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Weighted Elliptic Equations in Dimension N with Subcritical and Critical Double Exponential Nonlinearities

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In this paper, we prove the existence of nontrivial solutions for the following weighted problem without the Ambrosetti–Rabinowitz condition: $-\operatorname{div}(\sigma(x)|\nabla u|^{N-2}\nabla u) = f(x,u)$ and $u > 0$ in B, $u = 0$ on ∂B, where B is the unit ball of $\mathbb{R}^N$, $\sigma(x) = \left(\log\left(\frac{e}{|x|}\right)\right)^{N-1}$ is the singular logarithmic weight in the Trudinger–Moser embedding. The nonlinearity is a critical or subcritical growth in view of Trudinger–Moser inequalities. In order to obtain the existence result, we used minimax techniques combined with the Trudinger–Moser inequality. In the critical case, the associated energy does not satisfy the condition of compactness. We provide a new condition for growth and we stress its importance to avoid compactness level.

Key words: Trudinger–Moser inequality, nonlinearity of double exponential growth, critical exponents, compactness level

Mathematical Subject Classification 2020: 46E35, 35J20, 35J33, 35J60

1. Introduction and main results

In this paper, we study the existence of nontrivial solutions for the weighted problems without the Ambrosetti–Rabinowitz condition. More precisely, we consider the problem

$$-\operatorname{div}(\sigma(x)|\nabla u|^{N-2}\nabla u) = f(x,u) \quad \text{in } B \qquad (1.1a)$$

$$u > 0 \quad \text{in } B \qquad (1.1b)$$

$$u = 0 \quad \text{on } \partial B, \qquad (1.1c)$$

where B is the unit ball of $\mathbb{R}^N$ and the function $f(|x|,t)$ has a maximal growth in t with respect to the weighted gradient norm. The weight σ is given by

$$\sigma(x) = \left(\log\frac{e}{|x|}\right)^{N-1}. \qquad (1.2)$$

In recent years, a great attention has been focused on the study of the influence of weights on limiting inequalities of Trudinger–Moser type which has been a subject of great interest. As a consequence, the weights have had an important impact on the Sobolev norm.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\sigma \in L^1(\Omega)$ be a non-negative function. Let also the weighted Sobolev space be defined as

$$W_0^{1,N}(\Omega,\sigma) = \mathrm{cl}\left\{ u \in C_0^\infty(\Omega) \;\middle|\; \int_B |\nabla u|^N \sigma(x)\,dx < \infty \right\}.$$

A general embedding theory for these weighted Sobolev spaces was developed in [17].

It turns out that for weighted Sobolev spaces, logarithmic weights have a particular significance since they concern limiting situations of such embeddings. However, to obtain interesting results, one needs to restrict attention to radial functions. So, let us consider the subspace of radial functions

$$E = W_{0,\mathrm{rad}}^{1,N}(B,\sigma) = \mathrm{cl}\left\{ u \in C_{0,\mathrm{rad}}^\infty(B) \;\middle|\; \int_B |\nabla u|^N \sigma(x)\,dx < \infty \right\}$$

endowed with the norm

$$\|u\| = \left(\int_B |\nabla u|^N \sigma(x)\,dx \right)^{1/N}.$$

Since the logarithmic weights have a particular significance and are considered as the limiting situations of the embedding of the spaces $W_0^{1,N}(\Omega,\sigma)$, the choices of the weight induced in (1.2) and the space $W_{0,\mathrm{rad}}^{1,N}(B,\sigma)$ are also motivated by the following double exponential inequalities.

Theorem 1.1 ([7]). *Let σ be given by (1.2). Then*

$$\int_B \exp\left(e^{|u|^{N/(N-1)}}\right) dx < +\infty, \quad u \in W_{0,\mathrm{rad}}^{1,N}(B,\sigma), \tag{1.3}$$

and

$$\sup_{\substack{u \in W_{0,\mathrm{rad}}^{1,N}(B,\sigma) \\ \|u\| \le 1}} \int_B \exp\left(\beta e^{\alpha_N |u|^{N/(N-1)}}\right) dx < +\infty \;\Leftrightarrow\; \beta \le N, \tag{1.4}$$

where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ and ω_{N-1} denote the area of the unit sphere S^{N-1} in $\mathbb{R}^N$.

Inequality (1.4), also known as the Trudinger–Moser inequality, can be considered as a limiting case. Since inequality (1.4) and its variants have many applications in various aspects of analysis, generalization of (1.4) has already been a relevant research topic and a huge set of works have already been written within the last two decades.

Let N' be the Hölder conjugate of N, that is, $N' = \frac{N}{N-1}$. In view of inequality (1.4), the function f is said to have subcritical growth at $+\infty$ if

$$\forall \alpha > 0 \quad \lim_{s \to +\infty} \frac{|f(x,s)|}{\exp\left(e^{\alpha s^{N'}}\right)} = 0$$

and f has a critical growth at $+\infty$ if there exists some $\alpha_0 > 0$ such that

$$\lim_{s\to+\infty} \frac{|f(x,s)|}{\exp\left(Ne^{\alpha s^{N'}}\right)} = 0, \qquad \alpha > \alpha_0,$$
$$\lim_{s\to+\infty} \frac{|f(x,s)|}{\exp\left(Ne^{\alpha s^{N'}}\right)} = +\infty, \qquad \alpha < \alpha_0.$$

We recall the ARR condition, that is,

$$\exists t_0 > 0\ \exists M > 0\ \forall |u| \geq t_0\ \forall x \in B \quad 0 < F(x,u) = \int_0^u f(x,s)\,ds \leq M|f(x,u)|.$$

We point out that the special case $N = 2$ under the ARR condition on the linearity which have double exponential growth,

$$\begin{cases} L_{2,\sigma} := -\operatorname{div}(\sigma(x)\nabla u) = f(x,u) & \text{in } B \\ u > 0 & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases}$$

was studied by Calanchi et al. in [8].

Recently, Shengbing Deng in [12] treated the problem

$$\begin{cases} -\operatorname{div}(\sigma(x)|\nabla u|^{N-2}\nabla u) = \lambda \dfrac{e^u}{\log\left(\int_B e^{e^u}dx\right)} \dfrac{e^{e^u}}{\int_B e^{e^u}dx} & \text{in } B \\ u > 0 & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases}$$

where λ is a positive parameter, $N \geq 2$ and $\sigma(x) = \left(\log \frac{e}{|x|}\right)^{N-1}$. He proved that this problem has a positive weak radial solution for any $\lambda \in (0, \frac{w_{N-1}}{N})$. The case $N = 2$ was studied by Calanchi, Massa and Ruf in [9].

We also mention that the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{N-2}\nabla u) = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth domain of $\mathbb{R}^N$, $N \geq 2$ and the nonlinearity f behaves like $e^{t^{N/(N-1)}}$ as $t \to +\infty$, was studied by Adimurthi [1] and Ruf et al [14]. Furthemore, the problem (1.1) without weight ($w = \text{const.}$) have been extensively studied by several authors, see, for example, [1, 18, 20, 26] and references therein.

In recent years, Deng, Hu and Tang in [13] studied the problem

$$\begin{cases} -\operatorname{div}(\sigma(x)|\nabla u|^{N-2}\nabla u) = f(x,u) & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases}$$

where $N \geq 2$, the function $f(x,t)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp\left(e^{\alpha t^{N/(N-1)}}\right)$ as $t \to +\infty$, for some $\alpha > 0$. The authors proved that there is a non-trivial solution to this problem by using Mountain Pass theorem. They circumvented the loss of compactness of the associated energy function by an asymptotic condition on the nonlinearity and using appropriate Moser sequences. A similar result is proved in [27]. In the two works cited above, the authors imposed the the Ambrosetti–Rabinowitz condition on the nonlinearity f.

In this paper, we consider problem (1.1) with subcritical and critical growth nonlinearities $f(x,t)$. Furthermore, we also need to make some suitable assumptions on the behavior of f. More precisely, we will assume the following conditions:

(H1) The function $f : B \times \mathbb{R} \to \mathbb{R}$ is continuous, positive, radial in x, and $f(x,t) = 0$ for $t \leq 0$.

(H2) We have
$$\lim_{t\to+\infty} \frac{F(x,t)}{t^N} = +\infty \quad \text{uniformly in } x \in B,$$
where $F(x,t) = \int_0^t f(x,s)\,ds$.

(H3) There are $\bar{C} \geq 0$ and $\theta \geq 1$ such that $H(x,t) \leq \theta H(x,s) + \bar{C}$ for all $0 < t < s$ and $x \in B$, where
$$H(x,t) = tf(x,t) - NF(x,t).$$

(H4) We have
$$\limsup_{t\to 0} \frac{NF(x,t)}{t^N} < \lambda_1 \quad \text{uniformly in } x \in B.$$

(H5) We have
$$\lim_{t\to\infty} \frac{f(x,t)t}{e^{Ne^{\alpha_0 t^{N'}}}} \geq \gamma_0 \qquad \text{uniformly in } x \in B \text{ with } \gamma_0 > \frac{1}{\alpha_0^{N-1} e^N}.$$

(H6) For any $\{u_n\} \in E$, if $u_n \rightharpoonup 0$ weakly in E and $f(x,u_n) \to 0$ in $L^1(B)$, then
$$F(x,u_n) \to 0 \ \text{ in } \ L^1(B).$$

We denote by
$$\lambda_1 = \inf_{\substack{u\in W^{1,N}_{0,\mathrm{rad}}(B,\sigma)\\ u\neq 0}} \frac{\int_B |\nabla u|^N \sigma(x)dx}{\int_B |u|^N dx}$$
the first eigenvalue of $(L_{N,w}, W^{1,N}_{0,\mathrm{rad}}(B,\sigma))$. It is well known that λ_1 is an isolated simple positive eigenvalue and it has a positive bounded associated eigenfunction.

