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Korobov's Controllability Function as Motion Time: Extension of the Solution Set of the Synthesis Problem

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Dedicated to my teacher Professor Valerii Ivanovich Korobov at V.N. Karazin Kharkiv National University in recognition of his outstanding contributions to Control Theory

An extension of the solution set of the finite-time stabilization problem by bounded feedback controls, which is also called the synthesis problem for the canonical system via Korobov's nonunique controllability function, is given. We consider the case when the value of the controllability functions at the initial point is exactly the time of motion from the initial point to the origin. The family of positional controls resolving the synthesis problem is given in terms of a certain real parameter. We enlarge the interval of the parameters and explicitly compute its endpoints as functions of the dimension n of the given system.

Key words: synthesis problem, finite-time stabilization, bounded control, controllability function, canonical system

Mathematical Subject Classification 2020: 93D15, 34D20, 34D05, 34H05

1. Introduction

Consider the control system $\dot{x} = f(x, u)$ with a Lipschitz function f on a certain domain of $\mathbb{R}^n \times \mathbb{R}^r$ and a bounded control $u \in \Omega$, where $\Omega \subset \mathbb{R}^r$ is a compact set containing the origin. Given an initial position $x^0 \in \mathbb{R}^n$, the problem of finding positional controls $u = u(x)$ with values in Ω such that the trajectory $x(t)$ of the system

$$\dot{x} = f(x, u(x)) \tag{1.1}$$

starting at x^0 satisfies the equality $\lim\limits_{t \to T(x^0)} x(t) = 0$. Here $T(x^0)$ is a finite number. This problem is known as a *synthesis problem*.

The synthesis problem or the finite-time stabilization problem by bounded feedback controls is related to finite-time stability. In [14, 15, 21, 43], the finite-time stability of solutions of ordinary differential equations without control is studied. To the best of the author's knowledge, in [22] the synthesis problem was studied for the first time. For further references on the finite-time stabilization problem see [1, 14, 34, 37, 39].

For solving the synthesis problem, in [23] the controllability function method is proposed, which consists of the construction of positional controls $u(x, \Theta(x))$, where $\Theta(x)$ is a Lyapunov-type function. The controllability function differs from the classical Lyapunov function [31–33] in the following. First, the controllability function stabilizes the solution of a given controllable system in finite time instead of infinite time. Second, the controllability function can be applied to the nonequilibrium point or to an unstable equilibrium point, while the Lyapunov function deals with equilibrium points, see [29]. The controllability function has been used to solve a class of control problems, see [5,8,11,13,25–27] and references therein.

The controllability function $\Theta(x)$ satisfies the following inequality:

$$\dot{\Theta} \leq -\beta \Theta^{1-\frac{1}{\alpha}},\ \alpha > 0,\ \beta > 0. \tag{1.2}$$

The motion time from the initial point x^0 to the origin is estimated by the inequality $T \leq \frac{\alpha}{\beta}\Theta^{1/\alpha}(x^0)$. See also [25, 28, 29]. In [25], instead of (1.2), a more general condition is proposed: $\dot{\Theta} \leq -\phi(\Theta(x(t)))$ with $\phi(\Theta) > 0$ at $\Theta \neq 0$ and $\int_a^\Theta \frac{d\Theta}{\phi(\Theta)} < \infty$, $(a > 0)$.

In [28], for the first time the case

$$\dot{\Theta} = -1 \tag{1.3}$$

was studied for an arbitrary linear control system. With the controllability function $\Theta(x)$, the unique positive solution to the equation

$$2a_0\Theta = (x, K(\Theta)x) \tag{1.4}$$

was used, where $(\cdot,\cdot)$ is a scalar product, $K(\Theta)$ is a positive matrix for $\Theta > 0$, and a_0 a positive number. This equation we will refer as the Korobov equation. The physical meaning of (1.3) is remarkable as for an initial point x^0 belonging to a certain neighborhood of the origin, the solution Θ^0 to (1.4) for $x = x^0$ exactly represents the motion time from x^0 to the origin.

Roughly speaking, the controllability function as the motion time can be understood as a nonoptimal solution of the time-optimal control problem, which consists of finding a bounded control $u = u(t)$, $u \in \Omega$ such that the trajectory of the system

$$\dot{x} = f(x, u(t)) \tag{1.5}$$

starting at an initial state x^0 terminates at another state x^1 in minimum time. The time-optimal control problem can be solved with the help of the Bellman equation [2], [3]:

$$\min_{u\in\Omega} \sum_{i=1}^{n} \frac{\partial T(x)}{\partial x_i} f_i(x, u) = -1. \tag{1.6}$$

Here $T(x)$ represents the motion time from x to x^1 along the trajectory of system (1.1). Let $\dot{T}(x)|_{(1.1)}$ and $\dot{T}(x)|_{(1.5)}$ be the time derivatives of $T(x)$ along the trajectories of systems (1.1) and (1.5). Thus, the Bellman equation (1.6) can be written as $\min_{u\in\Omega} \dot{T}(x)|_{(1.5)} = -1$. On the other hand, equality (1.6) implies that $\dot{T}(x)|_{(1.1)} = -1$. The latter equality is valid in the points where the function $T(x)$ is differentiable.

In [3, p. 47], considering the time optimal control for the two-dimensional canonical system, the author explains that the function $T(x)$ is not differentiable on the switching curve $x_2 = \frac{1}{2}|x_1|x_1$. This phenomenon is caused by the bang-bang property of optimal control. A similar behavior occurs for time optimal control of linear systems of higher dimensions.

In [10], for the canonical control system

$$\dot{x}_1 = u, \quad \dot{x}_j = x_{j-1}, \quad j = 2, \cdots, n, \quad |u| \le d, \tag{1.7}$$

a family of positional bounded controls $u = u(x, a_{n,n})$ that depend on a certain real negative parameter $a_{n,n}$ was proposed. The parameter $a_{n,n}$ was chosen in such a way that the matrices

$$C_n(z) := \begin{pmatrix} z & \frac{1}{2\cdot 3} & \cdots & \frac{1}{n(n+1)} \\ \frac{1}{2\cdot 3} & \frac{1}{3\cdot 4} & \cdots & \frac{1}{(n+1)(n+2)} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{n(n+1)} & \frac{1}{(n+1)(n+2)} & \cdots & \frac{1}{(2n-1)2n} \end{pmatrix} \tag{1.8}$$

and

$$C_{n,1}(z) := \begin{pmatrix} 2z & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{pmatrix} \tag{1.9}$$

for $z = \xi_{2,n}a_{n,n} + \xi_{3,n}$, are both positive definite. Here the numbers $\xi_{2,n}$ and $\xi_{3,n}$ are quantities depending on n, see (2.8). The positiveness of C_n and $C_{n,1}$ guarantees a uniqueness of the solution $\Theta(x)$ of equation (1.4). In the sequel, we will use the notation $C_n(a_{n,n})$ and $C_{n,1}(a_{n,n})$ instead of $C_n(z)$ and $C_{n,1}(z)$.

In [10], it is proved that for $a_{n,n} \in (-\infty, \widehat{a}^1_{n,n})$ the positiveness of $C_n(a_{n,n})$ and $C_{n,1}(a_{n,n})$ is satisfied. The number $\widehat{a}^1_{n,n}$ is the root of the equation $\det(C_{n,1}(a_{n,n}) = 0$. In [10], the negative real value of $\widehat{a}^1_{n,n}$ is not given.

The construction of the controllability function resembles the construction of the Lyapunov function by using a quadratic form. The matrix associated with this quadratic form is a positive definite matrix. That is the reason why the matrix $C_n(a_{n,n})$, related to equation (1.4), in order to determine the controllability function $\Theta(x)$, should also be positive definite. On the other hand, the matrix $C_{n,1}(a_{n,n})$ is related to the calculation of the derivative of the function $\Theta(x)$ with respect to time. In the frame of Korobov's approach, the positive definiteness of the matrix $C_{n,1}(a_{n,n})$ is not imperative. The latter matrix is related to the quadratic form of the matrix $\frac{1}{\Theta}K - \frac{d}{d\Theta}K$, which is the coefficient at $\dot{\Theta}$, as taking the derivative with respect to time by virtue of system (1.7) of equation (1.4). In previous works on the controllability function, for example [10, p. 212], [27, Eq. (3.4)], the quantity $\dot{\Theta}$ was expressed as a quotient of two quadratic forms. The quadratic form of the matrix $\dfrac{1}{\Theta}K - \dfrac{d}{d\Theta}K$ appeared in the denominator of the mentioned quotient. In the present work, for some states $x \neq 0$, the latter quadratic form may vanish. Consequently, we can not express $\dot{\Theta}$ as a quotient. This fact explains the role of the matrix $\dfrac{1}{\Theta}K - \dfrac{d}{d\Theta}K$.

We mainly consider system (1.7) for three dimensions or higher. The two-dimensional case was studied in [7]. We extend the solution set of the synthesis problem to the case when the motion time coincides with the value of the controllability functions at the initial position x^0. This extension is achieved by admitting that the matrix $C_{n,1}(a_{n,n})$ is not necessarily positive definite, while the condition of positive definiteness of the matrix $C_n(a_{n,n})$ is maintained. Thus, in terms of the parameter $a_{n,n}$, the interval $(-\infty, \widehat{a}^1_{n,n})$ is enlarged to the interval $(-\infty, \widehat{a}_{n,n})$, where $\widehat{a}_{n,n}$ is the unique root of $\det C_n(a_{n,n}) = 0$ and $\widehat{a}^1_{n,n} < \widehat{a}_{n,n}$. Notice that the interval $(-\infty, \widehat{a}_{n,n})$ cannot be enlarged because the matrix C_n (equivalently, the matrix $K(\Theta)$) should be a positive definite matrix.

On other hand, for the n-dimensional system (1.7), the aforementioned extension means that for certain initial states x^0 there are up to $2n-1$ number of controllability functions Θ and the same number of positional controls $u(x, a_{n,n})$ for both fixed numbers $a_{n,n}$ and a_0. For the third and higher dimensions, this fact essentially differs from those of previous works on the method of the controllability function, where a unique positive controllability function $\Theta(x)$ was used [10]. In the two-dimensional case, non-unique solutions of equation (1.4) were considered in [7] and in [30]. In the latter work, the case when the end point is a non-equilibrium point was studied.

The main results of this work are described below.

a) The roots $\widehat{a}_{n,n}$ and $\widehat{a}^1_{n,n}$ of the equations $\det(C_{n,1}(a_{n,n})) = 0$ and $\det(C_{n,1}(a_{n,n})) = 0$ are precisely calculated.

b) With the help of the inverse of the Hilbert matrix, we propose an explicit form of the matrix $K(\Theta)$ which appears in (1.4). This result is relevant because equation (1.4) determines the controllability function $\Theta(x)$.

c) For $a_{n,n} \in [\widehat{a}^1_{n,n}, \widehat{a}_{n,n})$, we analyze the properties of the matrix $\dfrac{1}{\Theta}K - \dfrac{d}{d\Theta}K$. We prove that the set of points in $\mathbb{R}^n$, where the quadratic form of the matrix $\dfrac{1}{\Theta}K - \dfrac{d}{d\Theta}K$ vanishes, does not contain trajectories from x^0 to the origin.

d) We prove that for a fixed parameter $a_{n,n}$ belonging to the interval $[\widehat{a}^1_{n,n}, \widehat{a}_{n,n})$ and the certain fixed positive number a_0 appearing in (1.4), there are at most $2n-1$ different motion times from the given initial position x_0 to the origin. This means that there are at most $2n-1$ solutions of equation (1.4). Thus, there is the same number of positional controls that solve the synthesis problem. Note that if $a_{n,n} \in (-\infty, \widehat{a}_{n,n})$, then there is a unique positional control and a unique time that resolve the mentioned problem.

The results of this work can be used to solve the synthesis problem for nonlinear control systems with a linear part that is completely controllable as well as for systems which are mappable to canonical systems, see [22, Theorem on p. 552], [8, 13, 24, 27, 42] and references therein. The presented extension of the solution set for system (1.7) also extends the solution set of the mentioned nonlinear control systems. At the same time, the extension of the solution set of the synthesis problem proposed in the present work is a part of the solution set of the admissible control problem for the canonical system. See [9] and [6].

