Journal of Mathematical Physics, Analysis, Geometry
2024, Vol. 20, No. 2, pp. 195–204
doi: https://doi.org/10.15407/mag20.02.195

A Discrete Blaschke Theorem for Convex Polygons in 2-Dimensional Space Forms

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Let M be a 2-dimensional space form. Let P be a convex polygon in M. For these polygons, we define (and justify) a curvature κ_i at each vertex A_i of the polygon and prove the following Blaschke-type theorem: "If P is a convex polygon in M with curvature at its vertices $\kappa_i \geq \kappa_0 > 0$, then the circumradius R of P satisfies $\mathrm{ta}_\lambda(R) \leq \pi/(2\kappa_0)$ and the equality holds if and only if the polygon is a doubly covered segment".

Key words: Blachske theorem, circumradius, curvature at a vertex, convex polygon

Mathematical Subject Classification 2020: 52A10, 52A55, 51M10, 53C22

1. Introduction and the main result

We start recalling that an n-dimensional space form $\overline{M}^n_\lambda$ of curvature λ is a complete simply connected n-dimensional Riemannian manifold of constant sectional curvature λ. The only ones are: when $\lambda = 0$, the Euclidean space $\mathbb{R}^n$, when $\lambda > 0$, the n-dimensional sphere of radius $1/\sqrt{\lambda}$ in the Euclidean space $\mathbb{R}^{n+1}$, and when $\lambda < 0$, the hyperbolic space of sectional curvature λ that can be visualized as the upper connected component of the Minkowski sphere of radius $1/\sqrt{|\lambda|}$ in the Minkowski space $\mathbb{R}^{n+1}_1$.

In the book of Blaschke [1], it is proved that *if Γ is a closed convex regular curve in the Euclidean plane that bounds a compact convex region Ω and the curvature κ of Γ is bounded from below by some constant $\kappa_0 > 0$, then, for every point $p \in \Gamma$, the circle tangent to Γ at p, with radius $R = \frac{1}{\kappa_0}$ and with the unit normal vector that points to its center pointing also to the interior of Ω, bounds a disk that contains Ω.*

This result was extended by H. Karcher [14] for other space forms, and by A.D. Milka [15] for non regular curves. Before stating it, we recall a notation that allows us to describe the geometry of space forms in a unified way:

$$\mathrm{s}_\lambda(t) = \begin{cases} \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} & \text{if } \lambda > 0 \\ t & \text{if } \lambda = 0 \\ \frac{\sinh(\sqrt{|\lambda|}t)}{\sqrt{|\lambda|}} & \text{if } \lambda < 0 \end{cases}, \qquad \mathrm{c}_\lambda(t) = \begin{cases} \cos(\sqrt{\lambda}t) & \text{if } \lambda > 0 \\ 1 & \text{if } \lambda = 0 \\ \cosh(\sqrt{|\lambda|}t) & \text{if } \lambda < 0 \end{cases},$$

$$\mathrm{ta}_\lambda(t) = \frac{\mathrm{s}_\lambda(t)}{\mathrm{c}_\lambda(t)}, \qquad \mathrm{co}_\lambda(t) = \frac{\mathrm{c}_\lambda(t)}{\mathrm{s}_\lambda(t)}. \tag{1.1}$$

The functions above satisfy the following computational rules:

$$\mathrm{c}_\lambda' = -\lambda\,\mathrm{s}_\lambda, \quad \mathrm{s}_\lambda'(t) = \mathrm{c}_\lambda(t), \quad \mathrm{c}_\lambda^2 + \lambda\,\mathrm{s}_\lambda^2 = 1, \qquad \frac{1}{\mathrm{c}_\lambda^2(t)} = 1 + \lambda \mathrm{ta}_\lambda^2(t), \tag{1.2}$$
$$\mathrm{s}_\lambda(t+u) = \mathrm{s}_\lambda(t)\mathrm{c}_\lambda(u) + \mathrm{c}_\lambda(t)\mathrm{s}_\lambda(u) \text{ and } \mathrm{c}_\lambda(t+u) = \mathrm{c}_\lambda(t)\mathrm{c}_\lambda(u) - \lambda\,\mathrm{s}_\lambda(t)\mathrm{s}_\lambda(u),$$

where " $'$ " denotes the derivative with respect to t.

