Chronicle

In memory of Yuri Ilyich Lyubich

(April 22, 1931 – June 17, 2023)

The distinguished mathematician Yuri Ilyich Lyubich passed away in Livingston, New Jersey, on June 17, 2023, at the age of 92.

Yuri Ilyich had an incredibly difficult childhood and youth, growing up during that tragic epoch marked with Stalin's repressions, World War II, the Holocaust, and Soviet antisemitic policies.

Yura[1] was born April 22, 1931, in Krasnoyarsk, Siberia, where his parents had been exiled. His father, Ilya Arkad'evich, was imprisoned the following year (Yura never saw him again) and executed in 1938. The family was on the run from the NKVD[2] for about two years afterwards, eventually settling down in 1940 in Kharkov (now Kharkiv), Ukraine.

In June 1941 the German invasion of the Soviet Union began and by October their army reached the outskirts of Kharkov. Miraculously, the family was evacuated from Kharkov along with the staff of the hospital where his mother, Evgeniya Semenovna, had been recently employed, just days before the city was occupied by the Nazis. Almost all Jewish people remaining in Kharkov perished in the Holocaust.

They were brought to Abakan, Siberia, not too far away (in the Siberian scale) from the place of Yura's birth. The first winter there was harsh, marked

[1] In this way Yuri Ilyich was addressed when he was young, and later on by those close to him.

[2] This is a predecessor of the KGB.

by extreme cold and hunger. Yura's grandfather passed away just a month after their arrival, but the rest of the family (Yura's mother and his little cousin Vadik[3]) survived.

At that time, under those surreal circumstances, Yura's curiosity about mathematics started to emerge. He became fascinated with Euclid's Axiom of Parallels and attempted to prove it. His teachers could not find an error in his argument.

In 1944 the family returned to Kharkov and Yura enrolled in High School #36. It was an excellent school, on a par with the special math schools that proliferated decades later. There, Yura had his first research experience: jointly with his friend Lev Ronkin[4] he discovered many beautiful properties of the Fibonacci numbers.

In 1947 Yura became enrolled as a mathematics student in the Physics and Mathematics Department ("*physmat*") of the Kharkov State University.

There he met Lidia, a student in mechanics; they married in 1951.

Yet tragedy struck the family again: in 1949 Yura's mother was arrested and a few months later died in prison. 18 year-old Yura was left alone with his 12 year-old cousin Vadik, whom he raised like his son.

Yura, a student of the Kharkov University, in front of the old physmat building (fall 1949)

Yura's scientific mentors at the university were the world class mathematicians Naum Ilyich Akhiezer, Gershon Ihelevich Drinfeld[5], Alexander Yakovlevich Povzner, and Boris Yakovlevich Levin. Mathematical physics, function theory, and the then nascent field of functional analysis were flourishing. Yura was thinking non-stop about various problems in these areas. An anecdote tells that in Yura's third year, Professor Levin assigned to Yura, as an exam question, Gelfand's problem on the structure of invariant subspaces of the Volterra operator. And Yura solved it. B.Ya. never let him know the origin of the problem.

[3]Vadim Tkachenko, who eventually became a well-known mathematician in his own right. He passed away on July 24, 2023, just over one month after Yu.I., at the age of 85.

[4]Lev Ronkin later became a prominent expert in the theory of functions of several complex variables.

[5]Father of the future Fields Prize winner V. Drinfeld.

Eventually, it became known as Agmon's Theorem (which appeared at about that time).

Yura's Master thesis was dedicated to exploration of mean-periodic functions (which Yura re-discovered; they had been introduced by J. Delsarte).

Yura graduated from the university in 1952, at a time when Stalin's anti-semitic campaign was in full swing. Despite everybody recognizing his brilliance, he was not admitted to graduate school and ended up as a math teacher at High School #36, from where he had graduated just five years earlier. Despite the circumstances, Yura continued doing mathematics on his own. A couple of years later he asked Vladimir Alexandrovich Marchenko to give him a research problem. He solved it quickly and it became the basis for his PhD thesis, "Tauberian Theorems for Generalized Fourier Transforms". Yura recalled that he was writing his thesis in class while proctoring exams. He defended it in 1957.