We recall the Ambrosetti–Rabinowitz condition (AR) : there are $\theta > N$ and $t_0 > 0$ such that
$$\theta F(x,t) \leq f(x,t)t, \quad x \in B, \ |t| > t_0.$$
It is well known that the AR condition is quite important not only to ensure that the Euler–Lagrange functional associated to problem (1.1) has a mountain pass geometry, but also to guarantee that the Palais–Smale sequence of the Euler–Lagrange functional is bounded. However, this condition is very restrictive and

eliminates many interesting and important nonlinearities. In fact, the AR condition also implies that there exist positive constants θ, a_1, a_2 such that

$$F(x,t) \geq a_1|t|^\theta - a_2, \quad (x,t) \in B \times \mathbb{R},\ \theta > N.$$

Hence, for example, the function

$$f(x,t) = |t|^{N-2} t \log(1 + |t|)$$

does not satisfy the AR condition for any $\theta > N$. But it satisfies our conditions **(H2)**–**(H4)**.

Motivated by the works cited above, we try to get the existence of a nontrivial solution for problem (1.1) without the AR condition. In the subcritical double exponential growth, we have the following result.

Theorem 1.2. *Let $f(x,t)$ be a function that has a subcritical growth at $+\infty$ and satisfies* **(H1)**–**(H4)**. *Then problem* (1.1) *has a nontrivial radial solution.*

In the critical double exponential growth, we have the following result.

Theorem 1.3. *Assume that $f(x,t)$ has a critical growth at $+\infty$ for some α_0 and satisfies the conditions* **(H1)**–**(H3)** *with $\theta = 1$ and $\bar{C} = 0$. If in addition $f(x,t)$ satisfies* **(H4)**–**(H6)**, *then problem* (1.1) *has a nontrivial solution.*

Our approach is based on a suitable version of the Mountain Pass theorem introduced by G. Cerami [10]. Problem (1.1) has a variational structure. Finding weak solutions of (1.1) in the Banach space $E = W^{1,N}_{0,\mathrm{rad}}(B,\sigma)$ is equivalent to finding critical points of the $\mathcal{C}^1$ Euler–Lagrange functional $\mathcal{J} : E \to \mathbb{R}$ defined as follows:

$$\mathcal{J}(u) = \frac{1}{N}\int_B |\nabla u|^N \sigma(x)\, dx - \int_B F(x,u)\, dx. \tag{1.5}$$

The geometric requirements of the Mountain Pass theorem follow from the assumptions on the nonlinear reaction term f, but the difficulty is in the proof of the compactness condition. We will prove that when f has a subcritical growth, the functional $\mathcal{J}$ satisfies the compactness condition as required in the Ambrosetti–Rabinowitz theorem [3]. But in the critical growth case, the compactness is lost. To overcome the verification of compactness of the Euler–Lagrange functional at some suitable level, we choose testing functions, which are extremal to the weighted Trudinger–Moser inequality.

Finally, problem (1.1) is important and has several applications in non-Newtonian fluids, reaction diffusion problem, turbulent flows in porous media and image treatment [4, 5, 21, 24].

This paper is organized as follows. In Section 2, we give some useful knowledge and lemmas. In Section 3, we prove that the energy $\mathcal{J}$ satisfies the two geometric properties and we estimate the minimax level of the Euler–Lagrange functional associated to problem (1.1). In Section 4, the compactness analysis and the proof of the main results are given.

In this work, the constant C may change from one line to another and sometimes we index the constants in order to show how they change.

2. Preliminaries

We now discuss some definitions, notations and essential results which will be used in this paper. We denote by $\|u\|_p$ the usual norm in the Lebesgue space $L^p(B)$ for $1 \leq p < \infty$, given by

$$\|u\|_p = \left(\int_B |u|^p \, dx \right)^{1/p},$$

and by $\|u\|$, the norm defined in the weighted Sobolev space $E = W^{1,N}_{0,\mathrm{rad}}(B, \sigma)$ by

$$\|u\| = \left(\int_B |\nabla u|^N \sigma(x) \, dx \right)^{1/N}.$$

Definition 2.1. We say that $u \in E$ is a solution to problem (1.1) if $u \geq 0$ a.e., $f(x,u) \in L^1(B)$, and

$$\int_B \sigma(x) |\nabla u|^{N-2} \nabla u \cdot \nabla \varphi \, dx = \int_B f(x,u) \varphi \, dx$$

for all $\varphi \in \mathcal{C}_0^\infty(B)$.

Since the nonlinearity f is of critical or subcritical growth, there exist positive constants c_1 and c_2 such that

$$|f(x,t)| \leq c_1 e^{e^{c_2 t^{N'}}}, \quad x \in B, \ t \in \mathbb{R}. \tag{2.1}$$

By using **(H1)** and (2.1), the functional $\mathcal{J}$ given by (1.5) is well defined in E and it is of class $\mathcal{C}^1$.

Definition 2.2. Let (u_n) be a sequence in a Banach space E and $\mathcal{J} \in \mathcal{C}^1(E, \mathbb{R})$ and let $c \in \mathbb{R}$. We say that the sequence (u_n) is a Palais–Smale sequence at level c (or $(PS)_c$ sequence) for the functional $\mathcal{J}$ if

$$\mathcal{J}(u_n) \to c \quad \text{and} \quad \mathcal{J}'(u_n) \to 0 \qquad \text{in } E'.$$

We also say that the functional $\mathcal{J}$ satisfies the Palais–Smale condition $(PS)_c$ at the level c if every $(PS)_c$ sequence (u_n) is relatively compact in E.

A functional $\mathcal{J}$ is said to satisfy the Cerami condition $(C)_c$ at a level $c \in \mathbb{R}$ if any sequence $(u_n) \subset E$ such that

$$\mathcal{J}(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)\mathcal{J}'(u_n) \to 0$$

has a convergent subsequence.

In the critical point theory, there are some situations in which a Palais–Smale sequence does not lead to a critical point, but a Cerami sequence can lead to a critical point. This whole thing based on the concept of 'linking' (refer to [22] for more details and examples). The Cerami condition implies the Palais–Smale condition and hence the Cerami condition is a weaker than the Palais–Smale condition.

In the sequel, we need the following radial lemma introduced and proved in [7] which is of crucial importance.

Lemma 2.3 ([7]). *Let u be a radially symmetric function in $\mathcal{C}_0^1(B)$. Then we have*

$$|u(x)| \le \frac{1}{\omega_{N-1}^{1/N}} \log^{1/N'}\left(\log \frac{e}{|x|}\right) \|u\|,$$

where ω_{N-1} is the area of the unit sphere S_{N-1} in $\mathbb{R}^N$.

Since the function $\log\left(\log \frac{e}{|x|}\right)$ is in $W^{1,N}(B)$ and $W^{1,N}(B) \hookrightarrow L^q(B)$ for all $q \ge 1$, we deduce that there exists a constant $c > 0$ such that for all $u \in E$, $\|u\|_{N'q} \le c\|u\|$. Furthermore, we have that the embedding $E \hookrightarrow L^q(B)$ is compact for all $q \ge 1$.

The second important lemma.

Lemma 2.4 ([14]). *Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be a bounded domain and $f : \overline{\Omega} \times \mathbb{R}$ be a continuous function. Let $\{u_n\}_n$ be a sequence in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $f(x, u_n)$ and $f(x, u)$ are also in $L^1(\Omega)$. If*

$$\int_\Omega |f(x, u_n)u_n|\, dx \le C,$$

where C is a positive constant, then

$$f(x, u_n) \to f(x, u) \quad \text{in } L^1(\Omega).$$

3. The geometrical properties and the minimax level

3.1. The geometrical properties. In this section, we prove that the functional $\mathcal{J}$ has the mountain pass geometry. More precisely, we have the following result.

Proposition 3.1. *Suppose that* **(H1)**–**(H4)** *hold. Then there exist $a > 0$ and $\rho > 0$ such that $\mathcal{J}(u) \ge a$ for all $u \in E$ with $\|u\| = \rho$.*

Proof. By the hypothesis **(H4)**, there exists a small constant $\varepsilon_0 \in (0, 1)$ and $\delta > 0$ such that

$$F(x, t) \le \frac{1}{N}\lambda_1(1 - \varepsilon_0)|t|^N, \quad |t| \le \delta. \tag{3.1}$$

Indeed, from **(H4)**, we have

$$\limsup_{t \to 0} \frac{NF(x, t)}{t^N} < \lambda_1 \quad \text{uniformly in } x \in B$$

such that

$$\inf_{\beta > 0} \sup \left\{ \frac{NF(x, t)}{t^N} \;\middle|\; 0 < t < \beta \right\} < \lambda_1.$$

The last inequality is strict, so we can find $\varepsilon_0 > 0$ such that

$$\inf_{\beta > 0} \sup \left\{ \frac{NF(x, t)}{t^N} \;\middle|\; 0 < t < \beta \right\} < \lambda_1 - \varepsilon_0.$$

Hence, there exists $\delta > 0$ such that

$$\sup\left\{\frac{NF(x,t)}{t^N}\;\middle|\; 0<t<\delta\right\} < \lambda_1 - \varepsilon_0,$$

and thus

$$\forall |t|<\delta \quad F(x,t) \leq \frac{1}{N}\lambda_1(1-\varepsilon_0)t^N.$$

By inequality (2.1), for $q > N$, there exists a constant c_3 such that

$$F(x,t) \leq c_3|t|^q e^{e^{c_2\ t^{N'}}}, \quad |t| \geq \delta. \tag{3.2}$$

From (3.1) and (3.2), we conclude that

$$F(x,t) \leq \frac{1}{N}\lambda_1(1-\varepsilon_0)|t|^N + c_3|t|^q e^{e^{c_2 t^{N'}}}, \quad t \in \mathbb{R}. \tag{3.3}$$

