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Uniform Estimate of Potentials by Reflection Coefficients and its Application to KdV Flow

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Potentials of 1D Schrödinger operators are estimated by the moments of the reflection coefficients. Since the reflection coefficients are invariant under the KdV flow, the estimates provide information on some pre-compactness of solutions to the KdV equation starting from initial data having finite moments of the reflection coefficients.

Key words: Schroedinger operator, reflection coefficient, KdV equation

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1. Introduction

The *Korteweg-de Vries* (KdV) equation

$$\partial_t q = -\partial_x^3 q + 6q\partial_x q, \quad q(0,x) = q_0(x) \in \mathbb{R}$$

was proposed as a model describing the propagation of waves on shallow water. In the 1960s, it experienced a rapid development due to the discovery of infinitely many invariants by Gardner, Greene, Kruskal and Miura [9], which showed the spectrum of the Schrödinger operators with potentials of solutions to the KdV equation is invariant. Peter Lax [16] revealed this mystery. Since then, the inverse spectral technique was employed extensively to obtain solutions having periodic or decaying initial data.

It is known that these invariants cannot be exploited effectively if the initial data are neither periodic nor decaying. If the spectra of the associated Schrödinger operators consist of finitely many intervals and reflectionless there, there is an algebro-geometric approach and it is known that the solutions to the KdV equation are quasi-periodic in space as well as in time (see the book [10]). However, for infinite-gap setting the problem has been developed partially. The almost periodic initial data were investigated by Egorova [5] firstly, but virtually the class she discussed was a certain class of limit periodic initial data. Recently, almost-periodic initial data whose associated Schrödinger operators have purely absolutely continuous spectrum are considered as initial data [1, 3, 7]. Step-like initial data decaying on the right half axis have been investigated by [22] and [8] through the Hirota's Tau-function. More general initial data including ergodic initial data were studied recently by expressing Sato's Tau-functions by Weyl-functions [14, 15].

The purpose of the present article is to extend the result of Lundina [17], Marchenko [19]. We try to replace their reflectionless condition on $[0,\infty)$ by the finiteness of the moments of the reflection coefficients.

Let us first prepare some basic terminologies such as Herglotz functions, Weyl functions, reflection coefficients etc., and then state our main theorem. For real valued measurable function q of $L^1_{\text{loc}}(\mathbb{R})$, we consider the associated Schrödinger operator $L_q = -\partial_x^2 + q$ on $L^2(\mathbb{R})$. We assume the boundaries $\pm\infty$ are of limit point type, which means

$$\dim\left\{u \in L^2(\mathbb{R}_\pm) : \; -u'' + qu - zu = 0\right\} = 1$$

for any $z \in \mathbb{C}\backslash\mathbb{R}$. The *Weyl functions* $m_\pm(z)$ are defined by

$$m_\pm(z) = \pm\frac{u'(0)}{u(0)} \quad \text{with } u \in L^2(\mathbb{R}_\pm)\setminus\{0\} \text{ satisfying } L_q u = zu$$

for $z \in \mathbb{C}\backslash\mathbb{R}$. It is known that $m_\pm(z)$ are holomorphic on $\mathbb{C}\backslash\mathbb{R}$ and satisfy

$$\operatorname{Im} m_\pm(z) > 0 \quad \text{and} \quad \overline{m_\pm(\overline{z})} = m_\pm(z) \quad \text{on } \mathbb{C}_+.$$

A holomorphic function m on $\mathbb{C}_+$ having positive imaginary part is called a Herglotz function, and m has a non-vanishing finite limit

$$m(\lambda + i0) \equiv \lim_{\epsilon\downarrow 0} m(\lambda + i\epsilon)$$

for a.e. $\lambda \in \mathbb{R}$. The (right) reflection coefficient $R(z)$ is defined by

$$R(z) = \frac{\overline{m_+(z)} + m_-(z)}{m_+(z) + m_-(z)},$$

which trivially satisfies $|R(z)| \leq 1$. Reflection coefficients for 1D Schrödinger operators and Jacobi operators were considered by [11] and developed by [21] and [20] later as a generalization of conventional reflection coefficients for decaying potentials, namely the modulus $|R(\lambda + i0)|$ coincides with that of the conventional reflection coefficient if the potential decays sufficiently fast. This object is effectively used to measure the degree of absolutely continuous spectrum. Indeed, we have

$$\Sigma_{\text{ac}} \equiv (\text{the a.c. spectrum of } L_q) = \{\lambda \in \mathbb{R} : \; |R(\lambda + i0)| < 1\}$$

up to set of measure zero (see [20]).

We call $m_\pm(z)$ (or q) *reflectionless* on a Borel set $A \subset \mathbb{R}$ if

$$m_+(\lambda + i0) = -\overline{m_-(\lambda - i0)} \quad \text{for almost every } \lambda \in A. \tag{1.1}$$

Observe that identity (1.1) is valid if and only if $|R(\lambda + i0)| = 0$. Set

$$\Sigma_{\text{refl}} = \left\{\lambda \in \mathbb{R} : \; m_+(\lambda + i0, q) = -\overline{m_-(\lambda - i0, q)}\right\}.$$

Then, it clearly holds that

$$\Sigma_{\text{refl}} \subset \Sigma_{\text{ac}} \subset \operatorname{sp} L_q.$$

Denote a solution to the KdV equation with initial data q by q_t. It is known that L_q and L_{q_t} are unitarilly equivalent, hence the a.c. spectrum Σ_{ac} is invariant under the KdV equation. Moreover, $|R(\lambda + i0)|$ has a remarkable property that it is invariant under an action of the KdV flow the shift operation and the KdV equation. Therefore, Σ_{refl} turns to be also invariant under the shift operation and the KdV equation.

On the other hand, if the potential q is ergodic, namely $q_\omega(x) = Q(\theta_x\omega)$ on a probability space $\left(\Omega, \mathcal{F}, \mu, \{\theta_x\}_{x\in\mathbb{R}}\right)$, then it is known that the corresponding Schrödinger operator L_{q_ω} has a property $\Sigma^\omega_{\text{refl}} = \Sigma^\omega_{\text{ac}}$ (see [13]).

Fix $\lambda_0 \in \mathbb{R}$ and an integer $N \geq 1$. Set

$$\mathcal{Q}^{\mathcal{A}} = \left\{ q \in L^1_{\text{loc}}(\mathbb{R}) : \inf \operatorname{sp} L_q \geq \lambda_0,\ \int_0^\infty e^{c\lambda} |R(\lambda + i0)|\, d\lambda < \infty \text{ for a } c > 0 \right\},$$
$$\mathcal{Q}^{N} = \left\{ q \in L^1_{\text{loc}}(\mathbb{R}) : \inf \operatorname{sp} L_q \geq \lambda_0,\ \int_0^\infty \lambda^{N+1/2} |R(\lambda + i0)|\, d\lambda < \infty \right\},$$

and $\mathcal{Q}^\infty := \bigcap_{N=1}^\infty \mathcal{Q}^N$. Then obviously, $\mathcal{Q}^{\mathcal{A}} \subset \mathcal{Q}^\infty$ is valid. If the potentials are ergodic, then $q \in \mathcal{Q}^{\mathcal{A}}$ is equivalent to

$$\int_{[0,\infty)\setminus\Sigma_{\text{ac}}} e^{c\lambda}\, d\lambda < \infty$$

due to $\Sigma_{\text{ac}} = \Sigma_{\text{refl}}$. And for the same reason, $q \in \mathcal{Q}^N$ is equivalent to

$$\int_{[0,\infty)\setminus\Sigma_{\text{ac}}} \lambda^{N+1/2}\, d\lambda < \infty.$$

Let $c, d > 0$ be constants determined by

$$c^2(c+1) = 6, \quad d = \sqrt{c+1}. \tag{1.2}$$

Then we have

$$c > 1, \quad cd^2 \geq 2. \tag{1.3}$$

Theorem 1.1. *Assume $q \in \mathcal{Q}^N$ for $N \geq 2$ and $r > 0$ is any constant satisfying*

$$\frac{1}{2}\int_0^\infty \sum_{k=1}^{N-1} \frac{r^k \lambda^{k-1}}{k!} |R(\lambda + \lambda_0)|\, d\lambda \leq 1.$$