We shall recall also the following concept:

Given any convex closed curve Γ in $\overline{M}^2_\lambda$, the ***circumradius*** *of Γ is the minimal value of R such that a disk of radius R in $\overline{M}^2_\lambda$ contains the domain bounded by Γ.*

With this concept, the Blaschke–Karcher–Milka theorem can be stated in the following form:

Theorem 1.1 ([15])**.** *If Γ is a closed rectifiable curve in $\overline{M}^2_\lambda$ that bounds a compact convex region Ω and with specific curvature $\kappa \geq \kappa_0 > 0$ is $\lambda \geq 0$ and $\kappa_0 > \sqrt{-\lambda}$ if $\lambda < 0$, then the circumradius R of Γ satisfies $\mathrm{ta}_\lambda(R) \leq \frac{1}{\kappa_0}$.*

Understanding the statement of this theorem requires to explain the concept of specific curvature used in [15]. Its definition requires the following chain of definitions:

Definition 1.2 ([15])**.** Given a (non necessarily closed) polygon P in $\overline{M}^2_\lambda$, the sum of the supplementary of the internal angles $\widehat{A}_i$ of P, $\sum_{i=1}^n(\pi - \widehat{A}_i)$, is called the turning of P.

Given a rectifiable curve γ of length s in $\overline{M}^2_\lambda$, we shall denote by P_n a polygon with n vertices in γ. If γ is not closed, P_n is chosen so that its endpoints coincide with the endpoints of γ. The polygon is called *inscribed* into γ.

Definition 1.3 ([15])**.** The turning $\tau(\gamma)$ of γ is the upper limit of the turnings of inscribed polygons P_n when the length of the arc-segments of γ between any pair of consecutive vertices of P_n goes to zero as $n \to \infty$.

Definition 1.4 ([15])**.** The specific curvature of a curve γ of length s is the quotient $\tau(\gamma)/s$. A convex curve Γ is said to have specific curvature bounded from below by some constant κ_0 if every arc γ of Γ has the specific curvature no smaller than κ_0.

Remark 1.5. It follows from these definitions that every convex closed curve, which is piecewise C^2 and its C^2 arcs have curvature $\kappa > \kappa_0$, has the specific curvature bounded from below by κ_0.

Further developments of related Blaschke theorems are done by obtaining conditions under which a convex set in $\mathbb{R}^n$ can be included in other set [7, 9, 17] or its generalization to Riemannian manifolds, where the ball is the convex set which is included in another set [13].

In Theorem 1.1 (and in the other cited developments), the hypothesis of strong convexity ($\kappa \geq \kappa_0 > 0$) is necessary, the theorem is not true for $\kappa \geq 0$. Therefore it cannot be applied to closed convex polygons. Here we shall show that it is possible to have a version of the theorem for polygons once we give an appropriate definition of curvature at the vertices of a polygon. We shall take the following one:

Definition 1.6. Let A be a vertex of a convex polygon P in a space form $\overline{M}^2_\lambda$. When $\lambda > 0$, the lengths ℓ_i of the sides of P must satisfy $\ell_i < \pi/\sqrt{\lambda}$. Let $\widehat{A}$ be the interior angle of P at the vertex A, and let ℓ_1, ℓ_2 be the lengths of the sides of P that meet at vertex A. We define the "curvature of P at A" by the number

$$\kappa_A = \frac{(\pi - \widehat{A})}{\mathrm{ta}_\lambda(\ell_1/2) + \mathrm{ta}_\lambda(\ell_2/2)}. \tag{1.3}$$

When $\lambda = 0$, the Definition 1.3 becomes

$$\kappa_A = \frac{2(\pi - \widehat{A})}{\ell_1 + \ell_2}. \tag{1.4}$$

The reasons why we have chosen Definition 1.3 are given in the next section. The version of Theorem 1.1 that we prove for polygons is:

Theorem 1.7. *Let P be a closed convex polygon in $\overline{M}^2_\lambda$, with side lengths less than $\pi/\sqrt{\lambda}$ if $\lambda > 0$, and with curvature at each vertex A_i satisfying $\kappa_{A_i} \geq \kappa_0$, with $\kappa_0 > 0$ if $\lambda > 0$ and $2\kappa_0/\pi > \sqrt{-\lambda}$ if $\lambda < 0$. When $\lambda \neq 0$, we also assume that $\frac{\pi}{2\kappa_0} \geq \mathrm{ta}_\lambda(\ell_i/2)$ for every i. Then the circumradius R of P satisfies*

$$\mathrm{ta}_\lambda(R) \leq \pi/(2\kappa_0), \tag{1.5}$$

and the equality holds if and only if the polygon degenerates to a doubly covered segment.

Remark 1.8. Let us observe that for a polygon with n vertices, it follows from the Gauss–Bonnet formula and Definition 1.3 that $\kappa_{A_i} \geq \kappa_0$ implies that if we denote by S the area of the domain bounded by the polygon P and by L its perimeter, then

$$\kappa_0 \sum_{i=1}^{n} (\mathrm{ta}_\lambda(\ell_i/2) + \mathrm{ta}_\lambda(\ell_{i+1}/2)) \leq \sum_{i=1}^{n} (\pi - \widehat{A}_i) = 2\pi - \lambda S.$$

Therefore,

$$\frac{1}{\kappa_0} \geq \frac{2\sum_{1=1}^{n} \mathrm{ta}_\lambda(\ell_i/2)}{2\pi - \lambda S}.$$

If $\lambda = 0$, then $\sum_{1=1}^{n}(\ell_i/2) \geq \ell_M$, where $\ell_M := \max\{\ell_1, \dots, \ell_n\}$, and

$$\frac{\pi}{2\kappa_0} \geq \ell_M/2. \tag{1.6}$$

That is, the hypothesis "$\frac{\pi}{2\kappa_0} \geq \mathrm{ta}_\lambda(\ell_i/2)$ for every i" added when $\lambda \neq 0$ is automatically satisfied when $\lambda = 0$ as a consequence of the lower bound of κ_{A_i}.

Remark 1.9. For $\lambda \neq 0$, the hypothesis $\frac{\pi}{2\kappa_0} \geq \mathrm{ta}_\lambda(\ell_M/2)$ is satisfied if $\sum_{i=1}^n \mathrm{ta}_\lambda(\ell_i/2) \geq \frac{2\pi-\lambda S}{\pi}\mathrm{ta}_\lambda(\ell_M/2)$, which may be is true in general.

In the literature, a curve in $\overline{M}^2_\lambda$ satisfying $\kappa \geq \kappa_0$ is called κ_0-convex (see, for instance, [2–4, 10, 12]). For κ_0 big enough, these curves are characterized by the fact that at any point they are contained in a disc of radius R with $\kappa_0 = \mathrm{co}_\lambda(R)$. For polygons, Theorem 1.7 says that the concept is quantitatively different, here, when $\kappa_{A_i} \geq \kappa_0$, the relation between κ_0 and R is $(2/\pi)\kappa_0 = \mathrm{co}_\lambda(R)$.

For the Euclidean plane, Definition (1.4) was used in [5] in the study of approximations of surfaces by planar triangulations, and in [6] it was studied how good this definition is for approximating the curvature of a planar curve by a polygonal line. Other applications of this definition in the Euclidean case were done in [8]. Related but different definitions of curvature of a polygon in the Euclidean plane were used for other applications in [16] and [18].

Some people would prefer to take (1.4) as a definition for the curvature of a convex polygon for every $\overline{M}^2_\lambda$, without taking into account the value of λ. In the last section of the paper we give the corresponding result (Theorem 4.1) for this definition.

2. About Definition 1.6

If we consider a convex polygon as a limit of smooth curves approaching it and the curvature at a vertex as the limit of the curvature of the points at the curves whose limit is the vertex, then the curvature becomes infinite. Obviously, this is not a good definition for many geometric properties. We use a definition satisfying the following properties:

P1 the curvature of a vertex is bigger as the interior angle is lower;

P2 the curvature of a vertex is bigger as the lengths of the adjacent sides is shorter;

P3 if we have a regular polygon inscribed in a circle and we take the number of sides of the regular polygon increasing up to infinite, the curvature of the vertices approach the curvature of the circle.

Properties **P1** and **P3** correspond to a natural geometric intuition. Property **P2** is related to the fact that we want to generalize Theorem 1.1 which fails when $\kappa_0 = 0$ (in the Euclidean case) because with $\kappa_0 = 0$ you may have straight lines with arbitrary length which are the obstacle for upper bounds for the circumradius.

It is obvious that our definition 1.6 satisfies **P1** and **P2**. In the next proposition, we shall check that it also satisfies **P3**.

Proposition 2.1. *Let C be a circle of radius R ($< \pi/(2\sqrt{\lambda})$ if $\lambda > 0$) in $\overline{M}^2_\lambda$, and let P_n be a regular polygon of n sides inscribed in C. If κ_n denotes the curvature at any vertex A_n of the polygon P_n, then $\lim_{n\to\infty} \kappa_n = \mathrm{co}_\lambda(R)$, which is the curvature of C.*

Proof. Let us recall some trigonometric formulae of the space forms. Let Δ be a geodesic triangle with sides a, b, c and opposite vertices A, B, C. Let $\widehat{A}, \widehat{A}, \widehat{C}$ be the angles at these vertices. Then the following formulae hold:

$$\cos \widehat{A} = \frac{\mathrm{c}_\lambda(a) - \mathrm{c}_\lambda(b)\,\mathrm{c}_\lambda(c)}{\lambda\,\mathrm{s}_\lambda(b)\,\mathrm{s}_\lambda(c)}, \tag{2.1}$$

(when $\lambda = 0$, the quotient in the second term of (2.1) must be understood taking limits for $\lambda \to 0$, giving the standard cosine law in Euclidean plane),

$$\frac{\sin \widehat{A}}{\mathrm{s}_\lambda(a)} = \frac{\sin \widehat{A}}{\mathrm{s}_\lambda(b)} = \frac{\sin \widehat{C}}{\mathrm{s}_\lambda(c)}. \tag{2.2}$$

Let A_n, B_n be two consecutive vertices of the polygon bounding a side A_nB_n of length ℓ_n. Let M_n be the middle point of A_nB_n between A_n and B_n. Let O be the center of the circle. Consider the geodesic triangle OM_nB_n and denote by d_n the length of the geodesic OM_n. One has $\widehat{M_n} = \pi/2$. We can apply (2.1) and (2.2) to this triangle to obtain

$$\mathrm{c}_\lambda(R) = \mathrm{c}_\lambda(d_n)\mathrm{c}_\lambda(\ell_n/2), \qquad \mathrm{s}_\lambda(d_n) = \mathrm{s}_\lambda(R)\sin(\widehat{A}_n/2).$$

From these two equalities and the formulae (1.2), we obtain

$$\frac{\mathrm{c}^2_\lambda(R)}{\mathrm{c}^2_\lambda(\ell_n/2)} = \mathrm{c}^2_\lambda(d_n) = 1 - \lambda \mathrm{s}^2_\lambda(d_n) = 1 - \lambda\,\mathrm{s}^2_\lambda(R)\sin^2(\widehat{A}_n/2).$$

Then

$$\begin{aligned}\sin^2(\widehat{A}_n/2) &= \frac{1}{\lambda\,\mathrm{s}^2_\lambda(R)} - \frac{\mathrm{co}^2_\lambda(R)}{\lambda\,\mathrm{c}^2_\lambda(\ell_n/2)} = 1 + \frac{\mathrm{co}^2_\lambda(R)}{\lambda}\left(1 - \frac{1}{\mathrm{c}^2_\lambda(\ell_n/2)}\right)\\ \cos^2(\widehat{A}_n/2) &= 1 - \sin^2(\widehat{A}_n/2) = \mathrm{co}^2_\lambda(R)\mathrm{ta}^2_\lambda(\ell_n/2),\end{aligned}$$

and

$$\mathrm{ta}_\lambda(\ell_n/2) = \mathrm{ta}_\lambda(R)\cos(\widehat{A}_n/2) = \mathrm{ta}_\lambda(R)\sin(\pi/2 - \widehat{A}_n/2). \tag{2.3}$$

We apply now the definition (1.3) to the curvature κ_n of $\widehat{A}_n$,

$$\kappa_n = \frac{(\pi - \widehat{A}_n)}{2\mathrm{ta}_\lambda(\ell_n/2)} = \frac{2(\pi/2 - \widehat{A}_n/2)}{2\mathrm{ta}_\lambda(R)\sin(\pi/2 - \widehat{A}_n/2)}. \tag{2.4}$$

But $\lim_{n\to\infty} \widehat{A}_n = \pi$, then the limit of the quotient in (2.4) for $n \to \infty$ is $\mathrm{co}_\lambda(R)$, as claimed in Proposition 2.1. □

Definition 1.6 is not the unique one that satisfies properties **P1** to **P3**. If we take (1.4) as a definition of the curvature at A for any value of λ, it is obvious that it satisfies **P1** and **P2**. Moreover, **P3** follows from the same proof of Proposition 2.1, taking into account that $\lim_{n\to\infty} \frac{\mathrm{ta}_\lambda(\ell_n/2)}{(\ell_n/2)} = 1$. We prefer (1.3) because it gives a clean bound (1.5) in Theorem 1.7, but some people may prefer the other definition.

3. Proof of the theorem 1.7

Let $A_1, A_2, \ldots, A_n$ be the consecutive vertices of the polygon P. Let $\hat{A}_i$ be the angles at these vertices, and $\ell_1, \ldots, \ell_i, \ldots, \ell_n$ be the lengths of the sides $A_nA_1, \ldots, A_{i-1}A_i, \ldots, A_{n-1}A_n$, respectively (see Figure 3.1).

For every segment $A_{i-1}A_i$, we construct a segment of circle C_i of radius ρ_i with center O_i in a line orthogonal to $A_{i-1}A_i$ in its middle point and in the ray in the inward direction, and with boundary points A_i and A_{i+1}.

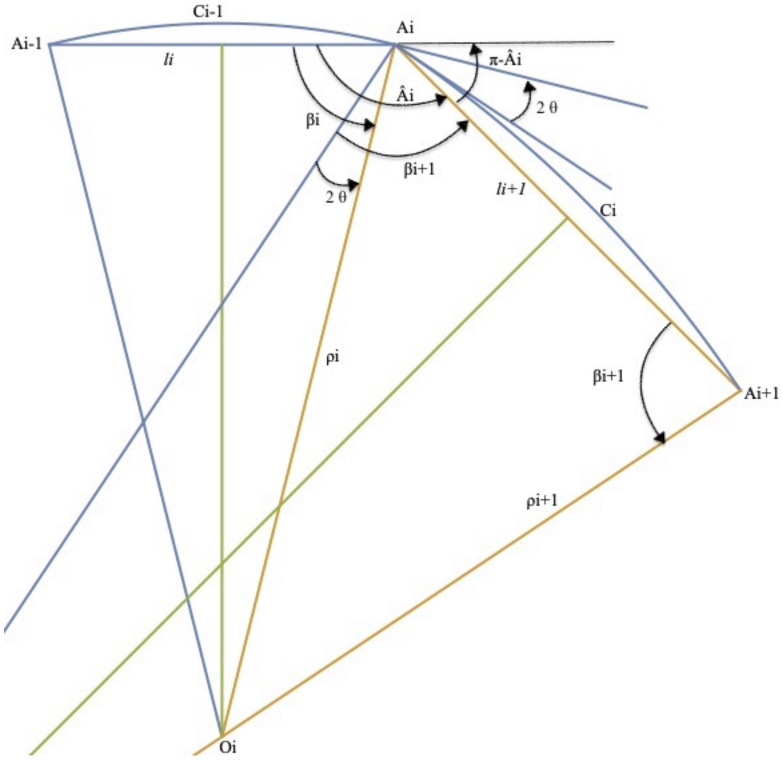

Fig. 3.1

Each angle β_i at A_i of the isosceles triangle $A_{i-1}A_iO_i$ satisfies $0 < \beta_i < \pi/2$, and the analogue of (2.3) for this triangle is

$$\mathrm{ta}_\lambda(\ell_i/2) = \mathrm{ta}_\lambda(\rho_i)\cos\beta_i = \mathrm{ta}_\lambda(\rho_i)\sin(\delta_i) \tag{3.1}$$

if we take $\delta_i = \pi/2 - \beta_i$.

