kappa_{2,0}|<1\ \Rightarrow\ q^{(2,0)}_{\alpha\varnothing}=q^{(0,2)}_{\varnothing\alpha}=0$ for all $\alpha\in\mathcal{I}_{m,2}$.
   Since the equality $\log x=x-1$ holds if and only if $x=1$, then we obtain from the equality of the right-hand side of (3.4) and (3.5) that

$$|z_0|^2+\lambda_j^2+|\kappa_{2,0}|\sum_{k=1}^{m}\left|q^{(2,0)}_{(j,k)\varnothing}\right|^2=1.$$

   Thus, for any j,

$$\lambda_j=\sqrt{1-|z_0|^2}.$$

2. $|\kappa_{2,0}|=1$ and $z_0\notin\mathbb{R}$.
   Equality in Hadamard's inequality is attained if and only if the columns of a matrix are orthogonal vectors. Hence, if equality is attained in (3.3), then the columns of the matrix $\tilde{F}$ are orthogonal. In particular, the orthogonality of the first and the $2m+2^{\text{nd}}$ yields

$$-(\mathcal{B}_{2,0})_{21}\overline{z}_0+z_0(\mathcal{B}_{2,0})_{21}=0.$$

   Since $z_0\neq\overline{z}_0$, the last identity implies

$$q^{(2,0)}_{(1,2)\varnothing}=\frac{1}{\sqrt{\kappa_{2,0}}}(\mathcal{B}_{2,0})_{21}=0.$$

   Using a similar argument, we get that all $q^{(2,0)}_{\alpha\varnothing}$ and $q^{(0,2)}_{\varnothing\alpha}$ are zeros. Next, similarly to the first case, we obtain $\lambda_1=\cdots=\lambda_m=\sqrt{1-|z_0|^2}$.

Totally, the assertion of the lemma is proven. □

To simplify the reading, the remaining steps are first explained in the case when the cumulants $\kappa_{p,s}$, $p+s>2$ are zeros.

3.1. Case of zero high cumulants. Now we proceed to the integral estimates. In a standard way the integration domain in (2.9) can be restricted as follows:

$$\mathsf{f}_m = Cn^{c_m}\int_{\Sigma_r}\triangle^2(\Lambda^2)\prod_{j=1}^m\lambda_j\times g(U\Lambda V^*,\hat{\boldsymbol{Q}})e^{(n-c_m)f(U\Lambda V^*,\hat{\boldsymbol{Q}})}d\mu(U)d\mu(V)d\Lambda d\hat{\boldsymbol{Q}} \\ +O(e^{-nr/2}),$$

where

$$\Sigma_r = \Big\{(\Lambda,U,V,\hat{\boldsymbol{Q}})\mid \|\Lambda\|+\|\hat{\boldsymbol{Q}}\|\le r\Big\}.$$

The next step is to restrict the integration domain by

$$\Omega_n = \Big\{(\Lambda,U,V,\hat{\boldsymbol{Q}})\mid \|\Lambda-\Lambda_0\|+\|\hat{\boldsymbol{Q}}\|\le \frac{\log n}{\sqrt{n}}\Big\}. \tag{3.7}$$

To this end, we need the estimate of $\Re f$ given by the following lemmas.

Lemma 3.2. *Let $\tilde{\Lambda}$ and $\hat{\tilde{\boldsymbol{Q}}}$ satisfy the condition $\|\tilde{\Lambda}\|+\|\hat{\tilde{\boldsymbol{Q}}}\|\le\log n$. Then uniformly in U and V,*

$$\begin{aligned} f(U(\Lambda_0+n^{-1/2}\tilde{\Lambda})V^*, n^{-1/2}\hat{\tilde{\boldsymbol{Q}}}) &= -m\lambda_0^2+n^{-1/2}\operatorname{tr}(\overline{z}_0\mathcal{Z}+z_0\mathcal{Z}^*)\\ &-\frac{1}{2n}\operatorname{tr}(2\lambda_0\tilde{\Lambda}+\overline{z}_0\mathcal{Z}_U+z_0\mathcal{Z}_V^*)^2+\frac{1}{n}\operatorname{tr}\mathcal{Z}_U\mathcal{Z}_V^*\\ &-\frac{1}{2n}\operatorname{tr}\Big[(1-|\kappa_{2,0}|\,\overline{z}_0^2)\tilde{B}_{2,0}^*\tilde{B}_{2,0}+(1-|\kappa_{2,0}|\,z_0^2)\tilde{B}_{0,2}^*\tilde{B}_{0,2}\\ &-|\kappa_{2,0}|\,\lambda_0^2\tilde{B}_{0,2}\tilde{B}_{2,0}-|\kappa_{2,0}|\,\lambda_0^2\tilde{B}_{2,0}^*\tilde{B}_{0,2}^*\Big]-\frac{1}{n}\left\|\tilde{\boldsymbol{Q}}_{>1}\right\|^2\\ &+O\big(n^{-3/2}\log^3 n\big), \end{aligned} \tag{3.8}$$

where $\mathcal{Z}_W = W^*\mathcal{Z}W$.

Proof. If $Q_1 = U(\Lambda_0+n^{-1/2}\tilde{\Lambda})V^*$, then F has the form

$$F = \begin{pmatrix} U & 0 & 0 & 0\\ 0 & V & 0 & 0\\ 0 & 0 & \overline{U} & 0\\ 0 & 0 & 0 & \overline{V}\end{pmatrix}\left(F_0+\frac{1}{\sqrt{n}}F_1\right)\begin{pmatrix} U^T & 0 & 0 & 0\\ 0 & V^T & 0 & 0\\ 0 & 0 & U^* & 0\\ 0 & 0 & 0 & V^*\end{pmatrix},$$

where

$$F_0 = \begin{pmatrix} 0 & A_0\\ -A_0^T & 0\end{pmatrix},\quad F_1 = \begin{pmatrix} B & A\\ -A^T & B^*\end{pmatrix},$$

$$A_0 = \begin{pmatrix} -z_0 I_m & \Lambda_0 \\ -\Lambda_0 & -\bar{z}_0 I_m \end{pmatrix}, \quad A = \begin{pmatrix} -\mathcal{Z}_U & \tilde{\Lambda} \\ -\tilde{\Lambda} & -\mathcal{Z}_V^* \end{pmatrix}, \quad B = \begin{pmatrix} \tilde{\mathcal{B}}_{2,0} & 0 \\ 0 & \tilde{\mathcal{B}}_{0,2}^* \end{pmatrix}. \tag{3.9}$$

