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Existence Results for Some Anisotropic Elliptic Problems Having Variable Exponent and L^1-data

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In this paper, we propose to study the existence of entropy solutions for the strongly nonlinear anisotropic elliptic equation

$$Au + H(x,u,\nabla u) = f \qquad \text{in} \quad \Omega,$$

where f belongs to $L^1(\Omega)$, A is a Leray–Lions operator and H is a nonlinear lower order term with nonstandard growth with respect to $|\nabla u|$ (i.e., such that $|H(x,s,\xi)| \leq c(x) + b(|s|)\sum_{i=1}^{N}|\xi_i|^{p_i(x)}$), but without assuming the sign condition $H(x,s,\xi)s \geq 0$. A concrete example is given to illustrate the existence result.

Key words: anisotropic, variable exponent, Sobolev spaces, nonlinear elliptic problem, penalization techniques, entropy solutions

Mathematical Subject Classification 2020: 35J60, 46E35, 35D35

1. Introduction

This paper is devoted to the study of some classes of anisotropic elliptic boundary value problems of the form

$$\begin{cases} Au + H(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

in a bounded open domain $\Omega \in \mathbb{R}^N$, $(N \geq 2)$. The interest in these problems relies on the fact that they are strongly nonlinear and non-homogeneous. Here, the anisotropic operator under considerarion is a differential operator involving partial derivatives with different powers $p_i(x) > 1$, that is,

$$Au = -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u),$$

where $D^i = \partial/\partial x_i$, for $i = 1,\ldots,N$. Therefore, in order to prove the existence results, we need to consider a different functional setting from the classical Sobolev space. Indeed, the appropriate space to capture such formulated problem is the anisotropic variable exponent Sobolev space $W_0^{1,\vec{p}(\cdot)}(\Omega)$ recently introduced by Mihalescu–Pucci–Raduslescu in [20].

Another relevant class of operators, for which a general and almost complete theory is now available, is without a doubt one of the equations with the so-called $p(\cdot)-$growth, i.e., the $p(\cdot)$-Laplacian equation

$$-\text{div }(|\nabla u|^{p(x)-2}\nabla u) = f,$$

where $p : \Omega \to (1, \infty)$ is a bounded and continuous function. We recall some papers (and references therein), in which this theory is developed: [3, 12, 15, 22].

We mention that partial differential equations and variational problems related to $p(\cdot)$-growth conditions have been extensively studied in the last decades. The reason is that they can model various phenomena arising from the study of elastic mechanics, electrorheological fluids or image restoration.

The interest of considering anisotropic problems with variable exponents is linked to a large scale of applications containing some non-homogeneous materials that have different behaviors in different space directions. It was established that for an appropriate treatment of these materials we can not rely on the classical Sobolev space and that we have to allow the exponent to vary instead. Furthermore, anisotropic fluids are widely applicable in the common life. Most of the modern electronic displays are liquid crystal based. We can mention also magnetorheological shock absorber of buildings or in the automotive industry, magnetorheological damper and electrorheological clutch.

Let us note that the definition of the variable exponent Lebesgue spaces and the variable exponent Sobolev spaces requires only the measurability of $p(\cdot)$, in this work we do not need to use Sobolev and Poincaré inequalities. Moreover, the sharp Sobolev inequality is proved for $p(\cdot)$ log-Hölder continuous, while the Poincaré inequality requires only the continuity of $p(\cdot)$, (for more details, we refer to [6, 13, 17]).

It should be mentioned that in [1] Benboubker et al studied the following problem which is quite close to (1.1):

$$\begin{cases} Au + H(x, u, \nabla u) + \delta |u|^{p_0(x)-2}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the nonlinear term H satisfies some growth condition without the sign condition. In [1], the authors proved the existence of solutions in the convex class $K_\psi = \{u \in W_0^{1,\vec{p}(\cdot)}(\Omega) \mid u \geq \psi \text{ a.e. in } \Omega\}$, where ψ is a fixed obstacle function such that $\psi^+ \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

The novelty of our work is in extending the results in [2] by taking into account a more general type of operator, that is, the anisotropic operator, and prove a new existence result without any sign condition on H.

The main difficulty in proving the existence of a solution stems from the fact that $H(x, u, \nabla u)$ does not assume the sign condition (i.e., $H(x, s, \xi)s \geq 0$). In other words, the term $H(x, u, \nabla u)$ is said to be an absorption term. In this case, a detailed picture of what happens is available (see, e.g., [2, 5, 9–11]). Secondly, we have to face the problem that the operator Au is not coercive in the anisotropic variable exponent Sobolev space $W_0^{1,\vec{p}(\cdot)}(\Omega)$. For this reason we try to overcome this difficulty by using a penalization term $\frac{1}{n}|u|^{p_0-2}u$ in the approximate problems.

Motivated by the ideas in [1,21], the method used here is to define approximate problems, then to obtain a priori estimates for their solutions using suitable test functions (exponential type) and finally to prove a new compactness property in order to pass to the limit.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on anisotropic variable exponent Sobolev spaces. In Section 3, we state the problem and formulate the main result. We also present some auxiliary results which will be used to prove the existence theorem in Section 4, and finally we give a concrete example of our main result.

2. Preliminaries

In this section, we will introduce some definitions and properties concerning the anisotropic variable exponent Sobolev space used for the study of our main existence result.

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$. We denote

$$\mathcal{C}_+(\Omega) = \{\text{measurable function} \quad p(\cdot) : \Omega \longmapsto \mathbb{R} \quad \text{such that} \quad 1 < p^- \leq p^+ < \infty\},$$

where

$$p^- = ess\inf\{p(x) \mid x \in \Omega\} \quad \text{and} \quad p^+ = ess\sup\{p(x) \mid x \in \Omega\}.$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \mapsto \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)} dx$$

is finite, and the expression

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 \mid \rho_{p(\cdot)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|.\|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1 < p^- \leq p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality

$$\left| \int_\Omega uv\ dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

An important role in manipulating the generalized Lebesgue spaces is played by the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result.

Proposition 2.1 (see [16, 23]). *If $u_n, u \in L^{p(\cdot)}(\Omega)$, then the following properties hold true:*

(i) $\|u\|_{p(\cdot)} < 1$ *(resp.*$, = 1, > 1)$ $\Leftrightarrow$ $\rho(u) < 1$ *(resp.*$, = 1, > 1)$,

(ii) $\|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)}^{p^-} \leq \rho(u) \leq \|u\|_{p(\cdot)}^{p^+}$ *and*

$$\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|_{p(\cdot)}^{p^+} \leq \rho(u) \leq \|u\|_{p(\cdot)}^{p^-},$$

(iii) $\|u_n\|_{p(\cdot)} \to 0 \Leftrightarrow \rho(u_n) \to 0$ *and* $\|u_n\|_{p(\cdot)} \to \infty \Leftrightarrow \rho(u_n) \to \infty$.

We notice that the norm convergence and the modular convergence are equivalent.
Now we define the variable exponent Sobolev space by

$$W^{1,p(\cdot)}(\Omega) = \{\, u \in L^{p(\cdot)}(\Omega) \mid |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space equipped with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}, \quad u \in W^{1,p(\cdot)}(\Omega).$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$, and we define the Sobolev exponent by

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{for } p(x) < N, \\ \infty & \text{for } p(x) \geq N. \end{cases}$$

Proposition 2.2 (see [14]).

(i) *Assuming $1 < p^- \leq p^+ < \infty$, the spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.*

(ii) *If $q(\cdot) \in \mathcal{C}_+(\Omega)$ and $q(x) < p^*(x)$ for a.e. $x \in \Omega$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.*

In order to study problem (1.1), let us introduce the functional spaces where it will be discussed. We will give just a brief review of some basic concepts and facts of the anisotropic variable exponent Sobolev spaces. For more details we refer the readers to [20].