2. Preliminaries and notations

In this section, we recall notations for the case of controllability function as the motion time. See [10]. Let us rewrite the canonical system (1.7) in the following form:

$$\dot{x} = A_n x + b_n u, \quad |u| \leq d, \tag{2.1}$$

where

$$A_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}, \quad b_n = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.2}$$

Now we reproduce some matrices and equations introduced in [10]. Let

$$D_n(\Theta) := \text{diag} \left(\Theta^{-\frac{2i-1}{2}} \right)_{i=1}^{n}, \quad J_n := \text{diag} \left((-1)^{j-1} \frac{1}{(j-1)!} \right)_{j=1}^{n} \tag{2.3}$$

and

$$\widetilde{P}_n := \begin{pmatrix} -1 & \frac{1}{3\cdot 4} & \dots & \frac{1}{n(n+1)} \\ 0 & \frac{1}{4\cdot 5} & \dots & \frac{1}{(n+1)(n+2)} \\ \vdots & \vdots & \dots & \vdots \\ 0 & \frac{1}{(n+1)(n+2)} & \dots & \frac{1}{(2n-2)(2n-1)} \end{pmatrix}. \tag{2.4}$$

Let also

$$\Delta_n := \det \ \widetilde{P}_n. \tag{2.5}$$

Furthermore, we will use the following notation.

Notation A. *The superscript* T *means the transpose of the matrix. Let*

$$\begin{aligned} d'_n &=: \left(-\frac{1}{(n+1)(n+2)}, -\frac{1}{(n+2)(n+3)}, \dots, -\frac{1}{(2n-1)2n} \right)^{\mathsf{T}}, \\ d''_n &=: \left(-\frac{3+a_{n,1}}{2\cdot 3}, -\frac{a_{n,1}}{3\cdot 4}, \dots, -\frac{a_{n,1}}{n(n+1)} \right)^{\mathsf{T}} \end{aligned} \tag{2.6}$$

see [10, p. 218]. *We denote by* $\Delta'_{n,j}$, $\Delta''_{n,j}$ $(j = 1, \dots, n-1)$ *(see* [10, p. 218]*) the determinants of the matrix* (2.4)*, where instead of its* j*-th column the columns* d'_n *and* d''_n *are inserted. Here (see* [10, Eq. (2.6)]*)*

$$a_{n,1} := -\frac{n(n+1)}{2}. \tag{2.7}$$

Let $\Delta'_{n,1}$, $\Delta''_{n,1}$ be as in Notation A and we denote

$$\xi_{2,n} := \frac{(-1)^{n-1}}{(n-1)!}\frac{\Delta'_{n,1}}{\Delta_n}, \quad \xi_{3,n} := \frac{\Delta''_{n,1}}{\Delta_n}. \tag{2.8}$$

In [10], with the help of (1.8), (2.3) and (2.8), for $a_{n,n} \in (-\infty, \widehat{a}^1_{n,n})$, the matrix

$$F_n(a_{n,n}) := (J_n C_n(a_{n,n}) J_n)^{-1} \tag{2.9}$$

was introduced. Note that the number $\widehat{a}^1_{n,n}$ was not explicitly given in [10]. The controllability function $\Theta(x)$, with $\Theta(0) = 0$, was proposed as the unique solution of the following equation:

$$2a_0\Theta = (K_n(\Theta, a_{n,n})x, x), \tag{2.10}$$

where

$$K_n(\Theta, a_{n,n}) := D_n(\Theta)F_n(a_{n,n})D_n(\Theta). \tag{2.11}$$

In the sequel, we will omit the dependence of $K_n(\Theta, a_{n,n})$ on $a_{n,n}$ and sometimes on Θ.

The quantity a_0 satisfies the inequalities

$$0 < a_0 \le \frac{d^2}{2(F_n^{-1}a_n, a_n)} \tag{2.12}$$

with

$$a_n := (a_{n,1}, a_{n,2}, \dots, a_{n,n})^{\mathsf{T}} \tag{2.13}$$

and

$$a_{n,j} := (-1)^j \frac{(j-1)!}{\Delta_n}\left(\Delta'_{n,j}\frac{(-1)^{n-1}}{(n-1)!}a_{n,n} + \Delta''_{n,j}\right), \quad j = 2, \dots, n-1. \tag{2.14}$$

The set of positional controls is given in the form

$$u(x, a_{n,n}) = \Theta^{-\frac{1}{2}}(x)a_n^{\mathsf{T}} D_n(\Theta(x))x = \sum_{j=1}^{n} \frac{a_{n,j}}{\Theta^j(x)}x_j. \tag{2.15}$$

In the following remark, we recall the relation between matrices (1.8) and (1.9).

Remark 2.1.

a) Equality (1.3) is verified [10] by taking the time derivative of equality (2.10) with positional control (2.15), $u = u(x, a_{n,n})$:

$$\left(x, \left(\frac{1}{\Theta}K_n - \frac{d}{d\Theta}K_n\right)x\right)\dot{\Theta} = (K_n(A_n x + b_n u), x) + (x, K_n(A_n x + b_n u)). \tag{2.16}$$

Let

$$H_n := \operatorname{diag}\left(-\frac{2i-1}{2}\right)_{i=1}^{n}. \tag{2.17}$$

Using (2.11) and (2.15), equality (2.16) can be written as

$$x^{\mathsf{T}} D_n(\Theta)\left(F_n - H_n F_n - F_n H_n\right) D_n(\Theta) x\, \dot{\Theta} = x^{\mathsf{T}} D_n(\Theta)\left(F_n A_n + A_n^{\mathsf{T}} F_n + F_n b_n a_n^{\mathsf{T}} + a_n b_n^{\mathsf{T}} F_n\right) D_n(\Theta) x. \quad (2.18)$$

b) Since (2.18) holds for all $x \in \mathbb{R}^n$, the following matrix equality is valid:

$$\left(F_n - H_n F_n - F_n H_n\right)\dot{\Theta} = F_n A_n + A_n^{\mathsf{T}} F_n + F_n b_n a_n^{\mathsf{T}} + a_n b_n^{\mathsf{T}} F_n. \quad (2.19)$$

This matrix equation is valid for $\dot{\Theta} = -1$ and does not depend on a state x.

c) Matrices (1.8) and (1.9) satisfy the equality

$$C_{n,1}(a_{n,n}) = C_n(a_{n,n}) - C_n(a_{n,n})H_n - H_n C_n(a_{n,n}), \quad (2.20)$$

the matrix $C_{n,1}(a_{n,n})$ is a factor of a decomposition of the matrix

$$\frac{1}{\Theta}K_n - \frac{d}{d\Theta}K_n = \frac{1}{\Theta}D_n(\Theta)F_n J_n C_{n,1} J_n F_n D_n(\Theta). \quad (2.21)$$

For the readers convenience, we paraphrase [10, Theorem 3.1] in terms of the notation used in the present work. For the positive parameter c_{2n-2} used in [10, Theorem 3.1], we set $c_{2n-2} = \frac{1}{2n(2n-1)}$.

Theorem 2.2. *Let A_n, b_n, C_n, Δ_n, $\Delta'_{n,1}$, $\Delta''_{n,1}$, and $u(x, a_{n,n})$ be as in (2.2), (1.8), (2.5), Notation A, and (2.15), respectively. Moreover, let*

$$\frac{\Delta'_{n,1}}{\Delta_n}\frac{(-1)^{n-1}a_{n,n}}{(n-1)!} + \frac{\Delta''_{n,1}}{\Delta_n} > \max\left\{\xi_0, \left(\frac{1}{2} + \frac{1}{2n}\right)\xi_0 + \frac{1}{4} - \frac{1}{4n}\right\}, \quad (2.22)$$

where ξ_0 is the root of the equation $\det(C_n(\xi)) = 0$. The number a_0 satisfies the condition (2.12) and the controllability function $\Theta(x)$ for $x \neq 0$ is defined by equality (2.10) and $\Theta(0) = 0$.

Thus, the control $u(x, a_{n,n})$ (2.15) transfers an arbitrary $x \in \mathbb{R}^n$ to the origin along the trajectory $\dot{x} = A_n x + b_n u(x, a_{n,n})$ in time $T(x) = \Theta(x)$ and satisfies the restriction $|u(x, a_{n,n})| \leq d$.

In Remark 3.9, we will see that in the right-hand side of inequality (2.22) it is sufficient to indicate one number. In other words, we will observe that Theorem 2.2 is suitable for $a_{n,n} \in (\infty, \widehat{a}^1_{n,n})$.

3. The interval of extension for the parameter $a_{n,n}$

In this section, we calculate the value of the parameter $a_{n,n}$ for which the determinant of the matrix $C_n(a_{n,n})$ appearing in (1.8) (respectively, $C_{n,1}(a_{n,n})$ and in (1.9)) is equal to zero. With these two values we form an interval in terms of the parameter $a_{n,n}$. This interval performs the extension of the solution set of the synthesis problem. For the calculation of the mentioned determinants, we crucially use the determinant of the Hilbert matrix from [19]. Let us introduce notations and assertions that will be relevant in the sequel.

Notation B. *Let $n \in \mathbb{N}$ and ℓ be a nonnegative integer number. Denote*

$$\mathcal{H}_{n,\ell} := \left(\frac{1}{k+i+\ell-1}\right)_{k,i=1}^{n}, \tag{3.1}$$

$$h_{[j,k]} := \left(\frac{1}{j}, \frac{1}{j+1}, \dots, \frac{1}{k}\right)^{\mathsf{T}}, \; j \le k, \tag{3.2}$$

$$\mathcal{K}_{n,\ell} := \left(\frac{1}{(i+j+\ell-1)(i+j+\ell)}\right)_{i,j=1}^{n}, \tag{3.3}$$

$$k_{[i,j]} = \left(\frac{1}{i(i+1)}, \frac{1}{(i+1)(i+2)}, \dots, \frac{1}{j(j+1)}\right)^{\mathsf{T}}, \; i \le j. \tag{3.4}$$

The matrix $\mathcal{H}_{n,0}$ is the well-known Hilbert matrix. The determinant of this matrix can be written as

$$|\mathcal{H}_{n,0}| = \frac{(c_n)^4}{c_{2n}}, \tag{3.5}$$

where $c_k = \prod_{j=1}^{k-1} j!$, see [4, 20].

In the following lemma, we will concentrate on the calculation of the determinant for certain submatrices of C_n and $C_{n,1}$ defined in (1.8) and (1.9). Our calculations will be given in terms of the value of the determinant of the Hilbert matrix. The equalities of the lemma below will be used in the sequel. In particular, with the help of these equalities we will compute the values of Δ_n, $\Delta'_{n,1}$ and $\Delta''_{n,1}$ defined in Notation A.

Lemma 3.1. *Let $\mathcal{H}_{n,\ell}$ and $\mathcal{K}_{n,\ell}$ be as in (3.1) and (3.3). Thus, the following equalities are valid:*

$$|\mathcal{H}_{n-1,0}| = \frac{(2n-1)!(2n-2)!}{((n-1)!)^4}|\mathcal{H}_{n,0}|, \tag{3.6}$$

$$|\mathcal{H}_{n-1,2}| = n^2|\mathcal{H}_{n,0}|, \tag{3.7}$$

$$|\mathcal{H}_{n-2,3}| = \frac{n^2(2n-1)!}{2((n-2)!)^2}|\mathcal{H}_{n,0}|, \tag{3.8}$$

$$|\mathcal{H}_{n-1,1}| = \frac{(2n-1)!}{((n-1)!)^2}|\mathcal{H}_{n,0}|, \tag{3.9}$$

$$|\mathcal{H}_{n-2,2}| = \frac{(2n-1)!}{((n-1)!)^2}\frac{(2n-2)!}{((n-2)!)^2}|\mathcal{H}_{n,0}|, \tag{3.10}$$

$$|\mathcal{K}_{n,0}| = \frac{(n!)^2}{(2n)!}|\mathcal{H}_{n,0}|, \tag{3.11}$$

$$|\mathcal{K}_{n-1,1}| = n|\mathcal{H}_{n,0}|, \tag{3.12}$$

$$|\mathcal{K}_{n-1,2}| = \frac{n!(n+1)!}{2(2n-1)!}|\mathcal{H}_{n,0}|, \tag{3.13}$$

$$|\mathcal{K}_{n-2,3}| = \frac{(n^2-1)n^3}{2}|\mathcal{H}_{n,0}|. \tag{3.14}$$

Proof. Equality (3.6) readily follows from (3.5). To prove equality (3.7), one uses the representation of the matrix $\mathcal{H}_{n,0}$ (resp. matrix $\mathcal{H}_{n-1,2}$) as a Cauchy matrix $C := \left(\frac{1}{x_j+y_k}\right)_{j,k=1}^{n}$ with $x_j = j$ and $y_k = k-1$ (resp. $x_j = j+1$ and $y_k = k$). The determinant of the matrix C is equal to

$$|C| = \frac{\prod\limits_{1\le i<j\le n} (x_i - x_j)(y_j - y_i)}{\prod\limits_{1\le i,j\le n} (x_i + y_j)}, \tag{3.15}$$

see [4, p. 306] or [38, p. 92]. By employing (3.15), for $x_j = j$ and $y_k = k-1$, the expression $\frac{|\mathcal{H}_{n,0}|}{|\mathcal{H}_{n-1,2}|}$ is equal to $\frac{\prod_{j=2}^{n}(x_1-x_j)(y_1-y_j)}{\prod_{j=2}^{n}(x_1+y_j)(y_1+x_j)}$, which in turn is equal to $\frac{1}{n^2}$. Thus, equality (3.7) is proven. In a similar manner, one can prove (3.8)–(3.10). To prove (3.11) for $1 \le i \le n-1$, add to each i-th row the rows from n-row to $i-1$, and extract $(n-i+1)$ from every i-th row. From each j-th column extract the value $1/(2n+j-1)$. Thus, equality (3.11) readily follows. Now we prove equality (3.13). For $i = 1$ to $n-1$, from each row i of the determinant $|\mathcal{K}_{n-1,2}|$ subtract the sum of the rows from $i-1$ to n. We attain that $|\mathcal{K}_{n-1,2}| = \frac{(n-1)!(n+1)!}{(2n)!}|\mathcal{H}_{n-1,2}|$. It remains to use (3.7). In the same manner, equalities (3.12) and (3.14) are proven. Here one uses the fact that $|\mathcal{K}_{n-2,3}| = \frac{(n+1)!(n-2)!}{(2n-1)!}|\mathcal{H}_{n-2,3}|$. □

We recall the definition of the Schur complement first introduced by E.V. Haynsworth [18].