By using inequality (3.3) and the Hölder inequality, we have

$$\mathcal{J}(u) \geq \frac{\varepsilon_0}{N}\|u\|^N - c_3\left(\int_B \exp\left(Ne^{c_2|u|^{N'}}\right)dx\right)^{1/N}\left(\int_B |u|^{N'q}\,dx\right)^{1/N'}.$$

From the TM-inequality (1.4), if we choose ρ to be a positive number satisfying $c_2\rho^{N'} \leq \omega_{N-1}^{\frac{1}{N-1}}$, we have for u such that $\|u\| = \rho$,

$$\int_B \exp\left(Ne^{c_2|u|^{N'}}\right)dx = \int_B \exp\left(N\exp\left(c_2\|u\|^{N'}\left(\frac{|u|}{\|u\|}\right)^{N'}\right)\right)dx \leq c_4.$$

By Lemma 2.3, we deduce that $\|u\|_{N'q} \leq c\|u\|$, and thus we have

$$\mathcal{J}(u) \geq \frac{\varepsilon_0}{N}\|u\|^N - c_5\|u\|^q \quad \text{for all } u \in E \text{ with } \|u\| = \rho.$$

Finally, after choosing $\rho > 0$ as the maximum point of the function $g(\rho) = \frac{\varepsilon_0}{N}\rho^N - c_5\rho^q$ on the interval $\left[0, \omega_{N-1}^{1/N}/c_2^{1/N'}\right]$ and letting $a = \mathcal{J}(\rho)$, Proposition 3.1 follows. □

As the second geometric property of the energy $\mathcal{J}$, we have the following result.

Proposition 3.2. *Suppose* **(H1)** *and* **(H2)** *hold. Then there exists $\overline{u} \in E$ such that $\|\overline{u}\| > \rho$ and $\mathcal{J}(\overline{u}) < 0$.*

Proof. Let $u_0 \in E \setminus \{0\}$, $u \geq 0$. By **(H2)**, for all $\varepsilon > 0$, there exists $D = D_\varepsilon$ such that for all $(x,t) \in B \times \mathbb{R}^+$,

$$F(x,t) \geq \varepsilon t^N - D.$$

Then

$$\mathcal{J}(tu_0) = \frac{t^N}{N}\|u_0\|^N - \int_B F(x,tu_0)\,dx.$$

So,

$$\begin{aligned}\mathcal{J}(tu_0) &\le \frac{|t|^N}{N}\|u_0\|^N - \varepsilon|t|^N\|u_0\|_N^N + \frac{1}{N}w_{N-1}D\\ &= |t|^N\left(\frac{\|u_0\|^N}{N} - \varepsilon\|u_0\|_N^N\right) + \frac{1}{N}w_{N-1}D.\end{aligned}$$

We chose $\varepsilon > \|u_0\|^N/\big(N\|u_0\|_N^N\big)$ to get

$$\mathcal{J}(tu_0) \to -\infty \quad \text{as } t \to +\infty.$$

Then Proposition 3.2 follows. □

3.2. Estimation of the minimax level C_M. According to Propositions 3.1 and 3.2, let

$$C_M := \inf_{\gamma\in\Lambda}\max_{t\in[0,1]}\mathcal{J}(\gamma(t)) > 0$$

and

$$\Lambda := \big\{\gamma \in \mathcal{C}([0,1],E) \mid \gamma(0) = 0 \text{ and } \mathcal{J}(\gamma(1)) < 0\big\}.$$

We are going to estimate the minimax value C_M of the functional $\mathcal{J}$. The idea is to construct a sequence of functions $(v_n) \in E$ and estimate $\max\{\mathcal{J}(tv_n) \mid t \ge 0\}$. For this goal, we consider the following Moser sequence:

$$\psi_n(t) = \begin{cases} \dfrac{\log(1+t)}{\big(\log(1+n)\big)^{1/N}} & \text{if } 0 \le t \le n,\\ \big(\log(1+n)\big)^{(N-1)/N} & \text{if } t \ge n.\end{cases}$$

Let $v_n(x)$ be a function defined by

$$\psi_n(t) = \omega_{N-1}^{\frac{1}{N}}v_n(x),$$

where $e^{-t} = |x|$. With this choice of ψ_n, the sequence (v_n) is normalized since

$$\|v_n\|^N = \frac{1}{\omega_{N-1}}\int_B |\nabla\psi_n|^N\left|\log\frac{e}{|x|}\right|^{N-1}dx = \int_0^{+\infty}|\psi'(t)|^N(1+t)^{N-1}\,dt = 1.$$

We have the following elementary crucial result.

Lemma 3.3. *We have*

$$\lim_{n\to+\infty}\int_0^{+\infty}\exp\left(Ne^{|\psi_n|^{N'}} - Nt\right)dt = \left(\frac{N+1}{N}\right)e^N.$$

Proof. We make the change of the variables $s = 1+t$ and $j = n+1$. So,

$$\int_0^{+\infty}\exp\left(Ne^{|\psi_n|^{N'}} - Nt\right)dt$$

$$
\begin{aligned}
&= \frac{e^N}{N} + \int_0^n \exp\left(N \exp \frac{\big(\log(1+t)\big)^{N'}}{\big(\log(1+n)\big)^{1/(N-1)}} - Nt\right) dt \\
&= \frac{e^N}{N} + \int_1^j \exp\left(N s^{\left(\frac{\log s}{\log j}\right)^{1(N-1)}} - N(s-1)\right) ds \\
&= \frac{e^N}{N} + e^N \int_1^j \exp\left(N s^{\left(\frac{\log s}{\log j}\right)^{1/(N-1)}} - Ns\right) ds.
\end{aligned}
$$

We claim that

$$\lim_{j\to+\infty} \int_1^j \exp\left(N s^{\left(\frac{\log s}{\log j}\right)^{1/(N-1)}} - Ns\right) ds = 1.$$

Indeed, for any $j > \left(\frac{N}{N-1}\right)^{2^N}$, we denote

$$\varphi_j(s) := N s^{\left(\frac{\log s}{\log j}\right)^{1/(N-1)}} - Ns \quad \text{with } s \geq 1.$$

The interval $[1, j]$ is then divided as follows:

$$\big[1, j\big] = \left[1, j^{1/2^{(N-1)}}\right] \cup \left[j^{1/2^{(N-1)}}, j - j^{1/2^{(N-1)}}\right] \cup \left[j - j^{1/2^{(N-1)}}, j\right].$$

First, we consider the interval $\left[1, j^{1/2^{(N-1)}}\right]$. Since

$$
\begin{aligned}
\chi_{\left[1, j^{1/2^{(N-1)}}\right]}(s) e^{\varphi_j(s)} &\leq e^{N s^{1/2} - Ns} \in L^1([1, +\infty)), \\
\chi_{\left[1, j^{1/2^{(N-1)}}\right]}(s) e^{\varphi_j(s)} &\to e^{N - Ns} \quad \text{as } \ j \to +\infty \text{ for a.a. } s \in [1, +\infty),
\end{aligned}
$$

using the Lebesgue dominated convergence theorem, we get

$$
\begin{aligned}
&\lim_{j\to+\infty} \int_1^{j^{1/2^{(N-1)}}} \exp\left(N s^{\left(\frac{\log s}{\log j}\right)^{1/(N-1)}} - Ns\right) ds \\
&\qquad\qquad = \lim_{j\to+\infty} \int_1^j \chi_{\left[1, j^{1/2^{(N-1)}}\right]}(s) e^{\varphi_j(s)}\, ds = \frac{1}{N}.
\end{aligned}
$$

Now we are going to study the limit of this integral on $\left[j^{1/2^{(N-1)}}, j - j^{1/2^{(N-1)}}\right]$ and $\left[j - j^{1/2^{(N-1)}}, j\right]$. So, we compute

$$\varphi_j\left(j^{1/2^{(N-1)}}\right) = -N j^{1/2^{(N-1)}} \left(1 - j^{-1/2^N}\right)$$

and

$$\varphi_j\left(j^{1/2^{(N-1)}}\right) \leq -j^{1/2^{(N-1)}} \quad \text{for all } j \geq \left(\frac{N}{N-1}\right)^{2^N}. \tag{3.4}$$

We also have

$$\begin{aligned}\varphi_j\left(j^{1/2^{(N-1)}}\right) &= N\exp\left(\frac{1}{\left(\log j\right)^{\frac{1}{N-1}}}\left[\log j+\log\left(1-j^{1/2^{(N-1)}-1}\right)\right]^{N'}\right)\\ &\qquad - N\left(j-j^{1/2^{(N-1)}}\right)\\ &= N\exp\left(\log j\left(1+\frac{\log\left(1-j^{1/2^{(N-1)}-1}\right)}{\log j}\right)^{N'}\right)\\ &\qquad - N\left(j-j^{1/2^{(N-1)}}\right)\\ &= N\exp\left(\log j - N' j^{1/2^{(N-1)}-1}+o\left(\frac{1}{j}\right)\right)-Nj+Nj^{1/2^{(N-1)}}\\ &= Nj\left(\exp\left(-N'j^{1/2^{(N-1)}-1}+o\left(\frac{1}{j}\right)\right)-1\right)+Nj^{1/2^{(N-1)}}.\end{aligned}$$

Therefore, for every $\varepsilon\in(0,1)$, there exists $j_\varepsilon\geq 1$ such that

$$\varphi_j\left(j^{1/2^{(N-1)}}\right)\leq Nj^{1/2^{(N-1)}}\big(1-(1-\varepsilon)N'\big)\quad \text{for every}\quad j\geq j_\varepsilon. \tag{3.5}$$