We now recall some facts on the anisotropic variable exponent Sobolev spaces used in the present paper. Let $p_0(x), p_1(x), \ldots, p_N(x)$ be $N+1$ variable exponents in $\mathcal{C}_+(\Omega)$. We denote

$$\vec{p}(\cdot) = \{p_0(\cdot), \ldots, p_N(\cdot)\}, \quad D^0 u = u, \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for} \quad i = 1, \ldots, N$$

and define

$$\underline{p} = \min(p_0^-, p_1^-, \ldots, p_N^-) \quad \text{then } \underline{p} > 1.$$

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follows:

$$W^{1,\vec{p}(\cdot)}(\Omega) = \{u \in L^{p_0(\cdot)}(\Omega) \mid D^i u \in L^{p_i(\cdot)}(\Omega) \text{ for } i = 1, 2, \ldots, N\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}(\cdot)} = \sum_{i=0}^{N} \|D^i u\|_{L^{p_i(\cdot)}(\Omega)}. \tag{2.1}$$

We define also $W_0^{1,\vec{p}(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}(\cdot)}(\Omega)$ with respect to the norm (2.1). The space $\big(W_0^{1,\vec{p}(\cdot)}(\Omega), \|u\|_{1,\vec{p}(\cdot)}\big)$ is a separable and reflexive Banach space (cf. [20]).

Lemma 2.3. *We have the following continuous and compact embeddings:*

- *if $\underline{p} < N$, then $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, for $q \in [\underline{p}, \underline{p}^*[$, where $\underline{p}^* = \frac{N\underline{p}}{N-\underline{p}}$;*
- *if $\underline{p} = N$, then $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, for $q \in [\underline{p}, +\infty[$;*
- *if $\underline{p} > N$, then $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$.*

The proof of this lemma follows from the fact that the embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow W_0^{1,\underline{p}}(\Omega)$ is continuous, and in view of the compact embedding theorem for Sobolev spaces.

The dual of $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is denoted by $W^{-1,\vec{p}\,'(\cdot)}(\Omega)$, where $\vec{p}\,'(\cdot) = \{p_0'(\cdot), \dots, p_N'(\cdot)\}$ with $\frac{1}{p_i'(\cdot)} + \frac{1}{p_i(\cdot)} = 1$, (cf. [8] for the constant exponent case).

Proposition 2.4. *For each $F \in W^{-1,\vec{p}\,'(\cdot)}(\Omega)$, there exists $F_i \in L^{p_i'(\cdot)}(\Omega)$ for $i = 0, 1, \dots, N$ such that $F = F_0 - \sum_{i=1}^N D^i F_i$. Moreover, for any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, we have*

$$\langle F, u\rangle = \sum_{i=0}^{N} \int_\Omega F_i\, D^i u \, dx.$$

We define a norm on the dual space by

$$\|F\|_{-1,\vec{p}\,'(\cdot)} = \inf \sum_{i=0}^{N} \|F_i\|_{p_i'(\cdot)}.$$

For any $k > 0$, we define the truncation function $T_k(\cdot)$ by : $T_k(s) := \max\{-k, \min\{k, s\}\}$. We set

$$\mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega) := \{u : \Omega \mapsto \mathbb{R} \mid u \text{ is measurable and } T_k(u) \in W_0^{1,\vec{p}(\cdot)}(\Omega) \text{ for any } k > 0\}.$$

Proposition 2.5. *Let $u \in \mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$. For any $i \in \{1, \dots, N\}$, there exists a unique measurable function $v_i : \Omega \mapsto \mathbb{R}$ such that*

$$\forall k > 0 \quad D^i T_k(u) = v_i \chi_{\{|u|<k\}} \quad a.e.\ x \in \Omega,$$

where χ_B denotes the characteristic function of a measurable set B. The functions v_i are called the weak partial derivatives of u and are still denoted by $D^i u$. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v_i coincides with the standard distributional derivative of u, that is, $v_i = D^i u$.

3. Statement of the problem and the main result

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ and $p_i(\cdot) \in \mathcal{C}_+(\Omega)$ for $i = 0, 1, \ldots, N$. We assume that

$$\forall x \in \Omega \quad p_0(x) \geq \max\{p_i(x) \mid i = 1, 2, \ldots, N\}. \tag{3.1}$$

Here A is an operator of Leray–Lions type acting from $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into its dual $W^{-1,\vec{p}'(\cdot)}(\Omega)$ defined by the formula

$$Au = -\sum_{i=1}^{N} D^i a_i(x, u, \nabla u),$$

where $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ is a *Carathéodory* function for $i = 1, \ldots, N$ (measurable with respect to x in Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) which satisfies the following conditions:

$$|a_i(x, s, \xi)| \leq \beta \left(K_i(x) + |s|^{p_i(x)-1} + |\xi_i|^{p_i(x)-1}\right) \qquad \text{for } i = 1, \ldots, N, \tag{3.2}$$

$$a_i(x, s, \xi)\xi_i \geq \alpha |\xi_i|^{p_i(x)} \qquad \text{for } i = 1, \ldots, N, \tag{3.3}$$

for all $\xi = (\xi_1, \ldots, \xi_N)$ and $\xi' = (\xi'_1, \ldots, \xi'_N)$, we have

$$(a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi'_i) > 0 \quad \text{with } \xi_i \neq \xi'_i, \tag{3.4}$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, with $K_i(x)$ being a nonnegative function lying in $L^{p'_i(\cdot)}(\Omega)$, and $\alpha, \beta > 0$.

As a consequence of (3.3) and the continuity of the function $a(x, s, \cdot)$ with respect to ξ, we have

$$a(x, s, 0) = 0.$$

The nonlinear term $H(x, s, \xi)$ is a Carathéodory function which satisfies only the growth condition

$$|H(x, s, \xi)| \leq c(x) + b(|s|) \sum_{i=1}^{N} |\xi_i|^{p_i(x)}, \tag{3.5}$$

where $b(\cdot) : \mathbb{R} \mapsto \mathbb{R}^+$ is a continuous positive function that belongs to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, while $c(\cdot) \in L^1(\Omega)$ is a nonnegative function.

We consider the following strongly nonlinear $\vec{p}(\cdot)-$elliptic problem:

$$\begin{cases} Au + H(x, u, \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where $f \in L^1(\Omega)$.

Definition 3.1. A function u is called an entropy solution of the strongly nonlinear $\vec{p}(\cdot)$-elliptic problem (3.6) if $u \in \mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$, $H(x,u,\nabla u) \in L^1(\Omega)$ and

$$\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u) D^i T_k(u-v)\,dx + \int_\Omega H(x,u,\nabla u) T_k(u-v)\,dx \leq \int_\Omega f T_k(u-v)\,dx$$

for every $v \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Theorem 3.2. *Assuming that* (3.2)–(3.5) *hold and* $f \in L^1(\Omega)$, *the problem* (3.6) *has at least one entropy solution.*

Remark 3.3. The assumption (3.1) is essential to ensure that $a_i(x,u,\nabla u)$ belongs to $L^{p_i'(\cdot)}(\Omega)$. In the case of $Au = -\sum_{i=1}^N D^i a_i(x,\nabla u)$, the existence of entropy solution is guaranteed without using this assumption.

To prove the existence theorem, we will need the following auxiliary results.

