Journal of Mathematical Physics, Analysis, Geometry
2024, Vol. 20, No. 2, pp. 205–220
doi: https://doi.org/10.15407/mag20.02.205

Ricci–Bourguignon Solitons on Sequential Warped Product Manifolds

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We study Ricci–Bourguignon solitons on sequential warped products. The necessary conditions are obtained for a Ricci–Bourguignon soliton with the structure of a sequential warped product to be an Einstein manifold when we consider the potential field as a Killing or a conformal vector field.

Key words: Ricci–Bourguignon soliton, warped product manifold, sequential warped product manifold, Killing vector field, conformal vector field

Mathematical Subject Classification 2020: 53E20, 53C21, 53C25

1. Introduction

Let (M, g) be a semi-Riemannian manifold and denote by Ric the Ricci tensor of (M, g). A semi-Riemannian manifold (M, g) is said to be a *Ricci soliton* [26] if there exists a smooth vector field X satisfying the equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g \tag{1.1}$$

for some constant λ and it is denoted by (M, g, X, λ), where $\mathcal{L}$ denotes the Lie derivative, and the vector field $X \in \mathfrak{X}(M)$ is called the *potential vector field*. If λ is a smooth function on (M, g), then (M, g, X, λ) is called an *almost Ricci soliton* [31].

Ricci solitons are a natural generalization of Einstein manifolds. They correspond to self-similar solutions of the Ricci flow equation

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric},$$

which was defined by Hamilton [25, 27]. Ricci solitons and their generalizations have been studied by many geometers in the recent years. See, for example, [2, 3, 7–9, 13, 16, 18, 20, 22, 30, 32] and the references therein.

If a potential vector field is the gradient of a smooth function u on M, then $(M, g, \nabla u, \lambda)$ is called a *gradient Ricci soliton* and equation (1.1) turns into

$$\mathrm{Ric} + \mathrm{Hess}\, u = \lambda g.$$

The study of the concept of Ricci–Bourguignon solitons was introduced by Dwiwedi [17]. They correspond to self-similar solutions of the Ricci–Bourguignon flow equation

$$\frac{\partial g}{\partial t} = -2(\mathrm{Ric} - \rho R g), \tag{1.2}$$

where R is the scalar curvature and $\rho \in \mathbb{R}$. The flow in equation (1.2) was introduced by J.-P. Bourguignon [6]. Equation (1.2) is precisely the Ricci flow for $\rho = 0$.

A *Ricci–Bourguignon soliton* (briefly RBS) ([6, 17]) is a semi-Riemannian manifold (M, g) endowed with a vector field X on M that satisfies

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g + \rho R g, \tag{1.3}$$

where $\lambda \in \mathbb{R}$ and it is denoted by (M, g, X, λ, ρ). If X is the gradient of a smooth function u on M, then $(M, g, \nabla u, \lambda, \rho)$ is called a *gradient Ricci–Bourguignon soliton* [17] and equation (1.3) turns into

$$\mathrm{Ric} + \mathrm{Hess}\, u = \lambda g + \rho R g.$$

When λ is a smooth function on (M, g), it is called a *Ricci–Bourguignon almost soliton* and a *gradient Ricci–Bourguignon almost soliton*, respectively [17]. In [17], Dwivedi proved some results for the solitons of the Ricci–Bourguignon flow generalizing the corresponding results for Ricci solitons. Later, in [33], Soylu gave classification theorems for Ricci–Bourguignon solitons and almost solitons with concurrent potential vector field. In [21], A. Ghosh studied Ricci–Bourguignon solitons and Ricci–Bourguignon almost solitons on a Riemannian manifold and proved some triviality results. In [11], Cunha, Lemos and Roing obtained conditions for a Ricci–Bourguignon soliton to be a Ricci soliton and some triviality cases. In [12], Cunha, Silva Junior, De Lima and De Lima investigated the triviality of gradient solitons of the Ricci–Bourguignon flow.

Warped product manifolds were defined by O'Neill and Bishop in [5] to construct manifolds with negative curvature. They have an important role in both geometry and physics. They are used in general relativity to model the space-time [10]. Doubly, multiply and sequential warped product manifolds are known as generalizations of the warped product manifolds ([15,36,37]). There are many papers in which Ricci solitons on some Riemannian manifolds or on warped product manifolds or on some generalizations of warped products have been studied (see, for example, [1, 4, 14, 19, 23, 24, 28, 29, 34, 35]. Motivated by the above studies, in this paper, we consider Ricci–Bourguignon solitons on sequential warped product manifolds. By considering the potential vector field as a Killing or a conformal vector field, we prove some results.

2. Preliminaries

Let (M_i, g_i) be semi-Riemannian manifolds, $1 \leq i \leq 3$, and $f : M_1 \to \mathbb{R}^+$, $h : M_1 \times M_2 \to \mathbb{R}^+$ be two smooth functions. The *sequential warped product* *manifold* M is the triple product manifold $M = (M_1 \times_f M_2) \times_h M_3$ endowed with the metric tensor $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$ [15]. Here the functions f, h are called the *warping functions*.

Throughout the paper, (M, g) will be considered as a sequential warped product manifold, where $M = M^n = (M_1^{n_1} \times_f M_2^{n_2}) \times_h M_3^{n_3}$ with the metric $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$. The restriction of the warping function $h : \overline{M} = M_1 \times M_2 \to \mathbb{R}$ to $M_1 \times \{0\}$ is $h^1 = h|_{M_1 \times \{0\}}$.

We use the notations ∇, ∇^i; Ric, Ric^i; Hess, Hess^i; Δ, Δ^i; $\mathcal{L}$, $\mathcal{L}^i$ for the Levi-Civita connections, Ricci tensors, Hessians, Laplacians and Lie derivatives of M, and M_i, respectively. The Hessian of $\overline{M}$ is denoted by $\overline{\mathrm{Hess}}$.

The following lemmas on sequential warped product manifolds are necessary to prove our results.

Lemma 2.1 ([15])**.** *Let (M, g) be a sequential warped product and $X_i, Y_i \in \mathfrak{X}(M_i)$ for $1 \leq i \leq 3$. Then*

1. $\nabla_{X_1} Y_1 = \nabla^1_{X_1} Y_1$;
2. $\nabla_{X_1} X_2 = \nabla_{X_2} X_1 = X_1(\ln f) X_2$;
3. $\nabla_{X_2} Y_2 = \nabla^2_{X_2} Y_2 - f g_2(X_2, Y_2) \nabla^1 f$;
4. $\nabla_{X_3} X_1 = \nabla_{X_1} X_3 = X_1(\ln h) X_3$;
5. $\nabla_{X_2} X_3 = \nabla_{X_3} X_2 = X_2(\ln h) X_3$;
6. $\nabla_{X_3} Y_3 = \nabla^3_{X_3} Y_3 - h g_3(X_3, Y_3) \nabla h$.

Lemma 2.2 ([15])**.** *Let (M, g) be a sequential warped product and $X_i, Y_i \in \mathfrak{X}(M_i)$ for $1 \leq i \leq 3$. Then*

1. $\mathrm{Ric}(X_1, Y_1) = \mathrm{Ric}^1(X_1, Y_1) - \frac{n_2}{f} \mathrm{Hess}^1 f(X_1, Y_1) - \frac{n_3}{h} \overline{\mathrm{Hess}}\, h(X_1, Y_1)$;
2. $\mathrm{Ric}(X_2, Y_2) = \mathrm{Ric}^2(X_2, Y_2) - f^\sharp g_2 (X_2, Y_2) - \frac{n_3}{h} \overline{\mathrm{Hess}}\, h(X_2, Y_2)$;
3. $\mathrm{Ric}(X_3, Y_3) = \mathrm{Ric}^3(X_3, Y_3) - h^\sharp g_3 (X_3, Y_3)$;
4. $\mathrm{Ric}(X_i, X_j) = 0$ *if* $i \neq j$, *where* $f^\sharp = \left(f \Delta^1 f + (n_2 - 1) \left\| \nabla^1 f \right\|^2 \right)$ *and* $h^\sharp = \left(h \Delta h + (n_3 - 1) \left\| \nabla h \right\|^2 \right)$.

Lemma 2.3 ([15])**.** *Let (M, g) be a sequential warped product manifold. A vector field $X \in \mathfrak{X}(M)$ satisfies the equation*

$$\begin{aligned}(\mathcal{L}_X g)(Y, Z) = {} & \left(\mathcal{L}^1_{X_1} g_1\right)(Y_1, Z_1) + f^2 \left(\mathcal{L}^2_{X_2} g_2\right)(Y_2, Z_2) + h^2 \left(\mathcal{L}^3_{X_3} g_3\right)(Y_3, Z_3) \\ & + 2f X_1(f) g_2(Y_2, Z_2) + 2h(X_1 + X_2)(h) g_3(Y_3, Z_3)\end{aligned}$$

for $Y, Z \in \mathfrak{X}(M)$.

A vector field V on a Riemannian manifold (M, g) is said to be *conformal* if there exists a smooth function on M satisfying the equation

$$\mathcal{L}_V g = 2fg.$$

If $f = 0$, then V is called a *Killing* vector field.

3. Main Results

In this section, we examine the properties of Ricci–Bourguignon solitons on sequential warped product manifolds.

Let ψ and σ be two smooth functions on a sequential warped product $M = (M_1 \times_f M_2) \times_h M_3$.

Firstly, we have the following theorem:

Theorem 3.1. *Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product equipped with the metric $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$. If (M, g, X, λ, ρ) is an RBS with potential vector field of the form $X = X_1 + X_2 + X_3$, where $X_i \in \mathfrak{X}(M_i)$ for $1 \leq i \leq 3$, then*

(i) *$(M_1, g_1, X_1, \lambda_1, \rho_1)$ is an RBS when $\operatorname{Hess} f = \sigma g$ and $\overline{\operatorname{Hess}}\, h = \psi g$ and $\lambda_1 = \lambda + \rho R + \frac{n_2}{f}\sigma + \frac{n_3}{h}\psi - \rho_1 R_1$ is a constant;*

(ii) *M_2 is an Einstein manifold when X_2 is a Killing vector field and $\overline{\operatorname{Hess}}\, h = \psi g$;*

(iii) *$(M_3, g_3, h^2 X_3, \lambda_3, \rho_3)$ is an RBS when $\lambda_3 = \lambda h^2 + \rho R h^2 + h^\sharp - h(X_1 + X_2)(h) - \rho_3 R_3$ is a constant.*

Proof. Assume that (M, g, X, λ, ρ) is an RBS with the structure of the sequential warped product. Then, for $Y, Z \in \chi(M)$, the equation

$$\operatorname{Ric}(Y, Z) + \frac{1}{2}\left(\mathcal{L}_X g\right)(Y, Z) = (\lambda + \rho R) g(Y, Z)$$

is satisfied. Using Lemma 2.2 and Lemma 2.3 for the vector fields Y and Z such that $Y = Y_1 + Y_2 + Y_3$ and $Z = Z_1 + Z_2 + Z_3$, we have

$$\begin{aligned}
&\operatorname{Ric}^1(Y_1, Z_1) - \frac{n_2}{f} \operatorname{Hess}^1 f(Y_1, Z_1) - \frac{n_3}{h} \overline{\operatorname{Hess}}\, h(Y_1, Z_1) \\
&\quad + \operatorname{Ric}^2(Y_2, Z_2) - f^\sharp g_2(Y_2, Z_2) - \frac{n_3}{h} \overline{\operatorname{Hess}}\, h(Y_2, Z_2) \\
&\quad + \operatorname{Ric}^3(Y_3, Z_3) - h^\sharp g_3(Y_3, Z_3) \\
&\quad + \frac{1}{2}\left(\mathcal{L}^1_{X_1} g_1\right)(Y_1, Z_1) + \frac{1}{2} f^2 \left(\mathcal{L}^2_{X_2} g_2\right)(Y_2, Z_2) + \frac{1}{2} h^2 \left(\mathcal{L}^3_{X_3} g_3\right)(Y_3, Z_3) \\
&\quad + f X_1(f) g_2(Y_2, Z_2) + h(X_1 + X_2)(h) g_3(Y_3, Z_3) \\
&= (\lambda + \rho R) g_1(Y_1, Z_1) + (\lambda + \rho R) f^2 g_2(Y_2, Z_2) + (\lambda + \rho R) h^2 g_3(Y_3, Z_3). \qquad (3.1)
\end{aligned}$$

Let $Y = Y_1$ and $Z = Z_1$. So, from equation (3.1), if $\operatorname{Hess} f = \sigma g$ and $\overline{\operatorname{Hess}}\, h = \psi g$, then we get

$$\begin{aligned}
\operatorname{Ric}^1(Y_1, Z_1) + \frac{1}{2}(\mathcal{L}^1_{X_1} g_1)(Y_1, Z_1) &= \lambda_1 g_1(Y_1, Z_1) \\
&\quad + [-\lambda_1 + \lambda + \rho R + \frac{n_2}{f}\sigma + \frac{n_3}{h}\psi] g_1(Y_1, Z_1) \\
&= \lambda_1 g_1(Y_1, Z_1) + \rho_1 R_1 g_1(Y_1, Z_1).
\end{aligned}$$

Hence $(M_1, g_1, X_1, \lambda_1, \rho_1)$ is an RBS when $\lambda_1 = \lambda + \rho R + \frac{n_2}{f}\sigma + \frac{n_3}{h}\psi - \rho_1 R_1$ is a constant.

Now, let $Y = Y_2$ and $Z = Z_2$. Then

$$\begin{aligned}\mathrm{Ric}^2(Y_2, Z_2) - f^\sharp g_2(Y_2, Z_2) - \frac{n_3}{h}\overline{\mathrm{Hess}}\,h(Y_2, Z_2)\\ + \frac{1}{2}f^2\left(\mathcal{L}^2_{X_2}g_2\right)(Y_2, Z_2) + fX_1(f)g_2(Y_2, Z_2)\\ = (\lambda + \rho R)f^2 g_2(Y_2, Z_2).\end{aligned}$$

Here, if X_2 is a Killing vector field and $\overline{\mathrm{Hess}}\,h = \psi g$, we get

$$\mathrm{Ric}^2(Y_2, Z_2) = (\lambda f^2 + \rho R f^2 + f^\sharp + \frac{n_3}{h}\psi f^2 - fX_1(f))g_2(Y_2, Z_2),$$

which implies that M_2 is an Einstein manifold.

Finally, let $Y = Y_3$ and $Z = Z_3$. Then

$$\begin{aligned}&\mathrm{Ric}^3(Y_3, Z_3) + \frac{1}{2}\left(\mathcal{L}^3_{h^2X_3}\right)g_3(Y_3, Z_3)\\ &\quad= \lambda_3 g_3(Y_3, Z_3) + [-\lambda_3 + \lambda h^2 + \rho R h^2 + h^\sharp - h(X_1 + X_2)(h)]g_3(Y_3, Z_3)\\ &\quad= \lambda_3 g_3(Y_3, Z_3) + \rho_3 R_3 g_3(Y_3, Z_3),\end{aligned}$$

which means that $(M_3, g_3, h^2X_3, \lambda_3, \rho_3)$ is an RBS when $\lambda_3 = \lambda h^2 + \rho R h^2 + h^\sharp - h(X_1 + X_2)(h) - \rho_3 R_3$ is a constant. □

In the following theorems, we provide some conditions for the manifolds M_i, $(1 \le i \le 3)$ to be Einstein manifolds.

Theorem 3.2. *Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product equipped with the metric $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$. If (M, g, X, λ, ρ) is an RBS and X is a Killing vector field, then*

(i) *M_1 is an Einstein manifold when $\mathrm{Hess}\,f = \sigma g$ and $\overline{\mathrm{Hess}}\,h = \psi g$;*
(ii) *M_2 is an Einstein manifold when $\overline{\mathrm{Hess}}\,h = \psi g$;*
(iii) *M_3 is an Einstein manifold.*

Proof. Let (M, g, X, λ, ρ) be an RBS with the structure of the sequential warped product and let X be a Killing vector field. Then, for all $Y, Z \in \chi(M)$, we have $\mathrm{Ric}(Y, Z) = (\lambda + \rho R)g(Y, Z)$. From equation (3.1), we may write:

$$\begin{aligned}\mathrm{Ric}^1(Y_1, Z_1) &= (\lambda + \rho R + \frac{n_2}{f}\sigma + \frac{n_3}{h}\psi)g_1(Y_1, Z_1),\\ \mathrm{Ric}^2(Y_2, Z_2) &= (\lambda f^2 + \rho R f^2 + f^\sharp + \frac{n_3}{h}\psi f^2)g_2(Y_2, Z_2),\\ \mathrm{Ric}^3(Y_3, Z_3) &= (\lambda h^2 + \rho R h^2 + h^\sharp)g_3(Y_3, Z_3),\end{aligned}$$

which imply that M_1, M_2 and M_3 are Einstein manifolds. □

Theorem 3.3. *Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product equipped with the metric $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$ and let (M, g, X, λ, ρ) be an RBS. Assume that $\mathrm{Hess}\,f = \sigma g$ and $\overline{\mathrm{Hess}}\,h = \psi g$. Then M_i $(1 \le i \le 3)$ are Einstein manifolds if one of the following conditions holds:*

(i) $X = X_1$ *and* X_1 *is Killing on* M_1;
(ii) $X = X_2$ *and* X_2 *is Killing on* M_2;
(iii) $X = X_3$ *and* X_3 *is Killing on* M_3.

Proof. Let (M, g, X, λ, ρ) be an RBS with the structure of the sequential warped product. Assume that $\operatorname{Hess} f = \sigma g$ and $\overline{\operatorname{Hess}}\, h = \psi g$. If $X = X_1$ and X_1 is Killing on M_1, using Lemma 2.3, we have

$$\mathcal{L}_X g = 2f X_1(f) g_2 + 2h X_1(h) g_3.$$

So, by using the above equation in (3.1), we get

$$\begin{aligned}
\operatorname{Ric}^1(Y_1, Z_1) &= (\lambda + \rho R + \frac{n_2}{f}\sigma + \frac{n_3}{h}\psi) g_1(Y_1, Z_1),\\
\operatorname{Ric}^2(Y_2, Z_2) &= (\lambda f^2 + \rho R f^2 + f^\sharp + \frac{n_3}{h}\psi f^2 - f X_1(f)) g_2(Y_2, Z_2).\\
\operatorname{Ric}^3(Y_3, Z_3) &= (\lambda h^2 + \rho R h^2 + h^\sharp - h X_1(h)) g_3(Y_3, Z_3).
\end{aligned}$$

Thus the manifolds M_1, M_2 and M_3 are Einstein. Using the same pattern, (ii) and (iii) can be verified. □

Theorem 3.4. *Let* $M = (M_1 \times_f M_2) \times_h M_3$ *be a sequential warped product equipped with the metric* $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$*, let* (M, g, X, λ, ρ) *be an RBS and* X *be a conformal vector field. Then*

(i) M_1 *is an Einstein manifold when* $\operatorname{Hess} f = \sigma g$ *and* $\overline{\operatorname{Hess}}\, h = \psi g$,
(ii) M_2 *is an Einstein manifold when* $\overline{\operatorname{Hess}}\, h = \psi g$,
(iii) M_3 *is an Einstein manifold.*

Proof. Assume that (M, g, X, λ, ρ) is an RBS with the structure of the sequential warped product and X is a conformal vector field with factor 2α. Then

$$\operatorname{Ric}(Y, Z) = (\lambda + \rho R - \alpha) g(Y, Z).$$

By using (3.1), the above equation implies

$$\begin{aligned}
\operatorname{Ric}^1(Y_1, Z_1) &- \frac{n_2}{f}\operatorname{Hess}^1 f(Y_1, Z_1) - \frac{n_3}{h}\overline{\operatorname{Hess}}\, h(Y_1, Z_1) + \operatorname{Ric}^2(Y_2, Z_2)\\
&- f^\sharp g_2(Y_2, Z_2) - \frac{n_3}{h}\overline{\operatorname{Hess}}\, h(Y_2, Z_2) + \operatorname{Ric}^3(Y_3, Z_3) - h^\sharp g_3(Y_3, Z_3)\\
&= (\lambda + \rho R - \alpha) g_1(Y_1, Z_1)\\
&+ (\lambda + \rho R - \alpha) f^2 g_2(Y_2, Z_2) + (\lambda + \rho R - \alpha) h^2 g_3(Y_3, Z_3).
\end{aligned}$$

If $\operatorname{Hess} f = \sigma g$ and $\overline{\operatorname{Hess}}\, h = \psi g$, then we get

$$\begin{aligned}
\operatorname{Ric}^1(Y_1, Z_1) &= (\lambda + \rho R - \alpha + \frac{n_2}{f}\sigma + \frac{n_3}{h}\psi) g_1(Y_1, Z_1),\\
\operatorname{Ric}^2(Y_2, Z_2) &= (\lambda f^2 + \rho R f^2 - \alpha f^2 + \frac{n_3}{h}\psi f^2 + f^\sharp) g_2(Y_2, Z_2),\\
\operatorname{Ric}^3(Y_3, Z_3) &= (\lambda h^2 + \rho R h^2 - \alpha h^2 + h^\sharp) g_3(Y_3, Z_3).
\end{aligned}$$

Hence, M_1, M_2 and M_3 are Einstein manifolds. □

Using Lemma 2.3, we can state the following theorem:

Theorem 3.5. *Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product equipped with the metric $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$. Then (M, g, X, λ, ρ) is an Einstein manifold if one of the following conditions holds:*

(i) *$X = X_3$ and X_3 is a Killing vector field on M_3.*

(ii) *X_1 is a Killing vector field on M_1, X_2 and X_3 are conformal vector fields on M_2 and M_3 with factors $-2X_1(\ln f)$ and $-2(X_1 + X_2)(\ln h)$, respectively.*

(iii) *$X = X_2 + X_3$, X_2 and X_3 are Killing on M_2 and M_3, respectively, and $X_2(h) = 0$.*

The next theorem gives the necessary condition for the components of the vector field X to be conformal vector fields.

Theorem 3.6. *Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product equipped with the metric $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$ and let (M, g, X, λ, ρ) be an RBS.*

(i) *If M_1 is an Einstein manifold, $\operatorname{Hess} f = \sigma g$ and $\overline{\operatorname{Hess}}\, h = \psi g$, then X_1 is a conformal vector field on M_1.*

(ii) *If M_2 is an Einstein manifold and $\overline{\operatorname{Hess}}\, h = \psi g$, then X_2 is a conformal vector field on M_2.*

(iii) *If M_3 is an Einstein manifold, then X_3 is a conformal vector field on M_3.*

Proof. Let (M_1, g_1), (M_2, g_2) and (M_3, g_3) be Einstein manifolds with factors μ_1, μ_2 and μ_3, respectively. Let (M, g, X, λ, ρ) be an RBS with the structure of the sequential warped product. If $\operatorname{Hess} f = \sigma g$ and $\overline{\operatorname{Hess}}\, h = \psi g$, then, from equation (3.1), we get

$$\begin{aligned}
&\mu_1 g_1(Y_1, Z_1) - \frac{n_2}{f}\sigma g_1(Y_1, Z_1) - \frac{n_3}{h}\psi g_1(Y_1, Z_1) + \mu_2 g_2(Y_2, Z_2) \\
&\quad - f^\sharp g_2(Y_2, Z_2) - \frac{n_3}{h}\psi f^2 g_2(Y_2, Z_2) + \mu_3 g_3(Y_3, Z_3) - h^\sharp g_3(Y_3, Z_3) \\
&\quad + \frac{1}{2}\left(\mathcal{L}^1_{X_1} g_1\right)(Y_1, Z_1) + \frac{1}{2} f^2 \left(\mathcal{L}^2_{X_2} g_2\right)(Y_2, Z_2) + \frac{1}{2} h^2 \left(\mathcal{L}^3_{X_3} g_3\right)(Y_3, Z_3) \\
&\quad + f X_1(f) g_2(Y_2, Z_2) + h(X_1 + X_2)(h) g_3(Y_3, Z_3) \\
&\quad = (\lambda + \rho R) g_1(Y_1, Z_1) + (\lambda + \rho R) f^2 g_2(Y_2, Z_2) + (\lambda + \rho R) h^2 g_3(Y_3, