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Exponential Stability for a Flexible Structure System with Thermodiffusion Effects and Distributed Delay

Madani Douib, Salah Zitouni, and Abdelhak Djebabla

In the paper, the well-posedness and asymptotic behavior of solutions to a flexible structure with thermodiffusion effects and distributed delay are studied. Under suitable assumptions on the weight of the damping and the weight of the distributed delay, we prove the existence and the uniqueness of the solution using the semigroup theory. Then, by using the perturbed energy method and constructing some Lyapunov functionals, we obtain the exponential decay of the solution.

Key words: flexible structure, thermodiffusion effects, distributed delay, well-posedness, exponential stability

Mathematical Subject Classification 2020: 37C75, 93D05

1. Introduction

In the paper, we consider a flexible structure system with thermodiffusion effects and distributed delay. The system is written as

$$
\begin{aligned}
& m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \mu_0 u_t \\
& \qquad + \int_{\tau_1}^{\tau_2} \mu(s)u_t\,(x, t-s)\,ds - \gamma_1\theta_x - \gamma_2 P_x = 0, && \text{(1.1a)} \\
& c\theta_t + dP_t - k\theta_{xx} - \gamma_1 u_{xt} = 0, && \text{(1.1b)} \\
& d\theta_t + rP_t - hP_{xx} - \gamma_2 u_{xt} = 0, && \text{(1.1c)}
\end{aligned}
$$

where $(x,t) \in (0,L)\times(0,+\infty)$, with the following initial and boundary conditions:

$$
\begin{aligned}
& u\,(x,0) = u_0\,(x)\,, \quad u_t\,(x,0) = u_1\,(x)\,, && x \in (0,L)\,, && \text{(1.2a)} \\
& \theta\,(x,0) = \theta_0\,(x)\,,\ P\,(x,0) = P_0\,(x)\,, && x \in (0,L)\,, && \text{(1.2b)} \\
& u\,(0,t) = u\,(L,t) = 0, && t > 0, && \text{(1.2c)} \\
& \theta_x\,(0,t) = \theta_x\,(L,t) = 0,\ P_x\,(0,t) = P_x\,(L,t) = 0, && t > 0, && \text{(1.2d)} \\
& u_t\,(x,-t) = f_0(x,t), && 0 < t \le \tau_2, && \text{(1.2e)}
\end{aligned}
$$

where $u = u(x,t)$ is the displacement of a particle at position $x \in (0,L)$ and time $t > 0$, $\theta = \theta(x,t)$ is the temperature difference, $P = P\,(x,t)$ is the chemical potential, k and h are heat and mass diffusion conductivity coefficients. The parameters $m(x)$, $\delta(x)$ and $p(x)$ are responsible for the non-uniform structure of the body, where $m(x)$ denotes mass per unit length of the structure, $\delta(x)$ is the coefficient of internal material damping and $p(x)$ is a positive function related to the stress acting on the body at a point x. We recall the assumptions of the functions $m(x)$, $\delta(x)$ and $p(x)$ from [2] such that

$$m, \delta, p \in W^{1,\infty}(0, L), \quad m(x), \delta(x), p(x) > 0, \quad x \in [0, L].$$

Physical positive constants γ_1, γ_2, r, c, and d satisfy

$$\lambda = rc - d^2 > 0. \tag{1.3}$$

The distributed delay considered in this paper is important because it is given by a nonlocal time-delay control. The history of nonlocal problems with integral conditions for partial differential equations goes back to [8]. See also [26] and references therein. This kind of delay,

$$\int_{\tau_1}^{\tau_2} \mu\,(s)\, u_t\,(x, t - s)\, ds,$$

is called nonlocal because the integral is not a pointwise relation. This condition provokes some mathematical difficulties which make the studying of the problem particularly interesting. For the last several decades, various types of equations have been employed as some mathematical models describing physical, chemical, ecological and biological systems. See, for example, [14].

The coefficients μ_0 are positive constants, and $\mu : [\tau_1; \tau_2] \to \mathbb{R}$ is a bounded function, where τ_1 and τ_2 are two real numbers satisfying $0 \le \tau_1 < \tau_2$. Here, we prove the well-posedness and stability results for the problem with the following parameters under the assumption:

$$\mu_0 > \int_{\tau_1}^{\tau_2} |\mu\,(s)|\, ds. \tag{1.4}$$

Condition (1.4) was previously assumed for viscoelastic waves with distributed delay: in [24], where the authors used the energy method, and in [29], where a different approach, the semigroup technique, was used.

One of the main issues concerning the vibrations in models of flexible structural systems is the question of the stabilization of the structure. The linear differential equation describing the vibrations of an inhomogeneous flexible structure with an exterior disturbing force can be described by the following equation:

$$m\,(x)\, u_{tt} - \left(p\,(x)\, u_x + 2\delta\,(x)\, u_{xt}\right)_x = f\,(x) \quad \text{in } (0, L) \times \mathbb{R}^+. \tag{1.5}$$