Lemma 3.4 (see [18, Theorem 13.47])**.** *Let* $(u_n)_n$ *be a sequence in* $L^1(\Omega)$ *and* $u \in L^1(\Omega)$ *such that*

(i) $u_n \to u$ *a.e. in* Ω,

(ii) $u_n \geq 0$ *and* $u \geq 0$ *a.e. in* Ω,

(iii) $\displaystyle\int_\Omega u_n\,dx \to \int_\Omega u\,dx$,

then $u_n \to u$ *in* $L^1(\Omega)$.

Lemma 3.5 (see [4])**.** *Let* $g \in L^{p(\cdot)}(\Omega)$ *and* $g_n \in L^{p(\cdot)}(\Omega)$ *with* $\|g_n\|_{p(\cdot)} \leq C$ *for* $1 < p(x) < \infty$.

If $g_n(x) \to g(x)$ *a.e. on* Ω, *then* $g_n \rightharpoonup g$ *in* $L^{p(\cdot)}(\Omega)$.

Lemma 3.6 (see [7])**.** *Assuming that* (3.2)–(3.4) *hold, let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* $W_0^{1,\vec{p}(\cdot)}(\Omega)$ *such that* $u_n \rightharpoonup u$ *in* $W_0^{1,\vec{p}(\cdot)}(\Omega)$ *and*

$$\int_\Omega (|u_n|^{p_0(x)-2}u_n - |u|^{p_0(x)-2}u)(u_n - u)\,dx \tag{3.7}$$

$$+\sum_{i=1}^{N} \int_\Omega (a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u))(D^i u_n - D^i u)\,dx \to 0, \tag{3.8}$$

then $u_n \longrightarrow u$ *in* $W_0^{1,\vec{p}(\cdot)}(\Omega)$ *for a subsequence.*

4. Proof of Theorem 3.2

***Step* 1: Approximate problems.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of smooth functions such that $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \leq |f|$. We consider the approximate problem

$$\begin{cases} A_n u_n + H_n(x,u_n,\nabla u_n) + \dfrac{1}{n}|u_n|^{p_0(x)-2}u_n = f_n \\ u_n \in W_0^{1,\vec{p}(\cdot)}(\Omega), \end{cases} \tag{4.1}$$

with $A_n v = -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, T_n(v), \nabla v)$ and $H_n(x, s, \xi) = T_n(H(x, s, \xi))$. Notice that

$$|H_n(x, s, \xi)| \le |H(x, s, \xi)| \quad \text{and} \quad |H_n(x, s, \xi)| \le n, \quad n \in \mathbb{N}^*.$$

We define the operator $R_n : W_0^{1,\vec{p}(\cdot)}(\Omega) \mapsto W^{-1,\vec{p}'(\cdot)}(\Omega)$ by

$$\langle R_n u, v\rangle = \int_\Omega H_n(x, u, \nabla u) v \, dx + \frac{1}{n} \int_\Omega |u|^{p_0(x)-2} u v \, dx, \quad u, v \in W_0^{1,\vec{p}(\cdot)}(\Omega).$$

Thanks to the generalized Hölder type inequality, for all $u, v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, we have

$$\begin{aligned}
\big|\langle R_n u, v\rangle\big| &= \left| \int_\Omega H_n(x, u, \nabla u) v \, dx + \frac{1}{n} \int_\Omega |u|^{p_0(x)-2} u v \, dx \right| \\
&\le \left(\frac{1}{p_0^-} + \frac{1}{(p_0')^-} \right) \left(\big\| H_n(x, u, \nabla u) \big\|_{p_0'(\cdot)} + \frac{1}{n} \big\| |u|^{p_0(x)-1} \big\|_{p_0'(x)} \right) \|v\|_{p_0(\cdot)} \\
&\le 2 \left(\left(\int_\Omega n^{p_0'(x)} \, dx + 1 \right)^{\frac{1}{(p_0')^-}} + \frac{1}{n} \left(\int_\Omega |u|^{p_0(x)} \, dx + 1 \right)^{\frac{1}{(p_0')^-}} \right) \|v\|_{1,\vec{p}(\cdot)} \\
&\le 2 \left(\left(n^{(p_0')^+} \operatorname{meas}(\Omega) + 1 \right)^{\frac{1}{(p_0')^-}} + \frac{1}{n} \left(\int_\Omega |u|^{p_0(x)} \, dx + 1 \right)^{\frac{1}{(p_0')^-}} \right) \|v\|_{1,\vec{p}(\cdot)} \\
&\le C_0 \|v\|_{1,\vec{p}(\cdot)}.
\end{aligned} \tag{4.2}$$

In view of Lemma A.1 (see Appendix) and the classical theorem of Lions (cf. [19, Theorem 2.7, page 180]), there exists at least one solution $u_n \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ of the problem (4.1).

***Step 2:* A priori estimates.** Let n be large enough ($n \ge k$). We define

$$B(s) = \frac{1}{\alpha} \int_0^s b(|\tau|) \, d\tau.$$

Note that since the function $b(\cdot)$ is integrable on $\mathbb{R}$, then $0 \le B(\infty) := \frac{1}{\alpha} \int_0^{+\infty} b(|t|) \, dt$ is finite. Thus, taking $T_k(u_n) e^{B(|u_n|)}$ as a test function in (4.1), we get

$$\begin{aligned}
&\frac{1}{\alpha} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i u_n b(|u_n|) |T_k(u_n)| e^{B(|u_n|)} dx \\
&+ \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n) e^{B(|u_n|)} \, dx \\
&+ \int_\Omega H_n(x, u_n, \nabla u_n) T_k(u_n) e^{B(|u_n|)} \, dx \\
&+ \frac{1}{n} \int_\Omega |u_n|^{p_0(x)-1} |T_k(u_n)| e^{B(|u_n|)} \, dx = \int_\Omega f_n T_k(u_n) e^{B(|u_n|)} \, dx.
\end{aligned}$$

Using (3.3) and (3.5), we have

$$\int_\Omega H_n(x, u_n, \nabla u_n) T_k(u_n) e^{B(|u_n|)} \, dx$$

$$
\begin{aligned}
&\leq \int_{\Omega} c(x)|T_k(u_n)|e^{B(|u_n|)}\,dx + \sum_{i=1}^{N}\int_{\Omega} b(|u_n|)|D^i u_n|^{p_i(x)}|T_k(u_n)|e^{B(|u_n|)}\,dx\\
&\leq k e^{B(\infty)}\int_{\Omega} c(x)\,dx\\
&\qquad + \frac{1}{\alpha}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_n),\nabla u_n)D^i u_n b(|u_n|)|T_k(u_n)|e^{B(|u_n|)}dx.
\end{aligned}
$$

As $e^{B(|u_n|)} \geq 1$ and $\frac{1}{n}\int_{\Omega}|u_n|^{p_0(x)-1}|T_k(u_n)|e^{B(|u_n|)}\,dx \geq 0$, it follows that there exists a constant C_1 that does not depend on n and k such that

$$
\sum_{i=1}^{N}\int_{\Omega}|D^i T_k(u_n)|^{p_i(x)}\,dx \leq \frac{k}{\alpha}e^{B(\infty)}(\|f(x)\|_1 + \|c(x)\|_1) \leq C_1 k, \tag{4.3}
$$

and we obtain

$$
\sum_{i=0}^{N}\int_{\Omega}|D^i T_k(u_n)|^{p_i(x)}\,dx \leq kC_1 + k^{p_0^+}\operatorname{meas}(\Omega). \tag{4.4}
$$

Thus, the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, and there exists a subsequence still denoted by $(T_k(u_n))_n$ and a function $\eta_k \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$
\begin{cases}
T_k(u_n) \rightharpoonup \eta_k & \text{in } W_0^{1,\vec{p}(\cdot)}(\Omega),\\
T_k(u_n) \to \eta_k & \text{in } L^{\underline{p}}(\Omega) \text{ and a.e. in } \Omega.
\end{cases} \tag{4.5}
$$