We now take the curve obtained as the union of the segments of circle C_i. This curve is convex if and only if, for every $i = 1, .., n$, the tangent vectors at A_i of the circles C_{i-1} and C_i with the curve C_{i-1} oriented from A_{i-1} to A_i and the curve C_i oriented from A_i to A_{i+1} form an angle 2 θ_i in the interval $[0, \pi]$. This angle is the same as the one formed by the normals at A_i to C_{i-1} and C_i pointing inward. These normals are A_iO_i and A_iO_{i+1}, and this angle is non negative if and only if $\beta_i + \beta_{i+1} \geq \hat{A}_i$, that is, $\pi - \delta_i - \delta_{i+1} \geq \pi - (\pi - \hat{A}_i)$,

$$\delta_i + \delta_{i+1} \leq \pi - \hat{A}_i. \tag{3.2}$$

For every i, let us choose ρ_i such that $\mathrm{ta}_\lambda(\rho_i) = \frac{\pi}{2\kappa_0}$. The choice is possible because, by (1.6), we have that $\frac{\pi}{2\kappa_0} \geq \mathrm{ta}_\lambda(\ell_i/2)$ for $\lambda = 0$. It is a hypothesis for $\lambda \neq 0$, and for $\lambda < 0$, $2\kappa_0/\pi > \sqrt{-\lambda}$ it is also a hypothesis. From the hypothesis of $\kappa_{A_i} \geq \kappa_0$, by using formula (3.1), we have

$$\begin{aligned}\kappa_0 \leq \frac{\pi - \hat{A}_i}{\mathrm{ta}_\lambda(\ell_i/2) + \mathrm{ta}_\lambda(\ell_{i+1}/2)} &= \frac{(\pi - \hat{A}_i)}{\mathrm{ta}_\lambda(\rho_i)\sin\delta_i + \mathrm{ta}_\lambda(\rho_{i+1})\sin\delta_{i+1}} \\ &= \frac{2,\kappa_0}{\pi}\,\frac{(\pi - \hat{A}_i)}{\sin\delta_i + \sin\delta_{i+1}},\end{aligned}$$

then

$$\pi - \hat{A}_i \geq \frac{\pi}{2}\,(\sin\delta_i + \sin\delta_{i+1}) \geq (\delta_i + \delta_{i+1}), \tag{3.3}$$

an inequality which coincides with (3.2). As a consequence, the closed curve C formed by the union of the C_i is convex and with curvature equal $\frac{2\kappa_0}{\pi}$ at every regular point. Then the specific curvature of C is bigger than $\frac{2\kappa_0}{\pi}$ (see Remark 1.5) and, by Milka's theorem, the circumradius satisfies (1.5).

The equality holds in (1.5) if and only if equalities hold in all the inequalities of the above argument. In particular, equality implies $\sin\delta_i = \frac{2}{\pi}\delta_i$, which happens if and only if $\delta_i = \pi/2$. The other equalities that we must have are $\pi - \hat{A}_i = c\,(\sin\delta_i + \sin\delta_{i+1}) = \pi$, that is $A_i = 0$, and $\mathrm{ta}_\lambda(\ell_i/2) = \mathrm{ta}_\lambda(\rho_i)\sin\delta_i = \pi/\kappa_0$ and it is satisfied only in a doubly covered segment of length π/κ_0. It is a degenerate polygon of curvature $\pi/(2\mathrm{ta}_\lambda(\ell_i/2)) = \kappa_0$. □

4. If we adopt definition (1.4)

In this section, we consider the curvature at a vertex of a convex polygon defined by (1.4).

Let us define R_0 by $\frac{1}{\kappa_0} = \mathrm{ta}_\lambda(R_0)$. From the inequality $\frac{2(\pi - \widehat{A}_i)}{\ell_i + \ell_{i+1}} \geq \kappa_0 = \mathrm{co}_\lambda(R_0)$ and the isoperimetric inequality $L^2 - 4\pi S + \lambda S^2 \geq 0$, (where L is the perimeter of the polygon and S is the area of the region enclosed by it), it follows that

$$\mathrm{s}_\lambda(R_0) \geq \frac{L}{2\pi} \geq \frac{\ell_M}{\pi}.$$

For definition (1.4) of the curvature at a vertex, Theorem 1.7 must be changed by:

Theorem 4.1. *Let P be a compact convex polygon in $\overline{M}^2_\lambda$ with $\kappa_{A_i} \geq \kappa_0$ and such that, if $\lambda > 0$, the sides satisfy $\ell_i \leq 2\,\mathfrak{e} < \pi/\sqrt{\lambda}$ and, if $\lambda < 0$, one has $\frac{2\kappa_0}{\pi} > \sqrt{-\lambda}$ and $\ell_i \geq 2\,\mathfrak{e} > 0$. When $\lambda \neq 0$, we will also assume that $\frac{\mathfrak{e}}{\mathrm{ta}_\lambda(\mathfrak{e})}\frac{\pi}{2\kappa_0} \geq \mathrm{ta}_\lambda(\ell_i/2)$ for every i. Then the circumradius R of P satisfies*