Definition 3.2. Let P, Q, W and R be $\ell\times\ell$, $\ell\times(m-\ell)$, $(m-\ell)\times\ell$ and $(m-\ell)\times(m-\ell)$, respectively. Additionally, let

$$M = \begin{pmatrix} P & Q \\ W & R \end{pmatrix}.$$

If R is invertible, then $P - WR^{-1}Q$ is the Schur complement of R in M.
If P is invertible, then $R - WP^{-1}Q$ is the Schur complement of P in M.

The following remark is to be used in the sequel.

Remark 3.3. [41, p. 217], [35, p. 188] Under the conditions of Definition 3.2, the following equality holds:

$$\det \begin{pmatrix} P & Q \\ W & R \end{pmatrix} = \det(P - WR^{-1}Q)\det R. \tag{3.16}$$

The following remark is readily proved by using the determinant of the Schur complement (3.16), equalities (3.11) and (3.13).

Remark 3.4. Let $\mathcal{K}_{n,\ell}$ and $k_{[j,n]}$ be as in (3.3) and (3.4). Thus, the following equalities are valid:

$$k_{[2,n]}^{\mathsf{T}}\mathcal{K}_{n-1,2}^{-1}k_{[2,n]} = \frac{1}{2} - \frac{1}{n(n+1)}. \tag{3.17}$$

The next proposition allows us to express the determinants Δ_n, $\Delta'_{n,1}$ and $\Delta''_{n,1}$ in terms of the determinant $|\mathcal{H}_{n,0}|$.

Proposition 3.5. *Let Δ_n, $\Delta'_{n,1}$ and $\Delta''_{n,1}$ be defined as in (2.5) and Notation A. Furthermore, let $\mathcal{H}_{n,0}$ denote the Hilbert matrix (3.1). The following equalities then hold:*

$$\Delta_n = -\frac{(n^2-1)n^3}{2}|\mathcal{H}_{n,0}|, \tag{3.18}$$

$$\Delta'_{n,1} = \frac{(-1)^{n+1}(n-1)!(n+1)!n^2}{(2n)!}|\mathcal{H}_{n,0}|, \tag{3.19}$$

$$\Delta''_{n,1} = -\frac{(n-2)n^2(n+1)^2}{4}|\mathcal{H}_{n,0}|. \tag{3.20}$$

Proof. Equality (3.18) follows directly from (2.4) and (3.14). To prove (3.19), we extract the minus sign from the first column and move it to the n-th column. We also use the obvious equality $\Delta'_{n,1} = (-1)^{n+1}|\mathcal{K}_{n-1,2}|$ and equality (3.13). Now we prove (3.20). We write (2.6) d''_n as

$$d''_n = -a_{n,1}\left(\frac{1}{2\cdot 3}, \frac{1}{3\cdot 4}, \ldots, \frac{1}{n(n+1)}\right)^{\mathsf{T}} - \left(\frac{1}{2}, 0, \ldots, 0\right)^{\mathsf{T}}.$$

Consequently,

$$\Delta''_{n,1} = -\,a_{n,1}|\mathcal{K}_{n-1,1}| - \frac{1}{2}|\mathcal{K}_{n-2,3}|.$$

Thus, (2.7), (3.12), and (3.14) imply (3.20). □

In the next lemma, we calculate the value of the $(1,1)$ entry of the matrix C_n by replacing the parameter $a_{n,n}$ by the numbers $\widehat{a}_{n,n}$ and $\widehat{a}^1_{n,n}$, which are introduced.

Lemma 3.6. *Let the quantities $\xi_{2,n}$ and $\xi_{3,n}$ be as in (2.8). Furthermore, let*

$$\widehat{a}_{n,n} := -\frac{(2n)!}{(n+1)!}, \tag{3.21}$$

$$\widehat{a}^1_{n,n} := -\frac{(n+1)(2n)!}{4nn!}. \tag{3.22}$$

Thus,

$$\widehat{a}_{n,n} > \widehat{a}^1_{n,n} \tag{3.23}$$

and the following equalities hold:

$$\xi_{2,n}\widehat{a}_{n,n} + \xi_{3,n} = \frac{1}{n+1} - \frac{1}{n} + \frac{1}{2}, \tag{3.24}$$

$$\xi_{2,n}\widehat{a}^1_{n,n} + \xi_{3,n} = \frac{1}{2}\left(1 - \frac{1}{n^2}\right). \tag{3.25}$$

Proof. Inequality (3.23) is obvious. Note that from (3.18)–(3.20), the next equalities hold:

$$\frac{\Delta'_{n,1}}{\Delta_n} =(-1)^n 2\frac{(n-2)!(n-1)!}{(2n)!}, \tag{3.26}$$

$$\frac{\Delta''_{n,1}}{\Delta_n} =\frac{(n-2)(n+1)}{2(n-1)n}. \tag{3.27}$$

We prove (3.24). By employing (2.8), (3.21), (3.26), and (3.27), we attain

$$\begin{aligned}\xi_{2,n}\widehat{a}_{n,n}+\xi_{3,n} &= \frac{\Delta'_{n,1}}{\Delta_n}\frac{(-1)^{n-1}\widehat{a}_{n,n}}{(n-1)!}+\frac{\Delta''_{n,1}}{\Delta_n}\\ &= -\left(2\frac{(n-2)!}{(2n)!}\widehat{a}_{n,n}-\frac{(n+1)(n-2)}{2(n-1)n}\right)\\ &= \frac{1}{n+1}-\frac{1}{n}+\frac{1}{2}.\end{aligned}$$

Equality (3.25) is proved in a similar manner. □

Remark 3.7. Let $a_{n,n}\in(-\infty,\widehat{a}_{n,n})$ and $C_n(a_{n,n})$ be as in (1.8). By (2.8), (3.26), and (3.27), the entry $(1,1)$ $c_{11}=\xi_{2,n}a_{n,n}+\xi_{3,n}$ of the matrix C_n can be written as

$$C_n(a_{n,n})=\begin{pmatrix} c_{11} & k^{\mathsf{T}}_{[2,n]}\\ k_{[2,n]} & \mathcal{K}_{n-1,2}\end{pmatrix}, \tag{3.28}$$

where

$$c_{11}=-\left(2\frac{(n-2)!}{(2n)!}a_{n,n}-\frac{(n+1)(n-2)}{2(n-1)n}\right). \tag{3.29}$$

We now state the main theorem of this section. We prove that the numbers $\widehat{a}_{n,n}$ and $\widehat{a}^1_{n,n}$ introduced in (3.21) and (3.22), are the ones, where the determinants of the matrices C_n and $C_{n,1}$ are 0.

Theorem 3.8. *Let C_n and $C_{n,1}$ be as in (1.8) and (1.9). Furthermore, let $\widehat{a}_{n,n}$ and $\widehat{a}^1_{n,n}$ be as in (3.21) and (3.22). Thus, the following assertions are valid:*

a) *The equalities $\det C_{n,1}(\widehat{a}^1_{n,n})=0$ and $\det C_n(\widehat{a}_{n,n})=0$ are satisfied.*

b) *For $a_{n,n}<\widehat{a}_{n,n}$ (respectively, $a_{n,n}<\widehat{a}^1_{n,n}$), matrix (1.8) (respectively, (1.9)) is positive definite.*

Proof. Taking into account (3.25), the first column of the matrix $C_{n,1}(\widehat{a}^1_{n,n})$ can be written as

$$\left(1,\ \frac{1}{2},\ \dots,\ \frac{1}{n}\right)^{\mathsf{T}}-\left(\frac{1}{n^2},\ 0,\ \dots,\ 0\right)^{\mathsf{T}}.$$

Thus, by using (3.7), we have that

$$\det(C_{n,1}(\widehat{a}^1_{n,n}))=|\mathcal{H}_{n,0}|-\frac{1}{n^2}|\mathcal{H}_{n-1,2}|=0.$$

To prove the equality $\det(C_n(\widehat{a}_{n,n})) = 0$, we express the first column of $C_n(\widehat{a}_{n,n})$ as

$$\left(\frac{1}{2}, \frac{1}{2\cdot 3}, \dots, \frac{1}{n(n+1)}\right)^{\mathsf{T}} - \left(\frac{1}{n(n+1)}, 0, \dots, 0\right)^{\mathsf{T}}.$$

Using (3.11) and (3.13), we readily have that

$$\det(C_n(\widehat{a}_{n,n})) = |\mathcal{K}_{n,0}| - \frac{1}{n(n+1)}|\mathcal{K}_{n-1,2}| = 0.$$

Now we prove part b). The fact that the matrix $C_n(a_{n,n})$ is positive for $a_{n,n} < \widehat{a}_{n,n}$ is proved in [10, Lemma 2.4]. In a similar manner, $C_{n,1}(a_{n,n})$ is a positive definite matrix for $a_{n,n} < \widehat{a}^1_{n,n}$, see [10, Lemma 2.5]. □

Remark 3.9. From part b) of Theorem 3.8, it is clear that inequality (2.22) can be replaced by

$$\frac{\Delta'_{n,1}}{\Delta_n}\frac{(-1)^{n-1}a_{n,n}}{(n-1)!} + \frac{\Delta''_{n,1}}{\Delta_n} > 1 - \frac{1}{n^2}.$$

4. Calculation of the matrix K_n via the inverse of the Hilbert matrix

In this section, we calculate the matrix K_n (2.11) that appears in equation (2.10). By considering the form of the matrix K_n in (2.11) and in turn the form of the matrix F_n as in (2.9), we see that we should calculate the inverse of the matrix $C_n(a_{n,n})$ with $a_{n,n} \in (-\infty, \widehat{a}_{n,n})$. Obtaining an explicit expression for the matrix K_n is crucial because, with the help of the solution $\Theta(x)$ of equation (2.10), we construct the positional control (2.15). On the other hand, with an explicit expression of the matrix K_n, we calculate the motion time from x^0 to the origin $T(x^0) = \Theta^0$, which is the solution of (2.10) for x equal to the initial condition x^0.

In the following remark, we reproduce [36] on the inverse of the Hilbert-type matrix defined in (3.1).

Remark 4.1. Let $\mathcal{H}_{n,\ell}$ be as in (3.1). The inverse of this matrix has the form

$$\mathcal{H}^{-1}_{n,\ell} = (\nu^{(\ell)}_{\alpha\beta})^n_{\alpha,\beta=1}, \tag{4.1}$$

where

$$\nu^{(\ell)}_{\alpha\beta} = \frac{\prod\limits_{\substack{j=1\\ j\neq\alpha}}^{n}(j+\beta+\ell-1)}{\prod\limits_{\substack{j=1\\ j\neq\beta}}^{n}(j-\beta)} \, \frac{\prod\limits_{k=1}^{n}(k+\alpha+\ell-1)}{\prod\limits_{\substack{k=1\\ k\neq\alpha}}^{n}(k-\alpha)}.$$

The next remark can be readily verified.