The distributed force $f : (0, L) \times \mathbb{R}^+ \to \mathbb{R}$ is the uncertain disturbance appearing in the model, which is assumed to be continuously differentiable for all $t \ge 0$. Indeed, one expects to prevent a system from resonance effects and wants to ensure a decay of the total energy, at least polynomial and hopefully exponential. It is therefore of interest to investigate the theory behind the stabilization processes in flexible structural systems and to control their vibrations. In [13], Gorain established the uniform exponential stability of problem (1.5). It is physically relevant to take into account thermal effects in flexible structures. In 2014, M. Siddhartha et al. [20] showed the exponential stability of the vibrations of an inhomogeneous flexible structure with thermal effect governed by the Fourier law,

$$\begin{aligned} &m(x)\,u_{tt} - \left(p(x)\,u_x + 2\delta(x)\,u_{xt}\right)_x + \kappa\theta_x = f, \\ &\theta_t - \theta_{xx} + \kappa u_{tx} = 0. \end{aligned}$$

In the above model, thermal waves propagate with infinite speed. This property of the model is not consistent with the reality, where the heating or cooling of a flexible structure usually takes some time. Many researches have thus been conducted in order to modify the model of thermal effect.

Delay effects arise in many applications and practical problems (see, for instance, [6,27]) due to the fact that many phenomena depend on their past. It has been established that a voluntary introduction of delay can benefit the control (see [1]). On the other hand, it may not only destabilize a system, which is asymptotically stable in the absence of delay, but may also lead to ill-posedness (see [9,28] and references therein). Moreover, it influences on the asymptotic behavior of the solution for different types of problems (see [4,5,7,10,15–19,22,23]). Therefore, the issues of well-posedness and the stability result of systems with delay are of practical and theoretical importance. In [12], the authors considered the vibrations of an inhomogeneous flexible structure system with a constant internal delay under the Cattaneo law of heat condition,

$$m(x)\,u_{tt} - \left(p(x)\,u_x + 2\delta(x)\,u_{xt}\right)_x + \eta\theta_x + \mu u_t(x, t-\tau_0) = 0, \tag{1.6a}$$
$$\theta_t + \kappa q_x + \eta u_{tx} = 0, \tag{1.6b}$$
$$\tau q_t + \beta q + \kappa\theta_x = 0, \tag{1.6c}$$

where $(x,t) \in (0,L) \times (0,+\infty)$, with the boundary and initial conditions

$$u(0,t) = u(L,t) = 0,\ \theta(0,t) = \theta(L,t) = 0, \qquad t \geq 0, \tag{1.7a}$$
$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \qquad x \in [0,L], \tag{1.7b}$$
$$\theta(x,0) = \theta_0(x),\ q(x,0) = q_0(x), \qquad x \in [0,L], \tag{1.7c}$$

and proved the well-posedness and the exponential stability. In [2], M.S. Alves et al. considered system (1.6), (1.7) without delay term and obtained an exponential stability result for one set of boundary conditions, and at least polynomial for another set of boundary conditions.

Thermodiffusion in an elastic solid is due to the coupling of the fields of strain, temperature and mass diffusion. The processes of heat and mass diffusion has been widely used in many engineering applications, such as satellites problems, returning space vehicles and aircraft landing on water or land. In 1921, Timoshenko [30] gave a distinguished model for vibrations of elastic beams, which is coupled by the shear force and the bending moment on the system. Aouadi et al. [3] considered the following Timoshenko system with thermodiffusion effects:

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x + \mu\varphi_t = 0, \tag{1.8a}$$
$$\rho_2\psi_{tt} - \alpha\psi_{xx} + \kappa(\varphi_x + \psi) - \gamma_1\theta_x - \gamma_2 P_x = 0, \tag{1.8b}$$
$$c\theta_t + dP_t - k\theta_{xx} - \gamma_1\psi_{xt} = 0, \tag{1.8c}$$
$$d\theta_t + rP_t - hP_{xx} - \gamma_2\psi_{xt} = 0, \tag{1.8d}$$

together with Dirichlet boundary conditions and Neumann boundary conditions. The lack of exponential stability for Neumann boundary conditions was proved for $\mu = 0$ and exponential stability for (1.8) without any restrictions on the coefficients was established for $\mu \neq 0$. In addition, some numerical results for the two cases, $\mu = 0$ and $\mu \neq$, were given. In [11], Feng studied the Timoshenko system only with thermodiffusion effects. He established the exponential energy decay of the system with two kinds of boundary conditions under the assumption of equal wave speeds. This result extends the last result obtained by Aouadi et al. in [3].

Motivated by the above results, in the present work, our aim is to prove that system (1.1), (1.2) is well-posed and exponentially stable. The main features of this paper are summarized as follows. In Section 2, we adopt the semigroup method and the Lumer–Philips theorem to obtain the well-posedness of system (1.1), (1.2). In Section 3, we use the perturbed energy method and construct some Lyapunov functionals to prove the exponential stability of system (1.1), (1.2).