Let j be fixed and large enough. A qualitative study of φ_j in $[1,+\infty)$ shows that there exists a unique $s_j\in(1,j)$ such that the derivative $\varphi_j'(s_j)=0$. Consequently,

$$\begin{aligned}&\int_{j^{1/2^{(N-1)}}}^{j-j^{1/2^{(N-1)}}} e^{\varphi_j(s)}\,ds\\ &\qquad\leq\left(j-2j^{1/2^{(N-1)}}\right)\exp\left(\max\left[\varphi_j\left(j^{1/2^{(N-1)}}\right),\varphi_j\left(j-j^{1/2^{(N-1)}}\right)\right]\right).\end{aligned}$$

In addition, from (3.4) and (3.5) with $\varepsilon=\frac{1}{N^2}$, we obtain

$$\max\left[\varphi_j\left(j^{1/2^{(N-1)}}\right),\varphi_j\left(j-j^{1/2^{(N-1)}}\right)\right]\leq -j^{1/2^{(N-1)}}$$

by the condition that j is large enough. Hence, there exists $\bar{j}\geq 1$ such that

$$\int_{j^{1/2^{(N-1)}}}^{j-j^{1/2^{(N-1)}}} e^{\varphi_j(s)}\,ds\leq\left(j-2j^{1/2^{(N-1)}}\right)e^{-j^{1/2^{(N-1)}}}\qquad\text{for all } j\geq\bar{j}.$$

Therefore,

$$\lim_{j\to+\infty}\int_{j^{1/2^{(N-1)}}}^{j-j^{1/2^{(N-1)}}}\exp\left(N\exp\left(s^{\left(\frac{\log s}{\log j}\right)^{1/(N-1)}}\right)-Ns\right)ds=0.$$

Finally, we will study the limit on the interval $\left[j-j^{1/2^{N-1}},j\right]$. We mention that for a fixed $j\geq 1$ large enough, φ_j is a convex function on $\left[j-j^{1/2^{(N-1)}},+\infty\right)$, $\varphi_j(j)=0$. Hence, we can get the estimate

$$\varphi_j(s)\leq\frac{j-s}{j^{1/2^{(N-1)}}}\varphi_j\left(j-j^{1/2^{(N-1)}}\right),\quad s\in\left[j-j^{1/2^{(N-1)}},j\right].$$

On the other hand, in view of (3.4) and (3.5), if $\varepsilon \in (0, 1/N^2)$ and $j \geq j_\varepsilon$, we have

$$\varphi_j(s) \leq N(1-(1-\varepsilon)N')(j-s), \quad s \in \left[j - j^{1/2^{(N-1)}}, j\right]. \tag{3.6}$$

Furtheremore, using the fact that ψ_j is convex on $\left[j - j^{1/2^{(N-1)}}, +\infty\right)$ and $\varphi_j'(j) = N'$, we get

$$\varphi_j(s) \geq \varphi_j(j) + \varphi_j'(j)(s-j) = N'(s-j), \quad s \in \left[j - j^{1/2^{(N-1)}}, j\right]. \tag{3.7}$$

Then, by bringing together (3.6) and (3.7), we deduce

$$\frac{1}{N'} \leq \lim_{j \to +\infty} \int_{j-j^{1/2^{(N-1)}}}^{j} e^{\varphi_j(s)}\, ds \leq -\frac{1}{N(1-(1-\varepsilon)N')}.$$

By ε tending to zero, we get

$$\lim_{j \to +\infty} \int_{j^{1/2^{(N-1)}}}^{j-j^{1/2^{(N-1)}}} \exp\left(N \exp\left(s^{\left(\frac{\log s}{\log j}\right)^{1/(N-1)}}\right) - Ns\right) ds = \frac{1}{N'}.$$

Our claim is proved. □

Finally, we give the desired estimate.

Lemma 3.4. *Assume that* **(H5)** *holds and that α_0 is the real given by the definition of critical growth. Then*

$$C_M < \frac{1}{N}\left(\frac{\omega_{N-1}}{\alpha_0^{N-1}}\right).$$

Proof. We have $v_n \geq 0$ and $\|v_n\| = 1$. Then, from Proposition 3.2,

$$\mathcal{J}(tv_n) \to -\infty \quad \text{as } t \to +\infty.$$

As a consequence,

$$C_M \leq \max_{t \geq 0} \mathcal{J}(tv_n).$$

We argue by contradiction and suppose that for all $n \geq 1$,

$$\max_{t \geq 0} \mathcal{J}(tv_n) \geq \frac{1}{N}\left(\frac{\omega_{N-1}}{\alpha_0^{N-1}}\right).$$

Since $\mathcal{J}$ possesses the mountain pass geometry, for any $n \geq 1$, there exists $t_n > 0$ such that

$$\max_{t \geq 0} \mathcal{J}(tv_n) = \mathcal{J}(t_n v_n) \geq \frac{1}{N}\left(\frac{\omega_{N-1}}{\alpha_0^{N-1}}\right).$$

Using the fact that $F(x,t) \geq 0$ for all $(x,t) \in B \times \mathbb{R}$, we get

$$\frac{1}{N}\left(\frac{\omega_{N-1}}{\alpha_0^{N-1}}\right) \leq \mathcal{J}(t_n v_n) \leq \frac{1}{N} t_n^N.$$

Thus,

$$t_n^N \geq \frac{\omega_{N-1}}{\alpha_0^{N-1}}. \tag{3.8}$$

On the other hand,

$$\frac{d}{dt}J(tv_n)\bigg|_{t=t_n} = t_n^{N-1} - \int_B f(x, t_n v_n) v_n \, dx = 0,$$

that is,

$$t_n^N = \int_B f(x, t_n v_n) t_n v_n \, dx. \tag{3.9}$$

Now we claim that the sequence (t_n) is bounded in $(0, +\infty)$.

Indeed, it follows from condition **(H5)** that for all $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$f(x,t)t \geq (\gamma_0 - \varepsilon)\exp\left(Ne^{\alpha_0 t^{N'}}\right) \quad \text{for all } |t| \geq t_\varepsilon \text{ uniformly in } x \in B. \tag{3.10}$$

From (3.9) and the definition of v_n, we have

$$t_n^N = \int_B f(x, t_n v_n) t_n v_n dx \geq \omega_{N-1} \int_n^{+\infty} f\left(e^{-s}, t_n \frac{\psi_n}{\omega_{N-1}^{1/N}}\right) t_n \frac{\psi_n}{\omega_{N-1}^{1/N}} e^{-Ns} \, ds.$$

Also, from (3.8), we get on $[n, +\infty)$,

$$t_n \frac{\psi_n}{\omega_{N-1}^{1/N}} = t_n \left(\frac{\log(1+n)}{\omega_{N-1}^{1/N-1}}\right)^{1/N'} \geq \left(\frac{\log(1+n)}{\alpha_0}\right)^{1/N'}.$$

Then it follows from (3.10) that for all $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$,

$$t_n^N \geq \omega_{N-1}(\gamma_0 - \varepsilon) \int_n^{+\infty} \exp\left(N \exp\left(\frac{\alpha_0}{\omega_{N-1}^{1/N-1}} (t_n \psi_n)^{N'}\right) - Ns\right) ds, \tag{3.11}$$

that is,

$$t_n^N \geq \frac{\omega_{N-1}}{N}(\gamma_0 - \varepsilon) \exp\left(N \exp\left(\frac{\alpha_0}{\omega_{N-1}^{1/N-1}} t_n^{N'} \log(1+n)\right) - Nn\right).$$

So, for n large enough, we obtain

$$1 \geq \frac{\omega_{N-1}}{N}(\gamma_0 - \varepsilon) \exp\left(N \exp\left(\frac{\alpha_0}{\omega_{N-1}^{1/N-1}} t_n^{N'} \log(1+n)\right) - Nn - N\log(t_n)\right)$$

Therefore, (t_n) is bounded in $\mathbb{R}$. Now, suppose that

$$\lim_{n \to +\infty} t_n^N > \frac{\omega_{N-1}}{\alpha_0^{N-1}}.$$

For n large enough, $t_n^N > \frac{\omega_{N-1}}{\alpha_0^{N-1}}$. In this case, the right-hand side of inequality (3.11) gives the unboundedness of the sequence (t_n). Since (t_n) is bounded, we get

$$\lim_{n\to+\infty} t_n^N = \frac{\omega_{N-1}}{\alpha_0^{N-1}}.$$

Now we are going to estimate the expression in (3.9). Let

$$B_{n,+} = \{x \in B \mid t_n v_n(x) \geq t_\varepsilon\} \quad \text{and} \quad B_{n,-} = \{x \in B \mid t_n v_n(x) < t_\varepsilon\}$$

We have

$$t_n^N \geq (\gamma_0 - \varepsilon) \int_{B_{n,+}} \exp\left(N e^{\alpha_0 t_n^{N'} v_n^{N'}}\right) dx + \int_{B_{n,-}} f(x, t_n v_n) t_n v_n \, dx.$$

Then

$$t_n^N \geq (\gamma_0 - \varepsilon) \int_B \exp\left(N e^{\alpha_0 t_n^{N'} v_n^{N'}}\right) dx - (\gamma_0 - \varepsilon) \int_{B_{n,-}} \exp\left(N e^{\alpha_0 t_n^{N'} v_n^{N'}}\right) dx$$
$$+ \int_{B_{n,-}} f(x, t_n v_n) t_n v_n \, dx. \tag{3.12}$$