Then, it holds that for $n = 0, 1, 2, \ldots, 2N-3$,

$$\left| q^{(n)}(x) \right| \leq 2cd^n r^{-(1+n/2)}(n+1)!.$$

Theorem 1.2. *Assume $q \in \mathcal{Q}^{\mathcal{A}}$ and $r > 0$ is any constant satisfying*

$$\frac{1}{2}\int_0^\infty \frac{e^{r\lambda}-1}{\lambda}|R(\lambda+\lambda_0)|\,d\lambda \le 1.$$

Then, the associated potential q is analytically extendable to a strip

$$\left\{z \in \mathbb{C},\ |\operatorname{Im} z| < d^{-1}\sqrt{r}\right\}$$

and $q(z)$ satisfies a uniform estimate

$$|q(z)-\lambda_0| \le cr^{-1}\left(1-dr^{-1/2}|\operatorname{Im} z|\right)^{-2}. \tag{1.4}$$

Let

$$\Gamma = \{g = e^h :\ h \text{ is a real odd polynomial}\}.$$

Then, Theorems 2 and 3 of [14] and the invariance of the $|R(\lambda, q_t)|$ imply the existence of KdV flow with initial data in $\mathcal{Q}^\infty$ and $\mathcal{Q}^{\mathcal{A}}$. Therefore we have

Corollary 1.3. *For any $g \in \Gamma$, the following uniform estimates are valid:*

$$\begin{aligned} \left|\left(K(g)q\right)^{(n)}(x)-\lambda_0\right| &\le 2cd^n r^{-(1+n/2)}(n+1)! && \text{if } q \in \mathcal{Q}^\infty, \\ \left|\left(K(g)q\right)(z)-\lambda_0\right| &\le cr^{-1}\left(1-dr^{-1/2}|\operatorname{Im} z|\right)^{-2} && \text{if } q \in \mathcal{Q}^{\mathcal{A}}. \end{aligned}$$

Remark 1.4. Marchenko [19, p. 294, Corollary] proved $q(z)$ is analytic on a strip $\left\{z \in \mathbb{C} :\ |\operatorname{Im} z| < (-\lambda_0)^{-1/2}\right\}$ and satisfies

$$|q(z)| \le 2(-\lambda_0)\left(1-(-\lambda_0)^{1/2}|\operatorname{Im} z|\right)^{-2}$$

by assuming the corresponding $m_\pm(z)$ are reflectionless on $(0,\infty)$ and $\operatorname{sp} L_q \subset [\lambda_0, \infty)$ (hence $\lambda_0 < 0$). Under this assumption, we see that

$$\frac{1}{2}\int_0^\infty \frac{e^{r\lambda}-1}{\lambda}|R(\lambda+\lambda_0)|\,d\lambda \le \frac{1}{2}\int_0^{-\lambda_0} \frac{e^{r\lambda}-1}{\lambda}\,d\lambda = \frac{1}{2}\int_0^{(-\lambda_0)r} \frac{e^x-1}{x}\,dx.$$

By numerical computation, this integral equals 1 approximately at $(-\lambda_0)r = 1.35$. Thus our estimate yields

$$|q(z)-\lambda_0| \le \frac{c}{1.35}(-\lambda_0)\left(1-\frac{d}{1.35^{1/2}}(-\lambda_0)^{1/2}|\operatorname{Im} z|\right)^{-2}$$

with

$$c1.35^{-1} = 1.14, \quad d^{-1}1.35^{1/2} = 0.73.$$

Therefore, (1.4) is slightly worse than the result obtained by Marchenko.

Remark 1.5. If the potentials are ergodic, then $q \in \mathcal{Q}^{\mathcal{A}}$ is equivalent to

$$\int_{[0,\infty)\setminus\Sigma_{\mathrm{ac}}} e^{c\lambda}\, d\lambda < \infty \tag{1.5}$$

due to the fact $\Sigma_{\mathrm{ac}} = \Sigma_{\mathrm{refl}}$ (Kotani [13]) up to measure 0 set for ergodic potentials. And particularly, for some quasi-periodic potentials, condition (1.5) is fulfilled according to [6,18]. More precisely, Eliasson [6] had proved that for any *Diophatine frequency*, there exists a sufficiently large constant E_0 such that on $[E_0, \infty)$, Schrödinger operator has ac spectrum, thus (1.5) reduced to

$$\int_{[E_0,\infty)\setminus\,\mathrm{sp}\, L_q} e^{c\lambda}\, d\lambda < \infty. \tag{1.6}$$

Recall that $\omega \in \mathbb{R}^2$ is called Diophantine if $|n\omega| \geq \alpha |n|^{\tau}$ for some $0 < \alpha < 1$, and $\tau > 2$. Besides, for sufficiently small quasi-periodic potentials satisfying Diophantine frequency, the authors showed that the length of spectral gaps decays exponentially [18]. Moreover, due to the Dinaburg–Sinai transformation [2], the result for any potentials can be reduced to only for sufficiently small potentials with sufficiently large energy. Therefore, (1.6) holds for some $c > 0$.

Notation. Throughout the paper we use the following notation: $\mathbb{R}$ and $\mathbb{C}$, denote the real line and the whole complex plane respectively,

$$\mathbb{R}_+ = \{x \in \mathbb{R} : \ x > 0\}, \ \ \mathbb{R}_- = \{x \in \mathbb{R} : x < 0\}, \ \ \mathbb{C}_+ = \{z \in \mathbb{C} : \ \mathrm{Im}\, z > 0\};$$

$\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$; $\sqrt{z}$ is defined as a holomorphic function on $\mathbb{C}\setminus\mathbb{R}_-$ satisfying $\sqrt{1} = 1$.

2. Reflection coefficients and Xi functions

As we have mentioned in the previous section, $|R(z)|$ on $\mathbb{R}$ is invariant under the KdV flow, since it is invariant under an action of some transfer matrices. Let $T \in SL(2, \mathbb{R})$ be

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$$

For $m \in \mathbb{C}$, define

$$T \cdot m = \frac{t_{11} m + t_{12}}{t_{21} m + t_{22}}.$$

Set

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, we easily have

Lemma 2.1. *For $m_\pm \in \mathbb{C}$ satisfying $\mathrm{Im}\, m_\pm 0$, set*

$$\widehat{m}_+ = T \cdot m_+, \quad \widehat{m}_- = (\sigma T \sigma) \cdot m_-.$$

Then, it holds that

$$\left|\frac{\overline{m}_+ + m_-}{m_+ + m_-}\right| = \left|\frac{\overline{\widehat{m}}_+ + \widehat{m}_-}{\widehat{m}_+ + \widehat{m}_-}\right|.$$

One can apply this Lemma to the shift operation. Define the shift operation θ_x on potentials by

$$(\theta_x q)(\cdot) = q(\cdot + x).$$

The dependence on q is denoted by $m_\pm(z,q)$, $\xi_j(z,q)$, $R(z,q)$. Assume that $\{\varphi_z(x,q), \psi_z(x,q)\}$ are solutions to

$$L_q f = zf$$

with initial values

$$\begin{aligned} f(0) &= 1, & f'(0) &= 0,\\ f(0) &= 0, & f'(0) &= 1, \end{aligned}$$

respectively. Since

$$\begin{aligned} m_+(z,\theta_x q) &= \frac{\varphi_z'(x,q) + m_+(z,q)\psi_z'(x,q)}{\varphi_z(x,q) + m_+(z,q)\psi_z(x,q)},\\ m_-(z,\theta_x q) &= -\frac{\varphi_z'(x,q) - m_-(z,q)\psi_z'(x,q)}{\varphi_z(x,q) - m_-(z,q)\psi_z(x,q)} \end{aligned}$$

hold and $\{\varphi_z(x,q), \psi_z(x,q)\}$ take real values if $z \in \mathbb{R}$, Lemma 2.1 implies the invariance of $|R(\lambda + i0, q)|$ under θ_x. For the KdV equation, Rybkin [21] also showed this invariance.

Lemma 2.2. *There exists a measure* 0 *set* A *in* $\mathbb{R}$ *such that*

$$|R(\lambda + i0, q)| = |R(\lambda + i0, \theta_x q)| = R(\lambda + i0, q_t)$$

is valid for any $\lambda \in \mathbb{R}\backslash A$, $x, t \in \mathbb{R}$, *where* q_t *is a solution to the KdV equation with initial data* q.