Taking into account that

$$\det F_0 = \left[\det \begin{pmatrix} z_0 & \lambda_0 \\ -\lambda_0 & \bar{z}_0 \end{pmatrix} \det \begin{pmatrix} -z_0 & \lambda_0 \\ -\lambda_0 & -\bar{z}_0 \end{pmatrix}\right]^m = 1,$$

one gets

$$\begin{aligned} \log\det F &= \operatorname{tr}\log(1 + n^{-1/2}F_0^{-1}F_1) \\ &= \frac{1}{\sqrt{n}} \operatorname{tr} F_0^{-1}F_1 - \frac{1}{2n}\operatorname{tr}(F_0^{-1}F_1)^2 + O\left(\frac{\log^3 n}{\sqrt{n^3}}\right) \end{aligned} \tag{3.10}$$

uniformly in U and V. Further,

$$F_0^{-1}F_1 = \begin{pmatrix} (A_0^T)^{-1}A^T & -(A_0^T)^{-1}B^* \\ A_0^{-1}B & A_0^{-1}A \end{pmatrix} \tag{3.11}$$

and

$$(F_0^{-1}F_1)^2 = \begin{pmatrix} \left((AA_0^{-1})^T\right)^2 - (A_0^T)^{-1}B^*A_0^{-1}B & * \\ * & -A_0^{-1}B(A_0^T)^{-1}B^* + (A_0^{-1}A)^2 \end{pmatrix}. \tag{3.12}$$

Moreover,

$$\begin{aligned} A_0^{-1}A &= \begin{pmatrix} \bar{z}_0\mathcal{Z}_U + \lambda_0\tilde{\Lambda} & -\bar{z}_0\tilde{\Lambda} + \lambda_0\mathcal{Z}_V^* \\ -\lambda_0\mathcal{Z}_U + z_0\tilde{\Lambda} & \lambda_0\tilde{\Lambda} + z_0\mathcal{Z}_V^* \end{pmatrix}, \\ (A_0^T)^{-1}B^*A_0^{-1}B &= \begin{pmatrix} \bar{z}_0^2\tilde{\mathcal{B}}_{2,0}^*\tilde{\mathcal{B}}_{2,0} + \lambda_0^2\tilde{\mathcal{B}}_{0,2}\tilde{\mathcal{B}}_{2,0} & * \\ * & \lambda_0^2\tilde{\mathcal{B}}_{2,0}^*\tilde{\mathcal{B}}_{0,2}^* + z_0^2\tilde{\mathcal{B}}_{0,2}\tilde{\mathcal{B}}_{0,2}^* \end{pmatrix}. \end{aligned} \tag{3.13}$$

Combining (3.10)–(3.13) and (2.2), we get

$$\begin{aligned} f(U(\Lambda_0 + n^{-1/2}\tilde{\Lambda})V^*, n^{-1/2}\hat{\tilde{\boldsymbol{Q}}}) &= -\operatorname{tr}\left[\Lambda_0^2 + 2n^{-1/2}\lambda_0\tilde{\Lambda} + n^{-1}\tilde{\Lambda}^2\right] \\ &- \frac{1}{2n}\operatorname{tr}[\tilde{B}_{2,0}^*\tilde{B}_{2,0} + \tilde{B}_{0,2}^*\tilde{B}_{0,2}] - \frac{1}{n}\left\|\tilde{\boldsymbol{Q}}_{>1}\right\|^2 + \frac{1}{n^{1/2}}\operatorname{tr}[2\lambda_0\tilde{\Lambda} + \bar{z}_0\mathcal{Z}_U + z_0\mathcal{Z}_V^*] \\ &- \frac{1}{n}\operatorname{tr}\left[(\lambda_0^2 - |z_0|^2)\tilde{\Lambda}^2 + 2\bar{z}_0\lambda_0\mathcal{Z}_U\tilde{\Lambda} + 2z_0\lambda_0\mathcal{Z}_V^*\tilde{\Lambda} + \frac{1}{2}(\bar{z}_0\mathcal{Z}_U + z_0\mathcal{Z}_V^*)^2 - \mathcal{Z}_U\mathcal{Z}_V^*\right] \\ &+ \frac{1}{2n}|\kappa_{2,0}|\operatorname{tr}[\bar{z}_0^2\tilde{B}_{2,0}^*\tilde{B}_{2,0} + \lambda_0^2\tilde{B}_{0,2}\tilde{B}_{2,0} + \lambda_0^2\tilde{B}_{2,0}^*\tilde{B}_{0,2}^* + z_0^2\tilde{B}_{0,2}\tilde{B}_{0,2}^*] \\ &+ O\big(n^{-3/2}\log^3 n\big). \end{aligned}$$

Hence the last expansion yields (3.8). ☐

Corollary 3.3. *Let the function* $f_0\colon \mathbb{R}^{2^{2m}} \to \mathbb{C}$ *be defined by* (2.10). *Then the following assertions are true:*

(i) *the point* $\boldsymbol{Q}_* = (\Lambda_0, 0)$ *is a stationary point of the function* $f_0(\Lambda, \hat{\boldsymbol{Q}})$*;*

(ii) *the Hessian matrix of the function* $\Re f_0(\Lambda, \hat{\boldsymbol{Q}})$ *(as a function of real argument) at the point* $\boldsymbol{Q}_*$ *is negative definite.*

