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Multidimensional Submanifolds with Metric of Revolution in Hyperbolic Space

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In the paper, the structure of submanifolds of low codimension with induced metric of revolution in a hyperbolic space is considered. The condition on extrinsic properties of such submanifolds to be submanifolds of revolution is found. This paper is a generalization of the result for submanifolds in Euclidean space.

Key words: metric of revolution, submanifolds of revolution, lines of curvature, sectional curvature

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In Euclidean space E^3, a 2-dimensional surface of revolution of constant Gaussian curvature admits a standard coordinate system (u^1, u^2) such that the metric of this surface is the metric of revolution

$$ds^2 = \left(du^1\right)^2 + \varphi^2(u^1)\left(du^2\right)^2.$$

On the other hand, from the fact that the induced metric on $F^2 \subset E^3$ is a metric of revolution it does not follow that F^2 is a surface of revolution. There is a locally isometric embedding F^2 in E^3 such that the geodesic line $u^2 = 0$ mapped onto a space curve with torsion is not equal to zero at any point. The following example can be constructed by using the Cauchy–Kowalewski theorem.

Therefore it is interesting to find the condition when a multidimensional submanifold F^l with induced metric of revolution in a space of constant curvature M^{l+p} is a submanifold of revolution.

A.A. Borisenko in [4] presented this condition for submanifolds in Euclidean space. Let F^l be a submanifold of low codimension in Euclidean space with induced metric of revolution of constant-sign sectional curvature. If the geodesic coordinate lines on F^l are the lines of curvature, then F^l is a submanifold of revolution.

In paper [11], it was considered a classification of submanifolds in the Euclidean space in terms of the indicatrix of normal curvature up to projective transformation. In [1] it was studied an isometric immersions of the Lobachevsky plane into 4-dimensional Euclidean space with flat normal connection.

In the present paper, we consider an isometric immersion of submanifolds F^l of low codimension with induced metric of revolution in hyperbolic space H^{l+p}.

We consider 3 cases, when the extrinsic sectional curvature of F^l is negative, zero, and positive.

The similar result is true for submanifolds of low codimension with induced metric of revolution in a spherical space S^{l+p}.

1. Main definitions

Let E_1^n be a pseudo Euclidean space of signature $(1, n)$. The scalar product of vectors $X(x^0, x^1, \dots, x^n)$ and $Y(y^0, y^1, \dots, y^n)$ in E_1^n is

$$\langle X, Y\rangle = -x^0y^0 + x^1y^1 + \dots + x^ny^n. \tag{1.1}$$

Consider a sheet of a hyperboloid in E_1^n

$$H^n = \{X(x^0, x^1, \dots, x^n) | \langle X, X\rangle = -1, x_0 > 0\}.$$

The pseudo Euclidean metric induces the metric of constant negative curvature -1 on H^n.

Definition 1.1. A multidimensional Riemannian metric on a manifold F^l is called a metric of revolution if there exists a regular coordinate system such that this Riemannian metric has the form

$$ds^2 = \left(du^1\right)^2 + \varphi^2(u^1)d\sigma^2, \tag{1.2}$$

where $\varphi(u^1) > 0$ is a regular function, $d\sigma^2$ is a Riemannian metric of constant sectional curvature.

Definition 1.2. A submanifold F^l in a hyperbolic space $H^{l+p} \subset E_1^{l+p}$ is called a submanifold of revolution if the radius vector of F^l equals

$$r(u^1, \dots, u^l) = \begin{cases} x^0 = \chi(u^1) \\ x^1 = \psi(u^1) \\ x^2 = \varphi(u^1)\rho^1(u^2, \dots, u^l) \\ \quad \dots \\ x^{l+p} = \varphi(u^1)\rho^{l+p-1}(u^2, \dots, u^l) \end{cases}, \tag{1.3}$$

where

$$-\chi^2 + \psi^2 + \varphi^2 = -1,$$
$$\left(\rho^1\right)^2 + \left(\rho^2\right)^2 + \dots + \left(\rho^{l+p-1}\right)^2 = 1,$$

and $d\sigma^2 = \left(d\rho^1\right)^2 + \dots + \left(d\rho^{l+p-1}\right)^2$ is a Riemannian metric of constant sectional curvature.

The curve $\gamma(u^1)$ with the radius vector

$$\gamma(u^1) = \begin{cases} x^0 = \chi(u^1) \\ x^1 = \psi(u^1) \\ x^2 = \varphi(u^1) \end{cases}$$

lies on the hyperbolic plane $H^2 \subset E^2_1$ and u^1 is the arc-length parameter of γ.

The submanifold F^l is obtained by the rotation of the curve $\gamma(u^1)$ along the submanifold $F^{l-1} \subset S^{l+p-2}$ in E^{l+p}_1. The radius vector of F^{l-1} is

$$\rho(u^2, \dots, u^l) = \left(0, 0, \rho^1(u^2, \dots, u^l), \dots, \rho^{l+p-1}(u^2, \dots, u^l)\right).$$

The submanifold F^{l-1} has the intrinsic Riemannian metric $d\sigma^2$ of constant sectional curvature.

From (1.3), it is easy to check that the submanifold of revolution F^l admits the metric of revolution (1.2).

Consider the reverse problem, when a multidimensional submanifold F^l with induced metric of revolution (1.2) of constant-sign sectional curvature is a submanifold of revolution in H^{l+p}.

Definition 1.3. A line $\gamma \subset F^l \subset E^{l+p}_1$ is called a line of curvature of a submanifold F^l if for any normal n from the normal space NF^l the tangent vector γ' is a principal direction of the second fundamental form with respect to the normal n.

Definition 1.4. A direction τ from the tangent space T_QF^l at a point Q of a submanifold F^l in Riemannian manifold M^{l+p} is called asymptotic if $B_n(\tau, \tau) = 0$ for any normal $n \in N_QF^l$ at this point, where B_n is the second fundamental form relative to the normal n.

2. Submanifolds of negative extrinsic sectional curvature in hyperbolic space

Lemma 2.1. *Let F^l be a submanifold in hyperbolic space H^{2l-1} with induced metric of revolution*

$$ds^2 = \left(du^1\right)^2 + \varphi^2(u^1)\, d\sigma^2, \tag{2.1}$$

where $\varphi(u^1) > 0$ is a regular function, $d\sigma^2$ is a Riemannian metric of constant sectional curvature. Let F^l have a negative extrinsic sectional curvature. Then

1. *If $d\sigma^2$ is a flat metric, then $\varphi'' - \varphi > 0$, $(\varphi')^2 - \varphi^2 > 0$.*

2. *If $d\sigma^2$ is the metric of unit sphere, then $\varphi'' - \varphi > 0$, $(\varphi')^2 - \varphi^2 > 1$, $\varphi' > 1$ when $u^1 > 0$, and $\varphi(0) = 0$, $\varphi'(0) = 1$.*

3. *If $d\sigma^2$ is the metric of hyperbolic space of curvature -1, then $\varphi'' - \varphi > 0$, $(\varphi')^2 - \varphi^2 > -1$.*

Proof. 1. Consider the case when $d\sigma^2$ is a flat metric. Then the metric of revolution (2.1) is

$$ds^2 = \left(du^1\right)^2 + \varphi^2(u^1)\left(\left(du^2\right)^2 + \cdots + \left(du^l\right)^2\right). \tag{2.2}$$

In this case, the nonzero Christoffel symbols of the metric ds^2 equal

$$\Gamma^j_{1i} = \delta^j_i \frac{\varphi'}{\varphi}, \quad \Gamma^1_{ij} = -\delta^j_i \varphi\varphi', \quad i, j = 2, \ldots, l. \tag{2.3}$$

The intrinsic sectional curvatures of F^l along coordinate 2-dimensional planes Π_{ij} are equal

$$K(\Pi_{1j}) = -\frac{\varphi''}{\varphi}, \quad K(\Pi_{ij}) = -\frac{(\varphi')^2}{\varphi^2}, \quad i, j = 2, \ldots, l.$$

From the Gauss equation, it follows that the extrinsic and intrinsic section curvatures of the submanifold F^l in H^{2l-1} satisfy

$$K_{\text{ext}} = K_{\text{int}} + 1. \tag{2.4}$$

Then the extrinsic sectional curvatures of F^l equal

$$K_{\text{ext}}(\Pi_{1j}) = -\frac{\varphi'' - \varphi}{\varphi}, \quad K_{\text{ext}}(\Pi_{ij}) = \frac{-(\varphi')^2 + \varphi^2}{\varphi^2}, \quad i, j = 2, \ldots, l.$$

Since F^l has negative extrinsic sectional curvature, it follows that

$$\varphi'' - \varphi > 0, \quad (\varphi')^2 - \varphi^2 > 0.$$

2. The case, when $d\sigma^2$ is a metric of constant sectional curvature 1. The metric (2.1) has the form

$$ds^2 = \left(du^1\right)^2 + \varphi^2(u^1)\frac{4\left(\left(du^2\right)^2 + \cdots + \left(du^l\right)^2\right)}{\left(1 + (u^2)^2 + \cdots + (u^l)^2\right)^2}.$$

By direct computations, we get

$$K(\Pi_{1j}) = -\frac{\varphi''}{\varphi}, \quad K(\Pi_{ij}) = \frac{1 - (\varphi')^2}{\varphi^2}, \quad i, j = 2, \ldots, l.$$

From the Gauss equation, we obtain the extrinsic sectional curvatures of F^l,

$$K_{\text{ext}}(\Pi_{1j}) = -\frac{\varphi'' - \varphi}{\varphi}, \quad K_{\text{ext}}(\Pi_{ij}) = \frac{1 - (\varphi')^2 + \varphi^2}{\varphi^2}, \quad i, j = 2, \ldots, l.$$

By the assumption, the metric of F^l is regular. From the singularity of polar coordinate system, it follows that $\varphi(0) = 0$, $\varphi'(0) = 1$. Since F^l has a negative extrinsic curvature, then for $u^1 > 0$ we get

$$\varphi'' - \varphi > 0, \quad (\varphi')^2 - \varphi^2 > 1. \tag{2.5}$$

3. Consider the case, when $d\sigma^2$ is a metric of constant sectional curvature -1. The metric (2.1) has the form

$$ds^2 = \left(du^1\right)^2 + \varphi^2(u^1)\frac{4\left(\left(du^2\right)^2 + \dots + \left(du^l\right)^2\right)}{\left(1-(u^2)^2 - \dots - (u^l)^2\right)^2}.$$

Then we get

$$K(\Pi_{1j}) = -\frac{\varphi''}{\varphi}, \quad K(\Pi_{ij}) = \frac{-1-(\varphi')^2}{\varphi^2}, \quad i,j = 2,\dots,l.$$

The extrinsic sectional curvatures of F^l equal

$$K_{\text{ext}}(\Pi_{1j}) = -\frac{\varphi''-\varphi}{\varphi}, \quad K_{\text{ext}}(\Pi_{ij}) = \frac{-1-(\varphi')^2+\varphi^2}{\varphi^2}, \quad i,j = 2,\dots,l.$$

Since F^l has a negative extrinsic curvature, it follows that

$$\varphi''-\varphi > 0, \quad (\varphi')^2 - \varphi^2 > -1. \qquad \square$$

Let $r = r(u^1,\dots,u^l)$ be a radius vector of the submanifold F^l in a hyperbolic space $H^{2l-1} \subset E_1^{2l-1}$. Denote $\partial r/\partial u^i$ by r_i and denote $\partial^2 r/\left(\partial u^i \partial u^j\right)$ by r_{ij}.

Lemma 2.2. *Let F^l be a C^3-regular submanifold in a hyperbolic space $H^{2l-1} \subset E_1^{2l-1}$ with the induced metric of revolution of negative extrinsic sectional curvature. If the coordinate lines u^1 are the lines of curvature of the submanifold F^l, then the rank of the map*

$$\tilde r = \frac{\varphi'}{\sqrt{(\varphi')^2-\varphi^2}}r - \frac{\varphi}{\sqrt{(\varphi')^2-\varphi^2}}r_1 \tag{2.6}$$

is equal to one.

Proof. Let b_{ij}^α be the coefficients of the second fundamental forms F^l with respect to the orthogonal basis of normals n^α, $\alpha = 0,\dots,l-1$. Since F^l locates on the hyperbolic space $H^{2l-1} \subset E_1^{2l-1}$, it follow that $n^0 = r$. Then

$$b_{11}^0 = -\langle r_{11}, n^0\rangle = \langle r_1, r_1\rangle = 1.$$

Since the coordinate lines u^1 are the lines of curvature of the submanifold F^l, it follows that

$$b_{1j}^\alpha = 0, \quad j = 2,\dots,l, \quad \alpha = 0,\dots,l-1.$$

Now we calculate the Jacobi matrix of the map (2.6):

$$\begin{aligned}\tilde r_1 = \left(\frac{\varphi'}{\sqrt{(\varphi')^2-\varphi^2}}\right)' r + \frac{\varphi'}{\sqrt{(\varphi')^2-\varphi^2}}r_1 \\ - \left(\frac{\varphi}{\sqrt{(\varphi')^2-\varphi^2}}\right)' r_1 - \frac{\varphi}{\sqrt{(\varphi')^2-\varphi^2}}r_{11},\end{aligned}$$

$$\tilde{r}_j = \frac{\varphi'}{\sqrt{(\varphi')^2 - \varphi^2}} r_j - \frac{\varphi}{\sqrt{(\varphi')^2 - \varphi^2}} r_{1j}, \quad j = 2, \dots, l.$$

From the Weingarten equations

$$r_{ij} = \Gamma^k_{ij} r_k + \sum_{\alpha=0}^{l-1} b^\alpha_{ij} n^\alpha$$

for $F^l \subset H^{l+1} \subset E^{l+1}_1$ (see [7, §64]) and (2.3), we have

$$r_{11} = r + \sum_{\alpha=1}^{l-1} b^\alpha_{11} n^\alpha, \quad r_{1j} = \frac{\varphi'}{\varphi} r_j, \quad j = 2, \dots, l.$$

Hence we get

$$\tilde{r}_1 = \frac{-\varphi^2\,(\varphi'' - \varphi)}{((\varphi')^2 - \varphi^2)^{3/2}} r + \frac{\varphi\varphi'(\varphi'' - \varphi)}{((\varphi')^2 - \varphi^2)^{3/2}} r_1 - \sum_{\alpha=1}^{l-1} \frac{\varphi}{\sqrt{(\varphi')^2 - \varphi^2}} b^\alpha_{11} n^\alpha \neq 0,$$
$$\tilde{r}_j = 0, \quad j = 2, \dots, l.$$

It follows that the rank of the Jacobi matrix of the map (2.6) is equal to 1 and $\tilde{r}$ depends only on the variable u^1, i.e., $\tilde{r} = \Phi(u^1)$. □

Theorem 2.3. *Let F^l be a C^3-regular submanifold in a hyperbolic space $H^{2l-1} \subset E^{2l-1}_1$ with the induced metric of revolution of negative extrinsic sectional curvature*

$$ds^2 = \left(du^1\right)^2 + \varphi^2(u^1)\, d\sigma^2, \tag{2.7}$$

where $\varphi(u^1)$ is a regular positive function and $d\sigma^2$ is a Riemannian metric of constant sectional curvature. If the coordinate lines u^1 are the lines of curvature of the submanifold F^l, then F^l is a submanifold of revolution.

Proof. 1. $d\sigma^2$ is a flat metric. Consider an ordinary differential equation

$$\frac{\varphi'}{\sqrt{(\varphi')^2 - \varphi^2}}\, r - \frac{\varphi}{\sqrt{(\varphi')^2 - \varphi^2}}\, r_1 = \Phi(u^1) \tag{2.8}$$

with respect to the vector function r. The solution of this equation is

$$r(u^1, \dots, u^l) = -\varphi(u^1) \int_0^{u^1} \frac{\sqrt{(\varphi'(t))^2 - \varphi^2(t)}}{\varphi^2(t)}\, \Phi(t) dt + \varphi(u^1) C(u^2, \dots, u^l).$$

Consider the constant $\lambda = \frac{\sqrt{(\varphi'(0))^2 - \varphi^2(0)}}{\varphi(0)\varphi'(0)}$, and rewrite r in the following way:

$$r = \varphi(u^1) \left(\lambda\Phi(0) - \int_0^{u^1} \frac{\sqrt{(\varphi'(t))^2 - \varphi^2(t)}}{\varphi^2(t)}\, \Phi(t) dt\right) + \varphi(u^1)\Big(C(u^2, \dots, u^l) - \lambda\Phi(0)\Big). \tag{2.9}$$

We set

$$\psi(u^1) = \varphi(u^1)\left(\lambda\Phi(0) - \int_0^{u^1} \frac{\sqrt{(\varphi'(t))^2 - \varphi^2(t)}}{\varphi^2(t)}\,\Phi(t)dt\right), \tag{2.10}$$
$$\rho(u^2, \ldots, u^l) = C(u^2, \ldots, u^l) - \lambda\Phi(0).$$

From the choice of constant λ, we get

$$\psi(0) = \frac{\sqrt{(\varphi'(0))^2 - \varphi^2(0)}}{\varphi'(0)}\Phi(0), \tag{2.11}$$
$$\psi'(0) = 0. \tag{2.12}$$

So the radius vector r of the submanifold F^l is

$$r = \psi(u^1) + \varphi(u^1)\rho(u^2, \ldots, u^l).$$

The vectors tangent to the coordinate lines of F^l have the form

$$r_1 = \psi'(u^1) + \varphi'(u^1)\rho(u^2, \ldots, u^l),$$
$$r_j = \varphi(u^1)\rho_j(u^2, \ldots, u^l), \quad j = 2, \ldots, l.$$

Since F^l has the induced metric of revolution (2.2), then

$$g_{11} = \langle\psi', \psi'\rangle + 2\varphi'\langle\psi', \rho\rangle + (\varphi')^2\langle\rho, \rho\rangle = 1, \tag{2.13}$$
$$g_{1j} = \varphi\langle\psi', \rho_j\rangle + \varphi\varphi'\langle\rho, \rho_j\rangle = 0, \tag{2.14}$$
$$g_{ij} = \varphi^2\langle\rho_i, \rho_j\rangle = \varphi^2\delta_i^j, \quad i, j = 2, \ldots, l. \tag{2.15}$$

Consider equation (2.13) when $u^1 = 0$. Using (2.12), we get that for any $u^2, \ldots, u^l$,

$$\langle\rho, \rho\rangle = \frac{1}{(\varphi'(0))^2}.$$

Then the submanifold F^{l-1} with the radius vector $\rho = \rho(u^2, \ldots, u^l)$ belongs to a sphere $S_R^{2l-3} \subset E^{2l-2}$ of radius $R = 1/\varphi'(0)$. From (2.15), it follows that F^{l-1} has the flat intrinsic metric.

Let us show that F^{l-1} does not belong to the Euclidean space E^{2l-3}. Assume the converse. Then F^{l-1} is a submanifold on a sphere $S_R^{2l-4} \subset E^{2l-3}$. From the Gauss equation, the extrinsic curvatures of the F^{l-1} are obtained

$$K_{ext}(F^{l-1}) = -(\varphi'(0))^2.$$

It is known that if a submanifold F^m of a Riemannian space M^{m+p} has a negative extrinsic sectional curvature, then $p \geq m - 1$ [5, Theorem 3.2.2]. In our case, $m = l - 1$, $M^{m+p} = S^{2l-4}$. We get that the codimension of F^{l-1} equals $p = l - 3 = m - 2$. From this contradiction one can conclude that F^{l-1} belongs to the sphere $S_R^{2l-3} \subset E^{2l-2}$.

Since $\langle r, r_j\rangle = 0$, then $\langle \psi, \rho_j\rangle = 0$. If $u^1 = 0$, then from (2.11) it follows that

$$\langle \Phi(0), \rho_j\rangle = 0.$$

From (2.14), we get that $\langle \psi', \rho_j\rangle = 0$. Differentiating this equation with respect to u^1, we have

$$\langle \psi'', \rho_j\rangle = 0, \quad \langle \psi''', \rho_j\rangle = 0, \quad \langle \psi^{(4)}, \rho_j\rangle = 0, \dots \tag{2.16}$$

Consider (2.16) at the point $u^1 = 0$. From (2.10), we obtain

$$\psi^{(k)}(0) \in \mathrm{Lin}\{\Phi(0), \Phi'(0), \Phi''(0), \dots, \Phi^{(k-1)}(0)\}.$$

We get that for all $u^2, \dots, u^l$ the following is true:

$$\langle \Phi'(0), \rho_j\rangle = 0, \quad \langle \Phi''(0), \rho_j\rangle = 0, \quad \langle \Phi'''(0), \rho_j\rangle = 0, \dots \tag{2.17}$$

These equations are true for all u^1 and we can rewrite (2.17) in the following way:

$$\langle \Phi'(u^1), \rho\rangle = c_0(u^1), \quad \langle \Phi''(u^1), \rho\rangle = c_1(u^1), \quad \langle \Phi'''(u^1), \rho\rangle = c_2(u^1), \dots . \tag{2.18}$$

From (2.8), it follows that $\langle \Phi(u^1), \Phi(u^1)\rangle = -1$. Consider the subspace L in E_1^{2l-1} such that

$$L = \mathrm{Lin}\{\Phi'(u^1), \Phi''(u^1), \Phi'''(u^1), \dots\}.$$

If $\dim L = 3$, then from (2.18) it follows that the submanifold F^{l-1} belongs to the Euclidean space E^{2l-3}. We have proved before that this is impossible. Then $\dim L = 2$ for any point on the curve $\Phi(u^1)$. Thus the curve $\Phi(u^1)$ belongs to the plane $E_1^1 \subset E_1^{2l-1}$ and the submanifold F^{l-1} is orthogonal to $\Phi(u^1)$.

Choose the orthogonal coordinate system such that the plane E_1^1 coincides with the plane x^0Ox^1, where O is the origin of coordinates. Then $\Phi(u^1) = \big(\mu(u^1), \nu(u^1), 0, \dots, 0.\big)$. Using (2.9), we obtain the radius vector of the submanifold F^l,

$$r = \begin{cases} x^0 = \chi(u^1) \\ x^1 = \psi(u^1) \\ x^2 = \varphi(u^1)\rho^1(u^2, \dots, u^l) \\ \dots \\ x^{l+p} = \varphi(u^1)\rho^{l+p-1}(u^2, \dots, u^l) \end{cases} .$$

It follows that the submanifold F^l is a submanifold of revolution. This completes the proof of Theorem 2.3, part 1.

2. $d\sigma^2$ is a metric of constant sectional curvature 1.

Consider u_0^1 such that $u^1 > u_0^1 > 0$. Assume that $u_0^1 = 0$. Every submanifold $u^1 = u_0^1$ belongs to the sphere $S_R^{2l-3} \subset E^{2l-2}$ where $R = 1/\varphi'(u_0^1)$. From Lemma 2.1, part 2, we get that $\varphi'(u^1) > 1$, $\varphi''(u^1) > 1$ for $u^1 > 0$. Therefore F^l lies inside the sphere S of radius 1. Moreover, this sphere S is the supporting sphere of F^l at the point $u^1 = 0$. The normal n of the sphere S coincides with the normal of F^l at the point $u^1 = 0$ in H^{2l-1}. It follows that the second form of F^l

with respect to the normal n is positive defined. Then there are no asymptotic direction at the point $u^1 = 0$. On the other hand, F^l has a negative extrinsic sectional curvature in H^{2l-1}. Then F^l has 2^{l-1} asymptotic directions at every point [3, Lemma 3.2.1]. From this contradiction we get $u_0^1 > 0$.