On the other hand, we have

$$
\sum_{i=1}^{N}\int_{\Omega}|D^i T_k(u_n)|^{p_i(x)}\,dx \geq \sum_{i=1}^{N}\int_{\Omega}(|D^i T_k(u_n)|^{\underline{p}} - 1)\,dx = \|\nabla T_k(u_n)\|_{\underline{p}}^{\underline{p}} - N|\Omega|.
$$

Thanks to (4.3), we deduce that there exists a constant C_2 that does not depend on k and n such that

$$
\|\nabla T_k(u_n)\|_{\underline{p}} \leq C_2 k^{\frac{1}{\underline{p}}} \quad \text{for } k \geq 1.
$$

By the Poincaré type inequality, we obtain

$$
\begin{aligned}
k\operatorname{meas}\{|u_n| > k\} &= \int_{\{|u_n|>k\}}|T_k(u_n)|\,dx \leq \int_{\Omega}|T_k(u_n)|\,dx\\
&\leq C_3\|T_k(u_n)\|_{\underline{p}} \leq C_4\|\nabla T_k(u_n)\|_{\underline{p}} \leq C_5 k^{\frac{1}{\underline{p}}}.
\end{aligned}
$$

Then we can conclude that

$$
\operatorname{meas}\big\{|u_n| > k\big\} \leq C_5\frac{1}{k^{1-\frac{1}{\underline{p}}}} \to 0 \quad \text{as} \quad k \to +\infty. \tag{4.6}
$$

For all $\delta > 0$, we have

$$\begin{aligned}\operatorname{meas}\{|u_n - u_m| > \delta\} \le \operatorname{meas}\{|u_n| > k\} + \operatorname{meas}\{|u_m| > k\} \\ + \operatorname{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}.\end{aligned}$$

Letting $\varepsilon > 0$, by using (4.6), we can choose $k = k(\varepsilon)$ large enough such that

$$\operatorname{meas}\{|u_n| > k\} \le \frac{\varepsilon}{3} \quad \text{and} \quad \operatorname{meas}\{|u_m| > k\} \le \frac{\varepsilon}{3}. \tag{4.7}$$

On the other hand, due to (4.5), we can assume that $(T_k(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Thus, for any $k > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\operatorname{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \le \frac{\varepsilon}{3} \qquad \text{for all } m, n \ge n_0(\delta, \varepsilon). \tag{4.8}$$

In view of (4.7) and (4.8), we deduce that

$$\forall \delta > 0 \ \forall \varepsilon > 0 \ \exists n_0 = n_0(\delta, \varepsilon) \ \forall n, m \ge n_0(\delta, \varepsilon) \quad \operatorname{meas}\{|u_n - u_m| > \delta\} \le \varepsilon,$$

which proves that the sequence $(u_n)_n$ is a Cauchy sequence in measure and then converges almost everywhere to some measurable function u. Consequently, we have

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{in } W_0^{1,\vec{p}(\cdot)}(\Omega),$$

and in view of Lebesgue's dominated convergence theorem, we obtain

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{in } L^{p_0(\cdot)}(\Omega) \text{ and a.e. in } \Omega.$$

***Step* 3: Strong convergence of truncations.** In the sequel, we will denote by $\varepsilon_i(n)$, $i = 1, 2, \dots$ various real-valued functions of real variables that converge to 0 as n tends to infinity.

Taking $h > k > 0$ and $M = 4k + h$, we set $b_k := \max\{b(s) : |s| \le k\}$. Let $\varphi_k(s) = s\exp(\gamma s^2)$, for $\gamma = \left(\frac{b_k}{2\alpha}\right)^2$. It is clear that

$$\varphi_k'(s) - \frac{b_k}{\alpha}|\varphi_k(s)| \ge \frac{1}{2}, \quad s \in \mathbb{R}.$$

For n large enough $(n \ge M)$, we define

$$z_n := u_n - T_h(u_n) + T_k(u_n) - T_k(u) \quad \text{and} \quad \omega_n := T_{2k}(z_n).$$

By taking $\varphi_k(\omega_n)e^{B(|u_n|)}$ as a test function in (4.1), we obtain

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i \omega_n \varphi_k'(\omega_n) e^{B(|u_n|)}\,dx \\ &\quad + \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i u_n \frac{b(|u_n|)}{\alpha} \varphi_k(\omega_n) \operatorname{sign}(u_n) e^{B(|u_n|)}\,dx\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega H_n(x,u_n,\nabla u_n)\varphi_k(\omega_n)e^{B(|u_n|)}\,dx+\frac{1}{n}\int_\Omega |u_n|^{p_0(x)-2}u_n\varphi_k(\omega_n)e^{B(|u_n|)}\,dx\\
&=\int_\Omega f_n\varphi_k(\omega_n)e^{B(|u_n|)}\,dx.
\end{aligned}
$$

It is easy to check that $\nabla\omega_n=0$ on $\{|u_n|\geq M\}$ and that $\varphi_k(\omega_n)$ has the same sign as u_n on the set $\{|u_n|>k\}$. Then, in view of (3.5), we obtain

$$
\begin{aligned}
&\sum_{i=1}^N\int_{\{|u_n|\leq M\}} a_i(x,T_n(u_n),\nabla u_n)D^i\omega_n\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&\qquad -\sum_{i=1}^N\int_{\{|u_n|\leq k\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\frac{b(|u_n|)}{\alpha}|\varphi_k(\omega_n)|e^{B(|u_n|)}\,dx\\
&\qquad -\sum_{i=1}^N\int_{\{|u_n|\leq k\}} b(|u_n|)|D^iu_n|^{p_i(x)}|\varphi_k(\omega_n)|e^{B(|u_n|)}\,dx\\
&\qquad +\frac{1}{n}\int_{\{|u_n|\leq k\}}|u_n|^{p_0(x)-2}u_n\varphi_k(\omega_n)e^{B(|u_n|)}\,dx\\
&\qquad \leq e^{B(\infty)}\int_\Omega(|f_n|+c(x))|\varphi_k(\omega_n)|\,dx.
\end{aligned}
$$

Now, using (3.3) and the fact that $\omega_n=T_k(u_n)-T_k(u)$ on $\{|u_n|\leq k\}$, we get

$$
\begin{aligned}
&\sum_{i=1}^N\int_{\{|u_n|\leq k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&\quad +\sum_{i=1}^N\int_{\{k<|u_n|\leq M\}} a_i(x,T_M(u_n),\nabla T_M(u_n))D^i\omega_n\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&\quad -\frac{2b_k}{\alpha}\sum_{i=1}^N\int_{\{|u_n|\leq k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u_n)|\varphi_k(\omega_n)|e^{B(|u_n|)}\,dx\\
&\quad +\frac{1}{n}\int_{\{|u_n|\leq k\}}|T_k(u_n)|^{p_0(x)-2}T_k(u_n)\varphi_k(\omega_n)e^{B(|u_n|)}\,dx\\
&\quad \leq e^{B(\infty)}\int_\Omega(|f_n|+c(x))|\varphi_k(\omega_n)|\,dx.
\end{aligned}\tag{4.9}
$$