Proof. Let us put $\mathcal{Z} = 0$ and $\mathtt{p}_c = 0$. Then

$$f_0(\Lambda, \hat{\boldsymbol{Q}}) = f(\Lambda, \hat{\boldsymbol{Q}}).$$

Therefore it is possible to consider the expansion (3.8) as the Taylor formula for $f_0(\Lambda, \hat{\boldsymbol{Q}})$ at the point $(\Lambda_0, 0)$. We obtain

$$\begin{aligned} f_0(\Lambda_0 + n^{-\frac{1}{2}}\tilde{\Lambda}, n^{-\frac{1}{2}}\hat{\tilde{\boldsymbol{Q}}}) = {}& -m\lambda_0^2 - n^{-1}2\lambda_0^2 \operatorname{tr}\tilde{\Lambda}^2 - n^{-1}\left\|\tilde{\boldsymbol{Q}}_{>1}\right\|^2 \\ & - \frac{1}{2n}\operatorname{tr}\Big[(1-|\kappa_{2,0}|\,\overline{z}_0^2)\tilde{B}_{2,0}^*\tilde{B}_{2,0} + (1-|\kappa_{2,0}|\,z_0^2)\tilde{B}_{0,2}^*\tilde{B}_{0,2} \\ & - |\kappa_{2,0}|\,\lambda_0^2\tilde{B}_{0,2}\tilde{B}_{2,0} - |\kappa_{2,0}|\,\lambda_0^2\tilde{B}_{2,0}^*\tilde{B}_{0,2}^*\Big] + O\big(n^{-\frac{3}{2}}\log^3 n\big). \end{aligned}$$

Thus the gradient of the function $f_0(\Lambda, \hat{\boldsymbol{Q}})$ is evidently zero at the point $(\Lambda_0, 0)$. Assertion (i) is proven. Note that

$$\begin{aligned} \frac{1}{2n}\operatorname{tr}\Big[&(1-|\kappa_{2,0}|\,\overline{z}_0^2)\tilde{B}_{2,0}^*\tilde{B}_{2,0} + (1-|\kappa_{2,0}|\,z_0^2)\tilde{B}_{0,2}^*\tilde{B}_{0,2} \\ & - |\kappa_{2,0}|\,\lambda_0^2\tilde{B}_{0,2}\tilde{B}_{2,0} - |\kappa_{2,0}|\,\lambda_0^2\tilde{B}_{2,0}^*\tilde{B}_{0,2}^*\Big] \\ &= \frac{1}{n}\sum_{\alpha\in\mathcal{I}_{m,2}}\Bigg[(1-|\kappa_{2,0}|\,\overline{z}_0^2)\left|q_{\alpha\varnothing}^{(2,0)}\right|^2 + (1-|\kappa_{2,0}|\,z_0^2)\left|q_{\varnothing\alpha}^{(0,2)}\right|^2 \\ & + |\kappa_{2,0}|\,\lambda_0^2\left(q_{\alpha\varnothing}^{(2,0)}q_{\varnothing\alpha}^{(0,2)} + \overline{q_{\alpha\varnothing}^{(2,0)}q_{\varnothing\alpha}^{(0,2)}}\right)\Bigg]. \end{aligned} \tag{3.14}$$

Hence, in order to prove assertion (ii), it is enough to show that the quadratic form of x_1 and x_2

$$(1-|\kappa_{2,0}|\,\Re z_0^2)x_1^2 + (1-|\kappa_{2,0}|\,\Re z_0^2)x_2^2 \pm 2\,|\kappa_{2,0}|\,\lambda_0^2 x_1 x_2$$

is positive definite. A straightforward check yields

$$\begin{aligned} 1-|\kappa_{2,0}|\,\Re z_0^2 &> 0; \\ (1-|\kappa_{2,0}|\,\Re z_0^2)^2 - |\kappa_{2,0}|^2\,\lambda_0^4 &\ge (1-\left|\kappa_{2,0}z_0^2\right|)^2 - |\kappa_{2,0}|^2\,\lambda_0^4 \ge 0. \end{aligned} \tag{3.15}$$

Besides, if parameters $\kappa_{2,0}$ and z_0 are such as those in the assertion of Theorem 1.1, then inequality (3.15) is strict. □

Lemma 3.4. *Let* $\tilde{f}(Q_1, \hat{\boldsymbol{Q}}) = f(Q_1, \hat{\boldsymbol{Q}}) - f(\Lambda_0, 0)$*. Then for sufficiently large* n*,*

$$\max_{\frac{\log n}{\sqrt{n}} \le \|\Lambda-\Lambda_0\| + \|\hat{\boldsymbol{Q}}\| \le r} \Re\tilde{f}(U\Lambda V^*, \hat{\boldsymbol{Q}}) \le -C\frac{\log^2 n}{n}$$

uniformly in U *and* V*.*

Proof. First, let us check that the first and the second derivatives of f_r are bounded in the δ-neighborhood of Λ_0, where f_r is defined in (2.12) and δ is n-independent. Indeed, since h and h_0 are polynomials and $h \rightrightarrows h_0$ on compacts,

$$\left|\frac{1}{\sqrt{n}}\frac{\partial \Re f_r}{\partial x}\right| \le \left|\frac{1}{\sqrt{n}}\frac{\partial f_r}{\partial x}\right| = \left|\frac{\partial (f-f_0)}{\partial x}\right| = \left|\frac{\partial (\log h - \log h_0)}{\partial x}\right| \\ \le \left|\frac{1}{h_0}\frac{\partial h_0}{\partial x} - \frac{1}{h}\frac{\partial h}{\partial x}\right| \le \frac{C}{\sqrt{n}},$$

where x is either λ_j or an entry of $Q_{p,s}$, $(p,s) \neq (1,1)$. Let Λ_E be a real diagonal matrix of unit norm and let $\hat{\boldsymbol{Q}}_E$, $\|\hat{\boldsymbol{Q}}_E\| = 1$, be a set of matrices whose sizes correspond to those of $\hat{\boldsymbol{Q}}$. Then for any Λ_E and $\hat{\boldsymbol{Q}}_E$ and for $\frac{\log n}{\sqrt{n}} \le t \le \delta$, we have