The submanifold F^{l-1} with the radius vector ρ has the intrinsic metric of constant curvature 1 and lies inside a sphere $S_R^{2l-3} \subset E^{2l-2}$ of radius $R = 1/\varphi'(u_0^1)$. The extrinsic curvature $K_{\text{ext}}(F^{l-1})$ of the submanifold F^{l-1} is

$$K_{\text{ext}}(F^{l-1}) = 1 - \left(\varphi'(u_0^1)\right)^2 < 0.$$

By the same argument as in the part 1, the curve $\Phi(u^1)$ belongs to the plane $E_1^1 \subset E_1^{2l-1}$, and $\Phi(u^1)$ is orthogonal to F^{l-1}.

Choose the orthogonal coordinate system such that the plane E_1^1 coincides with the plane x^0Ox^1. Thus we obtain that the submanifold F^l is a submanifold of revolution.

3. $d\sigma^2$ is a metric of constant sectional curvature $K_\sigma = -1$. In this case, the proof is similar to that of the part 1. □

3. Submanifolds of zero extrinsic sectional curvature in hyperbolic space

Let L^l be a hypersurface of constant curvature in H^{l+1} and F^{l-1} be a submanifold of L^l. Through every point of F^{l-1} construct the geodesics γ tangent to the normal of L^l in H^{l+1}. We get the surface F^l with one-dimensional generator over the submanifold F^{l-1} in H^{l+1}. Consider H^{l+1} in a Cayley-Klein model inside the unit ball. Then

1) If all geodesics γ intersect in the fixed point inside the ball, then F^l is *a cone*.

2) If all geodesics γ intersect in the fixed point on the absolute of the model, then F^l is called *an asymptotic cone*.

3) If all geodesics γ do not intersect each other either inside the ball or on the absolute, then F^l is *a cylinder* with one-dimensional generator.

Theorem 3.1. *Suppose F^l is a regular hypersurface in hyperbolic space H^{l+1} with the induced metric of revolution of zero extrinsic sectional curvature*

$$ds^2 = \left(du^1\right)^2 + \varphi^2(u^1)\, d\sigma^2, \tag{3.1}$$

where $\varphi(u^1) > 0$ is a regular function, $d\sigma^2$ is a Riemannian metric of constant sectional curvature. Let the coordinate lines u^1 be the lines of curvature of the submanifold F^l.

1. *If $d\sigma^2$ is a metric of constant sectional curvature -1, then F^l is a cylinder with one-dimensional generator over a local isometric immersion of a domain of hyperbolic space H^{l-1} into H^l.*

2. *If $d\sigma^2$ is a metric of constant sectional curvature 1, then F^l is a cone with one-dimensional generator over a local isometric immersion of a domain of the unit sphere S^{l-1} into the unit sphere $S^l \subset H^{l+1}$.*

3. *If $d\sigma^2$ is a metric of constant sectional curvature 0, then F^l is an asymptotic cone with one-dimensional generator over a local isometric immersion of a domain of Euclidean space E^{l-1} into the horosphere $E^l \subset H^{l+1}$.*

Proof. Consider the definition of null-index (see [2]).

Definition 3.2. The extrinsic null-index $\mu(Q)$ of a point Q of a submanifold F^l in the Riemannian manifold M^{l+p} is the maximal dimension of a subspace $L(Q)$ of the tangent space T_QF^l such that $B_nx = 0$ for any vector $x \in L(Q)$ and any normal $n \in N_QF^l$ at this point, where B_n is the linear transformation in T_QF^l corresponding to the second fundamental form relative to the normal n.

Chern and Kuiper (see [6]) proved that the null-index of a submanifold $F^l \subset M^{l+p}$ with zero extrinsic curvature satisfies the inequality

$$\mu \geq l - p.$$

In our case, we get that the null-index of F^l in H^{l+1} equals $\mu(Q) = l - 1$ for any point Q of submanifold F^l. Therefore the nullity foliation $L(Q)$ is integrable and the leaves $SL(Q)$ of this foliation are totally geodesic submanifolds of constant curvature -1 in H^{l+1}. A normal n is constant along the leaves (see [8]).

Let us consider two cases.

1) The totally geodesic leaves $SL(u^1) \subset F^l$ are orthogonal to the coordinate lines u^1.

Since F^l has the induced metric of revolution (3.1), it follows that the leaves of foliation $SL(u^1)$ have the intrinsic metric of sectional curvature $K_\sigma/\varphi^2(u^1)$, where K_σ is a constant of curvature of metric $d\sigma^2$. Since $SL(u^1)$ has a constant curvature -1, then $\varphi(u^1)$ should be a constant function. The extrinsic sectional curvature of $F^l \subset H^{l+1}$ along the coordinate plane Π_{1j} equals

$$K_{\text{ext}}(\Pi_{1j}) = -\frac{\varphi''}{\varphi} + 1.$$

If $\varphi(u^1)$ is constant, then F^l has non zero extrinsic sectional curvature. From this contradiction, we get that case 1) is impossible.

2) The leaves $SL(Q)$ of foliation contain the coordinate lines u^1.

2.1) Let $d\sigma^2$ be a Riemannian metric of constant negative curvature -1. Since the submanifold F^l has a zero extrinsic curvature, it follows

$$\varphi'' - \varphi = 0, \quad (\varphi')^2 - \varphi^2 = -1.$$

Then $\varphi = \cosh(u^1)$ and $\varphi(u^1) > 0$ for all $u^1 \geq 0$. The metric of F^l has the form

$$ds^2 = \left(du^1\right)^2 + \cosh^2(u^1)\, d\sigma^2. \tag{3.2}$$

Let $r = r(u^1, \dots, u^l)$ be a radius vector of F^l. Consider the map

$$\tilde{r} = -\sinh u^1\, r + \cosh u^1\, r_1. \tag{3.3}$$

Calculate the rank of this map

$$\tilde{r}_1 = \cosh u^1 \left(-r + \, r_{11}\right), \quad \tilde{r}_j = -\sinh u^1\, r_j + \cosh u^1\, r_{1j}. \tag{3.4}$$

Let n be a normal to F^l in H^{l+1} and b_{ij} be the coefficients of the second fundamental form. The coordinate lines u^1 are the lines of a curvature of the submanifold F^l, then $b_{1j} = 0$, $j = 2, \dots, l$. Since the lines u^1 belong to the leaves $SL(Q)$, it follows that $b_{11} = 0$.