The sequence (v_n) converges to 0 in B and $\chi_{B_{n,-}}$ converges to 1 a.e. in B. By using the dominated convergence theorem, we obtain

$$\lim_{n\to+\infty} \int_{B_{n,-}} f(x, t_n v_n) t_n v_n \, dx = 0$$

and

$$\lim_{n\to+\infty} \int_{B_{n,-}} \exp\left(N e^{\alpha_0 t_n^{N'} v_n^{N'}}\right) dx \leq \frac{\omega_{N-1}}{N} e^N.$$

We also have

$$\lim_{n\to+\infty} \int_B \exp\left(N \exp\left(\omega_{N-1}^{1/N-1} |v_n|^{N'}\right)\right) dx$$
$$= \lim_{n\to+\infty} \omega_{N-1} \int_0^{+\infty} \exp\left(N e^{|\psi_n|^{N'}} - Nt\right) dt.$$

By using (3.8) and Lemma 3.3, we get

$$\lim_{n\to+\infty} \int_B \exp\left(N e^{\alpha_0 t_n^{N'} v_n^{N'}}\right) dx \geq \lim_{n\to+\infty} \int_B \exp\left(N \exp\left(\omega_{N-1}^{1/(N-1)} |v_n|^{N'}\right)\right) dx$$
$$= \omega_{N-1} \left(\frac{N+1}{N}\right) e^N.$$

Passing to the limit in (3.12), we obtain

$$\frac{\omega_{N-1}}{\alpha_0^{N-1}} \geq (\gamma_0 - \varepsilon)\, \omega_{N-1} e^N$$

for all $\varepsilon > 0$. Thus we have

$$\gamma_0 \leq \frac{1}{\alpha_0^{N-1} e^{-N}},$$

which contradicts condition **(H5)**. The lemma is proved. □

4. The Cerami sequences and proof of the main results

4.1. Lions-type concentration lemma. To prove a compactness condition for the energy $\mathcal{J}$, we need a Lions type result [19] on the improved Trudinger–Moser inequality.

Lemma 4.1. *Let $\{u_k\}_k$ be a sequence in E. Suppose that $\|u_k\| = 1$, $u_n \rightharpoonup u$, weakly in E, $u_n(x) \to u(x)$ and $\nabla u_n(x) \to \nabla u(x)$ almost everywhere in B. Then*

$$\sup_k \int_B \exp\left(N e^{p\omega_{N-1}^{1/(N-1)}|u_k|^{N'}}\right) dx < +\infty,$$

for all $1 < p < P$, where

$$P := \begin{cases} \left(1 - \|u\|^N\right)^{-1/(N-1)} & \text{if } \|u\| < 1, \\ +\infty & \text{if } \|u\| = 1. \end{cases}$$

Proof. By the Young inequality, we have

$$\exp\left(N e^{a+b}\right) \le \frac{1}{q}\exp\left(N e^{qa}\right) + \frac{1}{q'}\exp\left(N e^{q'b}\right), \quad a, b \in \mathbb{R},\ q > 1 \text{ and } \frac{1}{q} + \frac{1}{q'} = 1.$$

We can also estimate $|u_k|^{N'}$, using the following inequality:

$$(1+a)^q \le (1+\varepsilon)a^q + (1 - \frac{1}{(1+\varepsilon)^{\frac{1}{q-1}}})^{1-q}, \quad a \ge 0,\ \varepsilon > 0,\ q > 1.$$

So, we get

$$|u_k|^{N'} \le (1+\varepsilon)|u_k - u|^{N'} + \left(1 - \frac{1}{(1+\varepsilon)^{N-1}}\right)^{-1(N-1)} |u|^{N'}.$$

Therefore, for any $p > 1$, using the above inequalities, we obtain

$$\begin{aligned} &\int_B \exp\left(N\exp\left(p\omega_{N-1}^{1/(N-1)}|u_k|^{N'}\right)\right) dx \\ &\quad\le \frac{1}{q}\int_B \exp\left(N\exp\left(pq\omega_{N-1}^{1/(N-1)}(1+\varepsilon)|u_k - u|^{N'}\right)\right) dx \\ &\quad+ \frac{1}{q'}\int_B \exp\left(N\exp\left(pq'\omega_{N-1}^{1/N-1}\left(1 - \frac{1}{(1+\varepsilon)^{N-1}}\right)^{-1/(N-1)}|u|^{N'}\right)\right) dx. \end{aligned}$$

From (1.3), the last integral is finite and to complete the proof, we should prove that for every p such that $1 < p < P$, we have

$$\sup_k \int_B \exp\left(N\exp\left(pq\omega_{N-1}^{1/(N-1)}(1+\varepsilon)|u_k - u|^{N'}\right)\right) dx < +\infty$$

for some $\varepsilon > 0$ and $q > 1$.

By the Brezis–Lieb lemma, we have

$$\|u_n - u\|^N = \|u_n\|^N - \|u\|^N + o_n(1),$$

where $o_n(1) \to 0$ as $n \to +\infty$. Then

$$\|u_n - u\|^N = 1 - \|u\|^N + o_n(1),$$

where $o_n(1) \to 0$ as $n \to +\infty$. We may assume that $\|u\| < 1$, and the proof for the case $\|u\| = 1$ is similar. If $\|u\| < 1$, then, for

$$p < \frac{1}{(1 - \|u\|^N)^{1/(N-1)}},$$

there exists $\nu > 0$ such that

$$p(1 - \|u\|^N)^{1/(N-1)}(1 + \nu) < 1.$$

On the other hand,

$$\lim_{k \to +\infty} \|u_k - u\|^N = 1 - \|u\|^N,$$

and thus

$$\lim_{k \to +\infty} \|u_k - u\|^{N'} = (1 - \|u\|^N)^{1/(N-1)}.$$

Therefore, for every $\varepsilon > 0$, there exists $k_\varepsilon \geq 1$ such that

$$\|u_k - u\|^{N'} \leq (1 + \varepsilon)\left(1 - \|u\|^N\right)^{1/(N-1)}, \quad k \geq k_\varepsilon.$$

Then, for $q = 1 + \varepsilon$ with ε such that $\varepsilon = \sqrt[3]{1 + \nu} - 1$ and for every $k \geq k_\varepsilon$, we get

$$pq(1 + \varepsilon)\|u_k - u\|^{N'} \leq 1.$$

From (1.4), this leads to

$$\begin{aligned}
&\int_B \exp\left(N \exp\left(pq\omega_{N-1}^{1/N-1}(1+\varepsilon)|u_k - u|^{N'}\right)\right) dx \\
&\quad \leq \int_B \exp\left(N \exp\left((1+\varepsilon)pq\omega_{N-1}^{1/(N-1)}\left(\frac{|u_k - u|}{\|u_k - u\|}\right)^{N'} \|u_k - u\|^{N'}\right)\right) dx \\
&\quad \leq \int_B \exp\left(N \exp\left(\omega_{N-1}^{1/(N-1)}\left(\frac{|u_k - u|}{\|u_k - u\|}\right)^{N'}\right)\right) dx \\
&\quad \leq \sup_{\|u\| \leq 1} \int_B \exp\left(N \exp\left(\omega_{N-1}^{1/(N-1)}|u|^{N'}\right)\right) dx < +\infty,
\end{aligned}$$

which completes the proof. □

4.2. Proof of Theorem 1.2. We have the following version of the Mountain Pass theorem.

Lemma 4.2 ([3])**.** *Let E be a real Banach space and $\mathcal{J} \in C^1(E, \mathbb{R})$. Assume that $\mathcal{J}$ satisfies the $(C)_c$ condition for any $c \in \mathbb{R}$ and the following geometric assumptions.*

1. *We have $\mathcal{J}(0) = 0$ and there exist positive constants R and α such that*
$$\mathcal{J}(u) \geq \alpha \quad \text{for all } u \in E \text{ with } \|u\| = R.$$
2. *There exists $u_0 \in E$ such that $\|u_0\| > R$ and $\mathcal{J}(u_0) \leq 0$.*

Then there exists $u \in E$ such that $\mathcal{J}(u) = c$ and $\mathcal{J}'(u) = 0$. Furthermore, the critical value c is characterized by

$$c := \inf_{g \in \Gamma} \max_{u \in g([0,1])} \mathcal{J}(u),$$

where

$$\Gamma := \big\{ g \in C([0,1], E) \mid g(0) = 0,\ g(1) = u_0 \big\}.$$

Now we prove that the functional $\mathcal{J}$ satisfies the Cerami condition at all levels $c \in \mathbb{R}$ in the subcritical case.