2. Well-posedness

In this section, we prove the existence and uniqueness of solutions for (1.1), (1.2) using the semigroup theory [25]. As in [24], we introduce a new variable

$$z(x,\rho,s,t) = u_t\left(x, t-\rho s\right), \quad x \in (0,L)\,,\ \rho \in (0,1)\,,\ s \in (\tau_1,\ \tau_2)\,,\ t > 0.$$

Therefore, problem (1.1) takes the form

$$\begin{aligned} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \mu_0 u_t \\ + \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s,t)\, ds - \gamma_1\theta_x - \gamma_2 P_x = 0, \end{aligned} \tag{2.1a}$$
$$c\theta_t + dP_t - k\theta_{xx} - \gamma_1 u_{xt} = 0, \tag{2.1b}$$
$$d\theta_t + rP_t - hP_{xx} - \gamma_2 u_{xt} = 0, \tag{2.1c}$$
$$sz_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, \tag{2.1d}$$

where $(x,t) \in (0,L)\times(0,+\infty)$, with the following initial and boundary conditions:

$$u\left(x,0\right) = u_0\left(x\right),\ u_t\left(x,0\right) = u_1\left(x\right), \qquad x \in (0,L)\,, \tag{2.2a}$$
$$\theta\left(x,0\right) = \theta_0\left(x\right),\ P\left(x,0\right) = P_0\left(x\right), \qquad x \in (0,L)\,, \tag{2.2b}$$

$$
\begin{aligned}
&u(0,t) = u(L,t) = 0, && t>0, && \text{(2.2c)}\\
&\theta_x(0,t) = \theta_x(L,t) = 0, && t>0, && \text{(2.2d)}\\
&P_x(0,t) = P_x(L,t) = 0, && t>0, && \text{(2.2e)}\\
&z(x,0,s,t) = u_t(x,t) && \text{in } (0,L)\times(0,\infty)\times(\tau_1,\tau_2), && \text{(2.2f)}\\
&z(x,\rho,s,0) = f_0(x,\rho s) && \text{in } (0,L)\times(0,1)\times(\tau_1,\tau_2). && \text{(2.2g)}
\end{aligned}
$$

Introducing the vector function $U = (u,v,\theta,P,z)^T$, where $v = u_t$, system (2.1), (2.2) can be written as

$$
\begin{aligned}
&U'(t) = \mathcal{A}U(t), \quad t>0, && \text{(2.3a)}\\
&U(0) = U_0 = (u_0,u_1,\theta_0,P_0,f_0)^T, && \text{(2.3b)}
\end{aligned}
$$

where the operator $\mathcal{A}$ is defined by

$$
\mathcal{A}U = \begin{pmatrix} v \\ \dfrac{1}{m(x)}\left[(p(x)u_x+2\delta(x)v_x)_x+\gamma_1\theta_x+\gamma_2 P_x-\mu_0 v-\displaystyle\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,s,t)ds\right] \\ \left(\dfrac{rk}{\lambda}\right)\theta_{xx}-\left(\dfrac{hd}{\lambda}\right)P_{xx}+\left(\dfrac{r\gamma_1-d\gamma_2}{\lambda}\right)v_x \\ \left(\dfrac{ch}{\lambda}\right)P_{xx}-\left(\dfrac{kd}{\lambda}\right)\theta_{xx}+\left(\dfrac{c\gamma_2-d\gamma_1}{\lambda}\right)v_x \\ -s^{-1}z_\rho \end{pmatrix}.
$$

Let

$$
\mathcal{H} = H_0^1(0,L)\times L^2(0,L)\times L^2(0,L)\times L^2(0,L)\times L^2((0,L)\times(0,1)\times(\tau_1,\tau_2))
$$

be the Hilbert space equipped with the inner product

$$
\begin{aligned}
\left\langle U,\widetilde{U}\right\rangle_{\mathcal{H}} &= \int_0^L p(x)u_x\widetilde{u}_x\,dx + \int_0^L m(x)v\widetilde{v}\,dx + \int_0^L c\theta\widetilde{\theta}\,dx \\
&+ \int_0^L dP\widetilde{\theta}\,dx + \int_0^L d\theta\widetilde{P}\,dx + \int_0^L rP\widetilde{P}\,dx \\
&+ \int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu(s)|\,z(x,\rho,s)\,\widetilde{z}(x,\rho,s)\,ds\,d\rho\,dx.
\end{aligned}
$$

Then the domain of $\mathcal{A}$ is given by

$$
\begin{aligned}
D(\mathcal{A}) = \big\{&U\in\mathcal{H} \mid u\in H^2(0,L)\cap H_0^1(0,L),\ v,\theta,P\in H_0^1(0,L), \\
&z,z_\rho\in L^2((0,L)\times(0,1)\times(\tau_1,\tau_2)),\ z(x,0,s)=v(x)\big\}.
\end{aligned}
$$

It is clear that $D(\mathcal{A})$ is dense in $\mathcal{H}$.

We have the following existence and uniqueness result.

Theorem 2.1. *Assume that $U_0 \in \mathcal{H}$ and (1.4) holds. Then, for problem (2.1), (2.2), there exists a unique solution $U \in C(\mathbb{R}^+; \mathcal{H})$. Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C\left(\mathbb{R}^+; D(\mathcal{A})\right) \cap C^1\left(\mathbb{R}^+; \mathcal{H}\right).$$

Proof. We use the semigroup approach to prove that $\mathcal{A}$ is a maximal monotone operator, which means that $\mathcal{A}$ is dissipative and $Id - \mathcal{A}$ is surjective. First, we prove that $\mathcal{A}$ is dissipative. For any $U = (u, v, \theta, P, z)^T \in D(\mathcal{A})$, by using the inner product and integrating by parts, we obtain

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = &-2\int_0^L \delta(x) v_x^2\, dx - \left(\mu_0 - \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu(s)|\, ds\right)\int_0^L v^2\, dx \\ &- k\int_0^L \theta_x^2 dx - h\int_0^L P_x^2\, dx - \frac{1}{2}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)\, ds\, dx \\ &- \int_0^L v \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s,t)\, ds\, dx. \end{aligned} \tag{2.4}$$

Using Young's inequality, the last term in (2.4), we have

$$\begin{aligned} &-\int_0^L v \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s,t)\, ds\, dx \\ &\quad \le \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu(s)|\, ds \int_0^1 v^2\, dx + \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)\, ds\, dx. \end{aligned} \tag{2.5}$$

Substituting (2.5) in (2.4) and using (1.4), we obtain

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \le &-2\int_0^L \delta(x) v_x^2\, dx - k\int_0^L \theta_x^2\, dx - h\int_0^L P_x^2\, dx \\ &- \left(\mu_0 - \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds\right)\int_0^L v^2\, dx \le 0. \end{aligned}$$

Hence the operator $\mathcal{A}$ is dissipative.