Concerning the second term on the left-hand side of (4.9), we have

$$
\begin{aligned}
&\sum_{i=1}^N\int_{\{k<|u_n|\leq M\}} a_i(x,T_M(u_n),\nabla T_M(u_n))D^i\omega_n\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\\
&\quad =\sum_{i=1}^N\int_{\{k<|u_n|\leq M\}\cap\{|z_n|\leq 2k\}}\Big(a_i(x,T_M(u_n),\nabla T_M(u_n))\\
&\qquad \times D^i(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)}\Big)\,dx
\end{aligned}
$$

$$\geq -e^{B(\infty)}\varphi_k'(2k)\sum_{i=1}^N\int_{\{k<|u_n|\leq M\}}|a_i(x,T_M(u_n),\nabla T_M(u_n))|\,|D^iT_k(u)|\,dx$$

since $(|a_i(x,T_M(u_n),\nabla T_M(u_n))|)_n$ is bounded in $L^{p_i'(\cdot)}(\Omega)$. Then there exists $\phi_i\in L^{p_i'(\cdot)}(\Omega)$ such that $|a_i(x,T_M(u_n),\nabla T_M(u_n))|\rightharpoonup\phi_i$ in $L^{p_i'(\cdot)}(\Omega)$. Therefore,

$$\int_{\{k<|u_n|\leq M\}}|a_i(x,T_M(u_n),\nabla T_M(u_n))|\,|D^iT_k(u)|\,dx \to \int_{\{k<|u|\leq M\}}\phi_i\,|D^iT_k(u)|\,dx=0. \tag{4.10}$$

It follows that

$$\sum_{i=1}^N\int_{\{k<|u_n|\leq M\}}a_i(x,T_M(u_n),\nabla T_M(u_n))D^i\omega_n\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\geq\varepsilon_1(n). \tag{4.11}$$

Having in mind (4.9), we obtain

$$\begin{aligned}&\sum_{i=1}^N\int_{\{|u_n|\leq k\}}a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\\&\quad-\frac{2b_k}{\alpha}\sum_{i=1}^N\int_{\{|u_n|\leq k\}}a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u_n)|\varphi_k(\omega_n)|e^{B(|u_n|)}\,dx\\&\quad+\int_{\{|u_n|\leq k\}}\frac{1}{n}|T_k(u_n)|^{p_0(x)-2}T_k(u_n)\varphi_k(\omega_n)e^{B(|u_n|)}\,dx\\&\quad\leq e^{B(\infty)}\int_\Omega(|f_n|+c(x))|\varphi_k(\omega_n)|\,dx+\varepsilon_2(n).\end{aligned} \tag{4.12}$$

Now we will study each term on the left-hand side of (4.12).

First estimate: For the first term on the left-hand side of (4.12), we have $a_i(x,s,0)=0$. Then

$$\begin{aligned}&\sum_{i=1}^N\int_{\{|u_n|\leq k\}}a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\\&=\sum_{i=1}^N\int_\Omega\big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\big)\\&\qquad\qquad\times(D^iT_k(u_n)-D^iT_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\\&+\sum_{i=1}^N\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx.\end{aligned} \tag{4.13}$$

For the second term on the right-hand side of (4.13), we have

$$l\left|\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)}\,dx\right|$$

$$\leq \varphi_k'(2k)e^{B(\infty)} \int_\Omega |a_i(x, T_k(u_n), \nabla T_k(u))| \, |D^i T_k(u_n) - D^i T_k(u)| \, dx,$$

Applying the Lebesgue dominated convergence theorem, we have $T_k(u_n) \to T_k(u)$ in $L^{p_i(\cdot)}(\Omega)$. Then $a_i(x, T_k(u_n), \nabla T_k(u)) \to a_i(x, T_k(u), \nabla T_k(u))$ in $L^{p_i'(\cdot)}(\Omega)$, Since $D^i T_k(u_n) \rightharpoonup D^i T_k(u)$ in $L^{p_i(\cdot)}(\Omega)$, we deduce that

$$\varepsilon_3(n) = \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u))(D^i T_k(u_n) - D^i T_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)} \, dx \to 0 \quad \text{as } n \to \infty. \tag{4.14}$$

It follows that

$$\begin{aligned} \sum_{i=1}^N \int_\Omega &\big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\big) \\ &\qquad \times (D^i T_k(u_n) - D^i T_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)} \, dx \\ &= \sum_{i=1}^N \int_{\{|u_n| \leq k\}} a_i(x, T_k(u_n), \nabla T_k(u_n)) \\ &\qquad \times (D^i T_k(u_n) - D^i T_k(u))\varphi_k'(\omega_n)e^{B(|u_n|)} \, dx + \varepsilon_4(n). \end{aligned} \tag{4.15}$$

Second estimate: For the second term on the left-hand side of (4.12), we have

$$\begin{aligned} &\frac{2b_k}{\alpha} \sum_{i=1}^N \int_{\{|u_n| \leq k\}} a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u_n) |\varphi_k(\omega_n)| e^{B(|u_n|)} \, dx \\ &\quad = \frac{2b_k}{\alpha} \sum_{i=1}^N \int_\Omega \big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\big) \\ &\quad \times \big(D^i T_k(u_n) - D^i T_k(u)\big) |\varphi_k(\omega_n)| e^{B(|u_n|)} \, dx \\ &\quad + \frac{2b_k}{\alpha} \sum_{i=1}^N \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u)) \big(D^i T_k(u_n) - D^i T_k(u)\big) |\varphi_k(\omega_n)| e^{B(|u_n|)} \, dx \\ &\quad + \frac{2b_k}{\alpha} \sum_{i=1}^N \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u) |\varphi_k(\omega_n)| e^{B(|u_n|)} \, dx. \end{aligned} \tag{4.16}$$

Similarly to (4.14), we prove that

$$\varepsilon_5(n) = \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u)) \big(D^i T_k(u_n) - D^i T_k(u)\big) |\varphi_k(\omega_n)| e^{B(|u_n|)} \, dx \to 0 \quad \text{as } n \to \infty. \tag{4.17}$$

For the last term on the right-hand side of (4.16), since the sequence $(a_i(x, T_k(u_n), \nabla T_k(u_n)))_{n \in \mathbb{N}}$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, then there exists $\psi_i \in L^{p_i'(\cdot)}(\Omega)$ such that $a_i(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \psi_i$ in $L^{p_i'(\cdot)}(\Omega)$. Using the fact that

$$D^i T_k(u) \, |\varphi_k(\omega_n)| \to D^i T_k(u) \, |\varphi_k(T_{2k}(u - T_h(u)))| \quad \text{in } L^{p_i(\cdot)}(\Omega),$$

it follows that

$$\varepsilon_6(n) = \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u) |\varphi_k(\omega_n)| \, dx$$
$$\to \int_\Omega \psi_i D^i T_k(u) |\varphi_k(T_{2k}(u - T_h(u)))| \, dx = 0. \qquad (4.18)$$

By combining (4.16)–(4.18), we deduce that

$$\begin{aligned} &\frac{2b_k}{\alpha} \sum_{i=1}^N \int_\Omega \big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\big) \\ &\qquad \times \big(D^i T_k(u_n) - D^i T_k(u)\big) |\varphi_k(\omega_n)| e^{B(|u_n|)} \, dx \\ &\qquad = \frac{2b_k}{\alpha} \sum_{i=1}^N \int_{\{|u_n| \le k\}} a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u_n) |\varphi_k(\omega_n)| e^{B(|u_n|)} \, dx \\ &\qquad + \varepsilon_7(n). \end{aligned} \qquad (4.19)$$

Third estimate: Concerning the third term on the left-hand side of (4.12), we have $T_k(u_n) \to T_k(u)$ in $L^{p_0(\cdot)}(\Omega)$. It follows that

$$\begin{aligned} \varepsilon_8(n) &= \frac{1}{n} \Big| \int_{\{|u_n| \le k\}} |T_k(u_n)|^{p_0(x)-2} T_k(u_n) \varphi_k(\omega_n) e^{B(|u_n|)} \, dx \Big| \\ &\le \frac{1}{n} \exp(4\gamma k) e^{B(\infty)} \int_\Omega |T_k(u_n)|^{p_0(x)-1} |T_k(u_n) - T_k(u)| \, dx \to 0 \\ &\qquad \text{as } n \to \infty. \end{aligned} \qquad (4.20)$$