This invariance was discussed by [20] from a general point of view.

Although the reflection coefficient has these beautiful properties, for our purpose, it is better to introduce other quantities which connect $R(z)$ and $m_\pm(z)$. For two Herglotz functions $m_\pm(z)$, one can define new Herglotz functions g_1, g_2 by

$$g_1(z) = \frac{-1}{m_+(z) + m_-(z)}, \quad g_2(z) = \frac{m_+(z)m_-(z)}{m_+(z) + m_-(z)},$$

and their xi functions by

$$\xi_1(z) = \frac{1}{\pi}\arg g_1(z), \quad \xi_2(z) = \frac{1}{\pi}\arg g_2(z) \in [0,1],$$

among which the first one was investigated by [12] systematically. For $\lambda \in \Sigma_{\text{refl}}$, (1.1) implies $\xi_1(\lambda + i0) = \xi_2(\lambda + i0) = 1/2$. More precisely, they are related as follows.

Lemma 2.3. *For z such that* $\operatorname{Im} z \geq 0$*, we have* $g_1, g_2 \in \mathbb{C}_+$ *and*

$$|R(z)| \leq 1, \quad |\xi_1(z) - \frac{1}{2}|, \quad |\xi_2(z) - \frac{1}{2}| \leq \frac{1}{2}|R(z)|.$$

Proof. These inequalities are valid for any two complex numbers $m_\pm \in \{\operatorname{Im} z \geq 0\}$ satisfying $m_+ + m_- \neq 0$. The first one is trivial. The others can be verified as follows: set

$$\frac{-1}{m_+ + m_-} = re^{i\pi\xi_1} = ire^{i\pi(\xi_1 - 1/2)}.$$

Note

$$|R|^2 = 1 - \left(\frac{2 \operatorname{Im} \dfrac{-1}{m_+ + m_-}}{\dfrac{1}{\operatorname{Im} m_+} + \dfrac{1}{\operatorname{Im} m_-}} \right)^2 = 1 - \frac{4r^2}{\left(\dfrac{1}{\operatorname{Im} m_+} + \dfrac{1}{\operatorname{Im} m_-} \right)^2} \cos^2(\pi\,(\xi_1 - 1/2)).$$

Observing $|\xi_1 - 1/2| \leq 1/2$ and

$$4r = 4 \left| \frac{-1}{m_+ + m_-} \right| \leq \frac{4}{\operatorname{Im} m_+ + \operatorname{Im} m_-} \leq \frac{1}{\operatorname{Im} m_+} + \frac{1}{\operatorname{Im} m_-},$$

we have

$$\begin{aligned} |R|^2 &\geq 1 - \cos^2(\pi\,(\xi_1 - 1/2)) \geq 1 - \cos(\pi\,(\xi_1 - 1/2)) \\ &= 2 \sin^2(\pi\,(\xi_1 - 1/2)\,/2) \geq 4\,|\xi_1 - 1/2|^2 \end{aligned}$$

since $\sin(\pi x/2) \geq \sqrt{2}x$ if $0 \leq x \leq 1/2$. To show the last inequality for ξ_2, set $\widehat{m}_\pm = (-m_\pm)^{-1} \in \overline{\mathbb{C}}_+$. Note

$$\frac{m_+ m_-}{m_+ + m_-} = -\frac{1}{\widehat{m}_+ + \widehat{m}_-}$$

and

$$\widehat{R} = \frac{\overline{\widehat{m}}_+ + \widehat{m}_-}{\widehat{m}_+ + \widehat{m}_-} = \frac{(-\overline{m}_+)^{-1} + (-m_-)^{-1}}{(-m_+)^{-1} + (-m_-)^{-1}} = \frac{m_+}{\overline{m}_+} R.$$

Then, one can apply the previous argument to $\widehat{m}_\pm$. □

The Herglotz functions $g_j\,(z)$ can be represented by $\xi_j(\lambda + i0)$. Assume $\operatorname{sp} L_q \subset [\lambda_0, \infty)$. Then, it is known that

$$m_\pm\,(z) = -\sqrt{-z} + o\,(1) \tag{2.1}$$

as $z \to \infty$ in a sector $\mathbb{C} \setminus \{|\arg z| < \epsilon\}$ for any $\epsilon > 0$.

Lemma 2.4. *Suppose*

$$\int_0^\infty \frac{|\xi_j(\lambda+i0)-1/2|}{\lambda+1}\,d\lambda<\infty$$

for $j=1,2$*. Then, we have*

$$g_1(z)=\frac{1}{2\sqrt{-z}}\exp\left(\int_{\lambda_0}^\infty \frac{\xi_1(\lambda+i0)-I_{\lambda>0}/2}{\lambda-z}d\lambda\right),$$
$$g_2(z)=\frac{-\sqrt{-z}}{2}\exp\left(\int_{\lambda_0}^\infty \frac{\xi_2(\lambda+i0)-I_{\lambda>0}/2}{\lambda-z}d\lambda\right).$$

Proof. The proof follows from the fact that $\xi_j(z)=(\operatorname{Im} g_j(z))/\pi$ are also of Herglotz and $g_1(\lambda)>0$, $g_2(\lambda)<0$ on $(-\infty,\lambda_0)$ and (2.1). □

3. Uniform estimate of potentials by reflection coefficients

In this section, uniform estimates for the derivatives of $q\in\mathcal{Q}^N$ or $\mathcal{Q}^{\mathcal{A}}$ are given in terms of $|R(\lambda+i0)|$.

3.1. Relations between $m_\pm(z)$ and $\xi_j(z)$. In this subsection we assume $\lambda_0=0$. Suppose $q\in C^M(\mathbb{R})$. Then, the asymptotic expansions

$$m_+(z,\theta_x q)=-\sqrt{-z}-\sum_{j=1}^{M+1} c_{j+1}(x)\sqrt{-z}^{-j}+o\left(\sqrt{-z}^{-M-1}\right),$$
$$m_-(z,\theta_x q)=-\sqrt{-z}-\sum_{j=1}^{M+1} (-1)^{j+1}c_{j+1}(x)\sqrt{-z}^{-j}+o\left(\sqrt{-z}^{-M-1}\right) \tag{3.1}$$

are valid in a sector $\mathbb{C}\setminus\{|\arg(z)|<\epsilon\}$ for any fixed $\epsilon>0$, and the coefficients $c_{j+1}(x)$ can be obtained by the recurrence relation

$$c_1=0,\quad c_2=q(x)/2,$$
$$c_j=\frac{1}{2}(c'_{j-1}-\sum_{\ell=1}^{j-1}c_\ell c_{j-\ell}),\quad j\geq 3 \tag{3.2}$$

(see [4, 23]). If we set

$$g_1(x,z)=\frac{-1}{m_+(z,\theta_x q)+m_-(z,\theta_x q)},$$
$$g_2(x,z)=\frac{m_+(z,\theta_x q)m_-(z,\theta_x q)}{m_+(z,\theta_x q)+m_-(z,\theta_x q)},$$

then expansions (3.1) can be translated to

$$g_1(x,z)=\frac{1}{2\sqrt{-z}}\left(1+\sum_{k=1}^{N-1}a_k(x)z^{-k}\right)+o\left(z^{-N+1/2}\right),$$

$$g_2(x,z)=\frac{-\sqrt{-z}}{2}\left(1+\sum_{k=1}^{N-1}b_k(x)z^{-k}\right)+o\left(z^{-N+1/2}\right) \tag{3.3}$$

if $M=2\,(N-1)-1$. On the other hand, for $q\in\mathcal{Q}^N$ Lemma 2.3 implies that the xi functions $\xi_j(\lambda,x)=\xi_j(\lambda,\theta_x q)$ satisfy

$$\int_0^\infty \lambda^{N+1/2}|\xi_j(\lambda,x)-1/2|\,d\lambda<\infty.$$

Then Lemma 2.4 shows

$$\begin{aligned} g_1(x,z)&=\frac{1}{2\sqrt{-z}}\exp\left(\int_0^\infty\frac{\xi_1(\lambda,x)-1/2}{\lambda-z}\,d\lambda\right),\\ g_2(x,z)&=\frac{-\sqrt{-z}}{2}\exp\left(\int_0^\infty\frac{\xi_2(\lambda,x)-1/2}{\lambda-z}\,d\lambda\right). \end{aligned} \tag{3.4}$$