Lemma 4.3. *Suppose that* **(H1)**–**(H4)** *hold. Assume that the function $f(x,t)$ has subcritical growth at $+\infty$. Then the functional $\mathcal{J}$ satisfies the $(C)_c$ condition for any $c \in \mathbb{R}$.*

Proof. Let (u_n) be a $(C)_c$ sequence in E for some $c \in \mathbb{R}$. Then

$$\mathcal{J}(u_n) = \frac{1}{N}\|u_n\|^N - \int_B F(x, u_n)\, dx \to c \quad \text{as } n \to +\infty, \tag{4.1}$$

and for all $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$,

$$(1 + \|u_n\|)|\mathcal{J}'(u_n)v| = (1 + \|u_n\|)\left| \int_B \sigma(x)|\nabla u_n|^{N-2}\nabla u_n \cdot \nabla v\, dx - \int_B f(x, u_n) v\, dx \right| \leq \varepsilon\|v\|, \tag{4.2}$$

for all $v \in E$. Hence, for $\varepsilon_n \to 0$, up to a subsequence,

$$(1 + \|u_n\|)|\mathcal{J}'(u_n)v| = (1 + \|u_n\|)\left| \int_B \sigma(x)|\nabla u_n|^{N-2}\nabla u_n \cdot \nabla v\, dx - \int_B f(x, u_n) v\, dx \right| \leq \varepsilon_n\|v\|, \tag{4.3}$$

for all $v \in E$. We will show that $\{u_n\}$ is bounded. We argue by contradiction and suppose that

$$\|u_n\| \to +\infty.$$

Let
$$v_n = \frac{u_n}{\|u_n\|},$$
then $\|v_n\| = 1$. We may suppose that $v_n \rightharpoonup v$ in E up to a subsequence. We will show that $v_n^+ \rightharpoonup 0$ in E, where $v_n^+ = \max(v_n, 0)$. By the Sobolev embedding, we have $v_n^+(x) \to v^+(x)$ a.e in B and $v_n^+ \to v^+$ a.e. in $L^p(B)$ for all ≥ 1. Let $B^+ = \{x \in B \mid v^+(x) > 0\}$ and suppose that $\mu(B^+) > 0$, where μ is the Lebesgue measure. Then in B^+, we have
$$\lim_{n\to+\infty} u_n^+(x) = \lim_{n\to+\infty} v_n^+(x)\|u_n\| = +\infty.$$
Using condition **(H2)**, we obtain
$$\lim_{n\to+\infty} \frac{F(x, u_n^+(x))}{(u_n^+(x))^N} = +\infty \quad \text{a.e. in } B^+.$$
So,
$$\lim_{n\to+\infty} \frac{F(x, u_n^+(x))}{(u_n^+(x))^N}(v_n^+(x))^N = +\infty \ \text{ a.e. in } \ B^+.$$
On the one hand, by (4.1), we have
$$\|u_n\|^N = Nc + N\int_B F(x, u_n^+)dx + o_n(1).$$
Consequently,
$$\lim_{n\to+\infty} \int_B F(x, u_n^+)dx = +\infty.$$
Since $F(x,t) \geq 0$, by the Fatou lemma, we get
$$\begin{aligned} +\infty = \int_{B^+} \liminf_{n\to+\infty} \frac{F(x, u_n^+)}{(u_n^+)^N}(v_n^+)^N dx &\leq \liminf_{n\to+\infty} \int_{B^+} \frac{F(x, u_n^+)}{(u_n^+)^N}(v_n^+)^N dx \\ &\leq \liminf_{n\to+\infty} \int_B \frac{F(x, u_n^+)}{\|u_n\|^N} dx = \frac{1}{N}, \end{aligned}$$
which is a contradiction.

Now, let $t_n \in [0,1]$ such that
$$\mathcal{J}(t_n u_n) = \max_{t\in[0,1]} \mathcal{J}(tu_n).$$
Since f is subcritical at $+\infty$, for any given $R > 0$, there exists $C = C(R) > 0$ such that
$$F(x,s) \leq Cs + \exp\left(N \exp\left(\frac{w_{N-1}^{1/(N-1)}}{R^{N/(n-1)}} s^{N/(N-1)}\right)\right), \ (x,s) \in B \times (0, +\infty). \quad (4.4)$$
Since $\|u_n\| \to +\infty$, we have
$$\mathcal{J}(t_n u_n) \geq \mathcal{J}\left(\frac{Ru_n}{\|u_n\|}\right) = \mathcal{J}(Rv_n). \quad (4.5)$$

By (4.6), and using the fact that $\int_B F(x,v_n)dx = \int_B F(x,v_n^+)dx$, we get

$$\begin{aligned} N\mathcal{J}(Rv_n) &\geq R^N - NCR\int_B |v_n^+(x)|\,dx \\ &\quad - N\int_B \exp\left(N\exp\left(w_{N-1}^{1/(N-1)}(v_n^+)^{N/(N-1)}\right)\right)dx \\ &\geq R^N - NCR\int_B |v_n^+(x)|dx \\ &\quad - N\int_B \exp\left(N\exp\left(w_{N-1}^{1/(N-1)}v_n^{N/(N-1)}\right)\right)dx. \end{aligned} \tag{4.6}$$

The last integral in the right-hand side is finite in view of Theorem 1.1. Moreover, $v_n^+ \rightharpoonup 0$ in E, then we have $\int_B |v_n^+(x)|dx \to 0$ as $n \to +\infty$. Letting $n \to +\infty$ in (4.6) and $R \to +\infty$, we get

$$\mathcal{J}(t_nu_n) \to +\infty. \tag{4.7}$$

Since $\mathcal{J}(0) = 0$ and $\mathcal{J}(u_n) \to c$, we can suppose that $t_n \in (0,1)$. On the one hand, we have that $\mathcal{J}'(t_nu_n)t_nu_n = 0$, then

$$t_n\|u_n\|^N = \int_B f(x,t_nu_n)t_nu_n\,dx.$$

On the other hand, by (4.1) and **(H3)**, we get

$$\begin{aligned} N\mathcal{J}(t_nu_n) &= t_n\|u_n\|^N - N\int_B F(x,t_nu_n)\,dx \\ &= \int_B \big(f(x,t_nu_n) - NF(x,t_nu_n)\big)\,dx \\ &\leq \theta\int_B \big(f(x,u_n) - NF(x,u_n)\big)\,dx + \bar{C}. \end{aligned}$$

By (4.2), we have

$$\int_B \big(f(x,u_n) - NF(x,u_n)\,dx = Nc + o_n(1),$$

which contradicts (4.7). Therefore, $\{u_n\}$ is bounded in E. Up to a subsequence, without loss of generality, we may assume that

$$\begin{aligned} &\|u_n\| \leq K && \text{in } E, \\ &u_n \rightharpoonup u && \text{weakly in } E, \\ &u_n \to u && \text{strongly in } L^q(B) \text{ for all } q \geq 1, \\ &u_n(x) \to u(x) && \text{almost everywhere in } B. \end{aligned}$$

Since f is subcritical at $+\infty$, there exists a constant $C_K > 0$ such that

$$f(x,s) \leq C_K \exp\left(\exp\left(\frac{w_{N-1}^{1/(N-1)}}{K^{N/(n-1)}}s^{N/(N-1)}\right)\right), \quad (x,s) \in B \times (0,+\infty).$$

Then, by the Hölder inequality,

$$\begin{aligned}
&\left|\int_B f(x,u_n)(u_n-u)dx\right| \le \int_B |f(x,u_n)(u_n-u)|\,dx\\
&\le \left(\int_B |f(x,u_n)|^2dx\right)^{1/2}\left(\int_B |u_n-u|^2dx\right)^{1/2}\\
&\le C\left(\int_B \exp\left(2\exp\left(\frac{w_{N-1}^{1/(N-1)}}{M^{N/(n-1)}}u_n^{N/(N-1)}\right)\right)dx\right)^{1/2}\|u_n-u\|_2\\
&\le C\left(\int_B \exp\left(2\exp\left(\frac{w_{N-1}^{1/(N-1)}}{K^{N/(n-1)}}\|u_n\|^{N/(N-1)}\frac{|u_n|^{N/(N-1)}}{\|u_n\|^{N/(N-1)}}\right)\right)dx\right)^{1/2}\|u_n-u\|_2\\
&\le C\|u_n-u\|_2 \to 0 \quad \text{as } n\to+\infty.
\end{aligned}$$

Also, using the fact that

$$\int_B f(x,u)(u_n-u)dx \to 0 \quad \text{as } u_n \rightharpoonup u \text{ in } E,$$

we get

$$\int_B \big(f(x,u_n)-f(x,u)\big)(u_n-u)dx \to 0.$$

From (4.3) with $v=u_n-u$, we get

$$\int_B \sigma(x)|\nabla u_n|^{N-2}\nabla u_n\cdot(\nabla u_n-\nabla u)\,dx - \int_B f(x,u_n)(u_n-u)\,dx = o_n(1). \quad (4.8)$$

On the other hand, since $u_n \rightharpoonup u$ weakly in E, then

$$\int_B \sigma(x)|\nabla u|^{N-2}\nabla u\cdot(\nabla u_n-\nabla u)\,dx = o_n(1). \quad (4.9)$$

Combining (4.8) and (4.9), we obtain

$$\begin{aligned}
\int_B \sigma(x)\Big(|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u\Big)\cdot(\nabla u_n-\nabla u)\,dx&\\
= \int_B f(x,u_n)(u_n-u)\,dx + o_n(1).&
\end{aligned}$$

Using the well-known inequality

$$\big(|x|^{N-2}x-|y|^{N-2}y\big)\,(x-y) \ge 2^{2-N}|x-y|^N, \quad x,y\in\mathbb{R}^N,\ N\ge 2,$$

we obtain

$$0 \le 2^{2-N}\int_B \sigma(x)|\nabla u_n-\nabla u|^N\,dx \le \int_B f(x,u_n)(u_n-u)\,dx + o_n(1).$$

Using the above results, we get

$$2^{2-N}\int_B \sigma(x)|\nabla u_n-\nabla u|^N\,dx \le \int_B f(x,u_n)(u_n-u)\,dx + o_n(1) \to 0.$$

So,

$$\|u_n - u\| \to 0 \quad \text{as} \quad n \to \infty.$$

Then $\mathcal{J}$ satisfies the $(C)_c$ condition for all $c \in \mathbb{R}$ and the proof of Lemma 4.3 is achieved. □

By Lemma 4.3, the functional $\mathcal{J}$ satisfies the $(C)_c$ condition (at each level c). So, by Proposition 3.1 and Proposition 3.2, we deduce that the functional $\mathcal{J}$ has a nonzero critical point u in E. The proof of Theorem 1.2 is complete.