$$\begin{aligned}\frac{d}{dt}\Re\tilde{f}(U(\Lambda_0 + t\Lambda_E)V^*, t\hat{\boldsymbol{Q}}_E) &= \langle \nabla_{\Lambda,\hat{\boldsymbol{Q}}}\Re f_0(U(\Lambda_0 + t\Lambda_E)V^*, t\hat{\boldsymbol{Q}}_E), v(E)\rangle \\ &\quad + n^{-1/2}\langle \nabla_{\Lambda,\hat{\boldsymbol{Q}}}\Re f_r(U(\Lambda_0 + t\Lambda_E)V^*, t\hat{\boldsymbol{Q}}_E), v(E)\rangle \\ &= \langle \nabla_{\Lambda,\hat{\boldsymbol{Q}}}\Re f_0(\Lambda_0 + t\Lambda_E, t\hat{\boldsymbol{Q}}_E), v(E)\rangle + O(n^{-1/2}),\end{aligned}$$

where $v(E)$ denotes a vector whose components are all the real variables of Λ_E and $\hat{\boldsymbol{Q}}_E$ and $\langle\cdot,\cdot\rangle$ is a standard real scalar product. Expanding the scalar product by the Taylor formula and considering that $\nabla_{\Lambda,\hat{\boldsymbol{Q}}} f_0(\Lambda_0, 0) = 0$, we obtain

$$\frac{d}{dt}\Re\tilde{f}(U(\Lambda_0 + t\Lambda_E)V^*, t\hat{\boldsymbol{Q}}_E) = t\langle(\Re f_0)''(\Lambda_0, 0)v(E), v(E)\rangle + r_1 + O(n^{-1/2}),$$

where $(\Re f_0)''$ is a matrix of second order derivatives of $\Re f_0$ with respect to Λ, $\Re\hat{\boldsymbol{Q}}$ and $\Im\hat{\boldsymbol{Q}}$ and $|r_1| \le Ct^2$. $(\Re f_0)''(\Lambda_0, 0)$ is negative definite according to Corollary 3.3. Hence $\frac{d}{dt}\Re\tilde{f}(U(\Lambda_0 + t\Lambda_E)V^*, t\hat{\boldsymbol{Q}}_E)$ is negative and

$$\begin{aligned}\max_{\frac{\log n}{\sqrt{n}} \le \|\Lambda - \Lambda_0\| + \|\hat{\boldsymbol{Q}}\| \le \delta} \Re\tilde{f}(U\Lambda V^*, \hat{\boldsymbol{Q}}) &= \max_{\|\Lambda - \Lambda_0\| + \|\hat{\boldsymbol{Q}}\| = \frac{\log n}{\sqrt{n}}} \Re\tilde{f}(U\Lambda V^*, \hat{\boldsymbol{Q}}) \\ &\le \Re f(U\Lambda_0 V^*, 0) - C\frac{\log^2 n}{n} - f(\Lambda_0, 0). \quad (3.16)\end{aligned}$$

Notice that f_r is bounded from above uniformly in n. This fact and Lemma 3.1 imply that δ in (3.16) can be replaced by r,

$$\max_{\frac{\log n}{\sqrt{n}} \le \|\Lambda - \Lambda_0\| + \|\hat{\boldsymbol{Q}}\| \le r} \Re\tilde{f}(U\Lambda V^*, \hat{\boldsymbol{Q}}) \le \Re f(U\Lambda_0 V^*, 0) - f(\Lambda_0, 0) - C\frac{\log^2 n}{n}.$$

It remains to deduce from Lemma 3.2 that $\Re f(U\Lambda_0 V^*, 0) - f(\Lambda_0, 0) = O(n^{-1})$ uniformly in U and V. □

Lemma 3.4 and (2.9) yield

$$\mathsf{f}_m = Cn^{c_m} e^{nf(\Lambda_0, 0)} \Bigg(\int_{\Omega_n} \triangle^2(\Lambda^2) \prod_{j=1}^{m} \lambda_j g(\boldsymbol{Q}) e^{-c_m f(U\Lambda V^*, \hat{\boldsymbol{Q}})}$$

$$\times\, e^{n\tilde{f}(U\Lambda V^*,\hat{\boldsymbol{Q}})} d\mu(U)d\mu(V)d\Lambda d\hat{\boldsymbol{Q}} + O(e^{-C_1\log^2 n})\Bigg),$$

where Ω_n is defined in (3.7). Changing the variables $\Lambda = \Lambda_0 + \frac{1}{\sqrt{n}}\tilde{\Lambda}$, $\hat{\boldsymbol{Q}} = \frac{1}{\sqrt{n}}\tilde{\hat{\boldsymbol{Q}}}$ and expanding f according to Lemma 3.2, we obtain

$$\begin{aligned}
\mathsf{f}_m = C\mathsf{k}_n \int_{\sqrt{n}\Omega_n} & \triangle^2(\tilde{\Lambda}) g(\boldsymbol{Q}_*) e^{-c_m f(\Lambda_0,0)} d\mu(U)d\mu(V)d\tilde{\Lambda}d\tilde{\hat{\boldsymbol{Q}}}(1+o(1))\\
&\times \exp\Big\{-\frac{1}{2}\operatorname{tr}(2\lambda_0\tilde{\Lambda} + \bar{z}_0\mathcal{Z}_U + z_0\mathcal{Z}_V^*)^2 + \operatorname{tr}\mathcal{Z}_U\mathcal{Z}_V^* - \Big\|\tilde{\boldsymbol{Q}}_{>1}\Big\|^2\\
&\frac{1}{2}\operatorname{tr}\Big[(1-|\kappa_{2,0}|\,\bar{z}_0^2)\tilde{B}_{2,0}^*\tilde{B}_{2,0} + (1-|\kappa_{2,0}|\,z_0^2)\tilde{B}_{0,2}^*\tilde{B}_{0,2}\\
&-|\kappa_{2,0}|\,\lambda_0^2\tilde{B}_{0,2}\tilde{B}_{2,0} - |\kappa_{2,0}|\,\lambda_0^2\tilde{B}_{2,0}^*\tilde{B}_{0,2}^*\Big]\Big\},
\end{aligned}\tag{3.17}$$

where

$$\mathsf{k}_n = n^{m^2/2} e^{-mn\lambda_0^2 + \sqrt{n}\operatorname{tr}(\bar{z}_0\mathcal{Z} + z_0\mathcal{Z}^*)}.\tag{3.18}$$