From the Weingarten equations for $F^l \subset H^{l+1}$ (see [7, §64]) we obtain

$$r_{11} = \Gamma^k_{11} r_k + r + b_{11}\, n, \quad r_{1j} = \Gamma^k_{1j} r_k + b_{1j}\, n.$$

By a direct computation, we have

$$\Gamma^k_{11} = 0, \quad \Gamma^1_{1j} = 0, \quad \Gamma^i_{1j} = \delta^j_i \frac{\sinh u^1}{\cosh u^1}, \quad k = 1, ..., l, \quad j, i = 2, ..., l.$$

Then

$$r_{11} = r, \qquad r_{1j} = \frac{\sinh u^1}{\cosh u^1} r_j. \tag{3.5}$$

Substitute (3.5) into (3.4). We get that the rank of the map (3.3) is equal to 0. It follows that $\tilde{r}$ is a constant vector and $\langle \tilde{r}, \tilde{r} \rangle = 1$. Choose in E^{l+1}_1 the orthogonal coordinate system such that the axis x^{l+1} coincides with $\tilde{r}$, so $\tilde{r} = (0, 0, \dots, 1) = e_{l+1}$.

Consider the differential equation

$$-\sinh u^1\, r + \cosh u^1\, r_1 = e_{l+1}.$$

Solving this equation with respect to the vector function r, we get

$$r = \sinh u^1 e_{l+1} + \cosh u^1 \rho(u^2, \dots, u^l), \tag{3.6}$$

where $\rho = \rho(u^2, \dots, u^l)$ is a vector in E^{l+1}_1.

The submanifold $F^l \subset H^{l+1}$ with the radius vector r has the metric (3.2). Consider the equations

$$\langle r, r \rangle = \sinh^2 u^1 + \cosh u^1 \sinh u^1 \langle e_{l+1}, \rho \rangle + \cosh^2 u^1 \langle \rho, \rho \rangle = -1. \tag{3.7}$$

$$\langle r_1, r_1 \rangle = \cosh^2 u^1 + \cosh u^1 \sinh u^1 \langle e_{l+1}, \rho \rangle + \sinh^2 u^1 \langle \rho, \rho \rangle = 1. \tag{3.8}$$

Subtracting (3.7) from (3.8) yields

$$1 + \left(\sinh^2 u^1 - \cosh^2 u^1\right) \langle \rho, \rho \rangle = 2.$$

Thus we obtain that $\langle \rho, \rho \rangle = -1$ for any u^1.

From $\langle r, r_1 \rangle = 0$, we get

$$\cosh u^1 \sinh u^1 + \sinh^2 u^1 \langle e_{l+1}, \rho \rangle + \cosh^2 u^1 \langle e_{l+1}, \rho \rangle + \cosh u^1 \sinh u^1 \langle \rho, \rho \rangle = 0;$$
$$\left(\sinh^2 u^1 + \cosh^2 u^1\right) \langle e_{l+1}, \rho \rangle = 0.$$

From this equation we get that $\rho^{l+1}(u^2, \ldots, u^l) = 0$ for any u^1.

Therefore, from (3.6) we obtain the radius vector of F^l,

$$r = \begin{cases} x^0 = \cosh u^1 \rho^0(u^2, \ldots, u^l) \\ x^1 = \cosh u^1 \rho^1(u^2, \ldots, u^l) \\ \quad \ldots; \\ x^l = \cosh u^1 \rho^l(u^2, \ldots, u^l) \\ x^{l+1} = \sinh u^1 \end{cases}.$$

Since F^l has the metric of revolution (3.2), then

$$\langle r_i, r_j \rangle = \cosh^2 u^1 \langle \rho_i, \rho_j \rangle = \cosh^2 u^1 \ \sigma_{ij}, \quad i, j = 2, ..., l.$$

where σ_{ij} are the coefficients of the metric $d\sigma^2$.

It follows that the intersection of F^l with the hyperbolic space H^l orthogonal to coordinate lines u^1 at $u^1 = 0$ is a submanifold F^{l-1} with the radius vector $\rho = \rho(u^2, \ldots, u^l)$ and F^{l-1} has the intrinsic metric of constant sectional curvature -1. The coordinate lines u^1 of F^l coincide with the geodesic lines of H^{l+1} that are orthogonal to the subspace H^l containing F^{l-1}. We obtain that the submanifold F^l in H^{l+1} is a cylinder with one-dimensional generator over F^{l-1}.

2.2) Let $d\sigma^2$ be a metric of constant sectional curvature 1.
Since the submanifold F^l has a zero extrinsic curvature, it follows

$$\varphi'' - \varphi = 0, \quad (\varphi')^2 - \varphi^2 = 1.$$

From the solution of these differential equations, we get $\varphi = \sinh(u^1)$ and $\varphi(u^1) > 0$ for all $u^1 > 0$. Thus the metric of the submanifold F^l has the form

$$ds^2 = \left(du^1\right)^2 + 4 \sinh^2 u^1 \frac{\left(\left(du^2\right)^2 + \cdots + \left(du^l\right)^2\right)}{\left(1 + (u^2)^2 + \cdots + (u^l)^2\right)^2}. \tag{3.9}$$

Similarly to the part 2.1), consider the map

$$\tilde{r} = \cosh u^1 \ r - \sinh u^1 \ r_1.$$

The rank of this map is equal to 0. Then $\tilde{r}$ is a constant vector and $\langle \tilde{r}, \tilde{r} \rangle = -1$. Choose in E_1^{l+1} an orthogonal coordinate system such that the axis x^0 coincides with $\tilde{r}$, so $\tilde{r} = (1, 0, \ldots, 0) = e_0$.

From the differential equation

$$\cosh u^1 \ r - \sinh u^1 \ r_1 = e_0,$$

we get that the radius vector of F^l is

$$r = \cosh u^1 e_0 + \sinh u^1 \rho(u^2, \dots, u^l), \tag{3.10}$$

where $\rho = \rho(u^2, \dots, u^l)$ is a vector in E_1^{l+1}.