After the death of Stalin, the system gradually became "softer," and in 1955 Yura was hired as an Assistant Professor of the Kharkov State University, to the Geometry "*kafedra*"[6] led by A.V. Pogorelov.

In 1965 Yuri Ilyich was awarded his second degree, the much harder to earn Doctor of Science[7], for his thesis "Studies in the Theory of Linear Operators in Banach Spaces." His committee consisted of the famous mathematicians B.Ya. Levin, S.G. Krein, and G.E. Shilov. Soon afterwards, he became the head of the newly created Algebra and Logic *kafedra*. He remained in this position until shortly before he left the Soviet Union for good in 1990.[8]

After a couple of years as a visitor at Stony Brook University, Yu.I. was offered a specially created Distinguished position at Haifa's Technion[9]. He worked there until his retirement in 2002.

Yuri Ilyich was encyclopedically educated, a true Renaissance man. He loved art, especially impressionist art, and sometimes used his vast knowledge of literature in the context of math seminars. Once, after a talk given by a tense and nervous student, Yu.I. said that while talks often follow various literary styles (such as realism, romanticism, classicism), this one was performed as a stream of consciousness. This remark reflects Yuri Ilyich's rare ability to make a critical remark in a unique, witty, and respectful way.

Yu.I. was a legendary teacher. Clarity, elegance, the revelation of hidden underlying structures, surprising applications of the general theory, a perfect sense of the level of his audience, the spontaneous spark of a joke – all these features appeared in a natural and effortless way.[10] Lyubich's students still remember

[6] The department was subdivided into smaller specialized units called "kafedras", e.g., Theory of Functions, Math Physics, etc.

[7] In the Soviet system there were two degrees, "Candidate of Science" (corresponding to Western PhD) and "Doctor of Science" (corresponding to French habilitation).

[8] In 1986 the kafedra was re-structured into the "Theory of Functions and Functional Analysis" one.

[9] Israel Institute of Technology. This story is described in V. Milman's essay about Yu.I. [31].

[10] Sadly, it was partly lost when Yu.I. moved to the West. In this respect, he liked to quote Hermann Weyl: "Gods imposed upon me fetters of the foreign language that did not sound over my cradle".

his courses with awe. And so do high school students who were lucky enough to attend the math circle that Yura was running during his early University years.

But his exams were tough, for students at both ends of the spectrum. A weak student could not pass without understanding the basics, and on the other hand a strong student would not receive the highest grade ("5") without solving a non-standard problem. This approach caused quite a bit of tension between Yu.I. and the administration (of the university and of the high school alike). But his students remembered those exams for many years with gratitude (at least, eventually).

Yu.I. advised two dozen ("official" and "unofficial") graduate students, including T. Achieser[11] (math genetics), G. Belitskii (singularity theory), A. Blokh (real one-dimensional dynamics), G. Feldman (spectral theory of representations), E. Glazman (spectral theory of differential operators), V. Grinberg (discrete math), V. Kirzhner and A. Krapivin (math genetics), M. Lyubich (holomorphic dynamics), G. Maystrovskii (numerical methods), O. Shatalova (finite dimensional Banach geometry), M. Tabachnikov (discrete math), V. Tkachenko (spectral theory of differential operators), A. Veitsblit (finite dimensional Banach spaces and complex geometry), Vu Quoc Phong (almost periodic representations)... The impressive spectrum of their research themes reflected his broad scientific interests, and sometimes even went well beyond the areas he worked in himself, for example into dynamical systems or singularity theory. Some of his students went on to successful mathematical careers, while some of the others — despite their high qualifications and dedication to mathematics — had to quit research, due to the antisemitic realities of Soviet life.

