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The Law of Multiplication of Large Random Matrices Revisited

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The paper deals with the eigenvalue distribution of the product of two $n \times n$ positive definite matrices B_τ, $\tau = \pm 1$, rotated with respect to each other by the random orthogonal and Haar distributed matrix. The problem has been considered in several works by using various techniques. We propose a streamlined approach based on the random matrix theory techniques and a certain symmetry of the problem. We prove the convergence with probability 1 as n tends to infinity of the Normalized Counting Measure (NCM) of eigenvalues of the product to a non-random limit, derive a functional equation that determines the Stieltjes transform of the limiting NCM of the product in terms of limiting NCMs of the factors B_τ, $\tau = \pm 1$, and consider an interesting example.

Key words: random matrices, orthogonal matrices, eigenvalue distribution

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1. Introduction

The objective of the paper is the eigenvalue distribution of the $n \times n$ positive definite random matrices

$$M_n = A_n^{1/2} O_n B_n O_n^* A_n^{1/2}, \tag{1.1}$$

where A_n and B_n are positive definite and $O_n \in SO(n)$ is random and Haar distributed. We are interested in the Normalized Counting Measure (NCM) of eigenvalues $\{\lambda_\alpha^{(n)}\}_{\alpha=1}^n$ (possible coinciding) of M_n

$$\nu_{M_n} = n^{-1} \sum_{\alpha=1}^{n} \delta_{\lambda_\alpha^{(n)}}, \quad \nu_{M_n}(\mathbb{R}) = 1. \tag{1.2}$$

More precisely, we assume, as usual in random matrix theory, that we have infinite sequences $\{A_n\}_n$, $\{B_n\}_n$, and $\{O_n\}_n$, hence, an infinite sequence $\{M_n\}_n$, and we want to find a description of ν_{M_n} in terms of ν_{A_n} and ν_{B_n} in the limit $n \to \infty$.

One may mention several motivations for the problem. First, the problem seems quite natural from the point of view of random matrix theory. It is also can be viewed as an analog of the sample covariance matrices of statistics where the role of the data matrix plays the orthogonal matrix instead of the matrix with i.i.d (often Gaussian) entries, see, e.g., [5, 7, 11, 14, 15] for the latter. The random matrix (2.3) appeared recently in the studies of random deep neural networks [4, 12, 17] and in certain models of quantum information science [13]. One more aspect is related to the general problem to describe the eigenvalues of the product of two positive definite matrices in terms of eigenvalues of two factors of the product (see, e.g., [2], Section III.4 for a review). It seems unlikely to expect in general a sufficiently simple and closed expression for eigenvalues of the product (sum) of two given matrices via eigenvalues of factors (terms). Hence, it is natural to look for a "generic" asymptotic answer, studying a randomized version of the problem. Such an approach to the analysis of eigenvalues of Hermitian matrices was considered in [9, 15, 16].

The problem has been treated by several authors who used various techniques, the most known are the free probability techniques [9] and the random matrix techniques [15, 18]. We will use a version of the latter taking into account a certain symmetry of the problem.

The paper is organized as follows. In the next Section 2 we prove our main result, Theorem 2.1. We also discuss there an interesting particular case of the theorem where matrices A_n and B_n of (1.1) are orthogonal projections. Various auxiliary results that are used in the proof of the theorem are given in Section 3.

2. Main results

The non-zero eigenvalues of (1.1) coincide with those of $B_n^{1/2}O_n^*A_nO_nB_n^{1/2}$. Thus, it is convenient to consider the both matrices simultaneously as can be seen from the theorem below.

Theorem 2.1. *Let $B_{\tau,n}$, $\tau=\pm1$, be $n\times n$ positive definite matrices such that*

$$\sup_n n^{-1}\mathrm{Tr}\, B_{\tau,n}^2 \le b_2 < \infty \tag{2.1}$$

and their Normalized Counting Measures $\nu_{B_{\tau,n}}$, $\tau=\pm1$, converge weakly to the measures

$$\lim_{n\to\infty} \nu_{B_{\tau,n}} = \nu_{B_\tau}, \quad \nu_{B_\tau}(\mathbb{R}_+) = 1, \quad \tau=\pm1, \tag{2.2}$$

which are not concentrated at zero.

Consider the $n\times n$ positive definite random matrices

$$M_{\tau,n} = B_{\tau,n}^{1/2} O_n^\tau B_{-\tau,n} O_n^{-\tau} B_{\tau,n}^{1/2}, \quad \tau=\pm1, \tag{2.3}$$

where $O_n\in SO(n)$ is the random Haar distributed orthogonal matrix, and denote by $\nu_{M_{\tau,n}}$ the NCM's of $M_{\tau,n}$ (see (1.2)).

Then:

(i) $\nu_{M_{\tau,n}}$ *do not depend on $\tau=\pm1$*

$$\nu_{M_{+1,n}} = \nu_{M_{-1,n}} =: \nu_{M_n}, \tag{2.4}$$

and we have with probability 1 the weak non-random limit

$$\lim_{n\to\infty} \nu_{M_{\tau,n}} = \lim_{n\to\infty} \overline{\nu}_{M_{-1,n}} =: \nu_M; \tag{2.5}$$

(ii) *the Stieltjes transform*

$$f_M(z) := \int_0^\infty \frac{\nu_M(d\lambda)}{\lambda - z}, \quad z \in \mathbb{C} \setminus \mathbb{R}_+, \tag{2.6}$$

of the limiting NCM ν_M in (2.5) solves the system

$$(1 + z f_M(z)) f_M(z) - h_{+1}(z) h_{-1}(z) = 0, \tag{2.7}$$
$$f_{B_\tau}(z f_M(z)/h_{-\tau}(z)) = h_{-\tau}(z), \quad \tau = \pm 1, \tag{2.8}$$

with respect to the triple (f_M, h_+, h_-), where

$$f_{B_\tau}(z) := \int_0^\infty \frac{\nu_{B_\tau}(d\lambda)}{\lambda - z}, \quad \tau = \pm 1,\ z \in \mathbb{C} \setminus \mathbb{R}_+, \tag{2.9}$$

are the Stieltjes transforms of limiting ν_{B_τ}, $\tau = \pm 1$, of (2.2) and $d(z) = dist(z, \mathbb{R}_+)$ is large enough;

(iii) *the solution of (2.7)–(2.8) is unique in the class of triples (f_M, h_+, h_-) of functions that belong to the class $\mathcal{N}_+$ of functions that are analytic in $\mathbb{C} \setminus \mathbb{R}_+$, continuous and positive on the open negative semi-axis and such that*

$$\Im s(z) \Im z > 0, \quad \Im z \neq 0, \quad \sup_{y \geq 1} y s(iy) < \infty, \quad s \in \mathcal{N}_+. \tag{2.10}$$

Theorem 2.1 is proved below in this section. The corresponding assertion for the unitary Haar distributed matrices is analogous, see [15, 18] for earlier results in this case. We will give a streamlined approach for orthogonal matrices. Being applied to unitary matrices, the approach proves to be simpler than those in [15, 18].

Remark 2.2. The system (2.7)–(2.8) can be viewed as that determining a binary operation in the set (cone) of functions of the class $\mathcal{N}_+$, allowing one to determine a unique f_M given f_{B_τ}, $\tau = \pm 1$. Recalling the one-to-one correspondence between $\mathcal{N}_+$ and the set (cone) of non-negative measures supported on $\mathbb{R}_+$, one can also say that the system (2.7)–(2.8) determines a binary operation in the latter set as well. The operation is known in free probability as the free multiplicative convolution of measures, see [9], Chapter 3 and [12].

Here is an interesting particular case of the theorem, see e.g [3, 6] for similar results.

Example 2.3. Consider the case of (2.3) where $B_{\tau,n} = P_{\tau,n}$ are the orthogonal projections with rank $P_{\tau,n} = r_\tau$ and

$$\lim_{n\to\infty} r_\tau / n = \rho_\tau \in (0, 1). \tag{2.11}$$

In this case the limiting measures ν_{B_τ} of (2.2) are

$$\nu_{B_\tau} = (1-\rho_\tau)\delta_0 + \rho_\tau\delta_1, \quad \overline{\rho}_\tau = 1-\rho_\tau.$$

Plugging them into (2.7)–(2.8), we obtain the quadratic equation

$$z^2(z-1)f^2 + z\,(z-\overline{\rho})\,f - \overline{\rho}_+\overline{\rho}_-, \quad \overline{\rho} = \overline{\rho}_+ + \overline{\rho}_-.$$

The equation and (2.10) yield

$$\begin{aligned} f(z) &= -\frac{\overline{\rho}}{2z} + \frac{1-\overline{\rho}}{2(1-z)} + \frac{\sqrt{(z-p_+)(z-p_-)}}{2z(1-z)}, \\ p_\pm &= \left(\sqrt{\rho_+\overline{\rho}_-} \pm \sqrt{\rho_-\overline{\rho}_+}\right)^2 \in (0,1), \end{aligned} \tag{2.12}$$

where the branch of square root is fixed by the condition to behave as $z + O(1)$, $z \to \infty$.

By using the inversion formula

$$\nu(\Delta) = \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\int_\Delta \Im f(\lambda + i\varepsilon)\,d\lambda,$$

relating a measure ν and its Stieltjes transform f, we obtain

$$\begin{aligned} \nu_M &= \max_\tau \overline{\rho}_\tau\,\delta_0 + (\overline{\rho}-1)_+\,\delta_1 + \nu_M^c, \quad x_+ = \max\{0,x\}, \\ \nu_M^c(d\lambda) &= \psi_M(\lambda)d\lambda, \quad \psi_M(\lambda) = \frac{\sqrt{(p_+-\lambda)(\lambda-p_-)}}{2\pi\lambda(1-\lambda)}\mathbf{1}_{[p_+,p_-]}. \end{aligned}$$

Proof of Theorem 2.1. The proof is essentially based on the tools of the branch of random matrix theory that deals with large random matrices whose "randomness" is due to classical compact groups, the group $S(n)$ in our case, see, e.g., Chapters 8–10 of [15].

Assume temporarily that $B_{\tau,n}$, $\tau = \pm 1$, are bounded uniformly in $n \to \infty$:

$$\|B_{\tau,n}\| \le b < \infty. \tag{2.13}$$

This assumption is removed at the end of the proof.

In view of the one-to-one correspondence between non-negative measures and their Stieltjes transforms (see, e.g., [15], Section 2.1), it suffices to prove that the Stieltjes transform

$$g_{M_{\tau,n}}(z) := \int_0^\infty \frac{\nu_{M_{\tau,n}}(d\lambda)}{\lambda - z}, \quad z \in \mathbb{C}\setminus\mathbb{R}_+, \tag{2.14}$$

of the NCM $\nu_{M_{\tau,n}}$ of (1.2) converges with probability 1 as $n \to \infty$ to a limit f_M of (2.6) on a compact set $\mathbb{K} \in \mathbb{C}\setminus\mathbb{R}_+$. It is convenient to assume that (see (2.13))

$$\mathbb{K} \subset \left\{z \in \mathbb{C}\setminus\mathbb{R}_+ :\ |z| \ge 2b^2\right\}. \tag{2.15}$$

It is important that since the sequences $\{g_{M_{\tau,n}}\}_n$, $\tau = \pm 1$, consists of functions analytic in $\mathbb{C} \setminus \mathbb{R}_+$, the convergence on $\mathbb{K}$ implies that everywhere in $\mathbb{C} \setminus \mathbb{R}_+$.

By using the definition (1.2) of the NCM and spectral theorem for $M_{\tau,n}$, we can write the representation

$$\begin{aligned} g_{M_{\tau,n}}(z) &= n^{-1}\mathrm{Tr}\ G_{M_{\tau,n}}(z), \quad z \in \mathbb{C} \setminus \mathbb{R}_+, \\ G_{M_{\tau,n}}(z) &= (M_{\tau,n} - z)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}_+, \\ \|G_{M_{\tau,n}}(z)\| &\le 1/d(z), \quad d(z) = \mathrm{dist}(z, \mathbb{R}_+) \end{aligned} \tag{2.16}$$

of $g_{M_{\tau,n}}$ via the resolvent $G_{M_{\tau,n}}$ of $M_{\tau,n}$.

Taking into account (2.15) and the bound (see (2.13))

$$\|M_{\tau,n}\| \le b^2, \quad \tau = \pm 1, \tag{2.17}$$

we have the norm convergent expansion

$$G_{M_{\tau,n}}(z) = -\sum_{l=0}^{\infty} \frac{M_{\tau,n}^l}{z^{l+1}}, \quad \tau = \pm 1, \tag{2.18}$$

in (2.15). This and the equalities (see (2.3))

$$\mathrm{Tr}\ M_{\tau,n}^l = \mathrm{Tr}\ M_{-\tau,n}^l, \quad \tau = \pm 1,\ l = 1, 2 \ldots, \tag{2.19}$$

yield

$$n^{-1}\mathrm{Tr}\ G_{M_{\tau,n}} = n^{-1}\mathrm{Tr}\ G_{M_{-\tau,n}}, \quad \tau = \pm 1, \tag{2.20}$$

implying in view of (2.16) the independence of (2.14) on $\tau = \pm 1$

$$g_{M_{\tau,n}}(z) =: g_{M_n}(z), \tag{2.21}$$

hence, the validity of (2.4).

We will use now Lemma 3.3 with $B_0 = \mathbf{1}_n$, $b_0 = 1$ and (2.21) to obtain the bound

$$\mathbf{Var}\{g_{M_n}(z)\} \le 8Cb^4/n^2 d^4(z) \tag{2.22}$$

with $d(z)$ defined in (2.16).

The bound and the Borel–Cantelli lemma imply that it suffices to find the $n \to \infty$ limit of the expectation

$$f_{M_n}(z) := \mathbf{E}\{g_{M_n}(z)\} = \int_0^{\infty} \frac{\overline{\nu}_{M_n}(d\lambda)}{\lambda - z}, \quad \overline{\nu}_{M_n} := \mathbf{E}\{\nu_{M_n}\}. \tag{2.23}$$

for $z \in \mathbb{K}$ of (2.15). Then, in view of the one-to one correspondence between non-negative measures and their Stieltjes transforms, (see, e.g., [15], Section 2.1), we obtain with probability 1 the weak limit

$$\lim_{n\to\infty} \nu_{M_n} = \lim_{n\to\infty} \overline{\nu}_{M_n} =: \nu_M \tag{2.24}$$

implying (2.5) under condition (2.13).

In what follows we do not write the subindex n in those matrices where this does not lead to misunderstanding.

Introduce the matrix (cf. (2.3))

$$\mathcal{M}_\tau = B_\tau O^\tau B_{-\tau} O^{-\tau}, \quad \tau = \pm 1. \tag{2.25}$$

It follows from (2.13) that $\|\mathcal{M}_\tau\| \le b^2$ (cf. (2.17)), hence we have the norm convergent series (cf. (2.18))

$$G_{\mathcal{M}_\tau}(z) := -\sum_{l=0}^{\infty} \frac{\mathcal{M}_\tau^l}{z^{l+1}}, \quad \tau = \pm 1,\ z \in \mathbb{K},$$
$$\|G_{\mathcal{M}_\tau}(z)\| \le 1/b^2. \tag{2.26}$$

We have obviously (cf. (2.19))

$$\mathrm{Tr}\, M_\tau^l = \mathrm{Tr}\, \mathcal{M}_\tau^l, \quad l = 1, 2, \dots \infty, \tag{2.27}$$

and then (2.16), (2.18), and (2.26) yield

$$g_{\mathcal{M}_{\tau,n}}(z) := n^{-1}\mathrm{Tr}\ G_{\mathcal{M}_\tau}(z) = n^{-1}\mathrm{Tr}\ G_{M_\tau}(z) = g_{M_n}(z). \tag{2.28}$$

We conclude that it suffices to deal with $\mathbf{E}\{n^{-1}\mathrm{Tr}\ G_{\mathcal{M}_\tau}(z)\}$ instead of $\mathbf{E}\{n^{-1}\mathrm{Tr}\ G_{M_\tau}(z)\}$ of (2.16).

We will use now the differentiation formulas (3.1) for $G_{\mathcal{M}_+}(z)$ of (2.26) as the map Φ in the first formula and $G_{\mathcal{M}_{-1}}(z)$ in the second formula respectively.

Let

$$O \to (1 + A)O + O(A^2), \quad O \to O(1 + A) + O(A^2)$$

be the infinitesimal left ($\tau = +1$) and right ($\tau = -1$) shifts in $SO(n)$ and let

$$\delta_\tau O^\tau := \tau A O, \quad \delta_\tau O^{-\tau} := -\tau A O^{-\tau} \tag{2.29}$$

be the corresponding linear variations with an infinitesimal antisymmetric matrix A (cf. (3.3)). We have then from (2.25) and (2.29)

$$\delta_\tau \mathcal{M}_\tau = \tau B_\tau [A, B_{-\tau}(O^{-\tau})], \quad B_{-\tau}(O^\tau) = O^\tau B_{-\tau} O^{-\tau},$$
$$\delta_\tau G_{M_\tau} = -G_{M_\tau} \delta M_\tau G_{M_\tau}, \tag{2.30}$$

where for any two matrices D_1 and D_2 we denote

$$[D_1, D_2] = D_1 D_2 - D_2 D_1 \tag{2.31}$$

their commutator.

This yields

$$\mathbf{E}\{G_{\mathcal{M}_\tau} B_\tau [A, B_{-\tau}(O^\tau)] G_{\mathcal{M}_\tau}\} = 0, \quad \tau = \pm 1. \tag{2.32}$$

Choosing here $A = A^{(jk)}$ of (3.3) for $j, k = 1, \dots, n$, we obtain the following entry-wise version of (2.32) for $a, b = 1, \dots, n$:

$$\mathbf{E}\{(G_{\mathcal{M}_\tau} B_\tau)_{aj} (B_{-\tau}(O) G_{\mathcal{M}_\tau})_{kb} - (\mathcal{M}_\tau G_{\mathcal{M}_\tau})_{aj} (G_{\mathcal{M}_\tau})_{kb}\} = (\mathcal{T}_\tau)_{ab}^{jk},$$

$$(\mathcal{T}_\tau)^{jk}_{ab} = \mathbf{E}\{(G_{\mathcal{M}_\tau}B_\tau)_{ak}(B_{-\tau}(O^\tau)G_{\mathcal{M}_\tau})_{jb} \\ -(G_{\mathcal{M}_\tau}B_\tau B_{-\tau}(O^\tau))_{ak}(G_{\mathcal{M}_\tau})_{jb}\}, \tag{2.33}$$

i.e., $\mathcal{T}_\tau$ is given by the left-hand side of the first line of above relation with the interchanged j and k.

Set here $b = k$ and apply to the result the operation

$$n^{-1}\sum_{k=1}^{n}, \tag{2.34}$$

taking into account that $\mathcal{M}_\tau G_{\mathcal{M}_\tau} = 1 + zG_{\mathcal{M}_\tau}$, $\tau = \pm 1$ (see (2.26)). This and (2.16) yields for $\tau = \pm 1$

$$\mathbf{E}\{G_{\mathcal{M}_\tau}B_\tau h^{B_{-\tau}(O^\tau)}_{\mathcal{M}_\tau} - (1 + zG_{\mathcal{M}_\tau})g_{M_n}\} = T_\tau, \\ h^{B_{-\tau}(O^\tau)}_{\mathcal{M}_\tau} := n^{-1}\mathrm{Tr}\ B_{-\tau}(O^\tau)G_{\mathcal{M}_\tau}, \\ T_\tau := n^{-1}\mathbf{E}\{G_{\mathcal{M}_\tau}B_\tau G_{\mathcal{M}_\tau}B_{-\tau}(O^\tau) - G_{\mathcal{M}_\tau}\mathcal{M}_\tau G_{\mathcal{M}_\tau}\}, \tag{2.35}$$

where we used (2.28) to identify $g_{\mathcal{M}_{\tau,n}}$ and g_{M_n}.

We will use now (2.3), (2.18), and (2.25)–(2.26) to write $h^{B_{-\tau}(O^\tau)}_{\mathcal{M}_\tau}$ as

$$h^{B_{-\tau}(O^\tau)}_{\mathcal{M}_\tau} := n^{-1}\mathrm{Tr}\ B_{-\tau}(O^\tau)G_{\mathcal{M}_\tau} \\ = n^{-1}\mathrm{Tr}\ B_{-\tau}G_{M_{-\tau}} =: h_{-\tau,n}, \quad \tau = \pm 1. \tag{2.36}$$

As a result, the first line of (2.35) becomes

$$\mathbf{E}\{G_{\mathcal{M}_\tau}B_\tau h_{-\tau,n} - (1 + zG_{\mathcal{M}_\tau})g_{M_n}\} = T_\tau. \tag{2.37}$$

This is our basic intermediate result. We will outline now an argument based on (2.37) and leading to the system (2.7)–(2.8).

Introduce the notation (see (2.14) and (2.23))

$$\delta g_{M_n} = g_{M_n} - f_{M_n},\ \delta h_{\tau,n} = h_{\tau,n} - \overline{h}_{\tau,n}, \\ \overline{h}_{\tau,n} := \mathbf{E}\{h_{\tau,n}\},\ \overline{G}_{\mathcal{M}_{\tau,n}} := \mathbf{E}\{G_{\mathcal{M}_{\tau,n}}\}, \tag{2.38}$$

allowing us to write (2.37) as

$$\overline{G}_{\mathcal{M}_\tau}\overline{h}_{-\tau,n}(B_\tau - zf_{M_n}/\overline{h}_{-\tau,n}) = f_{M_n} + E_\tau + T_\tau, \tag{2.39}$$

where

$$E_\tau = -\mathbf{E}\{G_{\mathcal{M}_\tau}B_\tau\delta h_{-\tau,n}\} + z\mathbf{E}\{G_{\mathcal{M}_n}\delta g_{M_n}\} \tag{2.40}$$

is the contribution of the deviation terms δg_{M_n} and $\delta h_{\tau,n}$ of (2.38).

Applying to (2.39) the operation $n^{-1}\mathrm{Tr}$ and using formulas (2.28) and

$$G_{\mathcal{M}_{\tau,n}}B_\tau = B^{1/2}_{\tau,n}G_{M_{\tau,n}}B^{1/2}_{\tau,n},$$

we obtain in view of (2.23), (2.36), and (2.38)

$$\overline{h}_{\tau,n}(z)\overline{h}_{-\tau,n}(z) - (1 + zf_{M_n}(z))f_{M_n}(z) = r^{(1)}_{\tau,n}(z),$$
$$r^{(1)}_{\tau,n}(z) = n^{-1}\mathrm{Tr}\, E_\tau + n^{-1}\mathrm{Tr}\ T_\tau, \quad \tau = \pm 1. \tag{2.41}$$

Next, denote $\{\lambda_\alpha\}_\alpha$ and $\{\psi_\alpha\}_\alpha$ the eigenvalues and eigenvectors of the positive definite matrix $M_{\tau,n}$ of (2.3). Then (2.36) and spectral theorem for $M_{\tau,n}$ imply

$$h_{\tau,n}(z) = \int_0^\infty \frac{\mu_{\tau,n}(d\lambda)}{\lambda - z}, \quad z \in \mathbb{C} \setminus \mathbb{R}_+,$$
$$\mu_{\tau,n} = n^{-1}\sum_\alpha \delta_{\lambda_\alpha}(B_\tau\psi_\alpha, \psi_\alpha), \quad \mu_{\tau,n}(\mathbb{R}_+) = n^{-1}\mathrm{Tr} B_\tau := \beta_{\tau,n}. \tag{2.42}$$

It follows from Lemma 3.2 that

$$z^*_{\tau,n}(z) := zf_{M_n}(z)/\overline{h}_{\tau,n}(z)$$
$$= (z/\beta_\tau)(1 + o(1), \quad d(z) \to \infty, \ n \to \infty. \tag{2.43}$$

We conclude that the matrix $(B_\tau - z^*_{\tau,n}(z))$ of (2.39) is invertible and if

$$G_{B_\tau}(\zeta) = (B_\tau - \zeta)^{-1}, \quad \|G_{B_\tau}(\zeta)\| \le 1/d(\zeta),$$

is the resolvent of the positive definite matrix B_τ, then we have in view of (2.43)

$$(B_\tau - z^*_{\tau,n}(z))^{-1} = G_{B_\tau}(z^*_{\tau,n}(z)), \quad \|G_{B_\tau}(z^*_{\tau,n}(z))\| \le 2\beta_\tau/d(z), \tag{2.44}$$

Hence, (2.39) is equivalent to

$$\overline{h}_{-\tau,n}(z)\overline{G}_{\mathcal{M}_\tau}(z) - f_{M_n}(z)G_{B_\tau}(z^*_{\tau,n}(z)) = (E_\tau + T_\tau)G_{B_\tau}(z^*_{\tau,n}(z)). \tag{2.45}$$

Applying to (2.45) the operation $n^{-1}\mathrm{Tr}$, we obtain in view of (2.16), (2.21), (2.23), and (2.36)

$$f_{B_{\tau,n}}(zf_{M_n}(z)/\overline{h}_{-\tau,n}(z)) = \overline{h}_{-\tau,n}(z) + r^{(2)}_{\tau,n}(z),$$
$$r^{(2)}_{\tau,n}(z) = -n^{-1}\mathrm{Tr}\ E_\tau G_{B_\tau}(z^*_{\tau,n}(z)/f_{M_n}(z)$$
$$- n^{-1}\mathrm{Tr}\ T_\tau G_{B_\tau}(z^*_{\tau,n}(z)/f_{M_n}(z), \quad \tau = \pm 1. \tag{2.46}$$

We observe that (2.41) and (2.46) are prelimit versions of system (2.7)–(2.8).

We will derive now (2.7)–(2.8) from (2.41)–(2.46). This is just a version of a standard argument of random matrix theory.

Consider first (2.41) and let us show that the error term $r^{(1)}_{\tau,n}$ vanishes as $n \to \infty$ under conditions (2.13) and (2.15).

Indeed, by using (2.28) and (2.40), we obtain for the contribution of the first term of the right-hand side of $r^{(1)}_{\tau,n}$ in (2.41)

$$n^{-1}\mathrm{Tr}\, E_\tau = -\mathbf{E}\{\delta h_{\tau,n}\delta h_{-\tau,n}\} - z\mathbf{E}\{(\delta g_{M_n})^2\}$$

and then Schwarz inequality for expectations and Lemma 3.3 with $B_0 = \mathbf{1}_n$, $B_0 = B_{\tau,n}$, hence $b_0 = 1$, $b_0 = b$, imply

$$\begin{aligned}|n^{-1}\mathrm{Tr}\, E_{\tau,n}| &= \mathbf{Var}^{1/2}\{h_{\tau,n}\}\mathbf{Var}^{1/2}\{h_{-\tau,n}\} + |z|\mathbf{Var}\{g_{M_n}\} \\ &\le 8Cb^4(b^2+|z|)/n^2d^4(z). \end{aligned} \tag{2.47}$$

To bound the second term in $r^{(1)}_{\tau,n}$ in (2.41) we use the inequality

$$|\mathrm{Tr}\, A| \le n\|A\| \tag{2.48}$$

valid for any $n \times n$ matrix A. The inequality, (2.17), (2.26), and (2.35) yield

$$\|T_\tau\| \le 2/nb^2, \tag{2.49}$$

and then (2.48) implies that the second term in $r^{(1)}_{\tau,n}$ in (2.41) admits the same bound $2/nb^2$.

We conclude that the error term $r^{(1)}_{\tau,n}$ in (2.41) vanishes as $n \to \infty$.

Furthermore, functions f_{M_n} and $\overline{h}_{\tau,n}$ are the Stieltjes transforms of expectations $\overline{\nu}_{M_n}$ and $\overline{\mu}_{\tau,n}$ of ν_{M_n} in (2.23) and $\mu_{\tau,n}$ in (2.42). According to (1.2) and Lemma 3.2, we have $\overline{\nu}_{M_n}(\mathbb{R}_+) = 1$ and $\overline{\mu}_{\tau,n}(\mathbb{R}_+) = \beta_\tau(1+o(1))$, $n \to \infty$ and then ((2.23) and (2.36) imply for sufficiently large n

$$|f_{M_n}(z)| \le 1/d(z), \quad |\overline{h}_{\tau,n}(z)| \le 2\beta_\tau/d(z). \tag{2.50}$$

Thus, the sequence $\{(f_{M_n}, \overline{h}_{+1,n}, \overline{h}_{-1,n})\}_n$ of triples of analytic and uniformly in n bounded in $\mathbb{C}\backslash\mathbb{R}_+$ functions has a subsequence that converge as $n \to \infty$ on $\mathbb{K}$ of (2.15) to certain (f_M, h_{+1}, h_{-1}) and, by the one-to-one correspondence between the nonnegative measures and their Stiltjes transforms, we have the triple of the corresponding limiting measures

$$(\nu_M, \mu_{+1}, \mu_{-1}). \tag{2.51}$$

This allows us to pass to the limit $n \to \infty$ in the right-hand side of (2.41) and to obtain (2.7).

Consider now (2.46) and show first that the error term $r^{(2)}_{\tau,n}$ vanishes. It follows from (2.40) and (2.48) that the first term in the right-hand side of $r^{(2)}_{\tau,n}$ (see (2.46)) is bounded by

$$\begin{aligned}&\|\mathbf{E}\{G_{\mathcal{M}_\tau}(z)\delta h_{-\tau,n}\}B_\tau G_{B_\tau}(z^*_{\tau,n}(z)\|\,|f_{M_n}(z)|^{-1} \\ &\qquad + z\|\mathbf{E}\{\delta g_{M_n} G_{\mathcal{M}_\tau}(z)\}G_{\mathcal{M}_\tau}(z)G_{B_\tau}(z^*_{\tau,n}(z)\|\,|f_{M_n}(z)|^{-1},\end{aligned}$$

which, in turn, is bounded by

$$\frac{2\beta_\tau|z|}{bd(z)}(\mathbf{E}\{|\delta h_{-\tau,n}|\} + |z|b^{-1}\mathbf{E}\{|\delta g_{M_n}|\}),$$

where we used (2.13), (2.26), (2.44), and Lemma 3.2. Now, Schwarz inequality for expectations and Lemma 3.3 imply that the first term in the right-hand side of $r^{(2)}_{\tau,n}$ (see (2.46)) is bounded by

$$\frac{2\beta_\tau|z|}{bd(z)}(\mathbf{Var}^{1/2}\{h_{-\tau,n}\}+|z|b^{-1}\mathbf{Var}^{1/2}\{g_{M_n}\})\le\frac{6\beta_\tau b|z|C^{1/2}}{nbd(z)}(b^2+|z|).$$

Likewise, it follows from (2.13), (2.48), (2.49), (2.44), and Lemma 3.2 that the second term in $r^{(2)}_{\tau,n}$ of (2.46)) is bounded by

$$\frac{2\beta_\tau|z|}{d(z)}\|T_\tau\|\le\frac{4\beta_\tau|z|}{nd(z)}.$$

Thus, according to the two last bounds, the error term $r^{(2)}_{\tau,n}$ in (2.46) vanishes as $n\to\infty$.