Since (3.14) the integral over $\tilde{\hat{\boldsymbol{Q}}}$ can be computed separately over real and imaginary parts of the entries of $\tilde{\hat{\boldsymbol{Q}}}$. Because $g(\boldsymbol{Q}_*)e^{-c_m f(\Lambda_0,0)} = 1 + o(1)$, the integration implies

$$\begin{aligned}
\mathsf{f}_m = C\mathsf{k}_n d_1(\kappa_{2,0})^{-m} \int & \triangle^2(\tilde{\Lambda}) d\mu(U)d\mu(V)d\tilde{\Lambda}(1+o(1))\\
&\times \exp\Big\{-\frac{1}{2}\operatorname{tr}(2\lambda_0\tilde{\Lambda} + \bar{z}_0\mathcal{Z}_U + z_0\mathcal{Z}_V^*)^2 + \operatorname{tr}\mathcal{Z}_U\mathcal{Z}_V^*\Big\},
\end{aligned}\tag{3.19}$$

where

$$d_1(\kappa_{2,0}) = \big|1 - |\kappa_{2,0}|\,z_0^2\big|^2 - |\kappa_{2,0}|^2\,\lambda_0^4.\tag{3.20}$$

Let us change the variables $V = WU$. Taking into account that the Haar measure is invariant with respect to shifts, we get

$$\begin{aligned}
\mathsf{f}_m &= C\mathsf{k}_n d_1(\kappa_{2,0})^{-m} \int_{\mathbb{R}^m}\int_{U(m)}\int_{U(m)} \triangle^2(\tilde{\Lambda}) d\mu(U)d\mu(W)d\tilde{\Lambda}(1+o(1))\\
&\quad\times \exp\left\{-\frac{1}{2}\operatorname{tr}(2\lambda_0\tilde{\Lambda} + U^*(\bar{z}_0\mathcal{Z} + z_0\mathcal{Z}_W^*)U)^2 + \operatorname{tr}\mathcal{Z}W^*\mathcal{Z}^*W\right\}\\
&= C\mathsf{k}_n d_1(\kappa_{2,0})^{-m} \int_{\mathbb{R}^m}\int_{U(m)}\int_{U(m)} \triangle^2(\tilde{\Lambda}) d\mu(U)d\mu(W)d\tilde{\Lambda}(1+o(1))\\
&\quad\times \exp\left\{-\frac{1}{2}\operatorname{tr}(2\lambda_0 U\tilde{\Lambda}U^* + (\bar{z}_0\mathcal{Z} + z_0\mathcal{Z}_W^*))^2 + \operatorname{tr}\mathcal{Z}W^*\mathcal{Z}^*W\right\}.
\end{aligned}$$

The next step is to change the variables $H = U\tilde{\Lambda}U^*$. The Jacobian is $\frac{\prod_{j=1}^{m-1} j!}{(2\pi)^{m(m-1)/2}}\triangle^{-2}(\tilde{\Lambda})$ (see, e.g., [40]). Thus,

$$\mathsf{f}_m = C\mathsf{k}_n d_1(\kappa_{2,0})^{-m} \int_{\mathcal{H}_m}\int_{U(m)} d\mu(W)dH(1+o(1))$$

$$\times \exp\left\{-\frac{1}{2}\operatorname{tr}(2\lambda_0 H + (\bar{z}_0\mathcal{Z} + z_0\mathcal{Z}_W^*))^2 + \operatorname{tr}\mathcal{Z}W^*\mathcal{Z}^*W\right\},$$

where $\mathcal{H}_m$ is a space of hermitian $m\times m$ matrices, and

$$dH = \prod_{j=1}^{m} d(H)_{jj}\prod_{j<k} d\Re(H)_{jk}d\Im(H)_{jk}.$$

The Gaussian integration over H implies

$$\mathsf{f}_m = C\mathsf{k}_n d_1(\kappa_{2,0})^{-m}\int_{U(m)}\exp\left\{\operatorname{tr}\mathcal{Z}W^*\mathcal{Z}^*W\right\}d\mu(W)(1+o(1)). \tag{3.21}$$

For computing the integral over the unitary group, the following Harish–Chandra/Itsykson–Zuber formula is used

Proposition 3.5. *Let A and B be normal $d\times d$ matrices with distinct eigenvalues $\{a_j\}_{j=1}^d$ and $\{b_j\}_{j=1}^d$, respectively. Then*

$$\int_{U(d)}\exp\{z\operatorname{tr}AU^*BU\}d\mu(U) = \left(\prod_{j=1}^{d-1} j!\right)\frac{\det\{\exp(za_jb_k)\}_{j,k=1}^d}{z^{(d^2-d)/2}\triangle(A)\triangle(B)},$$

where z is some constant, μ is a Haar measure, and $\triangle(A) = \prod_{j>k}(a_j - a_k)$.

For the proof, see, e.g., [44, Appendix 5].

Applying the Harish–Chandra/Itsykson–Zuber formula to (3.21), we obtain

$$\mathsf{f}_m = C\mathsf{k}_n e^{-m\log d_1(\kappa_{2,0})}\frac{\det\{e^{\zeta_j\bar{\zeta}_k}\}_{j,k=1}^m}{|\triangle(\mathcal{Z})|^2}(1+o(1)),$$

which in combination with (2.13) yields the result of Theorem 1.1.

3.2. General case. In the general case, the proof proceeds by the same scheme as in the case of zero high cumulants. In this subsection, we focus on the crucial distinctions from the partial case considered above and refine the corresponding assertions from the previous subsection.

At the point we are ready to generalize Lemma 3.2.