The influence of Yuri Ilyich extended much farther than the circle of his immediate students. An entire generation of Soviet mathematicians grew up on his wonderful books [1–7]. His style of mentoring young people is reflected in his problem book "Finite Dimensional Linear Analysis," written jointly with I.M. Glazman [2]. The idea is that students, by going through a series of small steps formulated as fairly simple, logically connected problems, gradually build up a deep theory by themselves. (It was modeled on the methods of Ludvig Spohr's violin school circa 1800, as was famously described in the preamble to the book.) In the early 1970s a student seminar was launched in Odessa, based solely on this book; it was running for many years. Participants were charmed by the penetrating logic and beauty of the book. One of them, Ilya Spitkovsky, recalls one fateful winter Sunday when the students had to dig their way through the piles of snow to get to the seminar room. On that day he learned about the *Hausdorff set of a matrix*, which became a major theme of his mathematical life.

The Glazman–Lyubich problem book was praised by Paul Halmos as "expository poetry". "A beautiful course could be given from this book (I would love to give one), and a student brought up in such a course could become an infant prodigy functional analyst in no time" [28].

One more illustration of the magic impact of Lyubich's books on the reader is given by V. Berkovich [26], whose thoughts about non-Archimedian analytic

[11] Tetjana's last name is spelled "Akhiezer" in the English translations of her papers.

geometry (leading, in particular to the important notion of *Berkovich space*) were inspired by Lyubich's book on Functional Analysis [6].

There were many more facets of Yu.I.'s interactions with young people. He was one of the major participants and lecturers at the famous Voronezh Winter Mathematical Schools (see, e.g. [32]). And without his help, a number of mathematicians (for instance, M. Brin, A. Figotin, and P. Kuchment), would not have been able to get their degrees under the Soviet system.

Yu.I. on a kayaking trip on lake Keret' in Karelia (summer 1979).

Yu.I. loved recreational activities: hiking, skiing, swimming, beach volleyball, badminton... (although he was not interested in the competitive side of the sport). Starting in the mid-1960s he regularly went, together with his family and friends, on kayaking trips along some picturesque river spiraling through dense forest. Svetlana Jitomirskaya fondly remembers how she, aged 17, and Yu.I, were discussing fundamental groups of classical Lie groups, while paddling in a double kayak. And he always carried along a recreational math book, often on number theory, which remained his life-long passion.

Yu.I. had a keen interest in world events. Growing up during WWII, he followed all the accounts of military actions and, having perfect memory, remembered the names and locations of all the villages involved. Early on, he learned how to read the Soviet press "between the lines," extracting germs of reality from thick layers of propaganda. Since the mid-1960s, he regularly listened, despite Soviet jamming, to "slanderous" Western radio stations (Voice of America, BBC, Radio Svoboda).

Yuri Ilyich had always kept the fate of Israel close to his heart. And he was deeply disturbed by the war in Ukraine ("I cannot believe that Kharkiv is being bombed again!"). During the last year and a half of his life he stopped doing anything but watching the news from Ukraine.

A brief overview of Yu.I. Lyubich's research

Yuri Ilyich Lyubich belongs to a select group of universal mathematicians with a broad vision of many, if not all, fundamental areas of mathematics, and with the rare skill of revealing hidden connections between them. He was able to pass effortlessly from traditional themes of functional analysis (spectral theory, representation theory, Banach geometry, ...) which was his main area of research, to topology, differential equations, combinatorics, automata theory, mathematical genetics, numerical methods, and their applications. Such universality requires a broad background, a vast knowledge of literature, and active communication with colleagues from all over the world. Unfortunately, Yu.I., like most other Soviet mathematicians, was deprived of contact with Western scientists. However, his role as a pioneering researcher and a mentor of younger generation was widely recognized, way beyond his immediate circle of colleagues and students.

Characteristic features of Yu.I. Lyubich's mathematics were not only its tremendous breadth, but also crystal clarity, the search for general structures behind specific analytical problems and for deep connections between Mathematics ("pure" and "applied") and the Natural Sciences.

Let us now briefly describe some of Lyubich's insights and results.