To describe the asymptotics of $g_j(x,z)$, define

$$\begin{aligned} \mu_k(x)&=-\int_0^\infty\lambda^{k-1}(\xi_1(\lambda,x)-1/2)\,d\lambda,\\ \nu_k(x)&=-\int_0^\infty\lambda^{k-1}(\xi_2(\lambda,x)-1/2)\,d\lambda. \end{aligned}$$

Here we change the variable z to $-k^2$ for later purpose. Then

$$\begin{aligned} \int_0^\infty\frac{\xi_1(\lambda,x)-1/2}{\lambda+k^2}\,d\lambda&=\sum_{j=1}^{N+1}\mu_j(x)\left(-k^2\right)^{-j}-\left(-k^2\right)^{-N-1}h_1(k),\\ \int_0^\infty\frac{\xi_2(\lambda,x)-1/2}{\lambda+k^2}\,d\lambda&=\sum_{j=1}^{N+1}\nu_j(x)\left(-k^2\right)^{-j}-\left(-k^2\right)^{-N-1}h_2(k) \end{aligned} \tag{3.5}$$

hold with

$$h_j(k)=\int_0^\infty\frac{\lambda^{N+1}\left(\xi_j(\lambda,x)-1/2\right)}{\lambda+k^2}\,d\lambda\quad\text{for } j=1,2.$$

Moreover, h_j has bounds

$$\begin{aligned} |h_j(k)|&\le\frac{1}{\operatorname{Re}k}\int_0^\infty\lambda^{N+1/2}\left|\xi_j(\lambda,x)-1/2\right|\,d\lambda,\\ \int_{-\infty}^\infty|h_j(t+iy)|^2\,dy&\le\frac{1}{2t}\left(\int_0^\infty\lambda^{N+1/2}\left|\xi_j(\lambda,x)-1/2\right|\,d\lambda\right)^2 \end{aligned} \tag{3.6}$$

if $\operatorname{Re}k>0$, $t>0$, since

$$h_j(k)=\int_0^\infty e^{-k\alpha}\,d\alpha\int_0^\infty\lambda^{N+1/2}\left(\xi_j(\lambda,x)-1/2\right)\sin\sqrt{\lambda}\,\alpha\,d\lambda$$

is valid. Hence $h_j\in H^{2,\infty}(\operatorname{Re}k>a)$ for any $a>0$. Here we define the Hardy spaces

$$H^2(\operatorname{Re}k>a)=\Big\{h:\ h\text{ is holomorphic on }\{\operatorname{Re}k>a\}\ \text{ and}$$

$$\text{satisfies } \sup_{t>c}\int_{-\infty}^{\infty}|h(t+iy)|^2\,dy<\infty\Big\},$$

$$H^{2,\infty}(\operatorname{Re}k>a)=\Big\{h\in H^2(\operatorname{Re}k>a):\sup_{\operatorname{Re}k>c}|h(k)|<\infty\Big\}. \tag{3.7}$$

$H^{2,\infty}(\operatorname{Re}k>c)$ is closed under product. More precisely we have

Lemma 3.1. *Suppose*

$$f(k)=\sum_{j=1}^{N}a_jk^{-j}+k^{-N}h(k)$$

with $a_j\in\mathbb{C}$, $h\in H^{2,\infty}(\operatorname{Re}k>a)$. *Let* $\phi(z)$ *be a power series*

$$\phi(z)=\sum_{n=1}^{\infty}\alpha_nz^n$$

with convergent radius $r>0$. *If* $\sup_{\operatorname{Re}k>c}|f(k)|<r$ *is satisfied, then*

$$\phi(f(k))=\sum_{j=1}^{N}b_jk^{-j}+k^{-N}h_1(k) \tag{3.8}$$

is valid with some $b_j\in\mathbb{C}$, $h_1\in H^{2,\infty}(\operatorname{Re}k>a)$.

Proof. If $N=0$, then the statement is trivially valid. For $N\geq 1$, observe

$$\begin{aligned}\phi(f(k))=&\sum_{n=1}^{N}\alpha_n\left(\sum_{j=1}^{N}a_jk^{-j}+k^{-N}h(k)\right)^n\\&+k^{-N-1}\sum_{n=N+1}^{\infty}\alpha_n\left(\sum_{j=1}^{N}a_jk^{-j+1}+k^{-N+1}h(k)\right)^n,\end{aligned}$$

and set

$$\sum_{n=1}^{N}\alpha_n\left(\sum_{j=1}^{N}a_jk^{-j}+k^{-N}h(k)\right)^n=\sum_{j=1}^{N}b_jk^{-j}+k^{-N}h_2(k)$$

$$h_1(k)=h_2(k)+k^{-1}\sum_{n=N+1}^{\infty}\alpha_n\left(\sum_{j=1}^{N}a_jk^{-j+1}+k^{-N+1}h(k)\right)^n$$

with h_1, $h_2\in H^{2,\infty}(\operatorname{Re}k>a)$. Then, (3.8) holds. □

To relate (3.3) with (3.5) for $q\in\mathcal{Q}^N$, we have to know $q\in C^M(\mathbb{R})$ for some M.

Lemma 3.2. *Suppose $q \in \mathcal{Q}^N$. Let J be the smallest j such that $\mu_j(0) + \nu_j(0) \neq 0$. If there is no such j, we set $J = N+1$. Then, there exist $c_j^{\pm} \in R$, $h_{\pm} \in H^{2,\infty}(\operatorname{Re} k > a)$ such that*

$$m_{\pm}\left(-k^2\right) = -k - \sum_{j=1}^{2N-J+1} c_j^{\pm} k^{-j} + k^{-2N+J-1} h_{\pm}(k)$$

hold for sufficiently large a.

Proof. Under a suitable definition of $\sqrt{1+4g_1(-k^2)g_2(-k^2)}$, $m_{\pm}$ are obtained from g_j by

$$m_{\pm}\left(-k^2\right) = -\frac{1}{2g_1(-k^2)} \pm \frac{\sqrt{1+4g_1(-k^2)g_2(-k^2)}}{2g_1(-k^2)}. \tag{3.9}$$

Set

$$\delta_1(k) = -\int_0^{\infty} \frac{\xi_1(\lambda) - 1/2}{\lambda + k^2} d\lambda = -\sum_{j=1}^{N+1} \mu_j(0)\left(-k^2\right)^{-j} + \left(-k^2\right)^{-N-1} h_1(k).$$

Then $|\delta_1(k)|$ has a bound

$$\sup_{\operatorname{Re} k \geq a} |\delta_1(k)| \leq \sum_{j=1}^{N+1} |\mu_j(0)| a^{-2j} + a^{-2(N+1)-1} \int_0^{\infty} \lambda^{N+1/2} |\xi_j(\lambda) - 1/2| \, d\lambda.$$

Hence Lemma 3.1 shows

$$e^{\delta_1(k)} = 1 + \sum_{j=1}^{N+1} \alpha_j k^{-2j} + k^{-2(N+1)} h_3(k)$$

with $\alpha_j \in \mathbb{R}$, $h_3 \in H^{2,\infty}(\operatorname{Re} k > a)$, and

$$-\frac{1}{2g_1(-k^2)} = -k e^{\delta_1(k)} = -k - \sum_{j=1}^{N+1} \alpha_j k^{-2j+1} - k^{-2(N+1)+1} h_3(k). \tag{3.10}$$

On the other hand, to estimate the second term of (3.9), note (3.4) and (3.5), for $x = 0$, show

$$1 + 4g_1(-k^2)g_2(-k^2) = 1 - e^{\delta(k)}$$

with

$$\begin{aligned}
\delta(k) &= \sum_{j=1}^{N+1} \left(\mu_j(0) + \nu_j(0)\right)\left(-k^2\right)^{-j} + \left(-k^2\right)^{-N-1}\left(h_1(k) + h_2(k)\right) \\
&= \left(\mu_J(0) + \nu_J(0)\right)\left(-k^2\right)^{-J} \left(1 + \sum_{j=J+1}^{N+1} \frac{\mu_j(0) + \nu_j(0)}{\mu_J(0) + \nu_J(0)} \left(-k^2\right)^{-j+J}\right.
\end{aligned}$$

$$+ \left(-k^2\right)^{-N-1+J} \frac{h_1(k)+h_2(k)}{\mu_J(0)+\nu_J(0)}\Bigg),$$

where J is the smallest j such that $\mu_j(0)+\nu_j(0)\neq 0$. Then, applying Lemma 3.1 yields

$$\begin{aligned}\sqrt{1+4g_1(-k^2)g_2(-k^2)} &= \sqrt{(-1)^J\left(\mu_J(0)+\nu_J(0)\right)}\\ &\quad\times k^{-J}\left(1+\sum_{j=1}^{N-J+1}\beta_j k^{-2j}+k^{-2(N-J+1)}h_4(k)\right)\end{aligned}\tag{3.11}$$

with some $\beta_j\in\mathbb{R}$, $h_4\in H^{2,\infty}(\operatorname{Re}k>a)$ if a is sufficiently large. Then, (3.10) and (3.11) show

$$\begin{aligned}m_+\left(-k^2\right) &= -k-\sum_{j=1}^{N+1}\alpha_j k^{-2j+1}-k^{-2(N+1)+1}h_3(k)\\ &\quad+k\left(1+\sum_{j=1}^{N+1}\alpha_j k^{-2j}+k^{-2(N+1)}h_3(k)\right)\\ &\quad\times\left(\sqrt{(-1)^J\left(\mu_J(0)+\nu_J(0)\right)}k^{-J}\right)\\ &\quad\times\left(1+\sum_{j=1}^{N-J+1}\beta_j k^{-2j}+k^{-2(N-J+1)}h_4(k)\right)\\ &= -k-\sum_{j=1}^{2N-J+1}c_j^+k^{-j}+k^{-2N+J-1}h_+(k)\end{aligned}\tag{3.12}$$

with some $c_j\in\mathbb{R}$, $h_+\in H^{2,\infty}(\operatorname{Re}k>a)$. If $\mu_j(0)+\nu_j(0)=0$ for all $j=1,2,\dots,N+1$, then

$$\begin{aligned}\sqrt{1+4g_1(-k^2)g_2(-k^2)} &= \sqrt{1-\exp\left(\left(-k^2\right)^{-N-1}\left(h_1(k)+h_2(k)\right)\right)}\\ &= k^{-N-1}h_6(k)\end{aligned}$$

holds with

$$h_6(k)=\sqrt{(-1)^{-N}\sum_{n=1}^{\infty}\frac{1}{n!}\left(-k^2\right)^{-(n-1)(N+1)}\left(h_1(k)+h_2(k)\right)^n}.$$

Since $h_6(k)=2g_1(-k^2)m_+\left(-k^2\right)+1$, one can define h_6 as a holomorphic function on $\operatorname{Re}k>0$, hence $h_6\in H^{2,\infty}(\operatorname{Re}k>a)$ holds. Therefore, in this case also, we have

$$m_+\left(-k^2\right)=-k-\sum_{j=1}^{N+1}\alpha_j k^{-2j+1}-k^{-2(N+1)+1}h_3(k)$$

$$
\begin{aligned}
&+k\left(1+\sum_{j=1}^{N+1}\alpha_j k^{-2j}+k^{-2(N+1)}h_3(k)\right)\times k^{-N-1}h_6(k)\\
&=-k-\sum_{j=1}^{N}c_j^{+}k^{-j}+k^{-N}h_+(k).
\end{aligned}
$$

Note that $m_-\left(-k^2\right)$ can be treated similarly. □

Lemma 3.3. *For $N\ge 0$, it holds that*

$$
\begin{aligned}
&q\in\mathcal{Q}^0\Rightarrow q\in L^2_{\mathrm{loc}}(\mathbb{R}),\\
&q\in\mathcal{Q}^1\Rightarrow q^{(1)}\in L^2_{\mathrm{loc}}(\mathbb{R}),\\
&q\in\mathcal{Q}^N\Rightarrow q^{(2(N-1))}\in L^2_{\mathrm{loc}}(\mathbb{R})\quad \text{if } N\ge 2.
\end{aligned}
$$

Proof. We apply Proposition 4.1 in Appendix. If $N=0$, the asymptotics of $m_\pm\left(-k^2\right)$ are

$$m_\pm\left(-k^2\right)=-k+h_\pm(k).$$

Then, letting $N_1=N_2=0$ in Proposition 4.1, we have $q\in L^2_{\mathrm{loc}}(\mathbb{R})$. If $N=1$, we have

$$m_\pm\left(-k^2\right)=-k-\sum_{j=1}^{3-J}c_j^{\pm}k^{-j}+k^{-3+J}h_\pm(k)$$

with $J=1$ or 2. Since $N_1=N_2=3-J\ge 1$ in either case, Proposition 4.1 implies

$$q\in C([0,\infty))\cap C((-\infty,0]).$$

Therefore, if we follow the above argument for $m_\pm\left(-k^2,\theta_x q\right)$ at $x\,(\neq 0)$, we can show $q\in C(\mathbb{R})$ and $q^{(1)}\in L^2_{\mathrm{loc}}(\mathbb{R})$. If $N\ge 3$, a similar argument shows $q\in C^1([0,\infty))\cap C^1((-\infty,0])$, and hence $q\in C^1(\mathbb{R})$. Unless q is constant, there exists $x\in\mathbb{R}$ such that $q'(x)\neq 0$. Without loss of generality, we can assume $q'(0)\neq 0$. Applying (3.1) for $M=1$ yields

$$
\begin{aligned}
m_+(-k^2)&=-k-c_2(0)\,k^{-1}-c_3(0)\,k^{-2}+o\left(k^{-2}\right),\\
m_-(-k^2)&=-k-c_2(0)\,k^{-1}+c_3(0)\,k^{-2}+o\left(k^{-2}\right),
\end{aligned}
$$

which implies

$$1+4g_1(-k^2)g_2(-k^2)=\left(\frac{m_+\left(-k^2\right)-m_-\left(-k^2\right)}{m_+\left(-k^2\right)+m_-\left(-k^2\right)}\right)^2=\frac{q'(0)^2}{16}k^{-6}+o\left(k^{-6}\right)$$

due to $c_3(0)=q'(0)/4$. Therefore, we see $J=3$ and Proposition 4.1 implies $q^{(k)}\in C((-\infty,0])\cap C([0,\infty))$ for any $k\le 2(N-1)-1$. One can replace 0 by other x which satisfies $q'(x)\neq 0$, and complete the proof. □

Now turning to (3.3) and (3.5) for $M = 2(N-1)-1$, we have

$$\log\left(1+\sum_{k=1}^{N-1} a_k(x)\, z^{-k}\right) + o\left(z^{-N+1/2}\right) = \sum_{k=1}^{N+1} \mu_k(x) z^{-k} + O\left(z^{-N-2}\right),$$

$$\log\left(1+\sum_{k=1}^{N-1} b_k(x) z^{-k}\right) + o\left(z^{-N+1/2}\right) = \sum_{k=1}^{N+1} \nu_k(x) z^{-k} + O\left(z^{-N-2}\right)$$

if $z \to \infty$. Generally, the relation

$$\log\left(1+\sum_{k=1}^{N-1} a_k z^{-k}\right) + o(z^{-N+1}) = \sum_{k=1}^{N-1} \mu_k z^{-k} + o\left(z^{-N+1}\right)$$

for large z implies

$$a_1 = \mu_1 \quad \text{and} \quad a_k = \mu_k + \sum_{\ell=1}^{k-1} \frac{\ell}{k} a_{k-\ell}\mu_\ell \quad \text{for } k \geq 2.$$

Set

$$f(x) = \frac{1}{2}\int_0^\infty \sum_{k=1}^{N-1} \frac{x^k \lambda^{k-1}}{k!} |R(\lambda)|\, d\lambda.$$

Let $r > 0$ be any constant number such that $f(r) \leq 1$. Then, we have

Lemma 3.4. *It holds that*

$$|a_k(x)|,\ |b_k(x)| \leq r^{-k} k! \quad \textit{for } k = 1, 2, \ldots, N-1. \tag{3.13}$$

