Journal of Mathematical Physics, Analysis, Geometry
2022, Vol. 18, No. 2, pp. 194–208
doi: https://doi.org/10.15407/mag18.02.194

On a Characterization of Frames for Operators in Quaternionic Hilbert Spaces

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In the present paper, we introduce the concepts of atomic systems for operators and K-frames in separable quaternionic Hilbert spaces. These concepts lead to a generalization of frames that have recently been studied in [18], and allow us to reconstruct elements from the range of a linear bounded operator in a separable quaternionic Hilbert space.

Key words: frames, atomic systems, K-frames, quaternionic Hilbert spaces

Mathematical Subject Classification 2010: 42C15, 41A58

1. Introduction

Frames, which are systems that provide robust, stable and usually non-unique representations of vectors, have been well studied in literature. In fact, the notion of frames dates backs to 1952 and was introduced in the pioneeristic paper of R.J. Duffin and A.C. Schaeffer [12] in the context of nonharmonic Fourier series. After some decades, I. Daubechies et al. [10] announced formally the characterization of frames in the abstract Hilbert spaces. This characterization has attracted a significant interest and has been generalized in many remarkable ways by several authors such as O. Christensen [6–9], A. Jeribi [17], and R.M. Young [19].

In fact, frames can be viewed as generalization of orthonormal and Riesz bases studied in [2–5, 13, 14, 17]. They are a redundant set of vectors which yield a representation for each vector in the space and have nice properties making them very useful in wavelet analysis, irregular sampling theory, signal processing and many other fields.

In 2012, frames were generalized by L. Găvruţa [15] who introduced the notion of K-frames in order to study the atomic systems with respect to a bounded linear operator K in a separable Hilbert space. This generalization of frames allows to reconstruct elements from the range of a linear bounded operator which is not a closed subspace. Moreover, many properties for ordinary frames may not hold for K-frames, such as the corresponding synthesis operator for K-frames is not surjective, the frame operator for K-frames is not isomorphic and the alternate dual reconstruction pair for K-frames is not interchangeable.

Recently, in [18] S.K. Sharma and S. Goel introduced and studied the existence of frames in separable quaternionic Hilbert spaces. They also gave a characterization of frames in terms of frame operators in a quaternionic Hilbert setting.

In this paper, we extend the study of frames in quaternionic Hilbert spaces by the notion of K-frames in the sense that the lower frame bound holds only for the elements in the range of a bounded linear operator K. The motivation of these systems is given by some specific problems where we may not find any possible frame, but we can find a K-frame because this notion is weaker and we would want to decompose just the range of K, $R(K)$. Indeed, if we consider in a three dimensional right quaternionic Hilbert space $V_R(\mathfrak{Q})$ and the operator $K \in \mathcal{L}(V_R(\mathfrak{Q}))$ defined by

$$Ke_1 = e_1, \quad Ke_2 = e_1, \quad Ke_3 = e_2,$$

where $\{e_n\}_{n=1}^3$ is a Hilbert basis, then, clearly, $\{f_n\}_{n=1}^3 := \{e_1, e_1, e_2\}$ is not a frame for $V_R(\mathfrak{Q})$ since it does not possess a lower frame bound. However, it is a K-frame (see Example 4.2).

The results in this paper are organized as follows. In Section 2, we recall basic definitions, known results in quaternionic Hilbert spaces. In Section 3, we present basic notions about atomic systems and we extend, in Section 4, the concept of K-frames to quaternionic Hilbert spaces. In the last section, we close this paper by applications of the obtained results in Sections 3 and 4, to give new characterizations of families of local atoms.

2. Quaternionic Hilbert space

As quaternions are non-commutative in nature, therefore there are two different types of quaternionic Hilbert spaces, the left quaternionic Hilbert space and the right quaternionic Hilbert space depending on positions of quaternions. This fact can entail several problems. For example, when a Hilbert space H is one-sided (either left or right), the set of linear operators acting on it does not have a quaternionic linear structure. In this section, we will study some basic notations about the algebra of quaternions, right quaternionic Hilbert space and operators on right quaternionic Hilbert spaces.

Next, we denote by $\mathfrak{Q}$ the skew field of quaternions, whose elements are in the form $q = x_0 + x_1 i + x_2 j + x_3 k$, where x_0, x_1, x_2 and x_3 are real numbers and i, j, k are called imaginary units and obey the following multiplication rules:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad \text{and} \quad ki = -ik = j.$$

For more information about the properties of quaternions, we refer the readers to [1, 16].

Let $V_R(\mathfrak{Q})$ be a linear vector space over the skew field of quaternions under right scalar multiplication. It is called a quaternionic pre-Hilbert space if there exists a Hermitian quaternionic scalar product, that is, a map

$$\langle \cdot, \cdot \rangle : V_R(\mathfrak{Q}) \times V_R(\mathfrak{Q}) \to \mathfrak{Q}$$

satisfying, for every $u, v, w \in V_R(\mathfrak{Q})$ and $p, q \in \mathfrak{Q}$, the following properties:

(i) $\langle u, v\rangle = \overline{\langle v, u\rangle}$, the quaternionic hermiticity,

(ii) $\langle u, u\rangle > 0$ unless $u = 0$,

(iii) $\langle u, vp + wq\rangle = \langle u, v\rangle p + \langle u, w\rangle q$.

Suppose that $V_R(\mathfrak{Q})$ is equipped with such a Hermitian quaternionic scalar product. Then we can define the quaternionic norm $\|\cdot\| : V_R(\mathfrak{Q}) \to \mathbb{R}_+$ by setting

$$\|u\| = \sqrt{\langle u, u\rangle}, \quad u \in V_R(\mathfrak{Q}). \tag{2.1}$$

It has been mentioned in [16] that the quaternionic norm satisfies all properties of a norm including Cauchy–Schwarz inequality and parallelogram identity.

The right quaternionic pre-Hilbert space $V_R(\mathfrak{Q})$ is said to be a right quaternionic Hilbert space if it is complete with respect to the norm (2.1).

Throughout this paper, we denote an index set by I and assume that the space $V_R(\mathfrak{Q})$ is always separable. Thus, it shares many of the standard properties of complex separable Hilbert spaces such as Hilbert basis. Let us recall the following results:

Proposition 2.1 ([16]). *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and N be a subset of $V_R(\mathfrak{Q})$ such that for $z, z' \in N$, $\langle z, z'\rangle = 0$ if $z \neq z'$ and $\langle z, z'\rangle = 1$. Then, the following assertions are equivalent:*

(i) *for every $u, v \in V_R(\mathfrak{Q})$, the series $\sum_{z\in N}\langle u, z\rangle\langle z, v\rangle$ converges absolutely and it holds that*
$$\langle u, v\rangle = \sum_{z\in N}\langle u, z\rangle\langle z, v\rangle,$$

(ii) $\|u\|^2 = \sum_{z\in N} |\langle z, u\rangle|^2$ *for every $u \in V_R(\mathfrak{Q})$,*

(iii) $N^\perp := \{v \in V_R(\mathfrak{Q}) : \quad \langle v, z\rangle = 0, \quad \forall z \in N\} = \{0\}$,

(iv) $\mathrm{span}(N)$ *is dense in $V_R(\mathfrak{Q})$.*

Remark 2.2. The subset N in Proposition 2.1 is called a Hilbert basis.

Proposition 2.3. *A quaternionic Hilbert space turns out to be separable as a metrical space if and only if it admits a finite or countable Hilbert basis.*

Proof. The necessary condition is clear. Now, we will show that the sufficient condition holds. Suppose that $V_R(\mathfrak{Q})$ is separable, then we can choose a sequence $\{f_k\}_{k\in I}$ in $V_R(\mathfrak{Q})$ such that $\overline{\mathrm{span}}\{f_k\}_{k\in I} = V_R(\mathfrak{Q})$. By extracting a subsequence, if necessary, we can assume that for each $n \in I, f_{n+1} \notin \mathrm{span}\{f_k\}_{k\in I_n}$, where $I_n \subseteq I$ and $|I_n| < +\infty$. By applying the Gram–Schmidt process to $\{f_k\}_{k\in I}$, we obtain an orthonormal system $\{e_k\}_{k\in I}$ in $V_R(\mathfrak{Q})$ for which $\overline{\mathrm{span}}\{e_k\}_{k\in I} = \overline{\mathrm{span}}\{f_k\}_{k\in I} = V_R(\mathfrak{Q})$. $\square$

Proposition 2.4 ([16]). *Every right quaternionic Hilbert space admits a Hilbert basis, and two Hilbert bases have the same cardinality. Furthermore, if N is a Hilbert basis of $V_R(\mathfrak{Q})$, then every $u \in V_R(\mathfrak{Q})$ can be uniquely decomposed as follows:*

$$u = \sum_{z \in N} z\langle z, u\rangle,$$

where the series $\sum_{z\in N} z\langle z, u\rangle$ converges absolutely in $V_R(\mathfrak{Q})$.

Remark 2.5. It is worth mentioning that the absolute convergence of the series given in Proposition 2.4 relies on the fact that absolute convergence is equivalent to unconditional convergence. For more details, see [16, 18].

Next, we will define right linear operators and recall some basis properties as needed for the development of this manuscript.

Definition 2.6. Let $V_R(\mathfrak{Q})$ and $U_R(\mathfrak{Q})$ be two quaternionic Hilbert spaces. A right linear operator is a map $T : \mathcal{D}(T) \subseteq V_R(\mathfrak{Q}) \to U_R(\mathfrak{Q})$ such that

$$T(up + v) = (Tu)p + Tv \quad \text{if } u, v \in \mathcal{D}(T) \text{ and } p \in \mathfrak{Q},$$

where the domain $\mathcal{D}(T)$ of T is a (not necessarily closed) right $\mathfrak{Q}$-linear subspace of $V_R(\mathfrak{Q})$.

As in the complex case, if $T : \mathcal{D}(T) \subseteq V_R(\mathfrak{Q}) \to U_R(\mathfrak{Q})$ is any right linear operator, we define $\|T\|$ by setting

$$\begin{aligned} \|T\| : &= \sup_{u\in\mathcal{D}(T)\setminus\{0\}} \frac{\|Tu\|_{U_R(\mathfrak{Q})}}{\|u\|_{V_R(\mathfrak{Q})}} \\ &= \inf\{K > 0 \mid \forall u \in \mathcal{D}(T) \quad \|Tu\|_{U_R(\mathfrak{Q})} \leq K\|u\|_{V_R(\mathfrak{Q})}\}. \end{aligned} \tag{2.2}$$

The right linear operator T is bounded if $\|T\| < +\infty$.

We close this section with the following definition of the notion of adjoint operator which is similar to that for complex Hilbert spaces.

Definition 2.7 ([16]). Let $V_R(\mathfrak{Q})$ and $U_R(\mathfrak{Q})$ be two right quaternionic Hilbert spaces and let $T : \mathcal{D}(T) \subseteq V_R(\mathfrak{Q}) \to U_R(\mathfrak{Q})$ be an operator with dense domain. The adjoint $T^* : \mathcal{D}(T^*) \subseteq U_R(\mathfrak{Q}) \to V_R(\mathfrak{Q})$ of T is the unique operator with the following properties:

$$\mathcal{D}(T^*) := \{u \in U_R(\mathfrak{Q}) \mid \exists w_u \in V_R(\mathfrak{Q})\ \forall v \in \mathcal{D}(T) \quad \langle w_u, v\rangle = \langle u, Tv\rangle\}$$

and

$$\langle T^*u, v\rangle = \langle u, Tv\rangle \quad \text{for all } v \in \mathcal{D}(T),\ u \in \mathcal{D}(T^*). \tag{2.3}$$

Until further notice we will consider only operators T with $\mathcal{D}(T) = V_R(\mathfrak{Q})$. We denote the set of all bounded right linear operators from $V_R(\mathfrak{Q})$ to $U_R(\mathfrak{Q})$ by $\mathcal{L}(V_R(\mathfrak{Q}), U_R(\mathfrak{Q}))$. Moreover, if $V_R(\mathfrak{Q}) = U_R(\mathfrak{Q})$, then $\mathcal{L}(V_R(\mathfrak{Q}), U_R(\mathfrak{Q}))$ is replaced by $\mathcal{L}(V_R(\mathfrak{Q}))$.

Remark 2.8.

(i) It was shown in [16] that the set of all bounded right linear operators is a complete normed space with the norm defined by (2.2).

(ii) If $T \in \mathcal{L}(V_R(\mathfrak{Q}), U_R(\mathfrak{Q}))$, then requirement (2.3) automatically determines T^* as an element of $\mathcal{L}(U_R(\mathfrak{Q}), V_R(\mathfrak{Q}))$.

3. Atomic systems in quaternionic Hilbert spaces

In this section, we define the concept of atomic system and present some properties relative to this notion. We begin with the following definitions of frames and Bessel sequences in separable right quaternionic Hilbert spaces $V_R(\mathfrak{Q})$.

Definition 3.1 ([18]). A family $\{f_n\}_{n\in I}$ is said to be a frame for $V_R(\mathfrak{Q})$ if there exist two positive constants $0 < A \leq B$ such that

$$A\|x\|^2 \leq \sum_{n\in I} |\langle f_n, x\rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in V_R(\mathfrak{Q}). \tag{3.1}$$

The numbers A and B are called a lower and an upper frame bounds.

Definition 3.2 ([18]). A family $\{f_n\}_{n\in I}$ is said to be a Bessel sequence for $V_R(\mathfrak{Q})$ with bound B if $\{f_n\}_{n\in I}$ satisfies the right-most inequality in the right-hand side of equation (3.1).

Now, let us define the space $l^2(\mathfrak{Q})$ by

$$l^2(\mathfrak{Q}) := \left\{ \{q_i\}_{i\in I} \subset \mathfrak{Q} \mid \sum_{i\in I} |q_i|^2 < +\infty \right\}$$

equipped with the quaternionic inner product

$$\langle p, q\rangle = \sum_{i\in I} \overline{p_i} q_i, \quad p = \{p_i\}_{i\in I}, \quad \text{and} \quad q = \{q_i\}_{i\in I} \in l^2(\mathfrak{Q}). \tag{3.2}$$

It is easy to see that $l^2(\mathfrak{Q})$ is a right quaternionic Hilbert space with respect to the above inner product.

Bellow, we will introduce the concept of atomic system.

Definition 3.3. A family $\{f_n\}_{n\in I}$ of $V_R(\mathfrak{Q})$ is called an atomic system for $K \in \mathcal{L}(V_R(\mathfrak{Q}))$ if the following statements hold:

(i) the series $\sum_{n\in I} f_n c_n$ converges unconditionally for all $c = \{c_n\}_{n\in I} \in l^2(\mathfrak{Q})$,

(ii) there exists $C > 0$ such that for every $x \in V_R(\mathfrak{Q})$ there exists $a_x = \{a_n\}_{n\in I} \in l^2(\mathfrak{Q})$ such that $\|a_x\|_{l^2(\mathfrak{Q})} \leq C\|x\|$ and $Kx = \sum_{n\in I} f_n a_n$.

Proposition 3.4. *If $\{f_n\}_{n\in I}$ is a sequence in $V_R(\mathfrak{Q})$ and the series $\sum_{n\in I} f_n c_n$ converges unconditionally for all $\{c_n\}_{n\in I} \in l^2(\mathfrak{Q})$, then $\{f_n\}_{n\in I}$ is a Bessel sequence.*

Proof. Let us consider the linear operator

$$T : l^2(\mathfrak{Q}) \to V_R(\mathfrak{Q}), \quad T\{c_k\}_{k\in I} := \sum_{k\in I} f_k c_k$$

and the sequence of bounded linear operators

$$T_n : l^2(\mathfrak{Q}) \to V_R(\mathfrak{Q}), \quad T_n\{c_k\}_{k\in I} := \sum_{k\in I_n} f_k c_k,$$

where $I_n \subseteq I$ and $|I_n| < +\infty$. Clearly $T_n \to T$ pointwise as $n \to \infty$. Therefore, by the uniform boundedness principle (which holds for quaternionic Hilbert spaces [16]), T is bounded.

To complete the proof of our result, we need to determine the adjoint operator T^* of T. To this interest, let $f \in V_R(\mathfrak{Q})$ and $\{c_n\}_{n\in I} \in l^2(\mathfrak{Q})$. Then we have

$$\langle T^* f, \{c_n\}_{n\in I}\rangle = \langle f, T\{c_n\}_{n\in I}\rangle = \left\langle f, \sum_{n\in I} f_n c_n \right\rangle = \sum_{n\in I} \langle f, f_n\rangle c_n. \tag{3.3}$$

On the other hand, $T^* \in \mathcal{L}(V_R(\mathfrak{Q}), l^2(\mathfrak{Q}))$ since $T \in \mathcal{L}(l^2(\mathfrak{Q}), V_R(\mathfrak{Q}))$. Hence, the k^{th} coordinate function is bounded from $V_R(\mathfrak{Q})$ to $\mathfrak{Q}$. Using the Riesz representation Theorem [16], we have

$$T^* f = \{\langle g_n, f\rangle\}_{n\in I} \quad \text{for some } \{g_n\}_{n\in I} \in V_R(\mathfrak{Q}).$$

Thus, we get

$$\langle T^* f, \{c_n\}_{n\in I}\rangle = \langle \{\langle g_n, f\rangle\}_{n\in I}, \{c_n\}_{n\in I}\rangle = \sum_{n\in I} \langle f, g_n\rangle c_n. \tag{3.4}$$

Consequently, (3.3) and (3.4) imply that $g_n = f_n$.

As $\|T\| = \|T^*\|$, we obtain

$$\sum_{n\in I} |\langle f_n, f\rangle|^2 = \|T^* f\|^2 \leq \|T^*\|^2 \|f\|^2 = \|T\|^2 \|f\|^2.$$

Therefore, $\{f_n\}_{n\in I}$ is a Bessel sequence for $V_R(\mathfrak{Q})$. □

The existence result of the atomic systems for an operator is presented in the following theorem.

Theorem 3.5. *Let $K \in \mathcal{L}(V_R(\mathfrak{Q}))$, then K has an atomic system.*

Proof. Let $\{e_n\}_{n\in I}$ be a Hilbert basis in $V_R(\mathfrak{Q})$ and $x \in V_R(\mathfrak{Q})$. Then we have

$$x = \sum_{n\in I} e_n \langle e_n, x\rangle.$$

Therefore, we get

$$Kx = \sum_{n\in I} K e_n \langle e_n, x\rangle.$$

Now, by setting $f_n = Ke_n$ and $a_n = \langle e_n, x\rangle$ for all $n \in I$, we obtain

$$\sum_{n\in I} |\langle f_n, x\rangle|^2 = \sum_{n\in I} |\langle Ke_n, x\rangle|^2 = \sum_{n\in I} |\langle e_n, K^*x\rangle|^2 = \|K^*x\|^2 \leq \|K^*\|^2\|x\|^2.$$

So, $\{f_n\}_{n\in I}$ is a Bessel sequence for $V_R(\mathfrak{Q})$. Further, we have

$$\sum_{n\in I} |a_n|^2 = \sum_{n\in I} |\langle e_n, x\rangle|^2 = \|x\|^2.$$

Hence, $\{f_n\}_{n\in I}$ is an atomic system for K. □

Next, we give the characterization of atomic systems.

Theorem 3.6. *Let $\{f_n\}_{n\in I} \subset V_R(\mathfrak{Q})$ and $K \in \mathcal{L}(V_R(\mathfrak{Q}))$. Then the following statements are equivalent:*

(i) *$\{f_n\}_{n\in I}$ is an atomic system for K,*

(ii) *there exists $A, B > 0$ such that*

$$A\|K^*x\|^2 \leq \sum_{n\in I} |\langle f_n, x\rangle|^2 \leq B\|x\|^2 \quad \text{for any } x \in V_R(\mathfrak{Q}),$$

(iii) *$\{f_n\}_{n\in I}$ is a Bessel sequence and there exists a Bessel sequence $\{g_n\}_{n\in I}$ such that*

$$Kx = \sum_{n\in I} f_n\langle g_n, x\rangle.$$

To prove the above theorem, we need the following lemma. We omit its proof as it follows the lines of the complex case given in [11, Theorem 1].

Lemma 3.7. *Let $L_1 \in \mathcal{L}(V_R(\mathfrak{Q})_1, V_R(\mathfrak{Q}))$, $L_2 \in \mathcal{L}(V_R(\mathfrak{Q})_2, V_R(\mathfrak{Q}))$ be two bounded operators. The following statements are equivalent:*

(i) *$R(L_1) \subset R(L_2)$,*

(ii) *$L_1L_1^* \leq \lambda^2 L_2L_2^*$ for some $\lambda \geq 0$,*

(iii) *there exists a bounded operator $M \in \mathcal{L}(V_R(\mathfrak{Q})_1, V_R(\mathfrak{Q})_2)$ such that $L_1 = L_2M$.*

Proof of Theorem 3.6. (i) $\Rightarrow$ (ii). Let $x \in V_R(\mathfrak{Q})$. As $\{f_n\}_{n\in I}$ is an atomic system for K, it follows from Definition 3.3 and Proposition 3.4 that $\{f_n\}_{n\in I}$ is a Bessel sequence. More precisely, there exists $B > 0$ such that

$$\sum_{n\in I} |\langle f_n, x\rangle|^2 \leq B\|x\|^2 \quad \text{for any } x \in V_R(\mathfrak{Q}). \tag{3.5}$$

Further, there exists $C > 0$ such that for every $g \in V_R(\mathfrak{Q})$ there exists $a_g = \{a_n\}_{n\in I} \in l^2(\mathfrak{Q})$ satisfying $\|a_g\|_{l^2(\mathfrak{Q})} \leq C\|g\|$ and

$$Kg = \sum_{n\in I} f_n a_n.$$

Therefore, we get

$$\begin{aligned}\|K^*x\| &= \sup_{\|g\|=1} |\langle K^*x, g\rangle| = \sup_{\|g\|=1} |\langle x, Kg\rangle| = \sup_{\|g\|=1} \left| \left\langle x, \sum_{n\in I} f_n a_n \right\rangle \right| \\ &= \sup_{\|g\|=1} \left| \sum_{n\in I} \langle x, f_n\rangle a_n \right| \le \sup_{\|g\|=1} \left(\sum_{n\in I} |\langle f_n, x\rangle|^2 \right)^{\frac{1}{2}} \left(|a_n|^2\right)^{\frac{1}{2}} \\ &\le C \sup_{\|g\|=1} \|g\| \left(\sum_{n\in I} |\langle f_n, x\rangle|^2 \right)^{\frac{1}{2}} = C \left(\sum_{n\in I} |\langle f_n, x\rangle|^2 \right)^{\frac{1}{2}}.\end{aligned}$$

So, we obtain

$$\frac{1}{C^2}\|K^*x\|^2 \le \sum_{n\in I} |\langle f_n, x\rangle|^2. \tag{3.6}$$

Combining (3.5) and (3.6), we get the first implication.

(ii) $\Rightarrow$ (iii). It follows from (ii) that $\{f_n\}_{n\in I}$ is a Bessel sequence for $V_R(\mathfrak{Q})$. Then, in view of [18, Theorem 3.4], there exists a bounded linear operator $T : l^2(\mathfrak{Q}) \to V_R(\mathfrak{Q})$ and a Hilbert basis $\{e_n\}_{n\in I}$ of $l^2(\mathfrak{Q})$, such that $f_n = Te_n$. Consequently, (ii) entails the estimate

$$A\|K^*x\|^2 \le \sum_{n\in I} |\langle Te_n, x\rangle|^2 = \sum_{n\in I} |\langle e_n, T^*x\rangle|^2 = \|T^*x\|^2 \;\text{ for any } x \in V_R(\mathfrak{Q}).$$

Hence, Lemma 3.7 implies that there exists a bounded linear operator $M : V_R(\mathfrak{Q}) \to l^2(\mathfrak{Q})$ such that $K = TM$.

Now we consider

$$F_n : V_R(\mathfrak{Q}) \to \mathfrak{Q}, \quad F_n x := \{Mx\}_n = a_n(x).$$

We denote $a := Mx = \{Mx\}_n$. Clearly,

$$|a_n| \le \left(\sum_{n\in I} |a_n|^2 \right)^{\frac{1}{2}} = \|a\|_{l^2(\mathfrak{Q})} \le \|M\|\|x\|.$$

Hence, we have

$$|a_n(x)| \le \|M\|\|x\|.$$

As the Riesz representation theorem holds true also for quaternionic Hilbert spaces, thus there exists $g_n \in V_R(\mathfrak{Q})$ such that $a_n = a_n(x) = \langle g_n, x\rangle$. Hence,

$$Kx = TMx = T(\{a_n\}) = \sum_{n\in I} f_n a_n = \sum_{n\in I} f_n \langle g_n, x\rangle.$$

Further,

$$\sum_{n\in I} |\langle g_n, x\rangle|^2 = \sum_{n\in I} |a_n|^2 \le \|M\|^2\|x\|^2$$

implies that $\{g_n\}_{n\in I}$ is a Bessel sequence.

(iii) $\Rightarrow$ (i). We have

$$\sum_{n\in I} |\langle g_n, x\rangle|^2 \le B\|x\|^2.$$

It suffices to take $x \in V_R(\mathfrak{Q})$, $a_x = \{\langle g_n, x\rangle\}$. $\square$

Corollary 3.8. *Let $\{f_n\}_{n\in I}$ be a frame for $V_R(\mathfrak{Q})$ with bounds $A, B > 0$ and $K \in \mathcal{L}(V_R(\mathfrak{Q}))$. Then $\{f_n\}_{n\in I}$ is an atomic system for K with bounds $\frac{1}{A^{-1}\|K\|^2}$ and B.*

Proof. Let S be the frame operator of $\{f_n\}_{n\in I}$. It follows from [18, Theorem 2.7] that $\{S^{-1}f_n\}_{n\in I}$ is a frame for $V_R(\mathfrak{Q})$ with bounds $B^{-1}, A^{-1} > 0$. Further, we have

$$x = \sum_{n\in I} S^{-1}f_n\langle f_n, x\rangle \quad \text{for all } x \in V_R(\mathfrak{Q}).$$

Thus, we get

$$\begin{aligned}
\|K^*x\| &= \sup_{\|y\|=1} |\langle K^*x, y\rangle| = \sup_{\|y\|=1} \left|\left\langle \sum_{n\in I} K^*S^{-1}f_n\langle f_n, x\rangle, y\right\rangle\right| \\
&= \sup_{\|y\|=1} \left|\left\langle \sum_{n\in I} K^*S^{-1}f_n\overline{\langle x, f_n\rangle}, y\right\rangle\right| = \sup_{\|y\|=1} \left|\sum_{n\in I} \langle x, f_n\rangle\langle K^*S^{-1}f_n, y\rangle\right| \\
&\le \sup_{\|y\|=1} \sum_{n\in I} |\langle x, f_n\rangle|\,|\langle K^*S^{-1}f_n, y\rangle| \\
&\le \sup_{\|y\|=1} \left(\sum_{n\in I} |\langle x, f_n\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{n\in I} |\langle K^*S^{-1}f_n, y\rangle|^2\right)^{\frac{1}{2}} \\
&\le \sup_{\|y\|=1} \left(\sum_{n\in I} |\langle f_n, x\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{n\in I} |\langle S^{-1}f_n, Ky\rangle|^2\right)^{\frac{1}{2}} \\
&\le \sqrt{A^{-1}} \sup_{\|y\|=1} \|Ky\| \left(\sum_{n\in I} |\langle f_n, x\rangle|^2\right)^{\frac{1}{2}} = \sqrt{A^{-1}}\|K\| \left(\sum_{n\in I} |\langle f_n, x\rangle|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Consequently,

$$\frac{1}{A^{-1}\|K\|^2}\|K^*x\|^2 \le \sum_{n\in I} |\langle f_n, x\rangle|^2 \le B\|x\|^2, \quad x \in V_R(\mathfrak{Q}).$$

Hence, Theorem 3.6 (ii) implies that $\{f_n\}_{n\in I}$ is an atomic system for K. $\square$

The converse of the above corollary holds if the operator K is onto.

Corollary 3.9. *Let $\{f_n\}_{n\in I}$ be an atomic system for K. If $K \in \mathcal{L}(V_R(\mathfrak{Q}))$ is onto, then $\{f_n\}_{n\in I}$ is a frame for $V_R(\mathfrak{Q})$.*

Proof. It is easy to see that $K \in \mathcal{L}(V_R(\mathfrak{Q}))$ is onto if and only if there is $M > 0$ such that

$$M\|x\| \leq \|K^*x\|, \quad x \in V_R(\mathfrak{Q}).$$

As $\{f_n\}_{n\in I}$ is an atomic system for K, Theorem 3.6 entails the existence of $A, B > 0$ such that

$$A\|K^*x\|^2 \leq \sum_{n\in I} |\langle f_n, x\rangle|^2 \leq B\|x\|^2.$$

Consequently, we get

$$AM^2\|x\|^2 \leq \sum_{n\in I} |\langle f_n, x\rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in V_R(\mathfrak{Q}). \qquad \square$$

4. K-frames in quaternionic Hilbert spaces

In this section, we extend the concept of K-frames from complex Hilbert spaces to separable quaternionic Hilbert spaces. To this interest, we begin with the following definition.