• The *Spectral Theory* of functions and operators was among the favorite motifs of Lyubich's math. One of his fundamental contributions to the theory, joint with V.I. Matsaev, was the *Theory of operators with separable spectrum.*

The classical Spectral Theorem implies that the spectrum of any self-adjoint operator A in a Hilbert space possesses the following strong *Spectrum Separation Property*:

– To any segment I of the real axis corresponds a *spectral subspace* $E(I)$ that carries the maximal portion of A whose spectrum lies in I;

– For any covering of the real axis by intervals, the family of the corresponding spectral subspaces is complete.

Lyubich and Matsaev put forward the paradigm: the Spectrum Separation Property provides a natural borderline for the spectral theory of operators; they then moved on to give efficient conditions for a real-spectrum operator (generally unbounded) in a Banach space to enjoy this spectral property [10]. They gave two conditions (that happen to be equivalent for bounded operators): one in terms of the growth of the semigroup e^{iAt} generated by A,

$$\int_{\mathbb{R}} \frac{\log \left\| e^{iAt} \right\|}{1+t^2}\, \mathrm{d}t < \infty ,$$

(reflecting Yu.I.'s general philosophy that an operator should be viewed through the lens of the (semi-)group it generates), and the other in terms of the growth of the resolvent $R_\lambda(A)$ near the spectrum (the so-called *Carleman–Levinson–Sjoberg–Wolf-loglog condition*).[12]

This work had a significant impact on the development of Spectral Theory. Yu.I. & V.I. repeatedly returned, together and separately, to this circle of ideas. In particular, the theory was further generalized, by Lyubich, Matsaev and Feldman,

[12] A similar theory was independently developed in the West by J. Wermer and F. Wolf & E. Bishop.

to group representations [18] based on the notion of representation spectrum introduced by Yu.I. [15] and on the Theory of Banach Algebras.

• *The Abstract Cauchy Problem and the Laplace Transform.* Another important Yu.I.'s contribution to Operator Theory motivated by problems in PDE and Probability, was his results on the Abstract Cauchy Problem, i.e., search for solutions of the following ordinary differential equation in a Banach space B:

$$\frac{dx(t)}{dt} = Ax(t), \quad t \in [0,T], \quad x(0) = x_0, \tag{1}$$

where A is a linear, closed, densely defined operator on B. Such a problem is called *well-posed* if A generates a strongly continuous operator semigroup $(e^{At})_{t\in\mathbb{R}_+}$ whose orbits are (uniquely defined) solutions of (1). A profound theory of well posed Cauchy problems became a subject of the 1000 pages treatise by Hille and Phillips (1957).

In the 1960s Yu.I. undertook the task of exploring ill-posed problems. The idea was to use the formal expression of the resolvent of A as the Laplace transform of the orbit,

$$R(\lambda, A)x_0 = \int_0^\infty e^{-\lambda t} x(t)\,\mathrm{d}t,$$

as the basis for a general theory. This insight led Yu.I. to a number of deep pioneering results [12] being precursors of subsequent important advances.

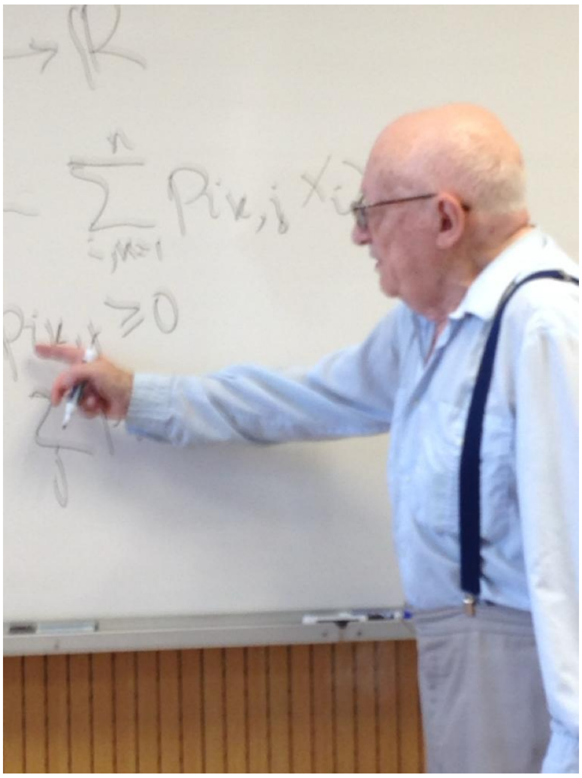
As solutions of (1) might a priori grow superexponentially, a direct application of the Laplace transform is impossible. To overcome this difficulty, Yu.I. worked out a regularized inversion formula for the Laplace transform. It allowed him, under the assumption that the resolvent $R(\lambda, A)$ grows polynomially in the right-half plane, to construct classical solutions for a dense set of initial points.