4.3. Proof of Theorem 1.3. We recall the following result.

Lemma 4.4 ([10])**.** *Let E be a real Banach space, $\mathcal{J} \in C^1(E, \mathbb{R})$ and $\mathcal{J}(0) = 0$. Assume that $\mathcal{J}$ satisfies the following geometric assumptions.*

1. *There exist positive constants R and α such that*

$$\mathcal{J}(u) \geq \alpha \quad \text{for all } u \in E \text{ with } \|u\| = R.$$

2. *There exists $u_0 \in E$ such that $\|u_0\| > R$ and $\mathcal{J}(u_0) \leq 0$.*

Let C_M be characterized by

$$C_M := \inf_{g \in \Gamma} \max_{u \in g([0,1])} \mathcal{J}(u),$$

where

$$\Gamma := \{ g \in C([0,1], E) \mid g(0) = 0,\ g(1) = u_0 \}.$$

Then $\mathcal{J}$ possesses a $(C)_{C_M}$ sequence.

Now, for the critical case, we will prove that the functional $\mathcal{J}$ satisfies the $(C)_{C_M}$ condition.

Lemma 4.5. *Suppose that* **(H1)–(H4)** *hold. Assume that the function $f(x,t)$ has critical growth at $+\infty$. Then the functional $\mathcal{J}$ satisfies the $(C)_{C_M}$ condition.*

Proof. According to Propositions 3.1 and 3.2, there exists a $(C)_{C_M}$ sequence $\{u_n\}$ in E, that is,

$$\mathcal{J}(u_n) = \frac{1}{N}\|u_n\|^N - \int_B F(x, u_n)\, dx \to C_M \quad \text{as } n \to +\infty, \tag{4.10}$$

and for $\varepsilon_n \to 0$, up to a subsequence,

$$\begin{aligned} &(1 + \|u_n\|)\big|\mathcal{J}'(u_n)v\big| \\ &= (1 + \|u_n\|)\left| \int_B \sigma(x)|\nabla u_n|^{N-2}\nabla u_n \cdot \nabla v\, dx - \int_B f(x, u_n)v\, dx \right| \leq \varepsilon_n \|v\| \end{aligned} \tag{4.11}$$

for all $v \in E$.

To show that $\{u_n\}$ is bounded, we argue by contradiction and suppose that

$$\|u_n\| \to +\infty. \tag{4.12}$$

Let

$$v_n = \frac{u_n}{\|u_n\|},$$

then $\|v_n\| = 1$. We may suppose that $v_n \rightharpoonup v$ in E up to a subsequence. We can similarly show that $v_n^+ \rightharpoonup 0$ in E, where $v_n^+ = \max(v_n, 0)$. Again, let $t_n \in [0,1]$ such that

$$\mathcal{J}(t_n u_n) = \max_{t\in[0,1]} \mathcal{J}(tu_n).$$

Let $R \in \left(0, \left(w_{N-1}^{1/(N-1)}/\alpha_0\right)^{(N-1)/N}\right)$ and choose $\varepsilon = w_{N-1}^{1/(N-1)}/R^{N/(N-1)} - \alpha_0 > 0$. By the criticality growth condition, we have

$$F(x,s) \le C|s| + \varepsilon \exp\left(N \exp\left((\alpha_0+\varepsilon)s^{N/(N-1)}\right)\right), \ (x,s) \in B\times(0,+\infty). \quad (4.13)$$

Since $\|u_n\| \to +\infty$, we have

$$\mathcal{J}(t_n u_n) \ge \mathcal{J}\left(\frac{R}{\|u_n\|}\right) = \mathcal{J}(Rv_n). \quad (4.14)$$

From (4.14) and using the fact that $\int_B F(x,v_n)dx = \int_B F(x,v_n^+)dx$, we get

$$\begin{aligned} N\mathcal{J}(Rv_n) &\ge R^N - NCR\int_B |v_n^+(x)|\,dx \\ &\quad - N\varepsilon\int_B \exp\left(N\exp\left((\alpha_0+\varepsilon)v_n^{N/(N-1)}\right)\right)dx \\ &\ge R^N - NCR\int_B |v_n^+(x)|\,dx \\ &\quad - N\varepsilon\int_B \exp\left(N\exp\left(w_{N-1}^{1/(N-1)}v_n^{N/(N-1)}\right)\right)dx. \end{aligned}$$

The last integral in the right-hand side is finite in view of Theorem 1.1. Moreover, if $v_n^+ \rightharpoonup 0$ in E, then we have $\int_B |v_n^+(x)|dx \to 0$ as $n \to +\infty$. Letting $n \to +\infty$ in (4.14) and $R \to \left(w_{N-1}^{1(N-1)}/\alpha_0\right)^{(N-1)/N}$, we get

$$\liminf_{n\to+\infty} \mathcal{J}(t_n u_n) \ge \frac{1}{N}\left(\frac{\omega_{N-1}}{\alpha_0^{N-1}}\right) > C_M. \quad (4.15)$$

We have $\mathcal{J}(0) = 0$ and $\mathcal{J}(u_n) \to C_M$. We can suppose that $t_n \in (0,1)$.

On the one hand, we have that $\mathcal{J}'(t_n u_n)t_n u_n = 0$. Then

$$t_n^N \|u_n\|^N = \int_B f(x, t_n u_n)t_n u_n\,dx.$$

In addition, by the hypothesis **(H3)** with $\theta = 1$ and $\bar{C} = 0$, we get

$$N\mathcal{J}(t_n u_n) = t_n^N\|u_n\|^N - N\int_B F(x,t_n u_n)\,dx$$

$$= \int_B \big(f(x,t_nu_n)t_nu_n - NF(x,t_nu_n)\big)\,dx$$
$$\le \int_B \big(f(x,u_n)u_n - NF(x,u_n)\big)\,dx.$$

Since

$$\int_B \big(f(x,t_nu_n)t_nu_n - NF(x,t_nu_n)\,dx = NC_M + o_n(1),$$

we reach a contradiction with (4.15). Therefore, $\{u_n\}$ is bounded in E. Up to a subsequence, without loss of generality, we may assume that

$$\begin{aligned} &\|u_n\| \le M && \text{in } E,\\ &u_n \rightharpoonup u && \text{weakly in } E,\\ &u_n \to u && \text{strongly in } L^q(B) \text{ for all } q \ge 1,\\ &u_n(x) \to u(x) && \text{almost everywhere in } B. \end{aligned}$$

We follow the schema of [2] to show the convergence almost everywhere of the gradient $\nabla u_n(x) \to \nabla u(x)$ for a.a. $x \in B$.

From (4.11), we obtain

$$0 < \int_B f(x,u_n)u_n\,dx \le C.$$

Also, from (4.10), we have

$$0 < \int_B F(x,u_n)\,dx \le C.$$

Consequently,

$$\begin{aligned} &\big(|\nabla u_n|^{N-2}\nabla u_n\big) && \text{is bounded in } (L^{N/(N-1)}(B,\sigma))^N,\\ &|\nabla u_n|^{N-2}\nabla u_n \rightharpoonup |\nabla u|^{N-2}\nabla u && \text{in } (L^{N/(N-1)}(B,\sigma))^N \text{ as } n \to +\infty. \end{aligned}$$

By Lemma 2.1 from [14], we have

$$f(x,u_n) \to f(x,u) \quad \text{in } L^1(B) \text{ as } n \to +\infty. \tag{4.16}$$

According to hypothesis **(H6)**, we have

$$F(x,u_n) \to F(x,u) \quad \text{in } L^1(B) \text{ as } n \to +\infty. \tag{4.17}$$

By (4.10), we obtain

$$\lim_{n\to+\infty} \|u_n\|^N = N(C_M + \int_B F(x,u)dx).$$

Therefore, passing to the limit in (4.11), we get

$$\int_B \sigma(x)|\nabla u|^{N-2}\nabla u \cdot \nabla\varphi\,dx = \int_B f(x,u)\varphi dx, \quad \varphi \in E. \tag{4.18}$$

Hence, u is a weak solution of problem (1.1).

Next, we are going to make some claims.

Claim 1. At this stage, we affirm that $u \neq 0$. Indeed, we argue by contradiction and suppose that $u \equiv 0$. Therefore, $\int_B F(x, u_n)dx \rightarrow 0$ and subsequently we get

$$\frac{1}{N}\|u_n\|^N \rightarrow C_M < \frac{1}{N}\frac{\omega_{N-1}}{\alpha_0^{N-1}}. \tag{4.19}$$

First, we claim that there exists $q > 1$ such that

$$\int_B |f(x, u_n)|^q \, dx \leq C. \tag{4.20}$$

By (4.11), we have

$$\left| \|u_n\|^N - \int_B f(x, u_n)u_n \, dx \right| \leq \frac{C\epsilon_n}{(1 + \|u_n\|)} \leq C\epsilon_n.$$

So,

$$\|u_n\|^N \leq C\epsilon_n + \left(\int_B |f(x, u_n)|^q\right)^{1/q} dx \left(\int_B |u_n|^{q'}\right)^{1/q'},$$

where q' is the conjugate of q. Since (u_n) converges to 0 in $L^{q'}(B)$,

$$\lim_{n\rightarrow+\infty} \|u_n\|^N = 0.$$

By the Brezis–Lieb lemma [6], $u_n \rightarrow 0$ in E. Therefore, $\mathcal{J}(u_n) \rightarrow 0$, which is in contradiction with $C_M > 0$.