Next, we prove that the operator $\mathrm{Id} - \mathcal{A}$ is surjective. Given $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{H}$, we prove that there exists $U = (u, v, \theta, P, z)^T \in D(\mathcal{A})$ satisfying

$$(\mathrm{Id} - \mathcal{A})\, U = F, \tag{2.6}$$

that is,

$$u - v = f_1, \tag{2.7a}$$

$$\begin{aligned} m(x)\, v - \Big[&(p(x)u_x + 2\delta(x)v_x)_x + \gamma_1\theta_x + \gamma_2 P_x - \mu_0 v \\ &- \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s,t) ds\Big] = m(x)\, f_2, \end{aligned} \tag{2.7b}$$

$$\lambda\theta - rk\theta_{xx} + hdP_{xx} - (r\gamma_1 - d\gamma_2)\, v_x = \lambda f_3, \tag{2.7c}$$

$$\lambda P - chP_{xx} + kd\theta_{xx} - (c\gamma_2 - d\gamma_1)\, v_x = \lambda f_4, \tag{2.7d}$$

$$sz + z_\rho = sf_5. \tag{2.7e}$$

Suppose that we have found u. Then equation (2.7a) yields

$$v = u - f_1. \tag{2.8}$$

It is clear that $v \in H_0^1(0, L)$.

Moreover, using the approach as in Nicaise and Pignotti's work [24], we obtain that (2.7e) with $z(x, 0, s, t) = v$ has a unique solution

$$z(x, \rho, s, t) = u(x)e^{-\rho s} - f_1(x)\, e^{-\rho s} + se^{-\rho s} \int_0^\rho f_5(x, \tau, s, t)e^{\tau s}\, d\tau.$$

In particular, $z(x, 1, s, t) = u(x)e^{-s} + z_0(x, s, t)$ with $z_0 \in L^2((0, L) \times (\tau_1, \tau_2))$ defined by

$$z_0(x, s, t) = -f_1(x)\, e^{-s} + se^{-s} \int_0^1 f_5(x, \tau, s, t)e^{\tau s}\, d\tau.$$

Inserting (2.8) into (2.7b)–(2.7d), we get

$$\mu_1 u - [(p(x)u_x + 2\delta(x)u_x)_x + \gamma_1\theta_x + \gamma_2 P_x] = g_1, \tag{2.9a}$$
$$\lambda\theta - rk\theta_{xx} + hdP_{xx} - (r\gamma_1 - d\gamma_2)\, u_x = g_2, \tag{2.9b}$$
$$\lambda P - chP_{xx} + kd\theta_{xx} - (c\gamma_2 - d\gamma_1)\, u_x = g_3, \tag{2.9c}$$

where

$$\begin{aligned}
\mu_1 &= m(x) + \mu_0 + \int_{\tau_1}^{\tau_2} \mu(s)e^{-s} ds,\\
g_1 &= \mu_0 f_1 + m(x)(f_1 + f_2) - (2\delta(x)f_{1x})_x - \int_{\tau_1}^{\tau_2} \mu(s)z_0(x, s, t)\, ds,\\
g_2 &= \lambda f_3 - (r\gamma_1 - d\gamma_2)\, f_{1x},\\
g_3 &= \lambda f_4 - (c\gamma_2 - d\gamma_1)\, f_{1x}.
\end{aligned}$$

Multiplying (2.9a) by $\widetilde{u}$, (2.9b) by $\dfrac{c}{\lambda}\tilde{\theta}$, (2.9c) by $\dfrac{r}{\lambda}\tilde{P}$, (2.9b) by $\dfrac{d}{\lambda}\tilde{P}$, and (2.9c) by $\dfrac{d}{\lambda}\tilde{\theta}$ and integrating their sum over $(0, L)$, we can obtain the following variational formulation:

$$\mathcal{B}\left((u, \theta, P)^T, \left(\widetilde{u}, \tilde{\theta}, \tilde{P}\right)^T\right) = \mathcal{G}\left(\widetilde{u}, \tilde{\theta}, \tilde{P}\right)^T, \tag{2.10}$$

where the bilinear form $\mathcal{B} : \left[H_0^1(0, L) \times L^2(0, L) \times L^2(0, L)\right]^2 \to \mathbb{R}$ is defined by

$$\begin{aligned}
&\mathcal{B}\left((u, \theta, P)^T, \left(\widetilde{u}, \tilde{\theta}, \tilde{P}\right)^T\right)\\
&= \mu_1 \int_0^L u\widetilde{u}\, dx + \int_0^L (p(x) + 2\delta(x))u_x\widetilde{u}_x\, dx + \gamma_1 \int_0^L \theta\widetilde{u}_x dx + \gamma_2 \int_0^L P\widetilde{u}_x\, dx
\end{aligned}$$