Proof. We prove the statement only for $a_k(x)$. Set

$$t_k = \frac{1}{2}\int_0^\infty \lambda^{k-1} |R(\lambda)|\, d\lambda.$$

Inequalities (3.13) hold for $k = 1$ since

$$|a_1(x)| = |\mu_1(x)| \leq \int_0^\infty \left|\frac{1}{2} - \xi_1(x,\lambda)\right| d\lambda \leq \frac{1}{2}\int_0^\infty |R(\lambda)|\, d\lambda = t_1,$$

and

$$1 \geq f(r) = \sum_{k=1}^{N-1} \frac{r^k}{k!} t_k \geq r t_1.$$

Assume (3.13) is valid up to k. Note that

$$|\mu_k(x)| \leq \left|\int_0^\infty \lambda^{k-1}\left(\frac{1}{2} - \xi_1(x,\lambda)\right) d\lambda\right| \leq \frac{1}{2}\int_0^\infty \lambda^{k-1}|R(\lambda)|\, d\lambda = t_k$$

for any $k \geq 1$. Hence Lemma 3.3 implies

$$|a_{k+1}(x)| \leq |\mu_{k+1}(x)| + \sum_{\ell=1}^{k} \frac{\ell}{k+1} |a_{k+1-\ell}(x)|\ |\mu_\ell(x)|$$

$$\leq \sum_{\ell=1}^{k+1} \frac{\ell}{k+1} r^{-k-1+\ell}(k+1-\ell)!t_\ell$$
$$= r^{-k-1}(k+1)! \sum_{\ell=1}^{k+1} \frac{\ell}{k+1} \frac{\ell!(k+1-\ell)!}{(k+1)!} r^\ell \frac{t_\ell}{\ell!}.$$

Since $\frac{\ell}{k+1}\frac{\ell!(k+1-\ell)!}{(k+1)!} \leq 1$, we see

$$|a_{k+1}(x)| \leq r^{-k-1}(k+1)! \sum_{\ell=1}^{k+1} r^\ell \frac{t_\ell}{\ell!} \leq r^{-k-1}(k+1)! f(r) \leq r^{-k-1}(k+1)!,$$

which shows the conclusion. □

3.2. Estimate of $q^{(n)}(x)$. In this subsection, still assume $\lambda_0 = 0$. First observe the estimates of $q^{(n)}(x)$ follows from those of $a_1^{(n)}(x)$ since $a_1(x) = q(x)/2$. To achieve this procedure, we obtain a recursive relation of $\{a_k(x), b_k(x)\}_{k\geq 1}$. It is known that $m_\pm(z, \theta_x q)$ satisfy Ricatti equations

$$\partial_x m_+(z, \theta_x q) = q(x) - z - m_+(z, \theta_x q)^2,$$
$$-\partial_x m_-(z, \theta_x q) = q(x) - z - m_-(z, \theta_x q)^2.$$

Then we have the following identities.

Lemma 3.5. *We have*

(i) $\partial_x g_1(x, z) = -\dfrac{m_+(z) - m_-(z)}{m_+(z) + m_-(z)}$,

(ii) $\partial_x g_2(x, z) = (q(x) - z)\partial_x g_1(x, z)$,

(iii) $\partial_x^2 g_1(x, z) = 2(q(x) - z)g_1(x, z) + 2g_2(x, z)$,

(iv) $\partial_x^3 g_1(x, z) = 2q'(x)g_1(x, z) + 4(q(x) - z)\partial_x g_1(x, z)$.

This lemma yields recurrence relations.

Lemma 3.6. *We have* $a_1(x) = -b_1(x) = q(x)/2$ *and*

(i) $\displaystyle\sum_{j=1}^{k-1} a'_{k-j}(x)a'_j(x) = 4\sum_{j=0}^{k+1} a_{k+1-j}(x)b_j(x)$,

(ii) $b'_{k+1}(x) = q(x)a'_k(x) - a'_{k+1}(x)$,

(iii) $a''_k(x) = 2q(x)a_k(x) - 2a_{k+1}(x) + 2b_{k+1}(x)$,

(iv) $a'''_k(x) = 2q'(x)a_k(x) + 4q(x)a'_k(x) - 4a'_{k+1}(x)$.

Here we set $a_0(x) = b_0(x) = 1$.

Proof. The identities $a_1(x) = -b_1(x) = q(x)/2$ come from (3.1), (3.2). To deduce (i), note that

$$(\partial_x g_1(x, x, q))^2 = 1 + 4g_1(x, z)g_2(x, z),$$

which implies

$$\begin{aligned}
&\frac{1}{-4z}\left(\sum_{k=1}^{N-1} a_k^{'}(x)z^{-k}+o(z^{-N+1})\right)^2 \\
&\quad = 1-\left(1+\sum_{k=1}^{N-1} a_k(x)z^{-k}+o(z^{-N+1})\right)\left(1+\sum_{k=1}^{N-1} b_k(x)z^{-k}+o(z^{-N+1})\right) \\
&\quad = -\sum_{k=1}^{N-1}(a_k(x)+b_k(x))z^{-k}-\sum_{k=1}^{N-1} a_k(x)z^{-k}\sum_{k=1}^{N-1} a_k(x)z^{-k}+O(z^{-N+1}).
\end{aligned}$$

Comparing the coefficients of the both sides of z^{-k} for each $k=1,2,\ldots,$ we obtain (i). Relations (ii) and (iii) are obtained by similar calculations. Relation (iv) follows from (ii) and (iii). □

Lemma 3.6 implies $a_k(x)$, $b_k(x)$ are polynomials of $\left\{q^{(j)}(x)\right\}_{0\le j\le 2(k-1)}$, hence $a_k(x)\in C^{2(N-k)-1}(\mathbb{R})$. Now we can show the key lemma.

Lemma 3.7. *For any k, n satisfying $1\le k\le N-1$ and $0\le n\le 2(N-k)-1$, it holds that*

$$\left|a_k^{(n)}(x)\right|\le cd^n r^{-k-n/2}(k+n)!, \tag{3.14}$$

which yields

$$\left|q^{(n)}(x)\right|\le 2cd^n r^{-1-n/2}(1+n)! \tag{3.15}$$

for $n\le 2N-3$.

Proof. We employ the method used by Marchenko [19, p. 22, the proof of Lemma 2.2]. It is easy to see that Lemma 3.4 shows (3.14) for $n=0$ due to (1.3) and $c>1$. Since Lemma 3.4 also shows $|b_k(x)|\le r^{-k}k!$, for $k\ge 1$, (iii) of Lemma 3.6 implies

$$\begin{aligned}
|a_k^{(2)}(x)| &\le 4|a_1(x)a_k(x)|+2|a_{k+1}(x)|+2|b_{k+1}(x)| \\
&\le 4r^{-1}r^{-k}k!+2r^{-k-1}(k+1)!+2r^{-k-1}(k+1) \\
&= 4r^{-k-1}k!(k+2)\le 2r^{-k-1}(k+2)!,
\end{aligned}$$

which yields (3.14) for $n=2$ due to (1.3). Here the reason why we employ estimate (3.14), a worse one than $\left|a_k^{(2)}(x)\right|\le 2r^{-k-1}(k+1)!$ shown above, is that (3.14) is attached to an induction. For $n=1$, it is not clear that Lemma 3.6 could induce the corresponding estimate, so we apply the Landau–Hadamard inequality for bounded smooth function f

$$\|f'\|_\infty^2\le 2\|f\|_\infty\|f''\|_\infty,$$

which implies

$$|a_k^{(1)}(x)|^2\le 2\|a_k\|_\infty\|a_k^{(2)}\|_\infty\le 2r^{-k}k!4r^{-k-1}k!(k+2)=8r^{-2k-1}(k!)^2(k+2).$$