Definition 4.1. Let $K \in \mathcal{L}(V_R(\mathfrak{Q}))$. A family $\{f_n\}_{n\in I}$ is said to be a K-frame for $V_R(\mathfrak{Q})$ if there exist positive constants $A, B > 0$ such that

$$A\|K^*x\|^2 \leq \sum_{n\in I} |\langle f_n, x\rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in V_R(\mathfrak{Q}).$$

Example 4.2. Let $\{e_n\}_{n=1}^3$ be a Hilbert basis for a three-dimensional right quaternionic Hilbert space $V_R(\mathfrak{Q})$ and let $K \in \mathcal{L}(V_R(\mathfrak{Q}))$ be defined as

$$Ke_1 = e_1, \quad Ke_2 = e_1, \quad Ke_3 = e_2.$$

Let also $\{f_n\}_{n=1}^3 = \{e_1, e_1, e_2\}$. We have

$$\sum_{n=1}^{3} |\langle f_n, f\rangle|^2 \leq 2\|f\|^2. \tag{4.1}$$

Further, by simple calculations, we can see that the adjoint of K is given by

$$K^*e_1 = e_1 + e_2, \quad K^*e_2 = e_3, \quad K^*e_3 = 0.$$

Therefore, we get

$$\frac{1}{4}\|K^*f\|^2 \leq \sum_{n=1}^{3} |\langle f_n, f\rangle|^2. \tag{4.2}$$

Hence, (4.1) and (4.2) imply that $\{f_n\}_{n=1}^3$ is a K-frame for $V_R(\mathfrak{Q})$. However, $\{f_n\}_{n=1}^3$ is not a frame for $V_R(\mathfrak{Q})$ since it does not possess a lower frame bound.

Now, using linear bounded operators, we characterize K-frames.

Theorem 4.3. *A sequence of vectors $\{f_n\}_{n\in I}$ is a K-frame if and only if there exists a linear bounded operator $L : l^2(\mathfrak{Q}) \to V_R(\mathfrak{Q})$ such that $f_n = Le_n$ and $R(K) \subset R(L)$, where $\{e_n\}_{n\in I}$ is a Hilbert basis for $l^2(\mathfrak{Q})$.*

Proof. Suppose that $\{f_n\}_{n\in I}$ is a K-frame. Then we get

$$A\|K^*x\|^2 \leq \sum_{n\in I} |\langle f_n, x\rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in V_R(\mathfrak{Q}). \tag{4.3}$$

Let us consider the mapping $\theta : V_R(\mathfrak{Q}) \to l^2(\mathfrak{Q})$ such that

$$\theta(x) = \sum_{n\in I} e_n\langle f_n, x\rangle. \tag{4.4}$$

Clearly, this mapping is a bounded linear operator. Further, for $y \in V_R(\mathfrak{Q})$ we have

$$\langle \theta^* e_n, y\rangle = \langle e_n, \theta y\rangle = \langle f_n, y\rangle.$$

Hence, we get that $\theta^* e_n = f_n$. Combining (4.3) and (4.4), we obtain

$$A\|K^*x\|^2 \leq \|\theta(x)\|^2.$$

Hence, $AKK^* \leq LL^*$, where $L = \theta^*$. Using Lemma 3.7, we get that $R(K) \subset R(L)$.

Now, suppose that $f_n = Le_n$, where $L \in \mathcal{L}(l^2(\mathfrak{Q}), V_R(\mathfrak{Q}))$ and $R(K) \subset R(L)$. We have

$$L^*x = \sum_{n\in I} e_n\langle f_n, x\rangle.$$

In fact,

$$\begin{aligned}\langle L^*x, g\rangle &= \left\langle L^*x, \sum_{n\in I} e_n c_n \right\rangle = \sum_{n\in I}\langle x, Le_n\rangle c_n = \sum_{n\in I}\langle x, f_n\rangle\langle e_n, g\rangle \\ &= \sum_{n\in I} \overline{\langle f_n, x\rangle}\langle e_n, g\rangle = \left\langle \sum_{n\in I} e_n\langle f_n, x\rangle, g \right\rangle, \quad g \in V_R(\mathfrak{Q}).\end{aligned}$$

On the other hand, we have

$$\sum_{n\in I} |\langle f_n, x\rangle|^2 = \sum_{n\in I} |\langle e_n, L^*x\rangle|^2 = \|L^*x\|^2 \leq \|L^*\|^2\|x\|^2.$$

So, $\{f_n\}_{n\in I}$ is a Bessel sequence. Further, as $R(K) \subset R(L)$, Lemma 3.7 yields the existence of a positive constant $A > 0$ such that $AKK^* \leq LL^*$. Therefore, we obtain

$$A\|K^*x\|^2 \leq \|L^*x\|^2 = \sum_{n\in I} |\langle f_n, x\rangle|^2.$$

This completes the proof. $\square$

The next theorem provides the sufficient condition assuring the construction of a frame from a K-frame.

Theorem 4.4. *Let $\{f_n\}_{n\in I}$ be a K-frame for $V_R(\mathfrak{Q})$ with bounds $A, B > 0$. If the operator K is onto, then $\{f_n\}_{n\in I}$ is a frame for $V_R(\mathfrak{Q})$.*

Proof. Obviously, $K \in \mathcal{L}(V_R(\mathfrak{Q}))$ is onto if and only if there is $M > 0$ such that

$$M\|x\| \leq \|K^*x\|, \quad x \in V_R(\mathfrak{Q}).$$

Since $\{f_n\}_{n\in I}$ is a K-frame, we get that

$$MA\|x\|^2 \leq A\|K^*x\|^2 \leq \sum_{n\in I} |\langle f_n, x\rangle|^2, \quad x \in V_R(\mathfrak{Q}). \qquad \square$$

Proposition 4.5. *A Bessel sequence $\{f_n\}_{n\in I}$ of $V_R(\mathfrak{Q})$ is a K-frame with bounds $A, B > 0$ if and only if $S \geq AKK^*$, where S is the frame operator for $\{f_n\}_{n\in I}$.*

Proof. $\{f_n\}_{n\in I}$ is a K-frame for $V_R(\mathfrak{Q})$ if and only if

$$\langle AKK^*x, x\rangle = A\|K^*x\|^2 \leq \sum_{n\in I} |\langle f_n, x\rangle|^2 = \langle Sx, x\rangle \leq B\|x\|^2, \quad x \in V_R(\mathfrak{Q}). \qquad \square$$

5. Families of local atoms as K-frames

In this section, we build a connection between the results developed in Sections 3 and 4 concerning atomic systems and K-frames and a new family of analysis and synthesis systems for a closed subspace H_0 of $V_R(\mathfrak{Q})$. Let us begin with the following formal definition.

Definition 5.1. Let $\{f_n\}_{n\in I} \in V_R(\mathfrak{Q})$ be a Bessel sequence and let H_0 be a closed subspace of $V_R(\mathfrak{Q})$. Then $\{f_n\}_{n\in I}$ is called a family of local atoms for H_0 if there exists a sequence of linear functionals $\{c_n\}_{n\in I}$ such that for all $x \in \mathrm{H}_0$ we have

(i) $\exists C > 0$ with $\sum_n |c_n(x)|^2 \leq C\|x\|^2$,

(ii) $x = \sum_n f_n c_n(x)$.

We say that the pair $\{f_n, c_n\}_{n\in I}$ provides an atomic decomposition for H_0 and C is an atomic bound of $\{f_n\}_{n\in I}$.

We close this section with some properties of families of local atoms.

Theorem 5.2. *Let $\{f_n\}_{n\in I} \subset V_R(\mathfrak{Q})$ be a Bessel sequence. Then the following assertions are equivalent:*

(i) *$\{f_n\}_{n\in I}$ is a family of local atoms for H_0,*

(ii) *$\{f_n\}_{n\in I}$ is an atomic system for P_{H_0}, where P_{H_0} is the orthogonal projection on H_0,*

(iii) *there exists* $A > 0$ *such that* $A\|P_{\mathrm{H}_0}x\|^2 \leq \sum_n |\langle f_n, x\rangle|^2$, $x \in V_R(\mathfrak{Q})$,

(iv) *there exists a Bessel sequence* $\{g_n\}_{n\in I} \subset V_R(\mathfrak{Q})$ *such that*

$$P_{\mathrm{H}_0}x = \sum_{n\in I} f_n\langle g_n, x\rangle \quad \textit{for any } x \in V_R(\mathfrak{Q}),$$

(v) *there exists a linear bounded operator* $L : l^2(\mathfrak{Q}) \to V_R(\mathfrak{Q})$ *such that* $f_n = Le_n$ *and* $\mathrm{H}_0 \subset R(L)$, *where* $\{e_n\}_{n\in I}$ *is a Hilbert basis for* $l^2(\mathfrak{Q})$.

Proof. (i) $\Rightarrow$ (ii). This implication is clear.

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). It is sufficient to apply Theorem 3.6 to get the desired results.

(iv) $\Rightarrow$ (i). Let $x \in \mathrm{H}_0$. In view of (iv), we have

$$x = \sum_{n\in I} f_n\langle g_n, x\rangle.$$

We denote by $c_n(x) = \langle g_n, x\rangle$. Since $\{g_n\}_{n\in I}$ is a Bessel sequence and c_n are linear functionals on H_0, we get

$$\sum_{n\in I} |c_n(x)|^2 = \sum_{n\in I} |\langle g_n, x\rangle|^2 \leq B\|x\|^2.$$

(ii) $\Leftrightarrow$ (v) This implication follows from Theorems 3.6 and 4.3. Indeed, it suffices to take $K = P_{\mathrm{H}_0}$. $\square$

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Received January 4, 2021, revised December 21, 2021.

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Про характеризацію фреймів для операторів у кватерніонному гільбертовому просторі

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У цій роботі ми вводимо поняття атомарних систем для операторів і K-фреймів у сепарабельних кватерніонних гільбертових просторах. Ці поняття призводять до узагальнення фреймів, які було нещодавно вивчено в [18], і дозволяють нам реконструювати елементи з образа лінійного і обмеженого оператора в кватерніонному гільбертовому просторі.

Ключові слова: фрейми, атомарні системи, K-фрейми, кватерніонний гільбертів простір