Relations (4.15), (4.19) and (4.20), and the fact that $e^{B(|u_n|)} \ge 1$, allow us to write

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega \big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\big) \\ &\qquad \times \big(D^i T_k(u_n) - D^i T_k(u)\big) \big(\varphi_k'(\omega_n) - \frac{2b_k}{\alpha} |\varphi_k(\omega_n)|\big) \, dx \\ &\qquad \le e^{B(\infty)} \int_\Omega (|f_n| + c(x)) |\varphi_k(\omega_n)| \, dx + \varepsilon_9(n). \end{aligned} \qquad (4.21)$$

As $f_n \to f$ in $L^1(\Omega)$ and $\varphi_k(\omega_n) \rightharpoonup 0$ weak-$\star$ in $L^\infty(\Omega)$, then

$$\int_\Omega (|f_n| + c(x)) |\varphi_k(\omega_n)| \, dx \longrightarrow 0 \qquad \text{as } n, h \to \infty. \qquad (4.22)$$

Therefore, by letting n and h tend to infinity on the right-hand side of (4.21), we deduce that

$$\lim_{n\to\infty} \sum_{i=1}^N \int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)))$$

$$\times(D^i T_k(u_n) - D^i T_k(u))\,dx = 0.$$

Since $T_k(u_n) \to T_k(u)$ in $L^{p_0(\cdot)}(\Omega)$, we obtain

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega &(a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u)))\\ &\times (D^i T_k(u_n) - D^i T_k(u))\,dx\\ &+\int_\Omega (|T_k(u_n)|^{p_0(x)-2}T_k(u_n) - |T_k(u)|^{p_0(x)-2}T_k(u))\\ &\times (T_k(u_n) - T_k(u))\,dx = 0. \end{aligned}\tag{4.23}$$

In view of Lemma 3.6, we conclude that

$$\begin{cases} T_k(u_n) \to T_k(u) \quad \text{strongly in} \quad W_0^{1,\vec{p}(\cdot)}(\Omega),\\ D^i u_n \to D^i u \quad \text{a.e. in } \Omega \text{ for } i = 1,\dots,N. \end{cases}\tag{4.24}$$

***Step 4:* The equi-integrability of the terms** $(H_n(x,u_n,\nabla u_n))_n$ **and** $(\frac{1}{n}|u_n|^{p_0(x)-2}u_n)_n$. In order to pass to the limit in the approximate equation, we will show that

$$H_n(x,u_n,\nabla u_n) \to H(x,u,\nabla u) \quad \text{and} \quad \frac{1}{n}|u_n|^{p_0(x)-2}u_n \to 0 \text{ strongly in } L^1(\Omega).\tag{4.25}$$

By using Vitali's theorem, it suffices to prove that $(H_n(x,u_n,\nabla u_n))_n$ and $(\frac{1}{n}|u_n|^{p_0(x)-2}u_n)_n$ are uniformly equi-integrable. Firstly, we define the function

$$\overline{B}(s) = \frac{2}{\alpha}\int_0^s b(|\tau|)\,d\tau.$$

By taking $(T_{h+1}(u_n) - T_h(u_n))e^{\overline{B}(|u_n|)}$ as a test function in (4.1), we have

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i(T_{h+1}(u_n) - T_h(u_n))e^{\overline{B}(|u_n|)}\,dx\\ &\quad+\frac{2}{\alpha}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i u_n b(|u_n|)|T_{h+1}(u_n) - T_h(u_n)|e^{\overline{B}(|u_n|)}dx\\ &\quad+\int_\Omega H_n(x,u_n,\nabla u_n)(T_{h+1}(u_n) - T_h(u_n))e^{\overline{B}(|u_n|)}\,dx\\ &\quad+\frac{1}{n}\int_\Omega |u_n|^{p_0(x)-2}u_n(T_{h+1}(u_n) - T_h(u_n))e^{\overline{B}(|u_n|)}\,dx\\ &\quad=\int_\Omega f_n(T_{h+1}(u_n) - T_h(u_n))e^{\overline{B}(|u_n|)}\,dx.\end{aligned}$$

According to (3.3) and (3.5), we obtain

$$\alpha\sum_{i=1}^{N}\int_{\{h<|u_n|\le h+1\}}|D^i u_n|^{p_i(x)}e^{\overline{B}(|u_n|)}\,dx$$

$$+ 2\sum_{i=1}^{N}\int_{\{h<|u_n|\}} |D^i u_n|^{p_i(x)} b(|u_n|)|T_{h+1}(u_n) - T_h(u_n)|e^{\overline{B}(|u_n|)}dx$$
$$+ \frac{1}{n}\int_{\{h<|u_n|\}} |u_n|^{p_0(x)-1} e^{\overline{B}(|u_n|)}|T_{h+1}(u_n) - T_h(u_n)|\, dx$$
$$\leq \int_{\{h<|u_n|\}} (|f_n| + c(x))e^{\overline{B}(|u_n|)}\, dx$$
$$+ \sum_{i=1}^{N}\int_{\{h<|u_n|\}} |D^i u_n|^{p_i(x)}|T_{h+1}(u_n) - T_h(u_n)|b(|u_n|)e^{\overline{B}(|u_n|)}dx.$$

Then

$$\sum_{i=1}^{N}\int_{\{h+1<|u_n|\}} b(|u_n|)|D^i u_n|^{p_i(x)}dx + \frac{1}{n}\int_{\{h+1<|u_n|\}} |u_n|^{p_0(x)-1}\, dx \leq e^{\overline{B}(\infty)}\int_{\{h<|u_n|\}} (|f| + c(x))\, dx.$$

Thus, for all $\eta > 0$, there exists $h(\eta) > 1$ such that

$$\sum_{i=1}^{N}\int_{\{h(\eta)<|u_n|\}} b(|u_n|)|D^i u_n|^{p_i(x)}\, dx + \frac{1}{n}\int_{\{h(\eta)<|u_n|\}} |u_n|^{p_0(x)-1}\, dx \leq \frac{\eta}{2}. \tag{4.26}$$

On the other hand, for any measurable subset $E \subseteq \Omega$, we have

$$\sum_{i=1}^{N}\int_{E} b(|u_n|)|D^i u_n|^{p_i(x)}\, dx + \frac{1}{n}\int_{E} |u_n|^{p_0(x)-1}\, dx$$
$$\leq \sum_{i=1}^{N}\int_{E} b(|T_{h(\eta)}(u_n)|)|D^i T_{h(\eta)}(u_n)|^{p_i(x)}\, dx + \frac{1}{n}\int_{E} |T_{h(\eta)}(u_n)|^{p_0(x)-1}\, dx$$
$$+ \sum_{i=1}^{N}\int_{\{h(\eta)<|u_n|\}} b(|u_n|)|D^i u_n|^{p_i(x)}\, dx + \frac{1}{n}\int_{\{h(\eta)<|u_n|\}} |u_n|^{p_0(x)-1}\, dx. \tag{4.27}$$

From (4.24), there exists $\beta(\eta) > 0$ such that for any $E \subset \Omega$ with $\text{meas}(E) \leq \beta(\eta)$, we have

$$\sum_{i=1}^{N}\int_{E} b(|T_{h(\eta)}(u_n)|)|D^i T_{h(\eta)}(u_n)|^{p_i(x)}\, dx + \frac{1}{n}\int_{E} |T_{h(\eta)}(u_n)|^{p_0(x)-1}\, dx \leq \frac{\eta}{2}. \tag{4.28}$$