To study the uniqueness of solutions of (1) Yu.I., in collaboration with V. Tkachenko [13], introduced a *local Laplace transform*, which can be applied to functions of arbitrary growth, while allows one to recover uniquely the original function. It led to an elegant uniqueness criterion asserting that the solutions of (1) are unique (even in the distributional sense) as long as the resolvent grows subexponentially along the real axis. This result is sharp.

The philosophy of Laplace transforms was adapted and elaborated in a fundamental paper by S. Agmon and L. Nirenberg (1963) and developed further by many other authors.

• *Genetics and Lyubich's Bernstein Algebras.* Genetics was a field brutally suppressed in the Soviet Union during the 30-some years of Lysenko's dictatorship in biology. In the second half of the 1950s the pressure relaxed, and it became possible to learn the basics of the field. Yuri Ilyich took full advantage of this opportunity, and quickly recognized that this field could be put on a solid mathematical foundation. He then dedicated several decades of his life to developing a rigorous Mathematical Genetics. He considered it a counterpart of classical Math Physics, where physical situations are formalized mathematically, and then rigorous theorems are proved to justify physical laws. Much of his work was eventually collected in the fundamental monograph "Mathematical Structures in Population Genetics" [3]. In this way, Yu.I. Lyubich became one of the founders of *Mathematical Genetics*.

A central theme in this story was played by the *Bernstein Problem* of describing the most general form of the Hardy–Weinberg Law governing all evolutionary operators that settle down to an equilibrium after one step. Mathematically it amounts to describing all quadratic maps V of the standard n-dimensional simplex such that $V^2 = V$. This problem was posed by S.N. Bernstein in a note published in 1924 in the Kharkov Math Journal, where it was solved for small n. (It was N.I. Akhiezer who pointed out this note to Yu.I.)



Yu.I. giving a talk on Bernstein's problem at a conference in Haifa's Technion (May 2015) celebrating V.Ya. Lin's 80th birthday. (This was probably the last Yu.I.'s public talk.)

Yu.I. singled out a biologically important *regularity condition*, whose meaning amounts to the existence of an underlying gene supply conserved under evolution. Then he fully solved Bernstein's problem under this condition. He insisted that beyond this restriction the problem loses its connection with biology (see [3]).

Moreover, Yu.I. revealed that behind the problem lay a novel algebraic structure, that he named *Bernstein Algebra*, which later on developed into a branch of Algebra in its own right.

Another fundamental theme thoroughly explored by Yu.I. is a description of *Population Dynamics* on the genes level. Using the algebraic formalism desgned by O. Reiersol (1962), Yu.I. *explicitly integrated* the system in the absence of selection [17]. He then demonstrated (jointly with V. Kirzhner [19]) that under these circumstances, all the orbits converge to finitely many equilibrium states. He followed up by establishing this property (jointly with G. Maystrovskii & Yu. Ol'hovskii [20], and L. Kun [21]) for populations subjected to *selection pressure* (assuming that each gene contributes additively to selection). [Such systems are described by rational, rather than polynomial, dynamics.] The latter results are based on the general theory of numerical *relaxation processes* developed earlier with G. Maystrovskii [16]. [In the early years of the computer era, Yu.I. grew interested in Numerical Methods, what would now be called "Computer Science." Ironically, dealing with real life computers always annoyed him.]