Whence, we have

$$\left|a_k^{(1)}(x)\right| \le 2^{3/2} r^{-k-1/2} k! \sqrt{(k+2)} \le c d r^{-k-1/2}(k+1)!,$$

which yields (3.14) for $n=1$ since $cd \ge \sqrt{6} \ge 2^{3/2}\sqrt{(k+2)}/(k+1)$. Now we assume (3.14) is valid up to n (≥ 2) and for all $k \ge 1$. Then (iv) of Lemma 3.6 shows

$$\begin{aligned} a_k^{(n+1)}(x) &= 4\sum_{j=0}^{n-2}\binom{n-2}{j} a_1^{(j+1)}(x) a_k^{(n-2-j)}(x) \\ &\quad + 8\sum_{j=0}^{n-2}\binom{n-2}{j} a_1^{(j)}(x) a_k^{(n-1-j)}(x) - a_{k+1}^{(n-1)}(x), \end{aligned}$$

which yields

$$\begin{aligned} &\left|a_k^{(n+1)}(x)\right| \\ &\le 4c^2 \sum_{j=0}^{n-2}\binom{n-2}{j} d^{j+1} r^{-1-(j+1)/2}(2+j)! d^{n-2-j} r^{-(k+(n-2-j)/2)}(k+n-2-j)! \\ &\quad + 8c^2 \sum_{j=0}^{n-2}\binom{n-2}{j} d^{j} r^{-1-j/2}(1+j)! d^{n-1-j} r^{-(k+(n-1-j)/2)}(k+n-1-j)! \\ &\quad + 4c\, d^{n-1} r^{-(k+1+(n-1)/2)}(k+n)! \\ &= 4c^2 d^{n-1} r^{-k-(n+1)/2} \sum_{j=0}^{n-2}\binom{n-2}{j}(2+j)!(k+n-2-j)! \\ &\quad + 8c^2 d^{n-1} r^{-k-(n+1)/2} \sum_{j=0}^{n-2}\binom{n-2}{j}(1+j)!(k+n-1-j)! \\ &\quad + 4c\, d^{n-1} r^{-k-(n+1)/2}(k+n)!. \end{aligned}$$

Here we note an identity

$$\frac{(k+n+1)!}{(k+1)!} = \sum_{j=0}^{n}\binom{n}{j}\frac{(k-\ell+n-j)!}{(k-\ell)!}\frac{(\ell+j)!}{\ell!} \tag{3.16}$$

for $0 \le \ell \le k$, which was obtained by Marchenko [19] by differentiating n times on the both sides of the identity $z^{-(\ell+2)} = z^{-(\ell-j+1)} z^{-(j+1)}$. Then replacing n, ℓ, k by $n-2$, 2, $k+2$, we have

$$\begin{aligned} &\sum_{j=0}^{n-2}\binom{n-2}{j}(2+j)!(k+n-2-j)! \\ &\quad = 2!k! \sum_{j=0}^{n-2}\binom{n-2}{j}\frac{(2+j)!(k+n-2-j)!}{2!k!} = \frac{2(k+n+1)!}{(k+1)(k+2)(k+3)}, \end{aligned}$$

and replacing n, ℓ, k by $n-2$, 1, $k+2$, we have

$$\sum_{j=0}^{n-2}\binom{n-2}{j}(1+j)!(k+n-1-j)!$$
$$= (k+1)!\sum_{j=0}^{n-2}\binom{n-2}{j}\frac{(1+j)!(k+n-1-j)!}{(k+1)!} = \frac{(k+n+1)!}{(k+2)(k+3)}.$$

Therefore, we can obtain that

$$\begin{aligned}
&\left|a_k^{(n+1)}(x)\right| \\
&\le 4cd^{n-1}r^{-k-(n+1)/2}\left(\frac{2c(k+n+1)!}{(k+1)(k+2)(k+3)} + \frac{c(k+n+1)!}{(k+2)(k+3)} + (k+n)!\right) \\
&= 4cd^{n-1}r^{-k-(n+1)/2}(k+n+1)!\left(\frac{2c}{(k+1)(k+3)} + \frac{1}{k+n+1}\right) \\
&\le 4cd^{n+1}r^{-k-(n+1)/2}(k+n+1)!d^{-2}(c+1),
\end{aligned}$$

which completes the induction due to (1.2). And estimate (3.15) follows from $a_1(x) = q(x)/2$. □

3.3. Proof of Theorem 1.1 and Theorem 1.2. For $q \in \mathcal{Q}^N$, let $\tilde{q}(x) = q(x) - \lambda_0$. Then $\operatorname{sp} L_{\tilde{q}} \subset [0,\infty)$ and $\inf \operatorname{sp} L_{\tilde{q}} = 0$. Theorem 1.1 can be obtained directly from Lemma 3.7. If $q \in \mathcal{Q}$, then for any $N \ge 1$ we have $\mathcal{Q} \subset \mathcal{Q}^N$ and

$$\begin{aligned}
\frac{1}{2}\int_0^\infty \sum_{k=1}^{N-1}\frac{r^k\lambda^{k-1}}{k!}|R(\lambda+\lambda_0)|\,d\lambda &\le \frac{1}{2}\int_0^\infty \sum_{k=1}^{\infty}\frac{r^k\lambda^{k-1}}{k!}|R(\lambda+\lambda_0)|\,d\lambda \\
&= \frac{1}{2}\int_0^\infty \frac{e^{r\lambda}-1}{\lambda}|R(\lambda+\lambda_0)|\,d\lambda \le 1.
\end{aligned}$$

Hence, we can choose the same r for any N in Theorem 1.1 and we have estimates

$$\left|q^{(n)}(x)\right| \le 2cd^n r^{-1-n/2}(1+n)!$$

for any $n \ge 0$ and $x \in \mathbb{R}$. Then for any $a \in \mathbb{R}$, applying (3.15) yields

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{|q^{(n)}(a)|}{n!}|z-a|^n &\le c\sum_{n=0}^{\infty}\frac{d^n r^{-(1+n/2)}(n+1)!}{n!}|z-a|^n \\
&= cr^{-1}\sum_{n=0}^{\infty}(n+1)(dr^{-1/2})^n|z-a|^n \\
&= cr^{-1}(1-dr^{-1/2}|z-a|)^{-2}.
\end{aligned}$$

Therefore, q can be analytically extendable to the strip

$$\left\{z \in \mathbb{C} : \ |\operatorname{Im} z| < d^{-1}\sqrt{r}\right\}.$$

Moreover, for z of this strip, taking $a = \operatorname{Re} z$, we see $|z-a| = |\operatorname{Im} z|$, hence

$$|q(z)-\lambda_0| = |\tilde{q}(z)| \le \frac{c}{r}\left(1 - d^{-1}\sqrt{r}|z-a|\right)^{-2} = \frac{c}{r}\left(1 - d^{-1}\sqrt{r}|\operatorname{Im} z|\right)^{-2},$$

which completes the proof of Theorem 1.2.

4. Appendix

Let q be a potential of a Schrödinger operator L_q and $m_+(z)$ be its (right) Weyl function. We assume here

$$\operatorname{sp} L_q \subset [\lambda_0, \infty) \tag{4.1}$$

with some $\lambda_0 > -\infty$. If the potential is in $C^N([0,a))$ for some $a > 0$, then it is known that $m_+(z)$ has asymptotics (3.1). As far as the authors know there exists no converse statement. In this appendix we give a partial answer to this issue.

For our purpose the A-function introduced by B. Simon [23] is quite suitable since it is directly related with m_+. For a bounded potential q the A-function is defined through an identity

$$m_+\left(-k^2\right) = -k - \int_0^\infty e^{-2\alpha k} A(\alpha)\, d\alpha. \tag{4.2}$$

Here $A(\alpha)$ satisfies

$$|A(\alpha)| \le \alpha^{-1} \|q\|_\infty^{1/2} \exp\left(2\alpha \|q\|_\infty^{1/2}\right). \tag{4.3}$$

Let $A(\alpha, x)$ be the A-function for the shifted potential $q(\cdot + x)$. Then, $A(\alpha, x)$ satisfies

$$\partial_x A(\alpha, x) = \partial_\alpha A(\alpha, x) + \int_0^\alpha A(\beta, x) A(\alpha - \beta, x)\, d\beta \tag{4.4}$$

in a distributional sense, and

$$q(x) = \lim_{\alpha \downarrow 0} A(\alpha, x). \tag{4.5}$$

Therefore, for a given initial value $A(\alpha, 0) = A(\alpha)$, solving equation (4.4) yields the potential q by (4.5).