$$R \leq \frac{\pi}{2}R_0 \qquad \text{if } \lambda = 0, \tag{4.1}$$

$$\mathrm{ta}_\lambda(R) \leq \frac{\mathrm{ta}_\lambda(\mathfrak{e})}{\mathfrak{e}}\frac{\pi}{2}\mathrm{ta}_\lambda(R_0) \qquad \text{if } \lambda \neq 0, \tag{4.2}$$

and the equality holds if and only if the polygon degenerates into a doubly covered segment.

We observe that, when $\lambda < 0$, we have $\frac{\mathrm{ta}_\lambda(\mathfrak{e})}{\mathfrak{e}} \frac{\pi}{2\ \kappa_0} < \pi/(2\kappa_0)$ for $\mathfrak{e} > 0$.

The proof is exactly the same for $\lambda = 0$. For $\lambda > 0$, one has that

$$\ell_i/2 = \frac{\ell_i/2}{\mathrm{ta}_\lambda(\ell_i/2)}\mathrm{ta}_\lambda(\ell_i/2) \geq \frac{\mathfrak{e}}{\mathrm{ta}_\lambda(\mathfrak{e})}\mathrm{ta}_\lambda(\ell_i/2).$$

Then, if for every i we choose ρ_i such that $\mathrm{ta}_\lambda(\rho_i) = \frac{\mathfrak{e}}{\mathrm{ta}_\lambda(\mathfrak{e})}\frac{\pi}{2\kappa_0}$, we obtain

$$\begin{aligned}\kappa_0 \leq \frac{\pi - \hat{A}_i}{(\ell_i/2) + (\ell_{i+1}/2)} &\leq \frac{(\pi - \hat{A}_i)}{\frac{\mathfrak{e}}{\mathrm{ta}_\lambda(\mathfrak{e})}\left(\mathrm{ta}_\lambda(\rho_i)\sin\delta_i + \mathrm{ta}_\lambda(\rho_{i+1})\sin\delta_{i+1}\right)} \\ &= 2\frac{\kappa_0}{\pi}\ \frac{(\pi - \hat{A}_i)}{\sin\delta_i + \sin\delta_{i+1}},\end{aligned} \tag{4.3}$$

from which $\pi - \hat{A}_i \geq (\delta_i + \delta_{i+1})$ as in the proof of Theorem 1.7, the union of the arcs C_i is convex, and the rest of the proof is the same as that for Theorem 1.7.

The proof for the case $\lambda < 0$ follows the same steps, with the unique change that now the function $\frac{\ell/2}{\mathrm{ta}_\lambda(\ell/2)}$ is decreasing and we have to bound it taking the minimal value of $\ell/2$.

Acknowledgments. We would like to thank the referee for some suggestions and corrections which have contributed to a more clear exposition of the paper. Especially we want to thank him/her for attracting our attention to paper [15]. That work was unknown to us and its use has greatly shortened our previous proof of Theorem 1.7.

This work was done when the first named author was under a contract of "Investigador Doctor Senior" in Valencia University in 2023 and he would like to thank the Department of Mathematics of that university for hospitality.

This research is partially supported by grant PID2019-105019GB-C21 funded by MCIN/AEI/ 10.13039/501100011033 and by "ERDF A way of makimg Europe", and by the grant AICO 2021 21/378.01/1 funded by the Generalitat Valenciana.

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Received June 16, 2023, revised December 12, 2023.

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Дискретна теорема Бляшке для опуклих багатокутників в 2-вимірних просторах постійної кривини

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Нехай M є 2-вимірною площиною постійної кривини, P є опуклим багатокутником в M. Для цих багатокутників дано визначення кривини κ_i в вершинах A_i і доведена дискретна теорема Бляшке: "якщо P є опуклий багатокутник в M з кривинами вершин $\kappa_i \geq \kappa_0 > 0$, то радіус R кола, описаного навколо P, задовольняє нерівність $\mathrm{ta}_\lambda(R) \leq \pi/(2\kappa_0)$, і рівність виконується тоді і лише тоді, коли багатокутник є 2-покритим сегментом".

Ключові слова: теорема Бляшке, кривина вершини, радіус описаного кола, опуклий багатокутник