Remark 4.2. Let $\mathcal{H}_{n-1,2}$, $h_{[n+1,2n-2]}$, $\mathcal{K}_{n-1,2}$ and $k_{[n+2,2n-2]}$ be as in (3.1)–(3.4), respectively. Furthermore, let

$$S_n := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix}, \qquad \lambda_n := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{4.2}$$

where S_n is an $n \times n$ matrix and λ_n is an $n \times 1$ matrix. Thus, the following equalities are valid:

$$\mathcal{K}_{n-2,2} = \mathcal{H}_{n-2,2} S_{n-2} + h_{[n+1,2n-2]} \lambda_{n-2}^{\mathsf{T}}, \tag{4.3}$$

$$k_{[n+1,2n-2]}^{\mathsf{T}} = h_{[n+1,2n-2]}^{\mathsf{T}} S_{n-2} + \frac{1}{2n-1} \lambda_{n-2}^{\mathsf{T}}. \tag{4.4}$$

Furthermore, the inverse matrix of S_n is equal to

$$S_n^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}.$$

In the next lemma, by using the inverse (4.1) of the Hilbert-type matrix (3.1), we calculate the inverse of the submatrix $\mathcal{K}_{n-1,2}$, which in turn will be used to compute the inverse of the matrix $C_n(a_{n,n})$, see (1.8). We use the fact that the Schur complement

$$\frac{1}{(2n-1)2n} - k_{[n+1,2n-2]}^{\mathsf{T}} \mathcal{K}_{n-2,2}^{-1} k_{[n+1,2n-1]}$$

of the matrix $\mathcal{K}_{n-1,2}$ is a positive number since $\mathcal{K}_{n-1,2}$ is a positive definite matrix.

We use $0_{p\times q}$ and I_q to denote the $p \times q$ zero matrix and the $q \times q$ identity matrix.

Lemma 4.3. *Let $\mathcal{H}_{n-1,2}$, $h_{[n+1,2n-2]}$, $\mathcal{K}_{n-1,2}$, $k_{[n+1,2n-2]}$, S_n and λ_n be as in (3.1)–(3.4) and (4.2), respectively. Let*

$$m_{n-2} := -\frac{\frac{1}{2n-1} - k_{[n+1,2n-2]}^{\mathsf{T}} \mathcal{K}_{n-2,2}^{-1} h_{[n+1,2n-2]}}{\frac{1}{(2n-1)2n} - k_{[n+1,2n-2]}^{\mathsf{T}} \mathcal{K}_{n-2,2}^{-1} k_{[n+1,2n-2]}},$$

$$d_{n-2} := \mathcal{K}_{n-2,2}^{-1}(h_{[n+1,2n-2]} - m_{n-2} k_{[n+1,2n-2]}).$$

Thus, the inverse of the matrix $\mathcal{K}_{n-1,2}$ is given by

$$\mathcal{K}_{n-1,2}^{-1} = \begin{pmatrix} I_{n-2} & d_{n-2} \\ 0_{1\times n-2} & m_{n-2} \end{pmatrix} S_{n-1}^{-1} \mathcal{H}_{n-1,2}^{-1}. \tag{4.5}$$

Proof. Since $\mathcal{K}_{n-1,2}$ is invertible, the number m_{n-2} is not equal to zero. Equality (4.5) is equivalent to the equality

$$\mathcal{K}_{n-1,2} = \mathcal{H}_{n-1,2} S_{n-1} \begin{pmatrix} I_{n-2} & -\frac{1}{m_{n-2}} d_{n-2} \\ 0_{1\times n-2} & \frac{1}{m_{n-2}} \end{pmatrix}. \tag{4.6}$$

Rewriting the matrices $\mathcal{H}_{n-1,2}$, $\mathcal{K}_{n-1,2}$ in the forms

$$\mathcal{H}_{n-1,2} = \begin{pmatrix} \mathcal{H}_{n-2,2} & h_{[n+1,2n-2]} \\ h^{\mathsf{T}}_{[n+1,2n-2]} & \frac{1}{2n-1} \end{pmatrix}, \quad \mathcal{K}_{n-1,2} = \begin{pmatrix} \mathcal{K}_{n-2,2} & k_{[n+1,2n-2]} \\ k^{\mathsf{T}}_{[n+1,2n-2]} & \frac{1}{(2n-1)2n} \end{pmatrix},$$

and using the equality

$$S_{n-1} = \begin{pmatrix} S_{n-2} & 0_{n-2\times 1} \\ \lambda^{\mathsf{T}}_{n-2} & 1 \end{pmatrix}$$

as well as (4.3) and (4.4), we readily verify (4.6). Consequently, equality (4.5) holds. □

In the following lemma, we calculate the $(1,1)$ entry of the inverse of matrix (1.8). Notice that this entry is equal to the corresponding Schur complement of the matrix $C_n(a_{n,n})$.

Lemma 4.4. *Let the matrix C_n be as in (3.28). Thus, the Schur complement $\widehat{c}_{11}(a_{n,n})$ of $\mathcal{K}_{n-1,2}$ in $C_n(a_{n,n})$ satisfies the equality*

$$\widehat{c}_n(a_{n,n}) := -2\left(\frac{(n-2)!}{(2n)!} a_{n,n} + \frac{1}{n^3-n}\right). \tag{4.7}$$

Proof. By Definition 3.2, the Schur complement of $\mathcal{K}_{n-1,2}$ in $C_n(a_{n,n})$ can be written as follows:

$$\begin{aligned} \widehat{c}_n(a_{n,n}) &= -\left(2\frac{(n-2)!}{(2n)!} a_{n,n} - \frac{(n+1)(n-2)}{2(n-1)n}\right) - k^{\mathsf{T}}_{[2,n]} \mathcal{K}^{-1}_{n-1,2} k_{[2,n]} \\ &= -\left(2\frac{(n-2)!}{(2n)!} a_{n,n} - \frac{(n+1)(n-2)}{2(n-1)n}\right) - \frac{1}{2} + \frac{1}{n(n+1)} \\ &= -2\left(\frac{(n-2)!}{(2n)!} a_{n,n} + \frac{1}{n^3-n}\right). \end{aligned}$$

In the second equality, we used identity (3.17). □

Remark 4.5. [35, Eq. (1.11)] Let M and R as in Definition 3.2 be both nonsingular. Then the Schur complement of R in M

$$T = P - QR^{-1}W$$

is also nonsingular and

$$M^{-1} = \begin{pmatrix} T^{-1} & -T^{-1}QR^{-1} \\ -R^{-1}WT^{-1} & R^{-1} + R^{-1}WT^{-1}QR^{-1} \end{pmatrix}.$$

Now we are ready to write the main result of this section. This result consists of the calculation of the matrix K_n appearing in equation (2.10). As seen from (2.9) and (2.11), the calculation of K_n returns to the calculation of the inverse of the matrix $C_n(a_{n,n})$. Notice that for the matrix K_n we calculate it by employing the inverse of Hilbert type matrices, see (4.1).

Theorem 4.6. *Let $k^{\mathsf{T}}_{[2,n]}$ and $\mathcal{K}^{-1}_{n-1,2}$ be as in (3.4) and (4.5). Furthermore, let $\widehat{a}_{n,n}$ be as in (3.21) and $a_{n,n} \in (-\infty, \widehat{a}_{n,n})$. Let $\widehat{c}_{11}(a_{n,n})$ be as in (4.7). Thus, the following assertions are valid:*

a) *The inverse of the matrix $C_n(a_{n,n})$ defined as in (1.8) is equal to*

$$\frac{1}{\widehat{c}_n(a_{n,n})}\begin{pmatrix} 1 & -k^{\mathsf{T}}_{[2,n]}\mathcal{K}^{-1}_{n-1,2} \\ -\mathcal{K}^{-1}_{n-1,2}k_{[2,n]} & \widehat{c}_n(a_{n,n})\mathcal{K}^{-1}_{n-1,2}+\mathcal{K}^{-1}_{n-1,2}k_{[2,n]}k^{\mathsf{T}}_{[2,n]}\mathcal{K}^{-1}_{n-1,2}\end{pmatrix}. \tag{4.8}$$

b) *Let $D_n(\Theta)$ and J_n be as in (2.3). The matrix $K_n(\Theta, a_{n,n})$ can be written as follows:*

$$K_n(\Theta, a_{n,n}) = D_n(\Theta)J_n^{-1}C_n^{-1}(a_{n,n})J_n^{-1}D_n(\Theta).$$

Proof. Write $C_n(a_{n,n})$ as in (3.28). By using [35, Eq. (1.11)] or [12, p. 1661], the inverse of the matrix $C_n(a_{n,n})$ can be written as in (4.8), where the Schur complement $\widehat{c}_{11}$ of $\mathcal{K}_{n-1,2}$ in $C_n(a_{n,n})$ is given by (4.7). Part b) readily follows from part a), (2.9), and (2.11). □

Let us write the matrix $C_n(a_{n,n})$ in the form

$$\begin{pmatrix} C_n^{n-1} & k_{[n,2n-2]} \\ k^{\mathsf{T}}_{[n,2n-2]} & \frac{1}{(2n-1)2n}\end{pmatrix}, \tag{4.9}$$

where

$$C_n^{n-1}(a_{n,n}) := \begin{pmatrix} c_{11} & k^{\mathsf{T}}_{[2,n-1]} \\ k_{[2,n-1]} & \mathcal{K}_{n-2,2}\end{pmatrix} \tag{4.10}$$

and c_{11} is as in (3.29).

Our next goal is to compute the $(1,1)$, $(1,n)$, $(n,1)$ and (n,n) entries of the inverse of the matrix $C_n(\widehat{a}^1_{n,n})$. Let

$$C_n^{-1}(a_{n,n}) = \begin{pmatrix} t_{11}(a_{n,n}) & \cdots & t_{1n}(a_{n,n}) \\ \vdots & \dots & \vdots \\ t_{1n}(a_{n,n}) & \cdots & t_{nn}(a_{n,n})\end{pmatrix}. \tag{4.11}$$

This matrix as the inverse of the Hankel matrix C_n is a symmetric matrix.

Using the adjoint of a matrix for calculating the matrix inverse, equalities (1.8), (3.3), (3.28) for $\mathcal{K}_{n-1,1}$ and (4.9), the following remark is readily verified.

Remark 4.7. Let the inverse of the matrix $C_n(a_{n,n})$ be as in (4.11). Thus, the following equalities hold:

$$t_{11}(a_{nn}) := \frac{|\mathcal{K}_{n-1,2}|}{|C_n(a_{n,n})|}, \tag{4.12}$$

$$t_{1n}(a_{nn}) := (-1)^{n+1}\frac{|\mathcal{K}_{n-1,1}|}{|C_n(a_{n,n})|}, \tag{4.13}$$

$$t_{nn}(a_{nn}) := \frac{|C_n^{n-1}(a_{n,n})|}{|C_n(a_{n,n})|}. \tag{4.14}$$

In the next lemma, we calculate the value of the entries (4.12)–(4.14) for $a_{nn} = \widehat{a}^1_{n,n}$.

Lemma 4.8. *Let the inverse of the matrix $C_n(a_{n,n})$ be as in (4.11). Furthermore, let $\widehat{a}^1_{n,n}$ be as in (3.22). Thus, the following equalities hold:*

$$t_{11}(\widehat{a}^1_{n,n}) := \frac{2n^2(n+1)}{n-1}, \tag{4.15}$$

$$t_{1n}(\widehat{a}^1_{n,n}) := (-1)^{n+1}\frac{2(2n)!}{(n-1)((n-1)!)^2}, \tag{4.16}$$

$$t_{nn}(\widehat{a}^1_{n,n}) := \frac{(n+1)(2n)!(2n-1)!}{(n-1)((n-1)!)^2(n!)^2}. \tag{4.17}$$

Proof. Using (3.25), we calculate the determinant of the matrix $C_n(\widehat{a}^1_{n,n})$ as in (1.8). To this end, we express the first column of this matrix as follows:

$$\left(\frac{1}{2}, \frac{1}{2\cdot 3}, \ldots, \frac{1}{n(n+1)}\right)^{\mathsf{T}} - \frac{1}{n^2}(1, 0, \ldots, 0)^{\mathsf{T}}. \tag{4.18}$$

Thus, using (3.11) and (3.13), we have that

$$|C_n(\widehat{a}^1_{n,n})| = |\mathcal{K}_{n,0}| - \frac{1}{2n^2}|\mathcal{K}_{n-1,2}| = \frac{(n-1)(n!)^2}{2n(2n)!}|\mathcal{H}_{n,0}|. \tag{4.19}$$

Using a similar decomposition as in (4.18) of the first column of (4.10) and (3.29), we obtain

$$\begin{aligned} |C_n^{n-1}(a_{n,n})| &= |\mathcal{K}_{n-1,0}| + \left(c_{11} - \frac{1}{2}\right)|\mathcal{K}_{n-2,2}| \\ &= -\frac{((n-1)!)^2(n-2)!n}{(2n-3)!(2n)!}a_{n,n}|\mathcal{H}_{n-1,0}| \\ &= -\frac{a_{n,n}}{(n-1)!}|\mathcal{H}_{n,0}|. \end{aligned} \tag{4.20}$$

For $a_{n,n} = \widehat{a}^1_{n,n}$, using (3.22) and (4.20), we obtain

$$|C_n^{n-1}(\widehat{a}^1_{n,n})| = \frac{(n+1)(2n-1)!}{2n((n-1)!)^2}|\mathcal{H}_{n,0}|. \tag{4.21}$$

To prove (4.15), we use (3.13), (4.12), and (4.19),

$$t_{11}(\widehat{a}^1_{n,n}) = \frac{|\mathcal{K}_{n-1,2}|}{|C_n(\widehat{a}^1_{n,n})|} = \frac{2n^2(n+1)}{n-1}.$$

Using (4.13), we prove equality (4.16),

$$t_{1n}(\widehat{a}^1_{n,n}) = (-1)^{n+1}\frac{|\mathcal{K}_{n-1,1}|}{|C_n(\widehat{a}^1_{n,n})|} = (-1)^{n+1}\frac{2(2n)!}{(n-1)((n-1)!)^2}.$$

In the second equality, we used (3.12) and (4.19).

Finally, to prove equality (4.17), we use (4.14),

$$t_{nn}(\widehat{a}^1_{n,n}) = \frac{|C^{n-1}_n(\widehat{a}^1_{n,n})|}{|C_n(\widehat{a}^1_{n,n})|} = \frac{(n+1)(2n)!(2n-1)!}{(n-1)((n-1)!)^2(n!)^2}.$$

In the second equality, we used (3.16), (4.19) and (4.21). □

5. Properties of the matrix $\frac{1}{\Theta}K_n - \frac{d}{d\Theta}K_n$

Recall that for $a_{nn} \in (-\infty, \widehat{a}^1_{nn})$ the matrix $\frac{1}{\Theta}K_n - \frac{d}{d\Theta}K_n$ is a positive definite matrix. See [10]. In this section, we focus on some properties of the matrix $\frac{1}{\Theta}K_n - \frac{d}{d\Theta}K_n$ for case when the parameter a_{nn} belongs to $[\widehat{a}^1_{nn}, \widehat{a}_{nn})$.

Lemma 5.1. *Let $\widehat{a}_{n,n}$ be as in (3.21) and $a_{n,n} < \widehat{a}_{n,n}$. Let K_n and $C_{n,1}$ be as in (2.11) and (1.9). Thus, equality*

$$\det\left(\frac{1}{\Theta}K_n(\Theta, a_{n,n}) - \frac{d}{d\Theta}K_n(\Theta, a_{n,n})\right) = 0$$

is equivalent to equality $\det C_{n,1}(a_{n,n}) = 0$.

Proof. By using equality $\frac{d}{d\Theta}K_n = -K_n\frac{d}{d\Theta}K_n^{-1}K_n$, (2.3), (2.9), (2.11), and (2.20), we have

$$\begin{aligned}\left(\frac{1}{\Theta}K_n - \frac{d}{d\Theta}K_n\right) &= K_n\left(\frac{1}{\Theta}K_n^{-1} + \frac{d}{d\Theta}K_n^{-1}\right)K_n\\ &= \frac{1}{\Theta}K_nD_n^{-1}(\Theta)J_n\left(C_n - C_nH_n - H_nC_n\right)J_nD_n^{-1}(\Theta)K_n\\ &= \frac{1}{\Theta}K_nD_n^{-1}(\Theta)J_nC_{n,1}J_nD_n^{-1}(\Theta)K_n. \qquad (5.1)\end{aligned}$$

From part b) of Theorem 3.8, for $a_{n,n} < \widehat{a}_{n,n}$, the inequality $\det(J_nD_n^{-1}(\Theta)K_n) \neq 0$ holds. Consequently, the assertion of the lemma is proven. □

Now we present one of the main results of this section.

Theorem 5.2. *Let K_n be defined as in (2.11). For $a_{n,n} = \widehat{a}^1_{n,n}$, the null space of the matrix $\frac{1}{\Theta}K_n - \frac{d}{d\Theta}K_n$ consists of all vectors of the form*

$$\bar{x} := \left(x_1, 0, \ldots, 0, -\frac{n!}{(2n-1)!}\Theta^{n-1}x_1\right)^{\mathsf{T}}. \qquad (5.2)$$

Proof. Let

$$\begin{pmatrix} \widetilde{k}_{11} & \cdots & \widetilde{k}_{1n} \\ \vdots & \cdots & \vdots \\ \widetilde{k}_{1n} & \cdots & \widetilde{k}_{nn} \end{pmatrix} := \frac{1}{\Theta}K_n - \frac{d}{d\Theta}K_n. \tag{5.3}$$

Using (2.3), (2.9), (2.11), and (4.15)–(4.17), we obtain

$$\begin{aligned} \widetilde{k}_{11} &= \frac{2n^2(n+1)}{(n-1)\Theta^2}, \\ \widetilde{k}_{1n} &= \frac{2(n+1)(2n)!}{(n-1)(n-1)!\Theta^{n+1}}, \\ \widetilde{k}_{nn} &= \frac{2n(n+1)(2n)!(2n-1)!}{(n-1)n!\Theta^{2n}}. \end{aligned}$$

By a direct calculation, we have

$$\begin{pmatrix} \widetilde{k}_{11} & \cdots & \widetilde{k}_{1n} \\ \vdots & \cdots & \vdots \\ \widetilde{k}_{1n} & \cdots & \widetilde{k}_{nn} \end{pmatrix} \bar{x} = 0.$$

Taking into account that the rank of the matrix $C_{n,1}(\widehat{a}^1_{n,n})$ is equal to $n-1$ and by (5.1) and (5.3), we have that the null space of the matrix $\frac{1}{\Theta}K_n - \frac{d}{d\Theta}K_n$ is given by vectors of the form (5.2). □

The next result allows us to express $C_{n,1}$ with the help of a diagonal matrix that has entries that are all positive numbers except the first entry, which can be zero or a negative number. We also express the matrix $\frac{1}{\Theta}K_n - \frac{d}{d\Theta}K_n$ with the help of a certain diagonal matrix.

Theorem 5.3. *Let $\widehat{a}_{n,n}$ and $\widehat{a}^1_{n,n}$ be as in (3.21) and (3.22). Furthermore, let $a_{n,n} \in [\widehat{a}^1_{n,n}, \widehat{a}_{n,n})$. Moreover, let $C_{n,1}(a_{n,n})$ be as in (1.9). The following equality then holds:*

$$P^{\mathsf{T}} C_{n,1}(a_{n,n}) P = \Lambda_n(a_{n,n}), \tag{5.4}$$

where P is a real orthogonal matrix and

$$\Lambda_n(a_{n,n}) := \begin{cases} \operatorname{diag}(0, \lambda_2^2(a_{n,n}), \ldots, \lambda_n^2(a_{n,n})), & a_{n,n} = \widehat{a}^1_{n,n} \\ \operatorname{diag}(-\lambda_1^2(a_{n,n}), \lambda_2^2(a_{n,n}), \ldots, \lambda_n^2(a_{n,n})), & a_{n,n} \in (\widehat{a}^1_{n,n}, \widehat{a}_{n,n}). \end{cases} \tag{5.5}$$

Additionally, let K_n be as in (2.11). Thus, the following equality is valid:

$$\frac{1}{\Theta}K_n - \frac{d}{d\Theta}K_n = \frac{1}{\Theta}D_n(\Theta)V_n^{\mathsf{T}}\Lambda_n V_n D_n(\Theta), \tag{5.6}$$

where V_n is an invertible matrix.

Proof. It is well known that since $C_{n,1}(a_{n,n})$ is a real symmetric matrix, there is an orthogonal matrix P such that the left-hand side of (5.4) is equal to a diagonal matrix that consists of the eigenvalues of the matrix $C_{n,1}(a_{n,n})$. See [17, Theorem 8.1.1]. Now we prove that the mentioned diagonal matrix has the form (5.5). By (2.8), (3.1), (3.2), and (3.29), the matrix $C_{n,1}(a_{n,n})$ can be written in the form

$$C_{n,1} = \begin{pmatrix} 2c_{11} & h_{[2,n+1]}^{\mathsf{T}} \\ h_{[2,n+1]} & \mathcal{H}_{n-1,2} \end{pmatrix} \tag{5.7}$$

Using part a) of Theorem 3.8 and the fact that the matrix $\mathcal{H}_{n-1,2}$ is a positive definite matrix, we attain the first part of (5.5). Now we prove the second part of (5.5). By [10, Lemma 2.2] and the continuity of the determinant with respect to its entries, we have that the determinant $\det C_{n,1}(a_{n,n})$ is negative for $a_{n,n} \in (\widehat{a}_{n,n}, \widehat{a}^1_{n,n})$. By employing Cauchy's interlacing theorem [16], we verify that the matrix Λ_n has the form as in (5.5). Here again we have used the fact that the matrix $\mathcal{H}_{n-1,2}$ is positive definite.

Now we prove (5.6). Note that the matrix on the left-hand side of (5.6) is a symmetric matrix. Using (2.21), (5.1), and (5.7), we have

$$\frac{1}{\Theta}K_n - \frac{d}{d\Theta}K_n = \frac{1}{\Theta}D_n(\Theta)F_nJ_nP^{-1^{\mathsf{T}}}\Lambda_nP^{-1}J_nF_nD_n(\Theta).$$

Denote $V_n := P^{-1}J_nF_n$. Clearly, this matrix is invertible. Consequently, equality (5.6) is proved. □

Next, we consider the set of fix states x described by the following definition.

Definition 5.4. Let $a_{n,n} \in [\widehat{a}^1_{n,n}, \widehat{a}_{n,n})$. The set $\mathcal{M}_\Theta$ of states x, where the equality

$$\left(x, \left(\frac{1}{\Theta}K_n - \frac{d}{d\Theta}K_n\right)x\right) = 0 \tag{5.8}$$

is satisfied for $x \neq 0$, and Θ being a solution to (2.10) is called the Θ-degenerated set.

In Section 6, in the case when $\dot{\Theta} = -1$, we prove that (5.8) for $n = 3$ does not contain complete trajectories of system (1.7).

In the three-dimensional case, the left-hand side of (5.8) can be written as

$$\frac{-1}{(a_{3,3}+30)\,\Theta^6}\left(720\Theta^4x_1^2 + 10800\Theta^3x_1x_2 + 28800\Theta^2x_1x_3\right.$$
$$\left.-480\,(a_{3,3}-45)\,\Theta^2x_2^2 - 3600\,(a_{3,3}-20)\,\Theta x_2x_3 - 7200a_{3,3}x_3^2\right). \tag{5.9}$$

The Korobov equation (2.10) for the maximal value of the quantity a_0, $a_0 = \dfrac{1080}{a_{3,3}^2 + 12a_{3,3} + 360}$ and $d = 1$, has the form

$$\Psi_6(x, \Theta) = 0, \tag{5.10}$$

where

$$\Psi_6(x,\Theta) := \frac{1080\Theta^6}{a_{3,3}^2+12a_{3,3}+360} + \frac{360\Theta^4 x_1^2}{a_{3,3}+30} + \frac{3600\Theta^3 x_1 x_2}{a_{3,3}+30} + \frac{7200\Theta^2 x_1 x_3}{a_{3,3}+30} + \frac{120\,(45-a_{3,3})\,\Theta^2 x_2^2}{a_{3,3}+30} - \frac{720\,(a_{3,3}-20)\,\Theta x_2 x_3}{a_{3,3}+30} - \frac{1200 a_{3,3} x_3^2}{a_{3,3}+30}.$$

Example 5.5. Let $a_{3,3} = -40$. For $x^0 = (1, 0, -37/45)^\mathsf{T}$, equation (5.10) has the form $\left(9\Theta^2 - 148\right)^3 = 0$. This equation has a positive root of multiplicity 3 equal to $\Theta^0 = \frac{2\sqrt{37}}{3}$. For the indicated initial positions x^0 and Θ^0, using (5.9), we see that equality (5.8) holds. The eigenvalues of the matrix $\frac{1}{\Theta}K_n - \frac{d}{d\Theta}K_n$ for $\Theta = \Theta_2^0$ are

$$\lambda_1 = 0, \quad \lambda_2 = \frac{81\left(16223 - \sqrt{261302059}\right)}{101306}, \quad \lambda_3 = \frac{81\left(16223 + \sqrt{261302059}\right)}{101306}.$$

Example 5.6. Let $a_{3,3} = -30.01$. For $x^0 = (-0.5442, 0.1916, -0.042)^\mathsf{T}$, equation (5.10) has the form

$$1.19936\Theta^6 - 10664.8\Theta^4 + 37557.2\Theta^3 - 49538.8\Theta^2 + 29004.7\Theta - 6360.27 = 0.$$

This equation has three positive roots: $\Theta_1^0 = 0.812683$, $\Theta_2^0 = 0.974682$, $\Theta_3^0 = 92.5117$. For x^0 and Θ_2^0, using (5.9), we verify that the value of the left-hand side of (5.8) is equal to -23.0429. The eigenvalues of the matrix $\frac{1}{\Theta}K_n - \frac{d}{d\Theta}K_n$ for $\Theta = \Theta_2^0$ are $\lambda_1 = -164987$, $\lambda_2 = 4.35248$, and $\lambda_3 = 29431300$.

In the following remark, we explain the value of the coefficient at $\dot\Theta$ appearing in (2.16) in the case when equality (5.8) holds.

Remark 5.7. We emphasize that if the left-hand side of (5.8) vanishes for some x and Θ, then the right-hand side of (2.16) also vanishes. Using (2.18), we see that equality (2.16) holds for all states $x \in \mathbb{R}^n$, including for the states x that satisfy (5.8) if $\dot\Theta = -1$. The reason for that is that the matrix equality (2.19) is satisfied for $\dot\Theta = -1$.

6. Trajectories on the Θ-degenerated set

In this section, we consider the question whether the Θ-degenerated set $\mathcal{M}_\Theta$ contains the trajectories $x(t)$ of system (1.7) with a positional control (2.15) such that $\dot\Theta = -1$.

Remark 6.1. In the two-dimensional case, for $\dot\Theta \neq -1$, the corresponding set $\mathcal{M}_\Theta$ is a parabolic arc. The motion time from the initial position x^0 to the origin is less than the value of the controllability function $\Theta(x)$ at x^0. The corresponding positional control is a bang-bang control having the values of $\pm d$. The trajectories starting at $x^0 \in \mathcal{M}_\Theta$ remain in $\mathcal{M}_\Theta$, see [7].
For $n \geq 3$, we leave open the question of whether there are solutions of the synthesis problem belonging to $\mathcal{M}_\Theta$ for the case when $a_{n,n} \in [\widehat{a}_{n,n}^1, \widehat{a}_{n,n})$ and $\dot\Theta \neq -1$.

The next remark allows us to claim that in the case when the equality $\dot{\Theta} = -1$ is satisfied for all $x \in \mathbb{R}^n$, including the states belonging to $\mathcal{M}_\Theta$ as in (5.8), the trajectory of system (1.7) achieves the origin at finite $T(x^0) = \Theta^0$. We partially reproduce a remark written on p. 121 of [25].

Remark 6.2. Let $V_\Theta(x) := (K_n(\Theta)x, x)$ be a family of positive definite quadratic forms. Using (2.11) and (2.16), the time derivative of V_Θ with respect to system (1.7) with positional control (2.15) has the form

$$\dot{V}_\Theta(x) = x^\mathsf{T}\left(K_nA_n + A_n^\mathsf{T}K_n + K_nb_na^\mathsf{T} + ab_n^\mathsf{T}K_n\right)x = -\frac{1}{\Theta}V_\Theta(x). \tag{6.1}$$

In the first equality, we used the notation $a = \Theta^{-\frac{1}{2}}D_n(\Theta)a_n$. From equality (6.1), it is seen that $V_\Theta(x)$ is a Lyapunov function for system (1.7) with positional control (2.15). Moreover,

$$V_\Theta(x(t)) = c_\Theta\, e^{-\frac{1}{\Theta}t}, \quad c_\Theta > 0, \quad 0 < \Theta \le \Theta^0. \tag{6.2}$$

Here $x(t)$ is the trajectory of system (1.7) with control (2.15). If one chooses the parameter Θ equal to $\Theta(x)$ such that $\Theta(x) \to 0$ as $x \to 0$, then the trajectory of system (1.7) with control (2.15) achieves the origin at a finite time from an arbitrary initial state x^0 including the states belonging to $\mathcal{M}_\Theta$ as in (5.8).

In the sequel, we will use the equalities $K_n = K_n(\Theta, a_{n,n})$ and $F_n = F_n(a_{n,n})$.

Remark 6.3. Let K_n be as in (2.11). Thus, the left-hand side of the equality $2a_0\Theta - (x^\mathsf{T}K_nx) = 0$, which is equivalent to (2.10), contains complete trajectories of system (1.7), i.e., the following identity is valid:

$$\begin{aligned}\frac{d}{dt}&\left(2a_0\Theta - (x^\mathsf{T}K_n, x)\right)\\ &= \frac{1}{\Theta}x^\mathsf{T}D_n(\Theta)\left((F_n - H_nF_n - F_nH_n)\dot{\Theta} - F_nA_n - A_n^\mathsf{T}F_n - F_nb_na_n^\mathsf{T} - a_nb_n^\mathsf{T}F_n\right)\\ &\qquad\qquad \times D_n(\Theta)x \equiv 0.\end{aligned} \tag{6.3}$$

This identity is verified by using (2.19) and $\dot{\Theta} = -1$.

Now, for $n = 3$, we explain that the Θ-degenerated set $\mathcal{M}_\Theta$ does not contain complete trajectories $x(t)$ of system (1.7). For dimension $n > 3$, the mentioned question will be considered elsewhere.

Let us denote by

$$\Gamma(x(t)) := \left(x, \left(\frac{1}{\Theta}K_3 - \frac{d}{d\Theta}K_3\right)x\right)\Big|_{x=x(t)} \tag{6.4}$$

the left-hand side of (5.8) on the trajectory $x = x(t)$ for some initial position x^0.

We will prove that

$$\frac{d}{dt}\Gamma(x(t)) \not\equiv 0 \tag{6.5}$$

for every initial position x^0. The derivative in (6.5) is calculated along the trajectory $x(t)$.

Let $a_{3,3} < -30$. Positional control (2.15) has the form

$$u(x, a_{3,3}) = -\frac{6x_1}{\Theta} + \frac{(a_{3,3} - 30)\, x_2}{3\Theta^2} + \frac{a_{3,3}x_3}{\Theta^3}. \tag{6.6}$$

Since the controllability function Θ satisfies (1.3), the right-hand side of (6.6) can be written as

$$u(x, a_{3,3}) = -\frac{6x_1}{\Theta^0 - t} + \frac{(a_{3,3} - 30)\, x_2}{3(\Theta^0 - t)^2} + \frac{a_{3,3}x_3}{(\Theta^0 - t)^3}.$$

Here $\Theta^0 = \Theta(x^0)$. As a consequence, with this control, system (1.7) is equivalent to Euler's differential equation

$$(\Theta^0 - t)^3 x_3^{(3)} + 6(\Theta^0 - t)^2 \ddot{x}_3 - (\Theta^0 - t)\frac{a_{3,3} - 30}{3}\dot{x}_3 - a_{3,3}x_3 = 0 \tag{6.7}$$

with initial conditions $x_3(0) = x_3^0$, $\dot{x}_3(0) = x_2^0$, and $\ddot{x}_3(0) = x_1^0$. Using the change of the variables $t = \Theta^0 - e^\tau$ and $y(\tau) = x_3(\Theta^0 - e^\tau)$, the Euler-type equation (6.7) is reduced to the equation

$$y''' - 9y'' - \frac{a_{3,3} - 54}{3}y' + a_{3,3}y = 0$$

with initial conditions $y(\tau_0) = x_3^0$, $y'(\tau_0) = -\Theta^0 x_2^0$ and $y''(\tau_0) = -\Theta^0 x_2^0 + (\Theta^0)^2 x_1^0$, where $\tau_0 = \ln\Theta^0$. The solution of the latter Cauchy problem has the form

$$y(\tau) = e^{3\tau}\left(c_1 + c_2\cos(\nu\tau) + c_3\sin(\nu\tau)\right)$$

with

$$\nu = \sqrt{-\frac{a_{3,3+27}}{3}}, \quad a_{3,3} < -30, \tag{6.8}$$

and

$$c_1 = \left(-\xi_1 + \frac{x_3^0}{(\Theta^0)^3}\right), \tag{6.9}$$

$$c_2 = \left(\xi_1\cos(\nu\ln\Theta^0) - \nu\xi_2\sin(\nu\ln\Theta^0)\right), \tag{6.10}$$

$$c_3 = \left(\xi_1\sin(\nu\ln\Theta^0) + \nu\xi_2\cos(\nu\ln\Theta^0)\right). \tag{6.11}$$

Here,

$$\xi_1 = -\frac{1}{\nu^2(\Theta^0)^3}\left((\Theta^0)^2 x_1^0 + 5\Theta^0 x_2^0 + 9x_3^0\right) \text{ and } \xi_2 = -\frac{1}{\nu^2(\Theta^0)^3}\left(\Theta^0 x_2^0 + 3x_3^0\right).$$

Taking into account the equalities $x_3(t) = y(\ln(\Theta^0 - t))$, $x_2(t) = \dot{x}_3(t)$ and $x_1(t) = \ddot{x}_3(t)$, we have

$$x_1(t) = \left(\Theta^0 - t\right)\left(-\left(c_3\left(\nu^2 - 6\right) + 5c_2\nu\right)\sin(\alpha(t))\right)$$

$$+\left(c_2\left(6-\nu^2\right)+5c_3\nu\right)\cos(\alpha(t))+6c_1\right), \tag{6.12}$$

$$x_2(t)=\left(\Theta^0-t\right)^2\left((c_2\nu-3c_3)\sin(\alpha(t))-(c_3\nu+3c_2)\cos(\alpha(t))-3c_1\right), \tag{6.13}$$

$$x_3(t)=\left(\Theta^0-t\right)^3\left(c_3\sin(\alpha(t))+c_2\cos(\alpha(t))+c_1\right), \tag{6.14}$$

where

$$\alpha(t)=\nu\ln(\Theta^0-t). \tag{6.15}$$

Clearly, $x(t)=(x_1(t),x_2(t),x_3(t))^{\mathsf{T}}$ approaches zero as $t\to\Theta^0$.

In the following result, for $n=3$, we prove that in the case when $\dot\Theta=-1$, the Θ-degenerated set defined as in (5.8) does not contain a trajectory of system (1.7). We emphasize that if $\dot\Theta=-1$, equality (2.18) is satisfied for $x\in\mathbb{R}^3$ including the states x belonging to the Θ-degenerated set defined as in (5.8). This fact is used in the following result.

Lemma 6.4. *Let $\widehat{a}_{3,3}$ and $\widehat{a}^1_{3,3}$ be as in (3.21) and (3.22). Let $a_{3,3}\in[\widehat{a}^1_{3,3},\widehat{a}_{3,3})$ and $\mathcal{M}_\Theta$ be as in Definition 5.4 with $\dot\Theta=-1$. For any initial state $x^0\in\mathbb{R}^3$, the set $\mathcal{M}_\Theta$ does not contain a complete trajectory of (1.7) under the influence of positional control (2.15).*

Proof. Using (6.8)–(6.14) and

$$\begin{aligned}
p_1&=6\left(a_{3,3}+24\right)c_1^2-\left(a_{3,3}+30\right)^2\left(c_2^2+c_3^2\right),\\
p_2&=6c_1\left(\left(a_{3,3}+25\right)c_2\nu-\frac{1}{3}\left(a_{3,3}+45\right)c_3\right),\\
p_3&=2c_1\left(-\left(a_{3,3}+45\right)c_2-3\left(a_{3,3}+25\right)c_3\nu\right),\\
p_4&=\frac{2}{3}\left(a_{3,3}+30\right)\left(\left(a_{3,3}+18\right)c_2c_3+3\left(c_3^2-c_2^2\right)\nu\right),\\
p_5&=\frac{1}{3}\left(a_{3,3}+30\right)\left(\left(a_{3,3}+18\right)c_2^2-\left(a_{3,3}+18\right)c_3^2+12c_3c_2\nu\right),
\end{aligned}$$

the expression (6.4) can be written in the following form:

$$\Gamma(x(t))=\frac{120\left(p_1+p_2\sin(\alpha(t))+p_3\cos(\alpha(t))+p_4\sin(2\alpha(t))+p_5\cos(2\alpha(t))\right)}{a_{3,3}+30}.$$

Notice that for $j=1,\dots,5$, p_j are not simultaneously equal to zero for $a_{3,3}\in[\widehat{a}^1_{3,3},\widehat{a}_{3,3})$ and $\sum_{k=1}^3 c_k^2\neq 0$. On the other hand, the Wronskian of the functions $\{1,\sin(\alpha(t)),\cos(\alpha(t)),\sin(2\alpha(t)),\cos(2\alpha(t))\}$ is equal to $-\dfrac{8\left(a_{3,3}+27\right)^5}{27\left(t-\Theta_0\right)^{10}}$. Thus, $\{1,\sin(\alpha(t)),\cos(\alpha(t)),\sin(2\alpha(t)),\cos(2\alpha(t))\}$ are linear independent functions on $[0,\Theta^0]$. Therefore, $\Gamma(x(t))\not\equiv 0$. Consequently, $\Gamma(x(t))$ does not contain complete trajectories of system (1.7). □

A similar result may be proved for the n-dimensional case.