The submanifold $F^l \subset H^{l+1}$ with the radius vector r has the metric (3.9). In a similar way, from $\langle r, r\rangle = -1$ and $\langle r_1, r_1\rangle = 1$ we obtain that $\langle \rho, \rho\rangle = 1$ for any u^1.

Then from the equation $\langle r, r_1\rangle = 0$, we get

$$\left(\sinh^2 u^1 + \cosh^2 u^1\right) \langle e_0, \rho\rangle = 0.$$

We obtain that $\rho^0(u^2, \dots, u^l) = 0$ for any u^1.

Therefore the radius vector of the submanifold F^l is

$$r = \begin{cases} x^0 = \cosh u^1 \\ x^1 = \sinh u^1 \rho^1(u^2, \dots, u^l) \\ \dots \\ x^l = \sinh u^1 \rho^l(u^2, \dots, u^l) \\ x^{l+1} = \sinh u^1 \rho^{l+1}(u^2, \dots, u^l) \end{cases}.$$

Since F^l has the metric of revolution (3.9), then

$$\langle r_i, r_j\rangle = \sinh^2 u^1 \langle \rho_i, \rho_j\rangle = \sinh^2 u^1\ \sigma_{ij}, \quad i, j = 2, \dots, l,$$

where σ_{ij} are the coefficients of the metric $d\sigma^2$.

We obtain that if $u^1 > 0$, then the submanifold F^{l-1} with the radius vector $\rho(u^2, \dots, u^l)$ is a locally isometric immersion of a domain of a sphere S^{l-1} into the sphere $S^l \subset H^{l+1}$. The geodesic coordinate lines u^1 coincide with the geodesics of the space H^{l+1} that are orthogonal to the sphere S^l. From singularity of polar coordinates, we get that all coordinate lines u^1 intersect at the origin of the coordinate system. Hence the submanifold F^l is a cone with one-dimensional generator over $F^{l-1} \subset S^l$.

2.3) Let $d\sigma^2$ be a flat metric. Since the submanifold F^l has a zero extrinsic curvature, it follows that

$$\varphi'' - \varphi = 0, \quad (\varphi')^2 - \varphi^2 = 0.$$

Then $\varphi(u^1) = e^{-u^1}$, $\varphi(0) = 1$. Thus the metric of the submanifold F^l has the form

$$ds^2 = \left(du^1\right)^2 + e^{-2u^1} \left(\left(du^2\right)^2 + \dots + \left(du^l\right)^2\right). \tag{3.11}$$

Similarly to the previous parts, consider the map

$$\tilde{r} = e^{-u^1} r + e^{-u^1} r_1.$$

The rank of this map is also equal to 0. It follows that $\tilde{r}$ is a constant vector and $\langle \tilde{r}, \tilde{r}\rangle = 0$. Take the coordinate system in E_1^{l+1} such that $\tilde{r} = (1, 1, 0, \dots, 0) = e_0 + e_1$ in E_1^{l+1}.

Then the radius vector of F^l is

$$r = \cosh u^1 e_0 + \sinh u^1 e_1 + e^{-u^1}\rho(u^2, \dots, u^l).$$

From $\langle r, r\rangle = -1$, by a direct computation, we get

$$e^{-2u^1}\langle \rho, \rho\rangle = -2e^{-u^1}\cosh u^1 \langle e_0, \rho\rangle - 2e^{-u^1}\sinh u^1 \langle e_1, \rho\rangle. \tag{3.12}$$

Since $\langle r, r_1\rangle = 0$, it follows that

$$e^{-u^1}\left(\sinh u^1 - \cosh u^1\right)\langle e_0, \rho\rangle + e^{-u^1}\left(\cosh u^1 - \sinh u^1\right)\langle e_1, \rho\rangle \\ - e^{-2u^1}\langle \rho, \rho\rangle = 0. \tag{3.13}$$

From (3.12) and (3.13), we obtain $\langle e_0, \rho\rangle + \langle e_1, \rho\rangle = 0$, which means that $\rho_0 = \rho_1$. Therefore, the submanifold F^{l-1} with the radius vector ρ belongs to the horosphere $E^l \subset H^{l+1}$.

Since F^l has the metric of revolution (3.11), it follows that F^{l-1} has the intrinsic flat metric.

The radius vector of the submanifold F^l has the form

$$r = \begin{cases} x^0 = \cosh u^1 + e^{-u^1}\rho^0(u^2, \dots, u^l) \\ x^1 = \sinh u^1 + e^{-u^1}\rho^0(u^2, \dots, u^l) \\ x^2 = e^{-u^1}\rho^2(u^2, \dots, u^l) \\ \dots \\ x^{l+1} = e^{-u^1}\rho^{l+1}(u^2, \dots, u^l) \end{cases}.$$

We obtain that if $u^1 > 0$, then the submanifold F^{l-1} with the radius vector $\rho(u^2, \dots, u^l)$ is a locally isometric immersion of a domain of a Euclidean space E^{l-1} into the horosphere $E^l \subset H^{l+1}$. The coordinate lines u^1 coincide with the geodesic lines of H^{l+1} that are orthogonal to the horosphere E^l. Consider H^{l+1} in a Cayley–Klein model inside the unit ball. Then all coordinate lines u^1 intersect at the fixed point on the absolute of the model (point on infinity). Thus we get that the submanifold F^l is an asymptotic cone with one-dimensional generator over $F^{l-1} \subset E^l$. $\square$

4. Submanifolds of positive extrinsic sectional curvature in hyperbolic space

Theorem 4.1. *Suppose that F^l is a regular hypersurface in a hyperbolic space $H^{l+1} \subset E_1^{l+1}$ with the induced metric of revolution of positive extrinsic sectional curvature.*

1. *If $l > 2$, then F^l is a hypersurface of revolution.*
2. *If $l = 2$ and the coordinate lines u^1 are the lines of curvature, then F^2 is a hypersurface of revolution in H^3.*

Proof. 1. A metric of revolution has the form

$$ds^2 = (du^1)^2 + \varphi^2(u^1)d\sigma^2, \tag{4.1}$$

where $d\sigma^2$ is the metric of constant curvature.

The proof is similar for all three cases, namely, when the curvature of $d\sigma^2$ is equal to -1, 0, 1. Consider the case when $d\sigma^2$ is a metric of constant sectional curvature 1. Then the function φ satisfies the following conditions: $\varphi(0) = 0$, $\varphi'(0) = 1$ and $\varphi''(u^1) - \varphi(u^1) < 0$, $(\varphi'(u^1))^2 - \varphi^2(u^1) < 1$ for $u^1 > 0$.

Consider a hypersurface of revolution $F^l \subset H^{l+1} \subset E_1^{l+2}$, $l > 2$, with the radius vector

$$r(u^1, \dots, u^l) = \begin{cases} x^0 = h(u^1) \\ x^1 = g(u^1) \\ x^2 = f(u^1)\rho^1(u^2, \dots, u^l) \\ x^3 = f(u^1)\rho^2(u^2, \dots, u^l) \\ \quad \dots \\ x^{l+1} = f(u^1)\rho^l(u^2, \dots, u^l) \end{cases}, \tag{4.2}$$

where $\rho = \big(\rho^1(u^2, \dots, u^l), \dots, \rho^l(u^2, \dots, u^l)\big)$ is a radius vector of the unite sphere S^{l-1},

$$(\rho^1)^2 + (\rho^2)^2 + \dots + (\rho^l)^2 = 1. \tag{4.3}$$

Since F^l belongs to the hyperboloid H^{l+1}, we get $\langle r, r\rangle = -1$. Then

$$-h^2 + g^2 + f^2 = -1. \tag{4.4}$$

Suppose F^l has the induced metric of revolution (4.1). Then

$$g_{11} = -(h')^2 + (g')^2 - (f)^2 = 1. \tag{4.5}$$

$$\varphi^2 d\sigma^2 = f^2 d\rho^2. \tag{4.6}$$

From equation (4.6), it follows that $f(u^1) = \varphi(u^1)$. Consider equations (4.4) and (4.5),

$$\begin{cases} -h^2 + g^2 + \varphi^2 = -1, \\ -(h')^2 + (g)^2 + (\varphi')^2 = 1. \end{cases} \tag{4.7}$$

The solution of the system (4.7) is

$$g(u^1) = \sqrt{1 + \varphi^2(u^1)}\ \sinh\alpha(u^1),$$
$$h(u^1) = \sqrt{1 + \varphi^2(u^1)}\ \cosh\alpha(u^1),$$

where

$$\alpha(u^1) = \int_0^{u^1} \frac{\sqrt{1 - (\varphi'(t))^2 + \varphi^2(t)}}{1 + \varphi^2(t)} dt.$$

We obtain a hypersurface of revolution F^l with the induced metric of revolution of positive extrinsic curvature in hyperbolic space H^{l+1}.

Now, let us take two isometric hypersurfaces of revolution F_1 and F_2 in H^{l+1} with induced metric of revolution of positive extrinsic sectional curvature. The radius vectors of these surfaces are

$$r_k = \begin{cases} x_k^0 = x_k^0(u^1, \dots, u^l) \\ x_k^1 = x_k^1(u^1, \dots, u^l) \\ \quad \dots \\ x_k^{l+1} = x_k^{l+1}(u^1, \dots, u^l) \end{cases}, \quad k = 1, 2.$$

Consider the Pogorelov transformation (see [9]) that maps the isometric hypersurfaces F_1^l, $F_2^l \subset H^{l+1}$ into the hypersurfaces $\widetilde{F}_1$, $\widetilde{F}_2^l$ in the Euclidean space E^{l+1}. The radius vectors of $\widetilde{F}_1$, $\widetilde{F}_2^l$ are equal to

$$\widetilde{F}_1: \quad \widetilde{r}_1 = \frac{r_1 + \langle r_1, e_0\rangle e_0}{x_1^0 + x_2^0}, \qquad \widetilde{F}_2: \quad \widetilde{r}_2 = \frac{r_2 + \langle r_2, e_0\rangle e_0}{x_1^0 + x_2^0},$$

where $e_0 = (1, 0, \dots, 0)$ is a coordinate vector along the axes x^0.

The hypersurfaces $\widetilde{F}_1$, $\widetilde{F}_2$ are also isometric (see [9, Theorem 2]). The coefficients of the second fundamental form of $\widetilde{F}_1$, $\widetilde{F}_2$ at the origin of coordinate system are equal to

$$\widetilde{b}_{ij}^1(0) = \frac{b_{ij}^1(0)}{2}, \quad \widetilde{b}_{ij}^2(0) = \frac{b_{ij}^2(0)}{2},$$

where b_{ij}^1, b_{ij}^2 are the coefficients of the second fundamental form of F_1, F_2, respectively.

Since the hypersurfaces F_1 and F_2 have the positive defined second fundamental forms of rank l, then the second fundamental forms of $\widetilde{F}_1$, $\widetilde{F}_2$ are also positive defined.

If $l \geq 3$, then the ranks of the second fundamental forms of $\widetilde{F}_1$, $\widetilde{F}_2$ in E^{l+1} are greater or equal to 3. Since $\widetilde{F}_1$, $\widetilde{F}_2$ are isometric, it follows that they coincide up to a rigid motion in the Euclidean space E^{l+1} [10, Theorem 6.2]. From the properties of Pogorelov's transformation, we get that the isometric hypersurfaces of revolution F_1^l, F_2^l also coincide up to a rigid motion in the hyperbolic space H^{l+1}.

2) For $l = 2$, the proof is similar to that of Theorem 2.3. □

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Багатовимірні підмноговиди з метрикою обертання у просторі Лобачевського

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У роботі розглянуто структуру підмноговидів малої ковимірності з індукованою метрикою обертання у просторі Лобачевського. Знайдено умову на зовнішні властивості таких підмноговидів, за яких підмноговид є підмноговидом обертання. Ця стаття є узагальненням результатів, одержаних для підмноговидів евклідового простору.

Ключові слова: метрика обертання, підмноговид обертання, лінії кривини, секційна кривина