Finally, by combining (4.26), (4.27) and (4.28), one easily has

$$\sum_{i=1}^{N}\int_{E} b(|u_n|)|D^i u_n|^{p_i(x)}\, dx + \frac{1}{n}\int_{E} |u_n|^{p_0(x)-1}\, dx \leq \eta \quad \text{for all } E \text{ such that } \text{meas}(E) \leq \beta(\eta). \tag{4.29}$$

By using (3.5), we deduce that $(H_n(x,u_n,\nabla u_n))_n$ and $(|u_n|^{p_0(x)-2}u_n)_n$ are equi-integrable, and in view of Vitali's theorem, we conclude that

$$H_n(x,u_n,\nabla u_n)\to H(x,u,\nabla u) \quad \text{and} \quad \frac{1}{n}|u_n|^{p_0(x)-2}u_n \to 0 \text{ in } L^1(\Omega). \tag{4.30}$$

***Step* 5: Passage to the limit.** Let $\varphi\in W_0^{1,\vec{p}(\cdot)}(\Omega)\cap L^\infty(\Omega)$ and $M=k+\|\varphi\|_\infty$. By taking $T_k(u_n-\varphi)$ as a test function in (4.1), we get

$$\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx+\int_\Omega H_n(x,u_n,\nabla u_n)T_k(u_n-\varphi)\,dx \\ +\frac{1}{n}\int_\Omega |u_n|^{p_0(x)-2}u_nT_k(u_n-\varphi)\,dx=\int_\Omega f_nT_k(u_n-\varphi)\,dx. \tag{4.31}$$

On the one hand, we have $\{|u_n-\varphi|\le k\}\subseteq\{|u_n|\le M\}$. Then

$$\begin{aligned}&\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx\\
&\qquad=\int_\Omega a_i(x,T_M(u_n),\nabla T_M(u_n))(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx\\
&\qquad=\int_\Omega (a_i(x,T_M(u_n),\nabla T_M(u_n))-a_i(x,T_M(u_n),\nabla\varphi))\\
&\qquad\quad\times(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx\\
&\qquad\quad+\int_\Omega a_i(x,T_M(u_n),\nabla\varphi)(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx.\end{aligned}$$

According to Fatou's lemma, we conclude that

$$\begin{aligned}&\liminf_{n\to\infty}\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n-\varphi)\,dx\\
&\quad\ge\sum_{i=1}^N\int_\Omega (a_i(x,T_M(u),\nabla T_M(u))-a_i(x,T_M(u),\nabla\varphi))\\
&\quad\quad\times(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\le k\}}\,dx\\
&\quad\quad+\lim_{n\to\infty}\sum_{i=1}^N\int_\Omega a_i(x,T_M(u_n),\nabla\varphi)(D^iT_M(u_n)-D^i\varphi)\chi_{\{|u_n-\varphi|\le k\}}\,dx\\
&\quad=\sum_{i=1}^N\int_\Omega a_i(x,T_M(u),\nabla T_M(u))(D^iT_M(u)-D^i\varphi)\chi_{\{|u-\varphi|\le k\}}\,dx\\
&\quad=\sum_{i=1}^N\int_\Omega a_i(x,u,\nabla u)D^iT_k(u-\varphi)\,dx.\end{aligned}$$

On the other hand, $T_k(u_n-\varphi)\rightharpoonup T_k(u-\varphi)$ being weak-$\star$ in $L^\infty(\Omega)$, thanks to (4.30), we deduce that

$$\int_\Omega H_n(x,u_n,\nabla u_n)T_k(u_n-\varphi)\,dx\to\int_\Omega H(x,u,\nabla u)\;T_k(u-\varphi)\,dx, \tag{4.32}$$

$$\frac{1}{n}\int_\Omega |u_n|^{p_0(x)-2}u_n T_k(u_n-\varphi)\,dx \to 0, \tag{4.33}$$

and

$$\int_\Omega f_n T_k(u_n-\varphi)\,dx \to \int_\Omega f\ T_k(u-\varphi)\,dx. \tag{4.34}$$

Hence, putting all the terms together, we conclude the proof of Theorem 3.2.

Example 4.1. We consider the following functions:

$$\begin{aligned} H(x,u,\nabla u) &= -e^{-|u|^2}\sum_{i=1}^N |D^i u|^{p_i(x)},\\ a_i(x,u,\nabla u) &= |D^i u|^{p_i(x)-2}D^i u \quad \text{for } i=1,\dots,N \end{aligned}$$

with $\int_{-\infty}^{+\infty} e^{-s^2}\,ds = \sqrt{\pi}$. It is clear that $a_i(x,u,\nabla u)$ and $H(x,u,\nabla u)$ verify the assumptions (3.2) – (3.5). Thanks to Theorem 3.2, the anisotropic quasilinear elliptic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^N D^i\big(|D^i u|^{p_i(x)-2}D^i u\big) = f + e^{-|u|^2}\sum_{i=1}^N |D^i u|^{p_i(x)} & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least one entropy solution $u \in T_0^{1,\vec{p}(\cdot)}(\Omega)$ for any $f \in L^1(\Omega)$.

A. Appendix

Lemma A.1. *The operator $G_n = A_n + R_n$ from $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into $W^{-1,\vec{p}'(\cdot)}(\Omega)$ is pseudo-monotone. Moreover, G_n is coercive in the following sense:*

$$\frac{\langle G_n v, v\rangle}{\|v\|_{1,\vec{p}(\cdot)}} \to +\infty \quad \textit{if} \quad \|v\|_{1,\vec{p}(\cdot)} \to +\infty \ \textit{for } v \in W_0^{1,\vec{p}(\cdot)}(\Omega).$$

Proof. Using the Hölder's inequality and the growth condition (3.2), we can show that the operator A_n is bounded, and by (4.2), we conclude that G_n is bounded. For the coercivity, for all $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, we have

$$\begin{aligned} \langle G_n u, u\rangle &= \sum_{i=1}^N \int_\Omega a_i(x,T_n(u),\nabla u)\ D^i u\,dx \\ &\quad + \int_\Omega H_n(x,u,\nabla u)u\,dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)}\,dx \\ &\geq \underline{\alpha}\sum_{i=0}^N \int_\Omega |D^i u|^{p_i(x)}\,dx - n\int_\Omega |u|\,dx \\ &\geq \underline{\alpha}\ \|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - \delta(N+1) - C_4\|u\|_{1,\vec{p}(\cdot)}, \end{aligned}$$

with $\underline{\alpha} = \min(\alpha, \frac{1}{n})$. It follows that

$$\frac{\langle G_n u, u\rangle}{\|u\|_{1,\vec{p}(\cdot)}} \longrightarrow +\infty \quad \text{as } \|u\|_{1,\vec{p}(\cdot)} \longrightarrow +\infty.$$

It remains to show that G_n is pseudo-monotone. Let $(u_k)_k$ be a sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } W_0^{1,\vec{p}(\cdot)}(\Omega), \\ G_n u_k \rightharpoonup \chi_n & \text{in } W^{-1,\vec{p}'(\cdot)}(\Omega), \\ \limsup\limits_{k\to\infty} \langle G_n u_k, u_k\rangle \leq \langle \chi_n, u\rangle. \end{cases} \tag{A.1}$$

We will prove that

$$\chi_n = G_n u \quad \text{and} \quad \langle G_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle \quad \text{as } k \to +\infty.$$