7. Nonunique solutions of the synthesis problem

Let $\widehat{a}_{n,n}$ and $\widehat{a}^1_{n,n}$ be as in (3.21) and (3.22). In this section, we present the main result of the present work. For the extension from $(-\infty, \widehat{a}^1_{n,n})$ to the interval $(-\infty, \widehat{a}_{n,n})$, we prove that equation (2.10) for system (1.7) may have more than one solution up to $2n-1$.

Theorem 7.1. *Let $a_{n,j}$, $j = 1, \ldots, n-1$ be as in (2.7), (2.14) and $a_{n,n} \in (-\infty, \widehat{a}_{n,n})$. Furthermore, let system (1.7) be influenced by positional control (2.15), where $\Theta(x)$ is the solution of equation (2.10). Thus, the following assertions are valid:*

a) *For $a_{n,n} \in (-\infty, \widehat{a}^1_{n,n})$, for any initial position $x^0 \in \mathbb{R}^n$, there is a unique solution $\Theta(x^0)$ of the Korobov equation (2.10) for a fixed number a_0 satisfying (2.12). The positional control $u(x, a_{n,n})$, constructed via $\Theta(x)$ as in (2.15), solves the synthesis problem of system (1.7).*

b) *Let $a_{n,n} \in [\widehat{a}^1_{n,n}, \widehat{a}_{n,n})$. For a given initial point x^0, the Korobov equation (2.10) has $k(x^0)$ number of solutions $\{\Theta_k(x^0)\}$, where $k(x^0) \in \{1, \ldots, 2n-1\}$ and a_0 is a fixed number satisfying (2.12). The set of solutions is performed by the positional controls $u(x, a_{n,n})$ as in (2.15), where $\Theta(x)$ is one of the functions $\Theta_k(x)$ such that $\dot{\Theta} = -1$.*

c) *In a) and b), the value of the controllability function at the initial position $\Theta(x^0)$ is exactly the motion time $T(x^0)$ from x^0 to the origin.*

Proof. Let
$$F_n = (f_{jk})^n_{j,k=0}. \tag{7.1}$$
The proof of part a) is given in the proof of [10, Theorem 3.1]. Now we prove part b). By using (2.11) and (7.1), equation (2.10) can be written in the form

$$2a_0\Theta^{2n} - f_{11}(a_{n,n})\Theta^{2n-2}x_1^2 - \cdots - f_{nn}(a_{n,n})x_n^2 = 0. \tag{7.2}$$

Taking into account the fact that $a_0 > 0$ and that the matrix $K_n(\Theta, a_{n,n})$ is positive definite for $a_{n,n} \in [\widehat{a}^1_{n,n}, \widehat{a}_{n,n})$ and consequently for $f_{nn}(a_{n,n}) > 0$, we have that equation (2.10) has at least one positive and one negative real root Θ for each fixed x. This fact is discussed in [25, p. 24] and [22, p. 540]. On the other hand, equation (7.2) or equality (2.10) has at most $2n$ real roots. Consequently, equation (2.10) may have $k(x^0) \in \{1, \ldots, 2n-1\}$ number of real positive roots. With each of these solutions, one can construct a positional control of the form (2.15). We attain a set of $k(x^0)$ number of solutions to the synthesis problem. Finally, part c) readily follows from the fact that the derivative of the controllability function $\Theta(x)$ on the trajectory satisfies the equality $\dot{\Theta} = -1$. By Remark 5.7, this equality holds for states in the Θ-degenerated set. □

The next example considers the three-dimensional case. We show that depending on the initial position x^0, there may be from 1 to 5 solutions to the synthesis problem such that the value of the controllability function Θ at x^0 represents the motion time from x^0 to the origin.

Example 7.2. By Theorem 7.1, the interval for the parameter $a_{3,3}$ is extended from $(-\infty,-40)$ [10] to the interval $(-\infty,-30)$. Depending on the initial position of system (2.1), there can be $k(x^0)$ different solutions, where $k(x^0)\in\{1,2,3,4,5\}$. In Tables 7.1 and 7.2, for $d=1$ and $a_{3,3}=-30.01$, we show the examples of nonunique solutions to equation (2.10).

Nonunique solutions of the Korobov equation for $a_{3,3}=-30.01$		
$x^0=(-1,1,4)^{\mathsf T}$	$x^0=(0.0118,-0.3368,1)^{\mathsf T}$	$x^0=(-1,1,-1)^{\mathsf T}$
$\Theta_1(x^0)=8.9984$ $\Theta_2(x^0)=9.6623$ $\Theta_3(x^0)=167.844$	$\Theta_1(x^0)=9.20328$ $\Theta_2(x^0)=5.56634$	$\Theta_1(x^0)=168.161$

Table 7.1: In the first column, for the initial position $x^0=(-1,1,4)^{\mathsf T}$, we have three roots of equation (5.10). In the second column, for the initial position $x^0=(0.0118,-0.3368,1)^{\mathsf T}$, we have two roots of equation (5.10). In the third column, for $x^0=(-1,1,-1)^{\mathsf T}$, we have a unique positive solution of equation (5.10).

Nonunique solutions of the Korobov equation for $a_{3,3}=-30.01$	
$x^0=(-1,1,-0.6)^{\mathsf T}$	$x^0=(1.3345,-1.4673,1)^{\mathsf T}$
$\Theta_1(x^0)=1.9746$ $\Theta_2(x^0)=2.03212$ $\Theta_3(x^0)=2.8659$ $\Theta_4(x^0)=3.1379$ $\Theta_5(x^0)=168.136$	$\Theta_1(x^0)=2.43243$ $\Theta_2(x^0)=2.70391$ $\Theta_3(x^0)=3.16287$ $\Theta_4(x^0)=225.609$

Table 7.2: In the first column, for the initial position $x^0=(-1,1,-0.6)$, we have five roots of equation (5.10). In the second column, for $x^0=(1.3345,-1.4673,1)^{\mathsf T}$, we have four roots of equation (5.10).

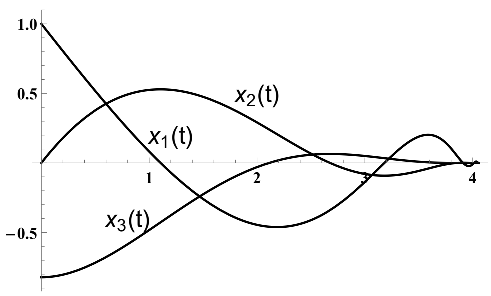
We do not establish the regions on $\mathbb{R}^3$, where equation (5.10) has a given number of positive roots. To compute the number of roots of (5.10) for the initial point $x^0=(-1,1,-0.6)^{\mathsf T}$ appearing in Table 7.2, we rewrite (5.10) in the equivalent form $\Phi_6(x,\Theta)=0$, where

$$\Phi_6(x,\Theta)=\Theta^6kw-k\left(\Theta^2x_2^2+6\Theta x_3x_2+10x_3^2\right)+3\left(\Theta^2x_1+5\Theta x_2+10x_3\right)^2$$

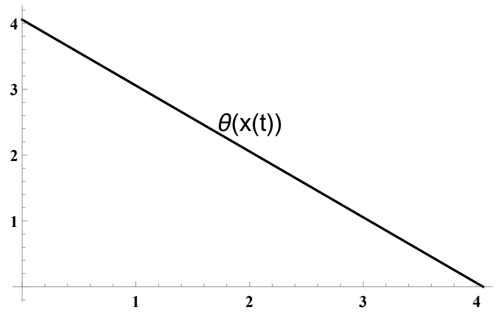
with $k:=30-a_{3,3}$ and $w:=\frac{9}{a_3^2+12a_3+360}$. Use the resultant $R(\Phi_6,\Phi_6',\Theta)$ of polynomials Φ_6 and Φ_6', where the prime means the derivative of Φ_6 with respect to Θ. See [40, p. 20]. By using Wolfram Mathematica, we attain the next expression for the resultant $R(\Psi_6,\Psi_6',\Theta)$:

$$\begin{aligned}R(\Psi_6,\Psi_6',\Theta)&=-64k^4w^2x_3^2\left(2430x_1^9\left(\left(31k^2-5175k+168750\right)x_2^6x_3-96000\right.\right.\\&\times\left(3k^2-100k+300\right)wx_3^5\big)-243\left(253k^2-113700k+3802500\right)x_2^4x_3^2x_1^{10}+81x_1^8\\&\times\left(3600\left(37k^3+11900k^2-451100k+1440000\right)wx_3^4x_2^2+(k-150)(k-75)^2x_2^8\right)\\&-48600\left(449k^3+119885k^2-4974000k+16312500\right)wx_2^4x_3^3x_1^7+6750kw^2x_3^3x_1^3\\&\times\left(\left(-1347k^4+434090k^3-42634500k^2+1613295000k-20709000000\right)x_2^6\right.\\&+86400(k-30)^3(3k-10)wx_3^4\big)-135wx_3^2x_1^6\left(432000\left(k^4-270k^3+9300k^2\right.\right.\end{aligned}$$

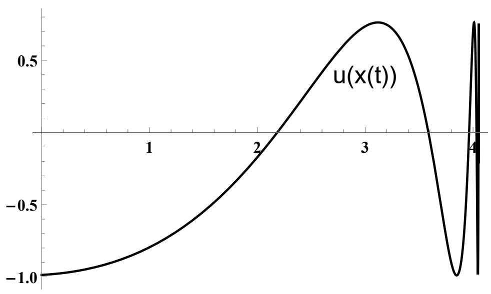
$$
\begin{aligned}
&- 65000k + 60000)\, wx_3^4 + \big(413k^4 - 67560k^3 - 35048250k^2 + 1507275000k \\
&- 5011875000)\, x_2^6\big) + 270wx_2^2x_3x_1^5\,\big(135000\,\big(19k^4 - 2302k^3 + 80212k^2 \\
&- 810600k + 120000)\cdot wx_3^4 + \big(226k^4 + 11775k^3 - 7228125k^2 + 302906250k \\
&- 1012500000)\, x_2^6\big) + 6750kw^2x_2^2x_3^2x_1^2\,\big(540(k-30)^2 \times \big(7k^3 - 1360k^2 + 73500k \\
&- 210000)\, wx_3^4 + \big(-2k^5 + 1703k^4 - 358905k^3 + 30801750k^2 - 1152787500k \\
&+ 15474375000)\, x_2^6\big) - 18000kw^2x_2^4x_3x_1\,\big(270\,\big(7k^5 - 1380k^4 + 110400k^3 \\
&- 4104000k^2 + 59850000k - 135000000)\, wx_3^4 - (k-75)^3\,\big(k^2 - 168k + 5100\big) \\
&\times x_2^6\big) + 9wx_2^4x_1^4\,\big((k-75)^2\,\big(8k^3 + 300k^2 - 230625k + 843750)\, x_2^6 + 6750\,\big(49k^5 \\
&- 33570k^4 + 3578600k^3 - 131106000k^2 + 1556850000k - 22500000\big)\, wx_3^4\big) \\
&+ kw^2\,\big(-3375\,\big(19k^6 - 6210k^5 + 775800k^4 - 50058000k^3 + 1718550000k^2 \\
&- 26122500000k + 54000000000)\, wx_3^4x_2^6 + 72900000(k-30)^5kw^2x_3^8 + 16 \\
&\times (k-150)\,(k-75)^5x_2^{12}\big) + 11664000(k-30)x_3^4x_1^{12} - 583200(49k - 1590) \\
&\times x_2^2x_3^3x_1^{11}\big)\,.
\end{aligned}
$$



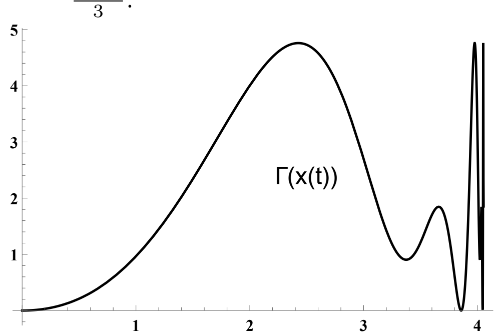
(a) Coordinates of the trajectory $x(t)$ for $x^0 = \big(1, 0, -\frac{37}{45}\big)^{\mathsf{T}}$, $a_{3,3} = -40$ and $\Theta^0 = \frac{2\sqrt{37}}{3}$.



(b) The controllability function on the trajectory $x(t)$ for $x^0 = \big(1, 0, -\frac{37}{45}\big)^{\mathsf{T}}$, $a_{3,3} = -40$ and $\Theta^0 = \frac{2\sqrt{37}}{3}$.



(c) The positional control on the trajectory $x(t)$ for $x^0 = \big(1, 0, -\frac{37}{45}\big)^{\mathsf{T}}$, $a_{3,3} = -40$ and $\Theta^0 = \frac{2\sqrt{37}}{3}$.



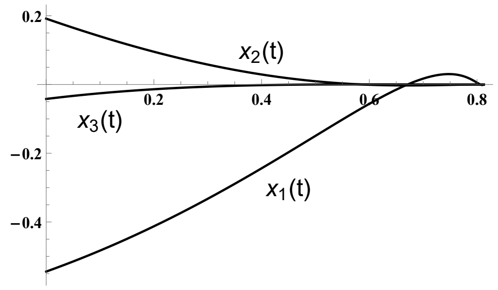
(d) Graph of the function $\Gamma(x(t))$ for $x^0 = \big(1, 0, -\frac{37}{45}\big)^{\mathsf{T}}$, $a_{3,3} = -40$ and $\Theta^0 = \frac{2\sqrt{37}}{3}$.

Fig. 7.1: The graphs of the coordinates of the trajectories, the controllability function, the positional control on the trajectory and the function (6.4).

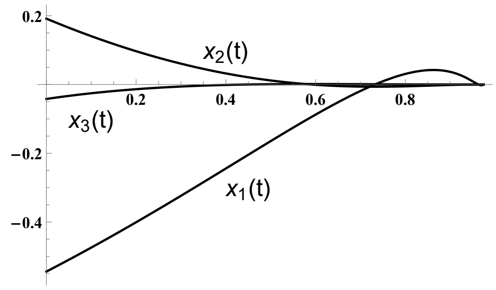
Substitute on left-hand side of the equality $R(\Phi_6, \Phi_6', \Theta) = 0$ the values $a_{3,3} = -30.01$, $x_1 = -1$, and $x_2 = 1$. Solve the resulting equation for the variable x_3. With the found real roots x_3, in the points $x = (-1, 1, x_3)^{\mathsf{T}}$ with $a_{3,3} = -30.01$ the polynomial Φ_6 or equivalently Ψ_6 on Θ will have from 1 to 5 positive real roots.

In the next examples, for the three-dimensional case, we provide graphs of the trajectory $x(t)$, the controllability function $\Theta(x(t))$, the positional control $u(x(t))$ on the trajectory and the function $\Gamma(x(t))$.

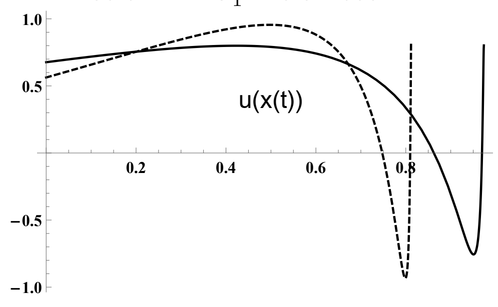
Example 7.3. We continue Example 5.5. As we have seen, for $x^0 = \left(1, 0, -\frac{37}{45}\right)^{\mathsf{T}}$, $a_{3,3} = -40$, there is a unique positive solution of the Korobov equation (5.10) $\Theta(x^0) = \dfrac{2\sqrt{37}}{3}$. The time motion $T(x^0)$ from x^0 to the origin is equal to $\Theta(x^0)$. The corresponding graphs of the coordinates of the trajectories, the controllability function, the positional control on the trajectory and the function (6.4) are given in Fig. 7.1.



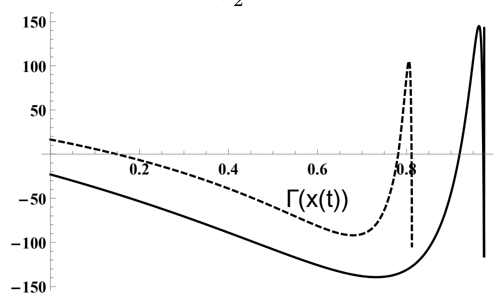
(a) Coordinates of the trajectory $x(t)$ for $x^0 = (-0.5442, 0.1916, -0.042)^{\mathsf{T}}$, $a_{3,3} = -30.01$ and $\Theta_1^0 = 0.812683$.



(b) Coordinates of the trajectory $x(t)$ for $x^0 = (-0.5442, 0.1916, -0.042)^{\mathsf{T}}$, $a_{3,3} = -30.01$ and $\Theta_2^0 = 0.974682$.



(c) The positional control on the trajectory $x(t)$ for $x^0 = (-0.5442, 0.1916, -0.042)^{\mathsf{T}}$ and $a_{3,3} = -30.01$. The dotted (resp. solid) curve corresponds to $\Theta_1^0 = 0.812683$ (resp. $\Theta_2^0 = 0.974682$).



(d) Graph of the function $\Gamma(x(t))$ for $x^0 = (-0.5442, 0.1916, -0.042)^{\mathsf{T}}$ and $a_{3,3} = -30.01$. The dotted (resp. solid) curve corresponds to $\Theta_1^0 = 0.812683$ (resp. $\Theta_2^0 = 0.974682$).

Fig. 7.2: The trajectory $x(t)$, the controllability function on the trajectory $\Theta(x(t))$, the control $u(x(t))$ and $\Gamma(x(t))$ for $x^0 = (-0.5442, 0.1916, -0.042)^{\mathsf{T}}$, $a_{3,3} = -30.01$ and $\Theta_1^0 = 0.812683$ or $\Theta_2^0 = 0.974682$.

In these graphs, the horizontal axis represents time whereas the vertical axis represents a certain value which is indicated at the caption of each figure.

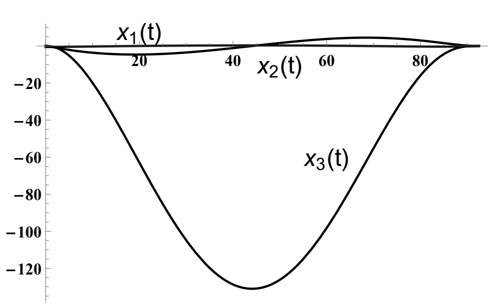
Notice that the initial position x^0 belongs to $\mathcal{M}_\Theta$. The function $\Gamma(x(t))$ is non negative and there are values of t in the interval $(0, T(x^0)]$ where $\Gamma(x(t)) = 0$.

Example 7.4. We continue Example 5.6. For $x^0 = (-0.5442, 0.1916, -0.042)^\mathsf{T}$, $a_{3,3} = -30.01$, $\Theta_1^0 = 0.812683$, $\Theta_2^0 = 0.974682$ and $\Theta_3^0 = 92.5117$, the corresponding graphs are given in Figures 7.2–7.3. Again, in these graphs, the horizontal axis represents time whereas the vertical axis represents a certain value which is indicated at the caption of each figure.

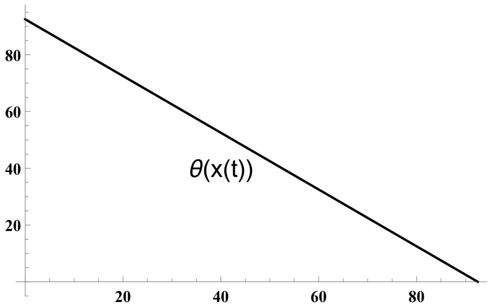
Observe that the function $\Gamma(x(t))$ takes negative and positive values and in certain values of t in the interval $[0, T(x^0)]$, $\Gamma(x(t)) = 0$.

The following four figures represent the graphs of the trajectory $x(t)$, the controllability function on the trajectory $\Theta(x(t))$, the control $u(x(t))$ and $\Gamma(x(t))$ for $x^0 = (-0.5442, 0.1916, -0.042)^\mathsf{T}$, $a_{3,3} = -30.01$ and $\Theta_3^0 = 92.5117$.

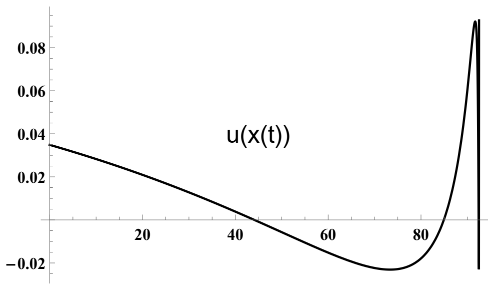
From Figures 7.2(c) and 7.3(d), we see that the positional control for Θ_1^0 (resp. Θ_2^0) takes its values in the interval $[-0.96, 0.96]$ (resp. $[-0.8, 0.8]$), whereas



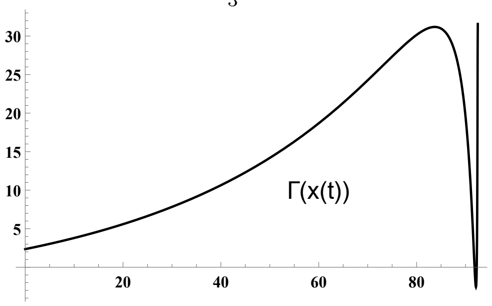
(a) Coordinates of the trajectory $x(t)$ for $x^0 = (-0.5442, 0.1916, -0.042)^\mathsf{T}$, $a_{3,3} = -30.01$ and $\Theta_3^0 = 92.5117$.



(b) The controllability function on the trajectory $x(t)$ for $x^0 = (-0.5442, 0.1916, -0.042)^\mathsf{T}$, $a_{3,3} = -30.01$ and $\Theta_3^0 = 92.5117$.



(c) The positional control on the trajectory $x(t)$ for $x^0 = (-0.5442, 0.1916, -0.042)^\mathsf{T}$, $a_{3,3} = -30.01$ and $\Theta_3^0 = 92.5117$.



(d) Graph of the function $\Gamma(x(t))$ for $x^0 = (-0.5442, 0.1916, -0.042)^\mathsf{T}$, $a_{3,3} = -30.01$ and $\Theta_3^0 = 92.5117$.

Fig. 7.3: The trajectory $x(t)$, the controllability function on the trajectory $\Theta(x(t))$, the control $u(x(t))$ and $\Gamma(x(t))$ for $x^0 = (-0.5442, 0.1916, -0.042)^\mathsf{T}$, $a_{3,3} = -30.01$ and $\Theta_3^0 = 92.5117$.

the values of the positional control for Θ_3^0 belong to $[-0.024, 0.093]$. From this observation, we conclude that the larger the interval $[0, \Theta_k^0]$, for $k = 1, 2, 3$, is, the "milder" is the influence of the positional control on system (1.7). Notice that Θ_3^0 is hundred times greater than Θ_1^0.

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Функція керованості Коробова як час руху: розширення множини розв'язків проблеми синтезу

A.E. Choque-Rivero

Знайдено розширення множини розв'язків проблеми стабілізації за скінченний час за допомогою обмеженого позиційного керування, яка також називається проблемою синтезу для канонічної системи за допомогою функції керованості Коробова. Ми розглядаємо випадок, коли значення функцій керованості в початковій точці є часом руху з цієї початкової точки до нуля. У термінах певних реальних параметрів знайдено сім'ю позиційних керувань, що розв'язують проблему синтезу. Ми збільшуємо інтервал параметрів і явно обчислюємо його кінцеві точки як функції від розмірності n системи, що розглядається.

Ключові слова: проблема сиснтезу, стабілізація за скінченний час, обмежене керування, канонічна система