Next, it was already explained while analyzing (2.41) how to pass to the $n\to\infty$ limit in f_{M_n} and $\overline{h}_{-\tau,n}$, hence, in and $z^*_{\tau,n}(z)$ of (2.46) and to obtain the limiting f_M, h_τ and $z^*_\tau(z)$. Thus, we can write for the left-hand side of (2.46) as

$$\begin{aligned}f_{B_{\tau,n}}(z^*_{-\tau,n}(z))&=\int_0^\infty\frac{\nu_{B_{\tau,n}}(d\lambda)}{\lambda-z^*_{-\tau,n}(z)}=\int_0^\infty\frac{\nu_{B_{\tau,n}}(d\lambda)}{\lambda-z^*_{-\tau}(z)}\\&+(z^*_{-\tau,n}(z)-z^*_{-\tau}(z))\int_0^\infty\frac{\nu_{B_{\tau,n}}(d\lambda)}{(\lambda-z^*_{-\tau,n}(z))(\lambda-z^*_{-\tau}(z))}.\end{aligned}$$

The limit of the first term of the right-hand side is obviously $f_{B_\tau}(z^*_{-\tau}(z))$, i.e., the left-hand side of (2.8) in view of the above and (2.2). The factor $(z^*_{\tau,n}(z)-z^*_\tau(z))$ in the second term of the right-hand side vanishes as $n\to\infty$, thus it suffices to show that the integral in the second term is bonded uniformly in $n\to\infty$. To this end recall that we assume that $d(z)$ of (2.16) is large enough, hence, (2.43) applies. This and Lemma 3.2 yield the bounds for $d(z)\to\infty$, $n\to\infty$

$$\begin{aligned}|\lambda-z^*_\tau(z)|&\ge d(z^*_\tau(z))=\beta_\tau^{-1}d(z)(1+o(1))\ge(2\beta_\tau)^{-1}d(z),\\|\lambda-z^*_{\tau,n}(z)|&\ge d(z^*_{\tau,n}(z))=\beta_\tau^{-1}d(z)(1+o(1))\ge(2\beta_\tau)^{-1}d(z)\end{aligned}$$

implying in view of (2.1) that the integral in the second term on the right is bonded by $4\beta_\tau\beta^2_{-\tau}/d^2(z)$.

This proves (2.8). In addition, it follows from (2.1) that measures μ_τ, $\tau=\pm1$, of (2.51) satisfy (3.21) and then Lemma 3.4 implies that the whole sequences $\{(f_{M_n},\overline{h}_{+1,n},\overline{h}_{-1,n})\}_n$ converges to a certain $(f_M,\overline{h}_{+1},\overline{h}_{-1})$, a unique solution of (2.7)–(2.8).

Thus, we proved the theorem under condition (2.13). Let us show that conditions (2.1) and (2.2) are already sufficient for the validity of the theorem.

Denote $\{b_{\tau,\alpha}\}_{\alpha=1}^n$ and $\{u_{\tau,\alpha}\}_{\alpha=1}^n$ the eigenvalues and the eigenvectors of B_τ in (2.1)–(2.3), denoted B_τ below. It follows from spectral theorem that

$$B_\tau=\sum_{\alpha=1}^n b_{\tau,\alpha}u_{\tau,\alpha}\otimes u_{\tau,\alpha}.$$

Introduce a sequence

$$\{b^{(r)}\}_{r=1}^{\infty}, \quad \lim_{r\to\infty} b^{(r)} = \infty. \tag{2.52}$$

and write for each $b^{(r)}$

$$B_\tau = B_\tau^r + C_\tau^r, \quad C_\tau^r := \sum_{b_{\tau,\alpha}\in I_r} b_{\tau,\alpha} u_{\tau,\alpha} \otimes u_{\tau,\alpha}, \quad I_r := [b^{(r)}, \infty], \tag{2.53}$$

and (cf. (2.3))

$$M_{\tau,n}^r = B_\tau^{1/2} O^\tau B_{-\tau}^r O^{-\tau} B_\tau^{1/2}, \quad M_{\tau,n}^{rr} = (B_\tau^r)^{1/2} O^\tau B_{-\tau}^r O^{-\tau} (B_\tau^r)^{1/2}. \tag{2.54}$$

It follows then from (2.53)–(2.54) that

$$M_{+,n}^r = M_{+,n}^{rr} + \sum_{b_{-,\alpha}\in I_r} y_{+,\alpha}^r \otimes y_{+,\alpha}^r, \; y_{+,\alpha}^r = b_{-,\alpha}^{1/2} (B_+^r)^{1/2} O u_{-,\alpha}, \tag{2.55}$$

i.e., $M_{+,n}^r$ is a perturbation of $M_{+,n}^{rr}$ of rank

$$\sharp\{b_{-,\alpha} \in I_r\} = n\nu_{B_{-,n}}(I_r). \tag{2.56}$$

This and the min-max principle of linear algebra (see [2], Section III.1) imply for any interval $\Delta \subset \mathbb{R}_+$

$$|\nu_{M_n^r}(\Delta) - \nu_{M_n^{rr}}(\Delta)| \le \nu_{B_{-,n}}(I_r), \tag{2.57}$$

where we took into account that the spectra, hence, the NCMs of $M_{\tau,n}^{rr}$, $\tau = \pm 1$, and $M_{\tau,n}^r$, $\tau = \pm 1$, do not depend on τ.

We then consider the pair (M_-, M_-^r) where M_- of (2.3) satisfies the conditions of the theorem. Using an argument analogous to that leading to (2.57), we obtain

$$|\nu_{M_n}(\Delta) - \nu_{M_n^r}(\Delta)| \le \nu_{B_{+,n}}(I_r), \quad \Delta \subset \mathbb{R}_+,$$

Combining this bound and (2.57), we get

$$|\nu_{M_n}(\Delta) - \nu_{M_n^{rr}}(\Delta)| \le \nu_{B_{+,n}}(I_r) + \nu_{B_{-,n}}(I_r), \quad \Delta \subset \mathbb{R}_+,$$

hence, in view of (2.1)

$$|\nu_{M_n}(\Delta) - \nu_{M_n^{rr}}(\Delta)| \le 2b_2/(b^{(r)})^2, \quad \Delta \subset \mathbb{R}_+. \tag{2.58}$$

It is important that the above bound holds for any realization of random matrices in question and is uniform in n.

According to the above proof, the theorem is valid with probability 1 for every $b^{(r)}$, hence, for the whole sequence $\{b^{(r)}\}_{r=1}^{\infty}$ with the same probability.

Denote $\nu_{M^{rr}}$ and ν_M the measures determined via the system (2.7)–(2.8) with the limiting measures $\nu_{B_\tau^r}$ and ν_{B_τ} respectively (see (2.2)) and write

$$\begin{aligned} |\nu_{M_n}(\Delta) - \nu_M(\Delta)| \le{}& |\nu_{M_n}(\Delta) - \nu_{M_n^{rr}}(\Delta)| \\ &+ |\nu_{M_n^{rr}}(\Delta) - \nu_{M^{rr}}(\Delta)| + |\nu_{M^{rr}}(\Delta) - \nu_M(\Delta)|. \end{aligned} \tag{2.59}$$

Replacing the first term on the right of (2.59) by (2.58) and taking into account that the $n \to \infty$ limit of the second term of the right-hand side of (2.59) vanishes with probability 1 for any r by the above proof (see (2.13)) and that the third term is non-random and independent of n, we obtain with probability 1:

$$\lim\sup_{n\to\infty} |\nu_{M_n}(\Delta) - \nu_{M_{+,n}}(\Delta)| \le 2b_2/(b^{(r)})^2 + |\nu_{M^r}(\Delta) - \nu_M(\Delta)|.$$

The right-hand side here vanishes as $r \to \infty$ in view of (2.52) and the continuity of solutions of (2.7)–(2.8) with respect to the weak convergence of ν_{B_τ}, see Lemma 3.4.