$$+ c\int_0^L \theta\tilde{\theta}\,dx + k\int_0^L \theta_x\tilde{\theta}_x\,dx + r\int_0^L P\tilde{P}\,dx + h\int_0^L P_x\tilde{P}_x\,dx$$
$$+ d\int_0^L \theta\tilde{P}\,dx + d\int_0^L P\tilde{\theta}\,dx - \gamma_2\int_0^L u_x\tilde{P}\,dx - \gamma_1\int_0^L u_x\tilde{\theta}\,dx,$$

and the linear form $\mathcal{G} : \left[H_0^1(0,L)\times L^2(0,L)\times L^2(0,L)\right]\to\mathbb{R}$ is defined by

$$\mathcal{G}\left(\widetilde{u},\tilde{\theta},\tilde{P}\right)^T = \int_0^L g_1\widetilde{u}\,dx + \frac{c}{\lambda}\int_0^L g_2\tilde{\theta}\,dx + \frac{r}{\lambda}\int_0^L g_3\tilde{P}\,dx$$
$$+ \frac{d}{\lambda}\int_0^L g_2\tilde{P}\,dx + \frac{d}{\lambda}\int_0^L g_3\tilde{\theta}\,dx.$$

Now we introduce the Hilbert space $\mathcal{V} = H_0^1(0,L)\times L^2(0,L)\times L^2(0,L)$ equipped with the norm

$$\|(u,\theta,P)\|_{\mathcal{V}}^2 = \|u\|_2^2 + \|u_x\|_2^2 + \|\theta\|_2^2 + \|\theta_x\|_2^2 + \|P\|_2^2 + \|P_x\|_2^2.$$

It is clear that $\mathcal{B}(\cdot,\cdot)$ and $\mathcal{G}(\cdot)$ are bounded. Furthermore, we can obtain that there exists a positive constant κ such that

$$\mathcal{B}\left((u,\theta,P)^T,(u,\theta,P)^T\right)$$
$$= \mu_1\int_0^L u^2\,dx + \int_0^L (p(x)+2\delta(x))u_x^2\,dx + c\int_0^L \theta^2\,dx$$
$$+ k\int_0^L \theta_x^2\,dx + r\int_0^L P^2\,dx + h\int_0^L P_x^2\,dx + 2d\int_0^L P\theta\,dx \geq \kappa\,\|(u,\theta,P)\|_{\mathcal{V}}^2,$$

which implies that $\mathcal{B}(\cdot,\cdot)$ is coercive.

Consequently, by the Lax–Milgram theorem, problem (2.10) has a unique solution

$$(u,\theta,P)\in H_0^1(0,L)\times L^2(0,L)\times L^2(0,L).$$

To obtain more regularity, we take $\left(\tilde{\theta},\tilde{P}\right) = (0,0)$ to obtain from (2.10)

$$\mu_1\int_0^L u\widetilde{u}\,dx + \int_0^L (p(x)+2\delta(x))u_x\widetilde{u}_x\,dx + \gamma_1\int_0^L \theta\widetilde{u}_x\,dx + \gamma_2\int_0^L P\widetilde{u}_x\,dx$$
$$= \int_0^L g_1\widetilde{u}dx,\quad \widetilde{u}\in H_0^1(0,L),$$

which implies

$$-(p(x)+2\delta(x))u_{xx} = -\mu_1 u + \gamma_1\theta_x + \gamma_2 P_x + g_1.$$

Thus, by the regularity theory for the linear elliptic equations, it follows that

$$u\in H^2(0,L)\cap H_0^1(0,L).$$

By the same arguments, we can get

$$\theta,P\in H_0^1(0,L).$$

Hence, there exists a unique $U = (u,v,\theta,P,z)^T \in D(\mathcal{A})$ such that (2.10) is satisfied. Therefore, the operator $\mathrm{Id}-\mathcal{A}$ is surjective. At last, the result of Theorem 2.1 follows from the Lumer–Phillips theorem. □

3. Exponential stability

In this section, we prove the exponential decay for system (2.1), (2.2). It is achieved by using the perturbed energy method. We define the energy functional $E(t)$ as

$$E(t) = \frac{1}{2}\int_0^L \left[m(x)\,u_t^2 + p(x)\,u_x^2 + c\theta^2 + 2d\theta P + rP^2\right] dx$$
$$+\frac{1}{2}\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu(s)|\,z^2(x,\rho,s,t)\,ds\,d\rho\,dx. \tag{3.1}$$

Noting (1.3), we have for $\theta, P \neq 0$,

$$c\theta^2 + 2d\theta P + rP^2 = \frac{\lambda}{r}\theta^2 + \left(\frac{d}{\sqrt{r}}\theta + \sqrt{r}P\right)^2 > 0.$$

Thus we get that the energy $E(t)$ is positive.

The stability result reads as follows.

Theorem 3.1. *Let (u, v, θ, P, z) be the solution to (2.1), (2.2) and let (1.4) hold. Then there exist two positive constants k_0 and k_1 such that*

$$E(t) \leq k_0 e^{-k_1 t}, \quad t \geq 0. \tag{3.2}$$

To prove the theorem we will use the following lemmas.