Firstly, since $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$, then

$$u_k \to u \quad \text{in } L^{\underline{p}}(\Omega) \text{ for a subsequence denoted again by } (u_k)_{k\in\mathbb{N}}.$$

As $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, then, by the growth condition $(a_i(x, T_n(u_k), \nabla u_k))_{k\in\mathbb{N}}$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, and there exists a function $\varphi_{i,n} \in L^{p_i'(\cdot)}(\Omega)$ such that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_{i,n} \quad \text{in } L^{p_i'(\cdot)}(\Omega) \text{ as } k \to \infty \tag{A.2}$$

and

$$\frac{1}{n}|u_k|^{p_0(x)-2}u_k \rightharpoonup \frac{1}{n}|u|^{p_0(x)-2}u \quad \text{in } L^{p_0'(\cdot)}(\Omega). \tag{A.3}$$

Similarly, we have that $(H_n(x, u_k, \nabla u_k))_{k\in\mathbb{N}}$ is bounded in $L^{\underline{p}'}(\Omega)$. Then there exists a function $\psi_n \in L^{\underline{p}'}(\Omega)$ such that

$$H_n(x, u_k, \nabla u_k) \rightharpoonup \psi_n \quad \text{in} \quad L^{\underline{p}'}(\Omega) \quad \text{as } k \to \infty. \tag{A.4}$$

Clearly, for all $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, we have

$$\begin{aligned} \langle \chi_n, v\rangle &= \lim_{k\to\infty} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i v\, dx + \lim_{k\to\infty} \int_\Omega H_n(x, u_k, \nabla u_k) v\, dx \\ &\quad + \lim_{k\to\infty} \frac{1}{n} \int_\Omega |u_k|^{p_0(x)-2} u_k v\, dx \\ &= \sum_{i=1}^N \int_\Omega \varphi_{i,n}\, D^i v\, dx + \int_\Omega \psi_n v\, dx + \frac{1}{n}\int_\Omega |u|^{p_0(x)-2} u v\, dx. \end{aligned} \tag{A.5}$$

From relations ($A.1$) and ($A.5$), we have

$$\limsup_{k\to\infty} \langle G_n(u_k), u_k\rangle = \limsup_{k\to\infty} \left\{ \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k\, dx \right.$$

$$
\begin{aligned}
&+\int_\Omega H_n(x,u_k,\nabla u_k)u_k\,dx+\frac{1}{n}\int_\Omega |u_k|^{p_0(x)}\,dx\Big\}\\
&\le \sum_{i=1}^N\int_\Omega \varphi_{i,n}\ D^i u\,dx+\int_\Omega \psi_n u\,dx+\frac{1}{n}\int_\Omega |u|^{p_0(x)}\,dx. \quad (A.6)
\end{aligned}
$$

Thanks to ($A.4$), we obtain

$$
\int_\Omega H_n(x,u_k,\nabla u_k)u_k\,dx\to\int_\Omega \psi_n u\,dx. \tag{A.7}
$$

Therefore,

$$
\begin{aligned}
\limsup_{k\to\infty}&\left(\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx+\frac{1}{n}\int_\Omega |u_k|^{p_0(x)}\,dx\right)\\
&\le \sum_{i=1}^N\int_\Omega \varphi_{i,n}\ D^i u\,dx+\frac{1}{n}\int_\Omega |u|^{p_0(x)}\,dx. \quad (A.8)
\end{aligned}
$$

On the other hand, by (3.4) we get

$$
\begin{aligned}
&\sum_{i=1}^N\int_\Omega (a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u))(D^i u_k-D^i u)\,dx\\
&\qquad+\frac{1}{n}\int_\Omega (|u_k|^{p_0(x)-2}u_k-|u|^{p_0(x)-2}u)(u_k-u)\,dx\ge 0. \quad (A.9)
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx+\frac{1}{n}\int_\Omega |u_k|^{p_0(x)}\,dx\\
&\quad\ge \sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u\,dx+\frac{1}{n}\int_\Omega |u_k|^{p_0(x)-2}u_k u\,dx\\
&\quad+\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u)(D^i u_k-D^i u)\,dx+\frac{1}{n}\int_\Omega |u|^{p_0(x)-2}u(u_k-u)\,dx.
\end{aligned}
$$

In view of the Lebesgue dominated convergence theorem, we have $T_n(u_k)\to T_n(u)$ in $L^{p_i(\cdot)}(\Omega)$. Then $a_i(x,T_n(u_k),\nabla u)\to a_i(x,T_n(u),\nabla u)$ in $L^{p_i'(\cdot)}(\Omega)$. By using ($A.2$) − ($A.3$), we get

$$
\begin{aligned}
\liminf_{k\to\infty}&\Big(\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx+\frac{1}{n}\int_\Omega |u_k|^{p_0(x)}\,dx\Big)\\
&\ge \sum_{i=1}^N\int_\Omega \varphi_{i,n}\ D^i u\,dx+\frac{1}{n}\int_\Omega |u|^{p_0(x)}\,dx.
\end{aligned}
$$

This implies by using ($A.8$) that

$$\lim_{k\to\infty}\left(\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx+\frac{1}{n}\int_\Omega |u_k|^{p_0(x)}\,dx\right)$$
$$=\sum_{i=1}^{N}\int_\Omega \varphi_{i,n}\;D^i u\,dx+\frac{1}{n}\int_\Omega |u|^{p_0(x)}\,dx. \qquad (A.10)$$

Relations ($A.5$), ($A.7$) and ($A.10$), give

$$\langle G_n u_k, u_k\rangle \to \langle \chi_n, u\rangle \quad \text{as } k\to+\infty.$$

Now, having in mind ($A.10$), we can prove that

$$\lim_{k\to+\infty}\Big(\sum_{i=1}^{N}\int_\Omega (a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u))(D^i u_k-D^i u)\,dx$$
$$+\int_\Omega (|u_k|^{p_0(x)-2}u_k-|u|^{p_0(x)-2}u)(u_k-u)\,dx\Big)=0.$$

Thus, by virtue of Lemma 3.6, we get

$$u_k\to u \ \text{ in } W_0^{1,\vec{p}(\cdot)}(\Omega) \quad\text{and}\quad D^i u_k\to D^i u \ \text{ a.e. in } \Omega.$$

Then

$$a_i(x,T_n(u_k),\nabla u_k)\rightharpoonup a_i(x,T_n(u),\nabla u) \quad \text{in } L^{p_i'(\cdot)}(\Omega) \quad \text{for } i=1,\dots,N$$

and

$$H_n(x,u_k,\nabla u_k)\rightharpoonup H_n(x,u,\nabla u) \quad \text{in } L^{p_0'(\cdot)}(\Omega),$$

which implies that $\chi_n=G_n u$. □

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Існування розв'язків деякої анізотропної еліптичної задачі зі змінними показниками і L^1-даними

Mohamed Badr Benboubker and Hassane Hjiaj

У роботі ми вивчаємо існування ентропійних розв'язків для сильно нелінійного анізотропного еліптичного рівняння

$$Au + H(x, u, \nabla u) = f \qquad \text{in} \quad \Omega,$$

де f належить $L^1(\Omega)$, A є оператором Лере–Ліонса, і H є нелінійним членом нижчого порядку зростання відносно $|\nabla u|$ (тобто таким, що $|H(x, s, \xi)| \leq c(x) + b(|s|) \sum_{i=1}^{N} |\xi_i|^{p_i(x)}$), але без припущення, що $H(x, s, \xi)s \geq 0$. Наведено конкретний приклад, що ілюструє результат існування розв'язків.

Ключові слова: анізотропний, змінний показник, простір Соболєва, нелінійна еліптична проблема, метод штрафних функцій, ентропійні розв'язки