Thus, we have proved that the theorem holds under conditions (2.1) and (2.2), but not necessarily (2.13). □

3. Auxiliary results

We will begin with a proposition that provides the tools of random matrix theory dealing with "randomness" due to Haar distributed orthogonal random matrices. Their proofs and discussion are given in [15].

Recall that we do not write the subindex n in various matrices where this does not lead to misunderstanding.

Proposition 3.1. *Given a positive integer n, consider the group $SO(n)$ of $n \times n$ orthogonal matrices with determinant 1 viewing it as the probability space with the normalized Haar measure of the group as the probability measure of the space.*

(i) *Let $\Phi : SO(n) \to \mathcal{M}_n(\mathbb{C})$ be a C^1 map admitting a C^1 continuation into an open neighborhood of $SO(n)$ in the whole algebra $\mathcal{M}_n(\mathbb{R})$. Denote $\mathbf{E}_n\{\cdots\}$ the integration (expectation) with respect to the normalized Haar measure of $SO(n)$. Then we have*

$$\mathbf{E}_n\{\Phi'(O_n) \cdot A_n O_n\} = \mathbf{E}_n\{\Phi'(O_n) O_n A_n\} = 0, \quad \forall A_n \in \mathcal{A}_n, \tag{3.1}$$

where $\mathcal{A}_n$ is the space of $n\times n$ real antisymmetric matrices and the derivative Φ' is viewed as a linear map from $\mathcal{M}_n(\mathbb{R})$ to $\mathcal{M}_n(\mathbb{C})$.

(ii) *We have in the above notation for a map $\varphi : SO(n) \to \mathbb{C}$ and a sufficiently large n*

$$\begin{aligned}\mathbf{Var}_n\{\varphi\} &:= \mathbf{E}_n\{|\varphi|^2\} - |\mathbf{E}_n\{\varphi\}|^2 \\ &\le \frac{C}{n}\mathbf{E}_n\Big\{\sum_{1\le j<k\le n}^{n} \Big|\varphi_{jk}^{(\tau)}\Big|^2\Big\}, \quad \tau = \pm 1,\end{aligned} \tag{3.2}$$

where C is an absolute constant and

$$\varphi_{jk}^{(+)}(O_n) = \varphi'(O_n) A_n^{(jk)} O_n = \lim_{\varepsilon\to\infty} (\varphi((\mathbf{1}_n + \varepsilon A_n^{(jk)})O_n) - \varphi(O_n))\varepsilon^{-1},$$
$$\varphi_{jk}^{(-)}(O_n) = \varphi'(O_n) O_n A_n^{(jk)} = \lim_{\varepsilon\to\infty} (\varphi(O_n(\mathbf{1}_n + \varepsilon A^{(jk)})) - \varphi(O_n))\varepsilon^{-1},$$

$$A^{(jk)} = \{A^{(jk)}_{ab}\}^n_{a,b=1}, \quad A^{(jk)}_{ab} = \delta_{aj}\delta_{bk} - \delta_{ak}\delta_{bj}, \tag{3.3}$$

i.e., $\{A^{(jk)}\}_{0\le j<k\le n}$ *is the basis of* $\mathcal{A}_n$*.*

(iii) *There exists an infinite-dimensional probability space* Ω_O *on which all* O_n*,* $n \ge 1$*, are simultaneously defined.*

Item (i) follows from the invariance of the Haar measure of $SO(n)$ with respect to the left $O \to e^{\varepsilon A}O$ and the right $O \to Oe^{\varepsilon A}$ group shifts with $\varepsilon \to 0$, see [15], Section 8.1. Item (ii) is a version of the Poincaré inequality for $SO(n)$, see [15], Section 8.1 and item (iii) is a structure property of $SO(n)$, see [10], Section 2.10 and [15], Section 8.1.

Next is an important bound for the fluctuations of various functions of the Haar distributed orthogonal matrices.

Lemma 3.2. *Consider the functions* f_n *and* $\overline{h}_{\tau,n}$ *given by* (2.23) *and* (2.36) *and the non-negative measures* $\overline{\nu}_{M_n}$ *and* $\overline{\mu}_{\tau,n}$ *which provide the Stieltjes representations* (2.23) *and* (*cf.* (2.42))

$$\overline{h}_{\tau,n}(z) = \int_0^\infty \frac{\overline{\mu}_{\tau,n}(d\lambda)}{\lambda - z}, \ z \in \mathbb{C}\setminus\mathbb{R}_+. \tag{3.4}$$

Then we have under condition (2.1)*:*

(i) *the uniform in* n *tail estimates*

$$\overline{\nu}_{M_n}([T,\infty]) \le b_2/T, \quad \overline{\mu}_{\tau,n}([T,\infty]) \le b_2^{3/2}/T, \tag{3.5}$$

where b_2 *is defined in* (2.1)*;*

(ii) *the bounds*

$$|zf_n(z)+1| \le b_2/d(z), \quad |\overline{h}_{\tau,n}(z) + \beta_{\tau,n}| \le b_2^{3/2}/d(z); \tag{3.6}$$

where $d(z)$ *are defined in* (2.16)*;*

(iii) *the limit*

$$\beta_\tau := \lim_{n\to\infty} \beta_{\tau,n} = \int_0^\infty \lambda\nu_{B_\tau}(d\lambda) > 0, \tag{3.7}$$

where

$$\beta_{\tau,n} = n^{-1}\mathrm{Tr}B_{\tau,n} = \int_0^\infty \lambda\nu_{B_{\tau,n}}(d\lambda), \tag{3.8}$$

and $\nu_{B_{\tau,n}}$ *and* ν_{B_τ} *is given by* (2.2)*.*

Proof. (i) It follows from the definition (1.2) of ν_{M_n} that

$$\int_0^\infty \lambda\overline{\nu}_{M_n}(d\lambda) = \mathbf{E}\{n^{-1}\mathrm{Tr}M_\tau\}. \tag{3.9}$$

By using the orthogonality relations for $SO(n)$

$$\mathbf{E}\{O_{j_1k_1}O_{j_2k_2}\} = n^{-1}\delta_{j_1j_2}\delta_{j_2k_2} \tag{3.10}$$

and Schwarz inequality for traces, we obtain in view of (2.3) and (2.1)

$$\begin{aligned}\mathbf{E}\{n^{-1}\mathrm{Tr}M_\tau\} &= \mathbf{E}\{n^{-1}\mathrm{Tr}B_\tau O^\tau B_{-\tau}O^{-\tau}\}\\ &= n^{-1}\mathrm{Tr}B_\tau\, n^{-1}\mathrm{Tr}B_{-\tau,n} \le b_2.\end{aligned} \tag{3.11}$$

Now a standard argument implies the first estimate in (3.5).

Likewise, we have from (2.42)

$$\int_0^\infty \lambda\overline{\mu}_{\tau,n}(d\lambda) = \mathbf{E}\{n^{-1}\mathrm{Tr}B_\tau M_\tau\}. \tag{3.12}$$

By using again (3.10) and (2.3), implying

$$\begin{aligned}\mathbf{E}\{n^{-1}\mathrm{Tr}B_\tau M_\tau\} &= \mathbf{E}\{n^{-1}\mathrm{Tr}B_\tau^2 O^\tau B_{-\tau}O^{-\tau}\}\\ &= n^{-1}\mathrm{Tr}B_\tau^2\, n^{-1}\mathrm{Tr}B_{-\tau,n} \le b_2^{3/2},\end{aligned} \tag{3.13}$$

we obtain the second estimates in (3.5) in view of (2.1) and Schwarz inequality for traces.

(ii) It follows from (2.23) that

$$|zf_{M_n}(z)+1| = \left|\int_0^\infty \frac{\lambda\overline{\nu}_{M_n}(d\lambda)}{\lambda - z}\right| \le d^{-1}(z)\int_0^\infty \lambda\overline{\nu}_{M_n}(d\lambda).$$

By using (3.9) and (3.13) in the right-hand side, we obtain the first bound in (3.6).

To get the second bound in (3.6), we use (2.42), (3.4), and (3.8) to write

$$|z\overline{h}_{\tau,n}(z)+\beta_{\tau,n}| = \left|\int_0^\infty \frac{\lambda\overline{\mu}_{\tau,n}(d\lambda)}{\lambda - z}\right| \le d^{-1}(z)\int_0^\infty \lambda\overline{\mu}_{\tau,n}(d\lambda).$$

By using (3.12) and (3.13), we obtain the second bound in (3.6).

(iii) If $\nu_{B_{\tau,n}}$ is the NCM of B_τ (see (2.2)), then we have for any $T>0$:

$$\begin{aligned}\int_0^T \lambda\nu_{B_\tau}(d\lambda) \le \beta_{\tau,n} &= \int_0^\infty \lambda\nu_{B_{\tau,n}}(d\lambda)\\ &= \int_0^T \lambda\nu_{B_{\tau,n}}(d\lambda) + \int_T^\infty \lambda\nu_{B_{\tau,n}}(d\lambda).\end{aligned} \tag{3.14}$$

Writing (2.1) as

$$\int_0^\infty \lambda^2\nu_{B_{\tau,n}}(d\lambda) \le b_2,$$

we can bound the second term in the right-hand side of (3.14) by b_2/T. Now, using (2.2) to pass first to the limit $n\to\infty$ in (3.14) and then passing to the limit $T\to\infty$, we obtain (3.7) with the strictly positive right-hand side, because ν_{B_τ} is not concentrated at zero according to the condition of the theorem. □

Next is an important bound for the fluctuations of various functions of the Haar distributed orthogonal matrices.

Lemma 3.3. *Assume that positive definite matrices $B_{\tau,n}$, $\tau = \pm 1$, in (2.3) satisfies (2.13) and let $B_{0,n}$ be a real symmetric $n \times n$ matrix such that*

$$\sup_n \|B_{0,n}\| \le b_0 < \infty. \tag{3.15}$$

Set

$$\varphi_{\tau,n} := n^{-1}\mathrm{Tr}\; B_{0,n} G_{M_{\tau,n}}(z), \tag{3.16}$$

where $G_{M_{\tau,n}}$ is defined by (2.3) and (2.16).

Then we have for sufficiently large n:

$$\mathbf{Var}\{\varphi_\tau\} \le 8Cb_0^2 b^4/n^2 d^4(z). \tag{3.17}$$

Proof. We will use an analog of the Poincaré inequality for orthogonal matrices given by (3.2)–(3.3), see, e.g., [8] for other approaches to estimate the fluctuations. To this end we choose the right-hand side of (3.16) as the map φ in (3.2). By using(2.25) and (2.29) (2.30)–(2.31), we obtain (cf. (2.30))

$$\delta M_\tau = \tau B_\tau^{1/2}[A, B_{-\tau}(O_\tau)]B_\tau^{1/2}$$

hence,

$$\begin{aligned}\delta_\tau \varphi_\tau &= n^{-1}\mathrm{Tr}\; B_0 \delta_\tau G_{M_\tau} = n^{-1}\mathrm{Tr}\; C_\tau A,\\ C_\tau &= [B_{-\tau}(O^\tau), H_\tau],\;\; H_\tau = B_\tau^{1/2} G_{M_\tau} B_0 G_{M_\tau} B_\tau^{1/2},\end{aligned} \tag{3.18}$$

implying for the derivative φ_{jk} of (3.3)

$$(\varphi_{\tau,n})_{jk} = (C_\tau)_{jk} - (C_\tau)_{kj} = 2(C_\tau)_{jk}, \quad 1 \le j < k \le n. \tag{3.19}$$

since C_τ is antisymmetric. The right-hand side of the above formula is well defined for all $1 \le j,k \le n$ and is antisymmetric in (j,k). This and a standard linear algebra argument, that takes into account that B_τ, $\tau = 0, \pm 1$, are symmetric, yield

$$\begin{aligned}\sum_{1\le j<k\le n} \mathbf{E}\left\{|(\varphi_{\tau,n})_{jk}|^2\right\} &= \frac{2}{n^2}\sum_{j,k=1}^{n} \mathbf{E}\left\{|(C_\tau)_{jk}|^2\right\}\\ &\le 4n^{-2}\mathrm{Tr}\; B_{-\tau}^2(O^\tau)(H_\tau H_\tau^* + H_\tau^* H_\tau).\end{aligned} \tag{3.20}$$

We will use now (2.48), (2.16), (2.13), and (3.15) implying that $\|C_\tau\| \le 2b^3/d^2(z)$. Combining this with (3.2), we obtain (3.17) for sufficiently large n. □

Next is the unique solvability of functional equations (2.7)–(2.8).

Lemma 3.4. *Consider the system (2.7)–(2.8) with the condition*

$$\int_0^\infty \lambda^2 \nu_{B_\tau}(d\lambda) < \infty, \quad \tau = \pm 1, \tag{3.21}$$

for the triple (f, h_{-+1}, h_{-1}) of functions that belong to the class $\mathcal{N}_+$ (see assertion (iii) of Theorem 2.1).

We have:

(i) *there exists at most one triple satisfying* (2.7)–(2.8) *for all sufficiently large* $d(z) := \mathrm{dist}(z, \mathbb{R}_+)$*;*

(ii) *if* $\{\nu^{(r)}_{B_\tau}\}_r$ *is a sequence of non-negative measures satisfying* (3.21) *and converging to a limit* ν_{B_τ} *also satisfying* (3.21) *and* $\{(f^{(r)}, h^{(r)}_+, h^{(r)}_-)\}_r$ *is the sequence of the corresponding solutions of* (2.7)–(2.8), *then the sequence converges as* $r \to \infty$ *to a triple* (f, h_+, h_-) *which is a unique of solution* (2.7)–(2.8) *with limiting* ν_{B_τ}.

Proof. (i) A function s of class admits the Stieltjes representation

$$s(z) = \int_0^\infty \frac{\sigma(d\lambda)}{\lambda - z}, \quad z \in \mathbb{C} \setminus \mathbb{R}_+, \tag{3.22}$$

with a non-negative and bounded measure σ, see [1], Section III.4.

Denote by $\widehat{\nu}$ and $\widehat{\mu}_\tau$ the measures corresponding to f and h_τ of the lemma and write

$$\begin{aligned} h_0(z) &:= zf(z) = -1 + \int_0^\infty \frac{\lambda \widehat{\nu}(d\lambda)}{\lambda - z}, \quad z \in \mathbb{C} \setminus \mathbb{R}_+, \\ h_\tau(z) &= \int_0^\infty \frac{\lambda \widehat{\mu}_\tau(d\lambda)}{\lambda - z}, \quad z \in \mathbb{C} \setminus \mathbb{R}_+. \end{aligned} \tag{3.23}$$

The system (2.7)–(2.8) written via $h = (h_0, h_{+1}, h_{-1})$ is

$$h_0(1 + h_0) - z h_{+1} h_{-1} = 0, \tag{3.24}$$

$$\int_0^\infty \frac{\nu_{B_{-\tau}}(d\lambda)}{\lambda h_\tau - h_0} = 1, \quad \tau = \pm 1. \tag{3.25}$$

Assume that there exists two solutions $h^{(a)} = (h^{(a)}_0, h^{(a)}_{-+1}, h^{(a)}_{-1})$, $a = 1, 2$ of (3.24)–(3.25) and denote

$$\delta h = h^{(1)} - h^{(2)} = \{\delta h_\tau\}_{\tau = 0, \pm 1}.$$

It follows then from (2.7)–(2.8) that δh satisfies the linear system

$$\begin{aligned} (h^{(1)}_0 + h^{(2)}_0 + 1)\delta h_0 - z h^{(1)}_- \delta h_+ - z h^{(2)}_+ \delta h_- &= 0, \\ I_+ \delta h_0 - J_+ \delta h_+ &= 0, \\ I_- \delta h_0 - J_- \delta h_- &= 0, \end{aligned}$$

where

$$\begin{aligned} I_\tau(z) &= \int_0^\infty \frac{\nu_{B_{-\tau}}(d\lambda)}{(\lambda h^{(1)}_\tau - h^{(1)}_0)(\lambda h^{(2)}_\tau - h^{(2)}_0)}, \\ J_\tau(z) &= \int_0^\infty \frac{\lambda \nu_{B_{-\tau}}(d\lambda)}{(\lambda h^{(1)}_\tau - h^{(1)}_0)(\lambda h^{(2)}_\tau - h^{(2)}_0)}. \end{aligned} \tag{3.26}$$

Hence, the solution of (2.7)–(2.8) is unique if the determinant $\Delta(z)$ of the above linear system is not zero. We have

$$\Delta = (h_0^{(1)} + h_0^{(2)} + 1)J_+J_- - zI_+J_-h_-^{(1)} - zI_-J_+h_{-+}^{(2)}.$$

Functions $h_\tau^{(a)}$, $a = 1, 2$, $\tau = 0, \pm 1$ are analytic in $\mathbb{C} \setminus \mathbb{R}_+$, hence, it suffices to prove that

$$h_\tau^{(1)}(-\xi) = h_\tau^{(2)}(-\xi), \quad \tau = 0, \pm 1, \ \xi \geq \xi_0,$$

where ξ_0 is large enough (cf. (2.15)).

Let $\widehat{\nu}^{(a)}$ and $\widehat{\mu}_\tau^{(a)}$ be the measures in the Stieltjes representations of $f^{(a)}$ and $h_\tau^{(a)}$. It follows from the representations with $z = -\xi$ (cf. (3.23))

$$f^{(a)}(-\xi) = \int_0^\infty \frac{\nu_M^{(a)}(d\lambda)}{\lambda + \xi}, \quad h_\tau^{(a)}(-\xi) = \int_0^\infty \frac{\lambda \nu_{B_\tau}^{(a)}(d\lambda)}{\lambda + \xi}$$

that

$$\begin{aligned} h_0^{(a)}(-\xi) &= -m_0^{(a)} + m_1^{(a)}\xi^{-1} + o(\xi^{-1}), \\ h_\tau^{(a)}(-\xi) &= m_{0,\tau}^{(a)}\xi^{-1}(1 + o(1)), \quad \xi \to \infty, \end{aligned} \tag{3.27}$$

where

$$m_s^{(a)} = \int_0^\infty \lambda^t \nu_M^{(a)}(d\lambda), \quad t = 0, 1, \quad m_{0,\tau}^{(a)} = \int_0^\infty \lambda \nu_{B_\tau}^{(a)}(d\lambda).$$

It follows then from (2.7)–(2.8) with $\xi \to \infty$ that

$$m_0^{(a)} = 1, \quad m_1^{(a)} = \beta_+\beta_-, \quad m_\tau^{(a)} = \beta_\tau, \quad \tau = \pm 1,$$

hence, (3.27) becomes

$$\begin{aligned} h_0^{(a)}(-\xi) &= -1 + \beta_+\beta_-\xi^{-1} + o(\xi^{-1}), \\ h_\tau^{(a)}(-\xi) &= \beta_\tau \xi^{-1}(1 + o(1)), \quad \xi \to \infty, \ \tau = \pm. \end{aligned} \tag{3.28}$$

This and (3.26) lead to the asymptotic formulas

$$I_\tau(-\xi) = 1 + o(1), \quad J_\tau(-\xi) = \beta_{-\tau} + o(1), \quad \xi \to \infty$$

and we obtain

$$\Delta(-\xi) = \beta_+\beta_{-1} + o(1) > 0, \quad \xi \to \infty.$$

(ii) Every triple $\left(f^{(r)}, h_+^{(r)}, h_-^{(r)}\right)$ consists of functions analytic and uniformly in r bounded in $\mathbb{C} \setminus \mathbb{R}_+$, hence, the sequence $\left\{\left(f^{(r)}, h_+^{(r)}, h_-^{(r)}\right)\right\}_r$ contains a subsequence converging uniformly in any compact of $\mathbb{C} \setminus \mathbb{R}_+$.

It is clear that the limiting triple satisfies (2.7) if $d(z)$ is large enough. Let us show that it satisfies (2.8) for the same $d(z)$. Indeed, use (2.8) for $\left(f^{(r)}, h_+^{(r)}, h_-^{(r)}\right)$ to write

$$1 - \int_0^\infty \frac{\nu_{B_\tau}(d\lambda)}{D_\tau(z)} = \int_0^\infty \frac{\nu_{B_{-\tau}}^{(b)}(d\lambda)}{D_\tau^{(r)}(z)} - \int_0^\infty \frac{\nu_{B_\tau}(d\lambda)}{D_\tau(z)}$$

$$= \int_0^\infty \frac{\nu^{(b)}_{B_{-\tau}}(d\lambda)}{D_\tau(z)} + (h_\tau(z) - h^{(r)}_\tau(z)) \int_0^\infty \frac{\lambda \nu_{B_\tau}(d\lambda)}{D_\tau(z) D^{(r)}_\tau(z)}$$
$$- (h_0(z) - h^{(r)}_0(z)) \int_0^\infty \frac{\nu_{B_\tau}(d\lambda)}{D_\tau(z) D^{(r)}_\tau(z)}, \tag{3.29}$$

where $D_\tau = \lambda h_\tau - h_0$, $D^{(r)}_\tau = \lambda h^{(r)}_\tau - h^{(r)}_0$. It follows from (3.23) and (3.27)–(3.28) for $d(z) \to \infty$

$$h^{(r)}_0(z) = -1 + O(d^{-1}(z)), \quad h^{(r)}_\tau(z) = -\beta^{(r)}_\tau / z(1 + O(d^{-1}(z)) \tag{3.30}$$

and analogous formulas for the $r \to \infty$ limits of these functions. Thus, we have

$$|D^{(r)}_\tau(z)| = |h^{(r)}_\tau(z)|\, |\lambda - h^{(r)}_0(z)/h^{(r)}_\tau(z)|$$
$$\geq |h^{(r)}_\tau(z)|\, |\Im h^{(r)}_0(z)/h^{(r)}_\tau(z)| = |\Im z|/|z|(1 + o(1)), \quad d(z) \to \infty,$$

and in passing to the last equality we used (3.30). This allows us to show that the right-hand side of (3.29) vanishes as $r \to \infty$ and $\Im z \neq 0$. If, however, $\Im z = 0$, then we have even a better bound $D^{(r)}_\tau(-\xi) = 1 + O(\xi^{-1})$, $\xi \to \infty$, that follows from (3.28). □

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Закон множення великих випадкових матриць: ще раз

Leonid Pastur

У статті розглядається розподіл власних значень добутку двох $n \times n$ додатно визначених матриць B_τ, $\tau = \pm 1$, які "повертаються" одна відносно одної випадковою ортогональною матрицею, що має міру Хаара ортогональної групи як її розподіл ймовірності. Задача розглядалася в кількох роботах з використанням різних методів. Ми пропонуємо спрощений підхід, оснований на техніці теорії випадкових матриць і певній симетрії задачі. Ми доводимо, що нормована міра розподілу власних значень добутку прямує до невипадкової граничної міри коли порядок матриці прямує до нескінченності, одержуємо функціональні рівняння, що однозначно визначають перетворення Стілтьєса граничної міри, через граничні міри множників B_τ, $\tau = \pm 1$, та розглядаємо цікавий приклад.

Ключові слова: випадкові матриці, ортогональні матриці, розподіл власних значень