Lemma 3.2 (Poincaré-type Scheeffer's inequality [21])**.** *Let $\phi \in H_0^1(0,L)$. Then*

$$\int_0^L |\phi|^2\,dx \leq \frac{L^2}{\pi^2}\int_0^L |\phi_x|^2\,dx.$$

Lemma 3.3 (Mean value theorem [2])**.** *Let (u, v, θ, P) be the solution to system (1.1), (1.2), with an initial data in $D(\mathcal{A})$. Then, for any $t > 0$, there exists a sequence of real numbers (depending on t), denoted by $\xi_i \in [0, L]$ $(i = 1, \ldots, 6)$, such that*

$$\int_0^L p(x)\,u_x^2 dx = p(\xi_1)\int_0^L u_x^2 dx, \qquad \int_0^L m(x)\,u^2 dx = m(\xi_2)\int_0^L u^2 dx,$$
$$\int_0^L m(x)\,u_t^2 dx = m(\xi_3)\int_0^L u_t^2 dx, \qquad \int_0^L \delta(x)\,u^2 dx = \delta(\xi_4)\int_0^L u^2 dx,$$
$$\int_0^L \delta(x)\,u_x^2 dx = \delta(\xi_5)\int_0^L u_x^2 dx, \qquad \int_0^L \delta(x)\,u_{xt}^2 dx = \delta(\xi_6)\int_0^L u_{xt}^2 dx.$$

Proof. Since $p(x)$, $m(x)$ and $\delta(x)$ are continuous functions on $x \in [0, L]$, the conclusion is straightforward using the mean value theorem. Moreover, it is obvious that $p(\xi_1)$, $m(\xi_2)$, $m(\xi_3)$, $\delta(\xi_4)$, $\delta(\xi_5)$ and $\delta(\xi_6)$ all are positive and bounded from above and below. □

Lemma 3.4. *Let (u, v, θ, P, z) be the solution to (2.1), (2.2) and let (1.4) hold. Then the energy functional, defined by equation (3.1), satisfies*

$$E'(t) \leq -2\int_0^L \delta(x)u_{tx}^2 dx - \left(\mu_0 - \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds\right)\int_0^L u_t^2 dx - k\int_0^L \theta_x^2 dx - h\int_0^L P_x^2 dx. \tag{3.3}$$

Proof. Multiplying (2.2a)–(2.2c) by u_t, θ, and P, respectively, and integrating over $(0, L)$, using integration by parts and the boundary conditions, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^L m(x)u_t^2\, dx + \frac{1}{2}\frac{d}{dt}\int_0^L p(x)u_x^2\, dx = -\int_0^L 2\delta(x)u_{tx}^2\, dx - \int_0^L \mu_0 u_t^2\, dx - \int_0^L u_t\int_{\tau_1}^{\tau_2} \mu(s)z(x,1,s,t)\, ds\, dx + \int_0^L \gamma_1\theta_x u_t\, dx + \int_0^L \gamma_2 P_x u_t\, dx, \tag{3.4}$$

$$\frac{1}{2}\frac{d}{dt}\int_0^L c\theta^2\, dx + \int_0^L dP_t\theta dx = -\int_0^L k\theta_x^2\, dx - \int_0^L \gamma_1 u_t\theta_x\, dx, \tag{3.5}$$

$$\frac{1}{2}\frac{d}{dt}\int_0^L rP^2 dx + \int_0^L d\theta_t P dx = -\int_0^L hP_x^2 dx - \int_0^L \gamma_2 u_t P_x dx. \tag{3.6}$$

On the other hand, multiplying (2.2d) by $|\mu(s)|\, z$ and integrating over $(0, L) \times (0, 1) \times (\tau_1, \tau_2)$, recalling that $z(x, 0, s, t) = u_t$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu(s)|\, z^2(x,\rho,s,t)\, ds\, d\rho\, dx = -\frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)\, ds\, dx + \frac{1}{2}\int_0^L u_t^2\int_{\tau_1}^{\tau_2} |\mu(s)|\, ds\, dx. \tag{3.7}$$

A combination of equations (3.4)–(3.7) gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^L \left[m(x)u_t^2 + p(x)u_x^2 + c\theta^2 + 2d\theta P + rP^2\right] dx \\
&\quad + \frac{1}{2}\frac{d}{dt}\int_0^L\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu(s)|\, z^2(x,\rho,s,t)\, ds\, d\rho\, dx \\
&= -\int_0^L 2\delta(x)u_{tx}^2 dx - \int_0^L \mu_0 u_t^2 dx - \int_0^L u_t\int_{\tau_1}^{\tau_2} \mu(s)z(x,1,s,t)\, ds\, dx \\
&\quad - \int_0^L k\theta_x^2\, dx - \int_0^L hP_x^2 dx - \frac{1}{2}\int_0^L\int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)\, ds\, dx \\
&\quad + \frac{1}{2}\int_0^L u_t^2\int_{\tau_1}^{\tau_2} |\mu(s)|\, ds\, dx.
\end{aligned} \tag{3.8}$$

Now, using Young's inequality, we obtain

$$
\begin{aligned}
&-\int_0^L u_t \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s,t)\,ds\,dx \\
&\qquad \leq \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu(s)|\,ds \int_0^1 u_t^2\,dx + \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)\,ds\,dx. \qquad (3.9)
\end{aligned}
$$