• *Automata Theory.* Perhaps the most important problem in modern computational complexity is the famous P vs NP question, which asks about the power of nondeterminism in efficient computation. But the notion of nondeterminism was studied even earlier in Automata Theory, where non-deterministic automata, unlike their deterministic counterparts, are able to explore multiple computation paths simultaneously. Yu.I. posed the problem of the asymptotical comparison between the sizes of deterministic and non-deterministic automata for unary languages. In a pioneering paper in Automata Theory, he proved that the "blow-up" has a surprising intermediate (strictly in between polynomial and exponential) rate: $e^{O(\sqrt{n \log n})}$ [11]. About twenty five years later, this estimate was slightly improved by M. Chrobak [27]: unlike Yu.I.'s original result, this paper has been widely cited.

• *Going back to Spectral Theory*, Yu.I. Lyubich did a pioneering work on *almost periodic representations of one-parameter groups* [9], i.e., representations whose orbits are precompact in a suitable topology. It provides a general framework for discreteness of the spectrum. A few years later (in the mid-1960s), a general theory of (weakly) almost periodic semigroup representations was developed by K. Jacobs, K. de Leeuw and I. Gliksberg. Altogether, these ideas illuminated the problem of convergence of semigroup orbits to an equilibrium. In the early 1980s, they were used by M. Lyubich to construct the measure of maximal entropy for rational endomorphisms of the Riemann sphere [30]. It also led to a delicate and useful spectral criterion for (asymptotic) stability of linear ODE in Banach spaces (with Vu Quoc Fong) [22] (discovered independently by W. Arendt and C.J.K. Batty[13]). It became widely known under the name ABLV and received a significant resonance. An elegant introduction to this subject appeared in Yu.I.'s book "Introduction to the Theory of Banach Representations of Groups" [5].

• *Finite-dimensional Banach Geometry and Lyubich's Last Book.* The idea of almost periodic representations led Yu.I. to a result on arithmetic properties of the boundary spectrum of contractions in finite-dimensional Banach spaces [14]. This, in turn, prompted a thought about Euclidean sections of such spaces, and

[13]The case of bounded generating operator was earlier treated by G. Sklyar and V. Shirman (1982).

more generally, about isometric embeddings between them (particularly, between the classical ℓ_p spaces). Much later, in 1990s, after having moved to the West, Yu.I., jointly with L.N. Vaserstein [23], discovered[14] the deep connection between this special topic and many other areas of mathematics: Number Theory (the Waring problem), Representation Theory (harmonic invariants of finite subgroups of the orthogonal group), Algebraic Combinatorics and Numerical Analysis (spherical designs and more general cubature formulas)... Yu.I. followed up by extending these results, jointly with his last graduate student O. Shatalova, to complex and quaternionic spaces. This work became the basis for Yu.I.'s last monograph "Spectral theory, isometric embeddings $\ell_q^m \to \ell_p^n$ and cubature formulas over the classical fields" [7], unfortunately never finished.

• An illustration of Yu.I.'s shrewd vision was his pioneering realization of the potential of *Holomorphic Dynamics* (partly motivated by his interest in the Newton root-finding algorithm and in the dynamics of the quadratic maps occurring in Mathematical Genetics). As early as 1979 he drew the attention of his son, Misha Lyubich, to this field; he suggested the initial problem (on the entropy of rational endomorphisms) and pointed out the classical work by Montel, Fatou and Julia. This was at a time when nobody appeared to be interested in the field and the Fatou-Julia theory had been almost completely forgotten. But Yu.I. made this theme (along with one-dimensional real dynamics) a central part of his seminar. That is where several young mathematicians, including A. Blokh, A. Eremenko and M. Lyubich, took their first steps in the mastery of these topics. V. Drinfeld who had just moved back to Kharkov at that time, also participated, and came up there with an idea that contributed to an important discovery, *Kirillov's complex structure* on the group of circle diffeomorphisms. It turned out that at the same time a similar seminar was being run in Paris, leading to a spectacular revival of the field in 1982.