Proposition 4.1. *Assume the Weyl function $m_+(z)$ takes a form of*

$$m_+\left(-k^2\right) = -k - \sum_{j=1}^{N_1} c_j k^{-j} - k^{-N_2} h(k) \tag{4.6}$$

for N_1, $N_2 \ge 0$ with h satisfying

$$\sup_{t \ge a} \int_{-\infty}^{\infty} |h(t + iy)|^2\, dy < \infty, \quad h \in H^2(\operatorname{Re} k \ge a), \tag{4.7}$$

for some $a > \|q\|_\infty^{1/2}$. Then, $q \in C^{N_2 - 1}([0,\infty))$ and $q^{(N_2)} \in L^2_{\mathrm{loc}}([0,\infty))$ hold.

Proof. The function $h(k)$ is analytic on $\left\{\operatorname{Re} k > \|q\|_\infty^{1/2}\right\}$ since $m_+(z)$ is analytic on $\mathbb{C} \backslash [-\|q\|_\infty, \infty)$. Set $f(\zeta) = h(a - i\zeta/2)$. Then, $f(\zeta)$ is analytic on $\mathbb{C}_+$

and condition (4.7) implies that f is an element of the Hardy space on $\mathbb{C}_+$, hence there exists $\phi \in L^2(\mathbb{R}_+)$ such that

$$f(\zeta) = \int_0^\infty e^{i\zeta\alpha}\phi(\alpha)\,d\alpha.$$

Let $\psi(\alpha)$ be the function defined by

$$\partial_\alpha^{N_2}\psi(\alpha) = e^{2a\alpha}\phi(\alpha), \quad \partial_\alpha^j\psi(0) = 0 \quad \text{for } j = 0, 1, 2, \ldots, N_2 - 1.$$

Then, we have

$$-k - \int_0^\infty e^{-2\alpha k}\left(2\sum_{j=0}^{N_1-1}\frac{c_{j+1}}{j!}(2\alpha)^j + 2^{N_2+1}\psi(\alpha)\right)d\alpha = m_+\left(-k^2\right),$$

hence

$$A(\alpha) = 2\sum_{j=0}^{N_1-1}\frac{c_{j+1}}{j!}(2\alpha)^j + 2^{N_2+1}\psi(\alpha). \tag{4.8}$$

For $x < \gamma$, set $C(\gamma, x) = A(\gamma - x, x)$. Then, (4.4) reads

$$C(\gamma, x) = A(\gamma) + \int_0^x dy\int_y^\gamma C(\lambda, y)\,C(\gamma - \lambda + y, y)\,d\lambda,$$

which is solvable by iteration $C_0(\gamma, x) = A(\gamma)$ and

$$C_n(\gamma, x) = A(\gamma) + \int_0^x dy\int_y^\gamma C_{n-1}(\lambda, y)\,C_{n-1}(\gamma - \lambda + y, y)\,d\lambda$$

for $n \geq 1$. If the initial value $A(\gamma)$ is $C^{N_2-1}([0,\infty))$ and $A^{(N_2)} \in L^2_{\mathrm{loc}}([0,\infty))$, then $C_n(\gamma, x)$ are of C^{N_2-1} in γ, x and their uniform limit $C(\gamma, x)$ is of C^{N_2-1}, hence $q(x) = C(x, x)$ is of C^{N_2-1} as well. The property $q^{(N_2)} \in L^2_{\mathrm{loc}}([0,\infty))$ is also verified similarly. $\square$

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References

[1] I. Binder, D. Damanik, M. Goldstein, and M. Lukic, *Almost periodicity in time of solutions of the KdV equation,* Duke Math. J. **167** (2018), 2633–2678.

[2] E.I. Dinaburg and Y.G. Sinai, *The one-dimensional Schrödinger equation with a quasiperiodic potential,* Funct. Anal. Its Appl. **9** (1975), 279–289.

[3] D. Damanik and M. Goldstein, *On the existence and uniqueness of global solutions of the KdV equation with quasiperiodic initial data,* J. Amer. Math. Soc. **29** (2016), 825–856.

[4] A.A. Danielya and B.M. Levitan, *On the asymptotic behavior of the Weyl-Titchmarsh m-function,* Math. USSR Izv. **36** (1991), 487–496.

[5] I.E. Egorova, *The Cauchy problem for the KdV equation with almost periodic initial data whose spectrum is nowhere dense,* Adv. Soviet Math. **19** (1994), 181–208.

[6] L.H. Eliasson, *Floquet solutions for the 1 dimensional quasi-periodic Schrödinger equation,* Commun. Math. Phys. **146** (1992), 447 C482.

[7] B. Eichinger, T. VandenBoom, and P. Yuditskii, *KdV hierarchy via Abelian coverings and operator identities,* Trans. Amer. Math. Soc. Ser. B **6** (2019), 1–44.

[8] S. Grudsky and A. Rybkin, *Soliton theory and Hankel operators,* SIAM J. Math. Anal. **47** (2015), 2283–2323.

[9] C.S. Gardner, J.M. Greene, M.D. Kruskal, and R.M. Miura, *A method for solving the Korteweg-de Vries equation,* Phys. Rev. Lett. **19** (1967), 1095–1097.

[10] F. Gesztesy and H. Holden, *Soliton Equations and Their Algebro-Geometric Solutions,* **I**: $(1+1)$*-Dimensional Continuous Models,* Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2003.

[11] F. Gesztesy and B. Simon, *The xi function,* Acta Math. **176** (1996), 49–71.

[12] F. Gesztesy and B. Simon, *Inverse spectral analysis with partial information on the potential, I. The case of an a.c. component in the spectrum,* Helv. Phys. Acta **70** (1997), 66–71.

[13] S. Kotani, *Ljapounov indices determine absolutely continuous spectra of stationary random Schrödinger operators,* Stochastic Analysis (Katata/Kyoto, 1982), 225–247, North-Holland Math. Library, **32**, North-Holland, Amsterdam, 1984.

[14] S. Kotani, *Construction of KdV flow, I. Tau function via Weyl function,* J. Math. Phys. Anal. Geom. **14** (2018), 297–335.

[15] S. Kotani, *Construction of KdV flow,* to appear in Peking Math. Journ., https://arxiv.org/abs/2107.05428.

[16] P.D. Lax, *Integrals of nonlinear equations of evolution and solitary waves,* Comm. Pure Appl. Math. **21** (1968), 467–490.

[17] D.S. Lundina, *Compactness of the set of reflectionless potentials,* Teor. Funkt. Funkt. Anal. Prilozh. **44** (1985), 55–66 (Russian). Available from: http://dspace.univer.kharkov.ua/bitstream/123456789/1776/4/Lundina.pdf

[18] M. Leguil, J. You, Z. Zhao, and Q. Zhou, Asymptotics ofspectral gaps of quasi-periodic Schrödinger operators, preprint, https://arxiv.org/abs/1712.04700.

[19] V.A. Marchenko, *The Cauchy problem for the KdV equation with non-decreasing initial data,* Springer series in Nonlinear Dynamics, What is Integrability? (Ed. V.E. Zakharov), 1990, 273–318.

[20] C. Remling, *Generalized reflection coefficients,* Comm. Math. Phys. **337** (2015), 1011–1026.

[21] A. Rybkin, *On the evolution of a reflection coefficient under the Korteweg-de Vries flow,* J. Math. Phys. **49** (2008),

[22] A. Rybkin, *The Hirota* τ*-function and well-posedness of the KdV equation with an arbitrary step like initial profile decaying on the right half line,* Nonlinearity **24** (2011), 2953–2990.

[23] B. Simon, *A new approach to inverse spectral theory, I. Fundamental formalism,* Ann. of Math. **150** (1999), 1029–1057.

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Однорідна оцінка потенціалів через коефіцієнти відбиття та її застосування до потоку Кортевега–де Фріза

Shinichi Kotani and Jinhui Li

Оцінено потенціали одновимірних операторів Шредінгера через моменти коефіцієнтів відбиття. Оскільки коефіцієнти відбиття є інваріантними відносно потоку Кортевега–де Фріза, оцінки надають інформацію про певну передкомпактність розв'язків рівняння Кортевега–де Фріза, починаючи з початкових значень, які мають скінченні моменти коефіцієнтів відбиття.

Ключові слова: оператор Шредінгера, коефіцієнт відвиття, рівняння Кортевега–де Фріза