The substitution of (3.9) into (3.8), by using (1.4), gives (3.3), which concludes the proof. □

Lemma 3.5. *Let* (u, v, θ, P, z) *be the solution to* (2.1), (2.2) *and let* (1.4) *hold. Then the functional*

$$
L_1(t) = \int_0^L m(x) u u_t\,dx + \int_0^L \delta(x) u_x^2\,dx
$$

satisfies, for any $\varepsilon > 0$, *the estimate*

$$
\begin{aligned}
L_1'(t) \leq &-\left(p(\xi_1) - \frac{L^2\varepsilon}{\pi^2}\right)\int_0^L u_x^2\,dx + \frac{\mu_0}{\varepsilon}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)\,ds\,dx \\
&+ \frac{\gamma_1^2}{\varepsilon}\int_0^L \theta_x^2\,dx + \frac{\gamma_2^2}{\varepsilon}\int_0^L P_x^2\,dx + \left(m(\xi_3) + \frac{\mu_0^2}{\varepsilon}\right)\int_0^L u_t^2\,dx. \qquad (3.10)
\end{aligned}
$$

Proof. Taking the derivative of $L_1(t)$ with respect to t, using (2.2a), we have

$$
\begin{aligned}
L_1'(t) = &-\int_0^L p(x) u_x^2\,dx - \mu_0 \int_0^L u_t u\,dx - \int_0^L u \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s,t)\,ds\,dx \\
&+ \gamma_1 \int_0^L \theta_x u\,dx + \gamma_2 \int_0^L P_x u\,dx + \int_0^L m(x) u_t^2\,dx. \qquad (3.11)
\end{aligned}
$$

By using Young's inequality, Lemma 3.2 and (1.4), we get for all $\varepsilon > 0$,

$$
-\mu_0 \int_0^L u_t u\,dx \leq \frac{\mu_0^2}{\varepsilon}\int_0^L u_t^2\,dx + \frac{L^2\varepsilon}{4\pi^2}\int_0^L u_x^2\,dx, \qquad (3.12)
$$

$$
\begin{aligned}
&-\int_0^L u \int_{\tau_1}^{\tau_2} \mu(s)\, z(x,1,s,t)\,ds\,dx \\
&\qquad \leq \frac{L^2\varepsilon}{4\pi^2}\int_0^L u_x^2\,dx + \frac{\mu_0}{\varepsilon}\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)\,ds\,dx, \qquad (3.13)
\end{aligned}
$$

$$
\gamma_1 \int_0^L \theta_x u\,dx \leq \frac{\gamma_1^2}{\varepsilon}\int_0^L \theta_x^2\,dx + \frac{L^2\varepsilon}{4\pi^2}\int_0^L u_x^2\,dx, \qquad (3.14)
$$

$$
\gamma_2 \int_0^L P_x u\,dx \leq \frac{\gamma_2^2}{\varepsilon}\int_0^L P_x^2\,dx + \frac{L^2\varepsilon}{4\pi^2}\int_0^L u_x^2\,dx. \qquad (3.15)
$$

From Lemma 3.3, we have

$$
-\int_0^L p(x) u_x^2\,dx = -p(\xi_1)\int_0^L u_x^2\,dx, \qquad (3.16)
$$

$$\int_0^L m(x)u_t^2\,dx = m(\xi_3)\int_0^L u_t^2\,dx. \tag{3.17}$$

Then (3.10) follows from (3.11)–(3.17). □

Lemma 3.6. *Let (u, v, θ, P, z) be a solution to (2.1), (2.2) and let (1.4) hold. Then the functions*

$$L_2(t) = \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho}\,|\mu(s)|\,z^2(x,\rho,s,t)\,ds\,d\rho\,dx$$

satisfy, for some positive constant n_1, the estimates

$$\begin{aligned} L_2'(t) \le{}& -n_1\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu(s)|\,z^2(x,\rho,s,t)dsd\rho dx \\ & - n_1\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\,z^2(x,1,s,t)dsdx + \mu_0\int_0^L u_t^2dx. \end{aligned} \tag{3.18}$$

Proof. By differentiating $L_2(t)$ with respect to t and using equation (2.2d), we obtain

$$\begin{aligned} L_2'(t) &= -2\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho}\,|\mu(s)|\,z(x,\rho,s,t)z_\rho(x,\rho,s,t)\,ds\,d\rho\,dx \\ &= -\frac{d}{d\rho}\int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho}\,|\mu(s)|\,z^2(x,\rho,s,t)\,ds\,d\rho\,dx \\ &\quad - \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho}\,|\mu(s)|\,z^2(x,\rho,s,t)\,ds\,d\rho\,dx \\ &= -\int_0^L \int_{\tau_1}^{\tau_2} |\mu_1(s)|\,[e^{-s}z^2(x,1,s,t) - z^2(x,0,s,t)]\,ds\,dx \\ &\quad - \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho}\,|\mu(s)|\,z^2(x,\rho,s,t)\,ds\,d\rho\,dx. \end{aligned}$$

Using the fact that $z(x,0,s,t) = u_t$ and $e^{-s} \le e^{-s\rho} \le 1$ for all $0 < \rho < 1$, we obtain

$$\begin{aligned} L_2'(t) \le{}& -\int_0^L \int_{\tau_1}^{\tau_2} e^{-s}\,|\mu(s)|\,z^2\,(x,1,s,t)\,ds\,dx + \int_{\tau_1}^{\tau_2} |\mu(s)|\,ds\int_0^L u_t^2\,dx \\ & - \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho}\,|\mu(s)|\,z^2(x,\rho,s,t)\,ds\,d\rho\,dx. \end{aligned}$$

Because $-e^{-s}$ is an increasing function, we have $-e^{-s} \le -e^{-\tau_2}$ for all $s \in [\tau_1, \tau_2]$.
Finally, setting $n_1 = e^{-\tau_2}$ and recalling (1.4), we obtain (3.18). □

Further, we turn to the proving of our main result in this section.