For the proof of the claim (4.20), since f has a critical growth, for every $\epsilon > 0$ and $q > 1$ there exist $t_\epsilon > 0$ and $C > 0$ such that for all $|t| \geq t_\epsilon$, we have

$$|f(x, t)|^q \leq C \exp\left(Ne^{\alpha_0(\epsilon+1)t^{N'}}\right).$$

Consequently,

$$\int_B |f(x, u_n)|^q dx = \int_{\{|u_n| \leq t_\epsilon\}} |f(x, u_n)|^q \, dx + \int_{\{|u_n| > t_\epsilon\}} |f(x, u_n)|^q \, dx$$
$$\leq \omega_{N-1} \max_{\overline{B}\times[-t_\epsilon, t_\epsilon]} |f(x, t)|^q + C\int_B \exp\left(Ne^{\alpha_0(\epsilon+1)|u_n|^{N'}}\right) dx.$$

Since $(NC_M)^{1/(N-1)} < \omega_{N-1}^{1/(N-1)}/\alpha_0$, there exist $\eta \in (0, 1/2)$ such that $(NC_M)^{1/(N-1)} = (1 - 2\eta)\omega_{N-1}^{1/(N-1)}/\alpha_0$. From (4.10), $\|u_n\|^{N'} \rightarrow (NC_M)^{1/(N-1)}$, so there exist $n_\eta \in \mathbb{N}$ such that $\alpha_0\|u_n\|^{N'} \leq (1 - \eta)\omega_{N-1}^{1/(N-1)}$, for all $n \geq n_\eta$. Therefore,

$$\alpha_0(1 + \epsilon)\left(\frac{|u_n|}{\|u_n\|}\right)^{N'} \|u_n\|^{N'} \leq (1 + \epsilon)(1 - \eta)\left(\frac{|u_n|}{\|u_n\|}\right)^{N'} \omega_{N-1}^{1/(N-1)}.$$

We choose $\epsilon > 0$ small enough to get

$$(1 + \epsilon)(1 - \eta) < 1.$$

Hence the second integral is uniformly bounded in view of (1.4).

Claim 2. We assert $u > 0$. Indeed, since (u_n) is bounded up to a subsequence, $\|u_n\| \to \rho > 0$. In addition, $\mathcal{J}'(u_n) \to 0$ leads to

$$\int_B \omega(x)|\nabla u|^{N-2}\nabla u \cdot \nabla\varphi \, dx = \int_B f(x,u)\varphi \, dx, \quad \varphi \in E.$$

By taking $\varphi = u^-$, with $w^\pm = \max(\pm w, 0)$, we get $\|u^-\|^N = 0$ and thus $u = u^+ \geq 0$. Since the nonlinearity has a critical growth at $+\infty$ and from the Trudinger–Moser inequality (1.4), $f(\cdot, u) \in L^p(B)$, for all $p \geq 1$. So, by elliptic regularity, $u \in W^{2,p}(B, \sigma)$ for all $p \geq 1$. Therefore, by the Sobolev embedding, $u \in C^{1,\gamma}(\overline{B})$.

Let us define $B_0 = \{x \in B \mid u(x) = 0\}$. The set $B_0 = \varnothing$. Indeed, by the contradiction, suppose that $B_0 \neq \varnothing$. Since $f(x,u) \geq 0$, by the Harnark inequality, we can deduce that B_0 is an open and closed set of B. In virtue of the connectedness of B, we reach a contradiction. Hence Claim 2 is proved.

We affirm that $\mathcal{J}(u) = C_M$. Indeed, by **(H3)** with $\theta = 1$ and $\bar{C} = 0$, we obtain

$$\mathcal{J}(u) \geq \frac{1}{N}\int_B \big(f(x,u)u - NF(x,u)\big)\, dx \geq 0. \tag{4.21}$$

Now, using the semicontinuity of the norm and (4.17), we get

$$\mathcal{J}(u) \leq \frac{1}{N}\liminf_{n \to \to \infty} \|u_n\|^N - \int_B F(x,u)\, dx = C_M.$$

Suppose that

$$\mathcal{J}(u) < C_M.$$

Then

$$\|u\|^N < \rho^N. \tag{4.22}$$

In addition,

$$\frac{1}{N}\lim_{n \to +\infty} \|u_n\|^N = \left(C_M + \int_B F(x,u)dx\right), \tag{4.23}$$

which means that

$$\rho^N = N\left(C_M + \int_B F(x,u)dx\right).$$

Set

$$v_n = \frac{u_n}{\|u_n\|} \quad \text{and} \quad v = \frac{u}{\rho}.$$

We have $\|v_n\| = 1$, $v_n \rightharpoonup v$ in E, $v \not\equiv 0$ and $\|v\| < 1$. So, by Lemma 4.1, we get

$$\sup_n \int_B \exp\left(N \exp\left(p\omega_{N-1}^{1/(N-1)}|v_n|^{N'}\right)\right) dx < \infty,$$

for $1 < p < \big(1 - \|v\|^N\big)^{-1/(N-1)}$.

By (4.17), (4.18) and (4.23), we have the following equality:

$$NC_M - N\mathcal{J}(u) = \rho^N - \|u\|^N.$$

From (4.21), (4.23) and the last equality, we obtain

$$\rho^N \leq NC_M + \|u\|^N < \frac{\omega_{N-1}}{\alpha_0^{N-1}} + \|u\|^N. \tag{4.24}$$

Since

$$\rho^{N'} = \Big(\frac{\rho^N - \|u\|^N}{1-\|v\|^N}\Big)^{\frac{1}{N-1}},$$

we deduce from (4.24) that

$$\rho^{N'} < \left(\frac{\frac{\omega_{N-1}}{\alpha_0^{N-1}}}{1-\|v\|^N}\right)^{1/(N-1)}. \tag{4.25}$$

On the one hand, we have

$$\int_B |f(x,u_n)|^q dx < C.$$

Indeed, for $\epsilon > 0$,

$$\begin{aligned}\int_B |f(x,u_n)|^q dx &= \int_{\{|u_n|\leq t_\epsilon\}} |f(x,u_n)|^q\, dx + \int_{\{|u_n|>t_\epsilon\}} |f(x,u_n)|^q\, dx \\ &\leq \omega_{N-1} \max_{B\times[-t_\epsilon,t_\epsilon]} |f(x,t)|^q + C\int_B \exp\left(Ne^{\alpha_0(1+\epsilon)|u_n|^{N'}}\right) dx \\ &\leq C_\epsilon + C\int_B \exp\left(Ne^{\alpha_0(1+\epsilon)\|u_n\|^{N'}|v_n|^{N'}}\right) dx \leq C\end{aligned}$$

provided $\alpha_0(1+\epsilon)\|u_n\|^{N'} \leq p\omega_{N-1}^{1/(N-1)}$, with $1 < p < (1-\|v\|^N)^{-1/(N-1)}$.

From (4.25), there exist $\delta \in (0,\frac{1}{2})$ such that

$$\rho^{N'} = (1-2\delta)\left(\frac{\omega_{N-1}}{\alpha_0^{N-1}(1-\|v\|^N)}\right)^{1/(N-1)}.$$

Since $\lim_{n\to+\infty} \|u_n\|^{N'} = \rho^{N'}$, then for n large enough,

$$\alpha_0(1+\epsilon)\|u_n\|^{N'} \leq (1+\epsilon)(1-\delta)\omega_{N-1}^{1/(N-1)}\left(\frac{1}{1-\|v\|^N}\right)^{1/(N-1)}.$$

We choose $\epsilon > 0$ small enough such that $(1+\epsilon)(1-\delta) < 1$ to have

$$\alpha_0(1+\epsilon)\|u_n\|^{N'} < \omega_{N-1}^{1/(N-1)}\left(\frac{1}{1-\|v\|^N}\right)^{1/(N-1)}.$$

So, the sequence $(f(x,u_n))$ is bounded in L^q, $q > 1$. Using the Hölder inequality, we deduce that

$$\left|\int_B f(x,u_n)(u_n-u)dx\right| \leq \left(\int_B |f(x,u_n)|^q\, dx\right)^{1/q}\left(\int_B |u_n-u|^{q'}\right)^{1/q'} dx$$

$$\leq C \left(\int_B |u_n - u|^{q'} \right)^{1/q'} dx \to 0 \quad \text{as } n \to +\infty,$$

where $1/q + 1/q' = 1$. Since $\mathcal{J}'(u_n)(u_n - u) = o_n(1)$, it follows that

$$\int_B \sigma(x) |\nabla u_n|^{N-2} \nabla u_n \cdot (\nabla u_n - \nabla u)\, dx \to 0.$$

On the other hand,

$$\int_B \sigma(x) |\nabla u_n|^{N-2} \nabla u_n \cdot (\nabla u_n - \nabla u)\, dx = \|u_n\|^N - \int_B \sigma(x) |\nabla u_n|^{N-2} \nabla u_n \cdot \nabla u\, dx.$$

Passing to the limit in the last equality, we get

$$\rho^N - \|u\|^N = 0.$$

Therefore $\|u\| = \rho$, which is in contradiction with (4.22). Therefore, $\mathcal{J}(u) = C_M$. As a consequence, again by the Bresis-Lieb lemma, $u_n \to u$ in E. We also have by (4.18), $\mathcal{J}'(u) = 0$. The proof of Theorem 1.3 is complete. □

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Вагові еліптичні рівняння розмірності N із субкритичними і критичними подвійними експоненціальними нелінійностями

Imed Abid and Rached Jaidane

У роботі доведено існування нетривіального розв'язку для такої вагової задачі без умови Амбросетті–Рабіновіца: $-\operatorname{div}(\sigma(x)|\nabla u|^{N-2}\nabla u) = f(x,u)$ і $u > 0$ в B, $u = 0$ на ∂B, де B є одиничною кулею в $\mathbb{R}^N$, $\sigma(x) = \left(\log\left(\frac{e}{|x|}\right)\right)^{N-1}$ є сингулярною логарифмічною вагою у вкладенні Трудінґера–Мозера. Нелінійність дає критичне або субкритичне зростання відносно нерівності Трудінґера–Мозера. Ми скористалися мінімакс технікою в комбінації з нерівністю Трудінґера–Мозера, щоб довести існування розв'язку. Ми запровадили нову умову для зростання та наполягаємо на її важливості для позбавлення рівня компактності.

Ключові слова: нерівність Трудінґера–Мозера, нелінійність подвійного експоненціального зростання, критичні експоненти, рівень компактності