Another, even earlier, example is Yu.I.'s insight into *Singularity Theory*, namely his approach to the classification problem for local smooth maps by means of the Topological Fixed Point Principle. This initial idea led to the deep theory developed by G. Belitskii [25].

• *The 1960 paper on Kolmogorov-type Inequalities.* Let us conclude our brief excursion into Lyubich's mathematics by describing two of his papers separated by half a century. They address some special problems in Operator Theory that illustrate beautifully Yu.I.'s. research style.

One of Yu.I.'s first papers [8] explored convexity inequalities of the form

$$\|A^k x\| \le C_{n,k} \|x\|^{1-k/n} \, \|A^n x\|^{k/n}, \quad 1 \le k \le n, \tag{2}$$

where A is a linear operator (generally unbounded) in a Banach space X. It was inspired by Kolmogorov's work on the differentiation operator $A = d/dt$ acting in the space of smooth functions $C^n(\mathbb{R})$. In this case, Kolmogorov proved (2), found optimal constants $C_{n,k}$ and identified extremal functions (which do not belong to $C^n(\mathbb{R})$).

[14] Independently, it was discovered by B. Reznik (1992).

Of course, Yu.I. looked at this concrete analytical problem from a bird's eye view of Operator Theory. In [8] he designed a *Resolvent Method* that can be applied in the Banach setting and a *Variational Method* specific to the Hilbert space. (He mentions that the latter goes back to the celebrated Hardy, Littlewood and Pólya Inequalities.) This techniques led him to general inequalities accompanied with non-trivial, sometimes sharp, bounds on the constants $C_{n,k}$. Eventually, the study of best constants drew a substantial interest of experts in Operator and Approximation Theories (see [29]), but in the end of the day, Yu.I.'s techniques appears to be unsurpassable.

• *The 2010 paper on the Stability Problem.* Lyapunov stability of the origin under the action of a linear operator A is equivalent to the uniform boundedness of the norms $\|A^n\|$. Moreover, the "quality" of stability is measured by the supremum $S(A)$ of these norms.

Let us finally describe one of Lyubich's last results, on the Stability Problem, that appeared in a remarkable paper in Studia Mathematica (2010), an article distinguished by masterly analysis and by the exceptional numerical sharpness of the estimates. The work examines operators of the form $A = \varphi(V)$, where φ is a non-constant function holomorphic near the origin, $\varphi(0) \neq 0$, and V is the classical Volterra operator acting in $L^p[0,1]$,

$$Vh(x) = \int_0^x h(t)dt.$$

Then under the normalization $\varphi(0) = 1$, the following amazing Dichotomy is established:

Either the origin is Lyapunov stable, which happens iff $p = 2$ and $\varphi'(0) < 0$ (moreover, in this case an explicit bound on $S(V)$ is given in terms of the Taylor coefficients of φ);

Or else, the norms $\|V^n\|$ grow at least with rate $\exp(cn^\gamma)$ for some $0 < \gamma \leq 1/2$.

The article continues with a variety of nice illustrations (orthopolynomials, etc.) improving previously known results.

Concluding with this 2010 paper, so beautiful mathematically and aesthetically, we can only recall another great mathematician, J.L. Littlewood, also famous for his prolific longevity, who remarked in his 1970 paper (when he was 85!): "*. . . the problem* [considered in this paper] *raised difficulties which defeated me for some time. I have now overcome them.*"

Final words

Yu.I.'s beloved wife Lidia had passed away in 2018. Though Yu.I. had been suffering from an incurable lung disease for several years, his death was sudden and unimaginable. His mind and memory remained sharp and clear until the very last moment.

He is survived by his daughter Zhenya (who has son Ilya and daughter Masha), his son Misha (who has two daughters, Eva and Chana), and his three great-

Yu.I. on the West Meadow beach in Stony Brook explaining some physics to his granddaughter Chana (Sept 2015).

grandchildren. Yu.I.'s research genes got split in between his children: Zhenya is a biologist, Misha is a mathematician.

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