Proof of Theorem 3.1. We define the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = NE(t) + L_1(t) + N_1 L_2(t), \tag{3.19}$$

where N and N_1 are positive constants that will be chosen later.

By differentiating $\mathcal{L}(t)$, exploiting (3.3), (3.10), and (3.18), and using Lemmas 3.2 and 3.3, we get

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\left(p(\xi_1) - \frac{L^2\varepsilon}{\pi^2}\right)\right]\int_0^L u_x^2\,dx \\
& - \left[2N\delta(\xi_6) + \left(\mu_0 - \int_{\tau_1}^{\tau_2} |\mu(s)|\,ds\right)\frac{L^2}{\pi^2}N\right. \\
& \left. - \left(m(\xi_3) + \frac{\mu_0^2}{\varepsilon}\right)\frac{L^2}{\pi^2} - \frac{\mu_0 L^2}{\pi^2}N_1\right]\int_0^L u_{tx}^2\,dx \\
& - \left[kN - \frac{\gamma_1^2}{\varepsilon}\right]\int_0^L \theta_x^2 dx - \left[hN - \frac{\gamma_2^2}{\varepsilon}\right]\int_0^L P_x^2\,dx \\
& - n_1 N_1 \int_0^L \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu(s)|\,z^2(x,\rho,s,t)\,ds\,d\rho\,dx \\
& - \left[n_1 N_1 - \frac{\mu_0}{\varepsilon}\right]\int_0^L \int_{\tau_1}^{\tau_2} |\mu(s)|\,z^2(x,1,s,t)\,ds\,dx.
\end{aligned}$$

At this point, taking

$$\varepsilon = \frac{\pi^2 p(\xi_1)}{2L^2},$$

we then choose N_1 large enough such that

$$N_1 > \frac{2L^2\mu_0}{n_1\pi^2 p(\xi_1)}.$$

After that we choose N sufficiently large such that

$$\begin{aligned}
N > \max & \left\{\frac{4L^2\gamma_1^2}{\pi^2 p(\xi_1)k}, \frac{4L^2\gamma_2^2}{\pi^2 p(\xi_1)h},\right. \\
& \left. \frac{n_1\pi^2 p(\xi_1) m(\xi_3) L^2 + 2L^4\mu_0^2 n_1 + 2L^4\mu_0^2}{n_1\pi^2 p(\xi_1)\left(2\delta(\xi_6)\pi^2 + \left(\mu_0 - \int_{\tau_1}^{\tau_2}|\mu(s)|\,ds\right)L^2\right)}\right\}.
\end{aligned}$$

Consequently, from the above, we deduce that there exists a positive constant α_0 such that

$$\mathcal{L}'(t) \leq -\alpha_0 E(t), \quad t \geq 0. \tag{3.20}$$

On the other hand, it is not hard to see that $\mathcal{L}(t) \sim E(t)$, i.e., there exist two positive constants α_1 and α_2 such that

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t), \quad t \geq 0. \tag{3.21}$$

Combining (3.20) and (3.21), we obtain that

$$\mathcal{L}'(t) \leq -k_1 \mathcal{L}(t), \quad t \geq 0, \tag{3.22}$$

where $k_1 = \dfrac{\alpha_0}{\alpha_2}$. A simple integration of (3.22) over $(0, t)$ yields

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-k_1 t}, \quad t \geq 0.$$

It gives the desired result, Theorem 3.1, when combined with the equivalence of $\mathcal{L}(t)$ and $E(t)$. □

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Madani Douib,

Department of Mathematics, Higher College of Teachers (ENS) of Laghouat, Algeria,
E-mail: madanidouib@gmail.com

Salah Zitouni,

Department of Mathematics and Informatics, University of Souk Ahras, P.O. Box 1553, Souk Ahras 41000, Algeria,
E-mail: zitsala@yahoo.fr

Abdelhak Djebabla,

Department of Mathematics, Faculty of Sciences, University of Annaba, P.O. Box 12, Annaba 23000, Algeria,
E-mail: adjebabla@yahoo.com

Експоненціальна стабільність для системи із гнучкою структурою з ефектами термодифузії та розподіленого загаювання

Madani Douib, Salah Zitouni, and Abdelhak Djebabla

У статті досліджується коректність та асимптотика розв'язків для гнучкої структури з ефектами термодифузії та розподіленого загаювання. За відповідних припущень щодо ваги демпфування та ваги розподіленого загаювання, доведено існування і єдиність розв'язку з використанням теорії півгруп. Далі за допомогою методу збуреної енергії та побудови деяких функціоналів Ляпунова доведено експоненціальну спадність розв'язку.

Ключові слова: гнучка структура, термодифузійні ефекти, розподілене загаювання, коректність, експоненціальна стабільність