Lemma 3.6. *Let $\|\tilde{\Lambda}\| + \|\hat{\tilde{\boldsymbol{Q}}}\| \le \log n$. Then uniformly in U and V,*

$$\begin{aligned} f(U(\Lambda_0 + n^{-1/2}\tilde{\Lambda})V^*, n^{-1/2}\hat{\tilde{\boldsymbol{Q}}}) &= -m\lambda_0^2 + n^{-1/2}\operatorname{tr}(\bar{z}_0\mathcal{Z} + z_0\mathcal{Z}^*) \\ &- \frac{1}{2n}\operatorname{tr}(2\lambda_0\tilde{\Lambda} + \bar{z}_0\mathcal{Z}_U + z_0\mathcal{Z}_V^*)^2 + \frac{1}{n}\operatorname{tr}\mathcal{Z}_U\mathcal{Z}_V^* \\ &- \frac{1}{2n}\operatorname{tr}\Big[(1-|\kappa_{2,0}|\,\bar{z}_0^2)\tilde{B}_{2,0}^*\tilde{B}_{2,0} + (1-|\kappa_{2,0}|\,z_0^2)\tilde{B}_{0,2}^*\tilde{B}_{0,2} \\ &- |\kappa_{2,0}|\,\lambda_0^2\tilde{B}_{0,2}\tilde{B}_{2,0} - |\kappa_{2,0}|\,\lambda_0^2\tilde{B}_{2,0}^*\tilde{B}_{0,2}^*\Big] - \frac{1}{n}\left\|\tilde{\boldsymbol{Q}}_{>1}\right\|^2 \end{aligned}$$

$$+ n^{-1} \quad \lambda_0^2\sqrt{\kappa_{2,2}}\,\mathrm{tr}\big[(\wedge^2 VU^*)\tilde{Q}_2 + \tilde{Q}_2^*(\wedge^2 UV^*)\big] + O\big(n^{-3/2}\log^3 n\big), \quad (3.22)$$

where we keep the notations of Lemma 3.2, and $\wedge^2 B$ is the second exterior power of a linear operator B (see [59] for the definition and properties of an exterior power of a linear operator).

Proof. Differently from the previous subsection, the function f has an additional term $n^{-1/2}\tilde{h}(\boldsymbol{Q}_1, \boldsymbol{Q}_2) + n^{-1}\mathtt{p}_c(\hat{\boldsymbol{Q}})$ under the logarithm, where $\tilde{h}$ is defined in (2.5) and $\mathtt{p}_c$ is a polynomial such that $\mathtt{p}_c(0) = 0$. Therefore, the contribution of the term $n^{-1}\mathtt{p}_c(n^{-1/2}\hat{\tilde{\boldsymbol{Q}}})$ is $O\big(n^{-3/2}\log n\big)$. Hence, it remains to determine the contribution of the term $n^{-1/2}\tilde{h}(\boldsymbol{Q}_1, \boldsymbol{Q}_2)$,

$$\begin{aligned} &n^{-\frac{1}{2}}\tilde{h}(n^{-\frac{1}{2}}\tilde{\boldsymbol{Q}}_1, n^{-\frac{1}{2}}\tilde{\boldsymbol{Q}}_2) = n^{-1}\tilde{h}(n^{-\frac{1}{2}}\tilde{\boldsymbol{Q}}_1, \tilde{\boldsymbol{Q}}_2) \\ &= -\frac{1}{n}\int \sum_{p+s=4}\Big(\mathrm{tr}\,\tilde{Y}_{p,s}\tilde{Q}_{p,s} + \mathrm{tr}\,\tilde{Q}^*_{p,s}Y_{p,s}\Big)\, e^{-\frac{1}{2}\boldsymbol{\rho}^T F\left(\frac{1}{\sqrt{n}}\boldsymbol{Q}_1\right)\boldsymbol{\rho}} d\boldsymbol{\phi}^+ d\boldsymbol{\phi} d\boldsymbol{\theta}^+ d\boldsymbol{\theta}, \end{aligned} \quad (3.23)$$

where $\boldsymbol{\rho}$ is defined in (2.6), F is defined in (2.4), $\tilde{Y}_{p,s}$ and $Y_{p,s}$ are defined by (2.7), (2.8).

Let us change the variables $\tilde{\boldsymbol{\phi}} = U^*\boldsymbol{\phi}$, $\tilde{\boldsymbol{\phi}}^+ = \boldsymbol{\phi}^+ U$, $\tilde{\boldsymbol{\theta}} = V^*\boldsymbol{\theta}$, $\tilde{\boldsymbol{\theta}}^+ = \boldsymbol{\theta}^+ V$. We have

$$\begin{aligned} \frac{1}{\sqrt{\kappa_{p,s}}}\, y^{(p,s)}_{\alpha\beta} &= \prod_{q=1}^{p}\phi_{\alpha_q}\prod_{r=1}^{s}\theta^*_{\beta_r} = \prod_{q=1}^{p}(U\tilde{\boldsymbol{\phi}})_{\alpha_q}\prod_{r=1}^{s}(\tilde{\boldsymbol{\theta}}^+ V^*)_{\beta_r} \\ &= \prod_{q=1}^{p}\sum_{\gamma_q=1}^{m} u_{\alpha_q\gamma_q}\tilde{\phi}_{\gamma_q}\prod_{r=1}^{s}\sum_{\delta_r=1}^{m}\tilde{\theta}^*_{\delta_r}\bar{v}_{\beta_r\delta_r} \\ &=: \sum_{\gamma\in\mathcal{I}_{m,p}}\sum_{\delta\in\mathcal{I}_{m,s}} a^{(p,s)}_{\alpha\beta\gamma\delta}\prod_{q=1}^{p}\tilde{\phi}_{\gamma_q}\prod_{r=1}^{s}\tilde{\theta}^*_{\delta_r}, \end{aligned} \quad (3.24)$$

where $a^{(p,s)}_{\alpha\beta\gamma\delta}$ just denotes the coefficient at $\prod_{q=1}^{p}\tilde{\phi}_{\gamma_q}\prod_{r=1}^{s}\tilde{\theta}^*_{\delta_r}$. Similarly,

$$\frac{1}{\sqrt{\kappa_{p,s}}}\,\tilde{y}^{(p,s)}_{\beta\alpha} = \sum_{\gamma\in\mathcal{I}_{m,p}}\sum_{\delta\in\mathcal{I}_{m,s}}\tilde{a}^{(p,s)}_{\beta\alpha\delta\gamma}\prod_{r=s}^{1}\tilde{\theta}_{\delta_r}\prod_{q=p}^{1}\tilde{\phi}^*_{\gamma_q} \quad (3.25)$$

Besides,

$$\boldsymbol{\rho}^T F\boldsymbol{\rho} = \tilde{\boldsymbol{\rho}}^T F_0\tilde{\boldsymbol{\rho}} + O(n^{-1/2}\log n), \quad (3.26)$$

where F_0 is defined in (3.1) and

$$\tilde{\boldsymbol{\rho}} = \begin{pmatrix} (\tilde{\boldsymbol{\phi}}^+)^T \\ (\tilde{\boldsymbol{\theta}}^+)^T \\ \tilde{\boldsymbol{\phi}} \\ \tilde{\boldsymbol{\theta}} \end{pmatrix}.$$

The “measure” changes as follows:

$$\begin{aligned} d\boldsymbol{\phi}^+ d\boldsymbol{\phi} d\boldsymbol{\theta}^+ d\boldsymbol{\theta} &= \det{}^{-1} U \det{}^{-1} U^* \det{}^{-1} V \det{}^{-1} V^* d\tilde{\boldsymbol{\phi}}^+ d\tilde{\boldsymbol{\phi}} d\tilde{\boldsymbol{\theta}}^+ d\tilde{\boldsymbol{\theta}} \\ &= d\tilde{\boldsymbol{\phi}}^+ d\tilde{\boldsymbol{\phi}} d\tilde{\boldsymbol{\theta}}^+ d\tilde{\boldsymbol{\theta}}. \end{aligned} \tag{3.27}$$

Eventually, substitution of (3.24)–(3.27) into (3.23) yields

$$\begin{aligned} n^{-1}\tilde{h}(n^{-\frac{1}{2}}\tilde{\boldsymbol{Q}}_1, \tilde{\boldsymbol{Q}}_2) = -\frac{1}{n}\int & e^{-\frac{1}{2}\tilde{\boldsymbol{\rho}}^+ F_0 \tilde{\boldsymbol{\rho}}} d\tilde{\boldsymbol{\phi}}^+ d\tilde{\boldsymbol{\phi}} d\tilde{\boldsymbol{\theta}}^+ d\tilde{\boldsymbol{\theta}} \\ & \times \sum_{p+s=4} \sum_{\substack{\alpha,\gamma\in\mathcal{I}_{m,p}\\ \beta,\delta\in\mathcal{I}_{m,s}}} \Big(\sqrt{\kappa_{p,s}} \tilde{a}^{(p,s)}_{\beta\alpha\delta\gamma} \tilde{q}^{(p,s)}_{\alpha\beta} \prod_{r=s}^{1} \tilde{\theta}_{\delta_r} \prod_{q=p}^{1} \tilde{\phi}^*_{\gamma_q} \\ & + \sqrt{\kappa_{p,s}} \bar{\tilde{q}}^{(p,s)}_{\alpha\beta} a^{(p,s)}_{\alpha\beta\gamma\delta} \prod_{q=1}^{p} \tilde{\phi}_{\gamma_q} \prod_{r=1}^{s} \tilde{\theta}^*_{\delta_r} \Big) \\ & + O\big(n^{-3/2}\log^3 n\big) \end{aligned} \tag{3.28}$$

uniformly in U and V.

The integration in (3.28) can be performed over $\tilde{\phi}_j$, $\tilde{\theta}_j$ separately for every j due to the structure of F_0. Thus it remains to compute the integrals of the form

$$\int \prod_{q=1}^{p} \tilde{\phi}_{\gamma_q} \prod_{r=1}^{s} \tilde{\theta}^*_{\delta_r} \exp\Big\{ \bar{z}_0 \tilde{\theta}_j \tilde{\theta}^*_j + \lambda_0 \tilde{\theta}_j \tilde{\phi}^*_j + z_0 \tilde{\phi}_j \tilde{\phi}^*_j - \lambda_0 \tilde{\phi}_j \tilde{\theta}^*_j \Big\} d\tilde{\phi}^*_j d\tilde{\phi}_j d\tilde{\theta}^*_j d\tilde{\theta}_j$$

Furthermore, expanding the exponent into series, one can observe that all the integrals are non-zero only if $p = s = 2$ and $\gamma = \delta$. Moreover,

$$\begin{aligned} \int \tilde{\phi}_j \tilde{\theta}^*_j e^{z_0\tilde{\theta}_j\tilde{\theta}^*_j + \lambda_0\tilde{\theta}_j\tilde{\phi}^*_j + z_0\tilde{\phi}_j\tilde{\phi}^*_j - \lambda_0\tilde{\phi}_j\tilde{\theta}^*_j} d\tilde{\phi}^*_j d\tilde{\phi}_j d\tilde{\theta}^*_j d\tilde{\theta}_j &= -\lambda_0, \\ \int \tilde{\theta}_j \tilde{\phi}^*_j e^{z_0\tilde{\theta}_j\tilde{\theta}^*_j + \lambda_0\tilde{\theta}_j\tilde{\phi}^*_j + z_0\tilde{\phi}_j\tilde{\phi}^*_j - \lambda_0\tilde{\phi}_j\tilde{\theta}^*_j} d\tilde{\phi}^*_j d\tilde{\phi}_j d\tilde{\theta}^*_j d\tilde{\theta}_j &= \lambda_0, \\ \int e^{z_0\tilde{\theta}_j\tilde{\theta}^*_j + \lambda_0\tilde{\theta}_j\tilde{\phi}^*_j + z_0\tilde{\phi}_j\tilde{\phi}^*_j - \lambda_0\tilde{\phi}_j\tilde{\theta}^*_j} d\tilde{\phi}^*_j d\tilde{\phi}_j d\tilde{\theta}^*_j d\tilde{\theta}_j &= 1. \end{aligned}$$

The last thing we need is the values of $a^{(2,2)}_{\alpha\beta\gamma\delta}$ and $\tilde{a}^{(2,2)}_{\beta\alpha\delta\gamma}$. The formula (3.24) implies

$$a^{(2,2)}_{\alpha\beta\gamma\delta} = (\wedge^2 U)_{\alpha\gamma} (\wedge^2 V^*)_{\delta\beta}.$$

Similarly,

$$\tilde{a}^{(2,2)}_{\beta\alpha\delta\gamma} = (\wedge^2 V)_{\beta\gamma} (\wedge^2 U^*)_{\delta\alpha}.$$

Finally,

$$\begin{aligned} \frac{1}{n}\tilde{h}(n^{-\frac{1}{2}}\tilde{\boldsymbol{Q}}_1, \tilde{\boldsymbol{Q}}_2) &= \frac{1}{n}\lambda_0^2 \sqrt{\kappa_{2,2}} (\operatorname{tr}(\wedge^2 U^*) \tilde{Q}_2 (\wedge^2 V) + \operatorname{tr}(\wedge^2 V^*) \tilde{Q}^*_2 (\wedge^2 U)) + o\Big(\frac{1}{n}\Big) \\ &= n^{-1}\lambda_0^2 \sqrt{\kappa_{2,2}} (\operatorname{tr}(\wedge^2 V U^*) \tilde{Q}_2 + \operatorname{tr} \tilde{Q}^*_2 (\wedge^2 U V^*)) \\ &\quad + O\big(n^{-3/2}\log^3 n\big). \end{aligned}$$

The above relation completes the proof of (3.22). □

Lemma 3.4 is still valid in the general case, despite the proof needs some insignificant changes due to a non-zero term $n^{-1/2}\tilde{h}(\boldsymbol{Q}_1, \boldsymbol{Q}_2)$.

Following the proof in the Gaussian case, one can see that (3.17) transforms into

$$\begin{aligned}
\mathsf{f}_m = C\mathsf{k}_n \int_{\sqrt{n}\Omega_n} & \triangle^2(\tilde{\Lambda}) g(\boldsymbol{Q}_*) e^{-c_m f(\Lambda_0,0)} d\mu(U) d\mu(V) d\tilde{\Lambda} d\hat{\tilde{\boldsymbol{Q}}}(1+o(1)) \\
& \times \exp\Big\{ -\frac{1}{2}\operatorname{tr}(2\lambda_0\tilde{\Lambda} + \bar{z}_0\mathcal{Z}_U + z_0\mathcal{Z}_V^*)^2 + \operatorname{tr}\mathcal{Z}_U\mathcal{Z}_V^* - \Big\| \tilde{\boldsymbol{Q}}_{>1} \Big\|^2 \\
& - \frac{1}{2}\operatorname{tr}\Big[(1 - |\kappa_{2,0}|\,\bar{z}_0^2)\tilde{B}_{2,0}^*\tilde{B}_{2,0} + (1 - |\kappa_{2,0}|\,z_0^2)\tilde{B}_{0,2}^*\tilde{B}_{0,2} \\
& - |\kappa_{2,0}|\,\lambda_0^2 \tilde{B}_{0,2}\tilde{B}_{2,0} - |\kappa_{2,0}|\,\lambda_0^2\tilde{B}_{2,0}^*\tilde{B}_{0,2}^* \Big] \\
& + \lambda_0^2\sqrt{\kappa_{2,2}}\operatorname{tr}\big[(\wedge^2 VU^*)\tilde{Q}_2 + \tilde{Q}_2^*(\wedge^2 UV^*) \big] \Big\},
\end{aligned}$$

where k_n is defined in (3.18). The Gaussian integration over $\hat{\tilde{\boldsymbol{Q}}}$ yields

$$\begin{aligned}
\mathsf{f}_m = C\mathsf{k}_n d_1(\kappa_{2,0})^{-m} & \int \triangle^2(\tilde{\Lambda}) d\mu(U) d\mu(V) d\tilde{\Lambda}(1+o(1)) \\
& \times \exp\Big\{ -\frac{1}{2}\operatorname{tr}(2\lambda_0\tilde{\Lambda} + \bar{z}_0\mathcal{Z}_U + z_0\mathcal{Z}_V^*)^2 + \operatorname{tr}\mathcal{Z}_U\mathcal{Z}_V^* \\
& + \lambda_0^4\kappa_{2,2}\operatorname{tr}\big[(\wedge^2 VU^*)(\wedge^2 UV^*) \big] \Big\}.
\end{aligned}$$

Notice that $\wedge^2 VU^*$ and $\wedge^2 UV^*$ are mutually inverse matrices. Therefore,

$$\begin{aligned}
\mathsf{f}_m = C\mathsf{k}_n d_1(\kappa_{2,0})^{-m} \exp\left\{ \frac{m^2 - m}{2}\lambda_0^4\kappa_{2,2} \right\} & \int \triangle^2(\tilde{\Lambda}) d\mu(U) d\mu(V) d\tilde{\Lambda}(1+o(1)) \\
\times \exp\Big\{ -\frac{1}{2}\operatorname{tr}(2\lambda_0\tilde{\Lambda} + \bar{z}_0\mathcal{Z}_U + z_0\mathcal{Z}_V^*)^2 + \operatorname{tr}\mathcal{Z}_U\mathcal{Z}_V^* \Big\}. &
\end{aligned}$$

The last formula differs from (3.19) only by a factor $\exp\left\{\frac{m^2-m}{2}\lambda_0^4\kappa_4\right\}$. Hence, there are no differences in further proof up to this factor.

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Про кореляційні функції характеристичних поліномів випадкових матриць з незалежними елементами: інтерполяція між комплексним і дійсним випадками

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У роботі розглянуто кореляційні функції характеристичних поліномів випадкових матриць з незалежними комплексними елементами. Ми дослідили те, як асимптотична поведінка кореляційних функцій залежить від другого моменту спільного закону розподілу ймовірностей для матричних елементів, при цьому другий момент можна трактувати як свого роду "міру дійсності" елементів. Показано, що кореляційні функції ведуть себе таким же чином, як і у випадку комплексного ансамблю Жинібра, з точністю до множника, що залежить лише від другого моменту та абсолютного четвертого моменту спільного розподілу ймовірностей матричних елементів.

Ключові слова: теорія випадкових матриць, ансамбль Жинібра, кореляційні функції характеристичних поліномів, моменти характеристичних поліномів, суперсиметрія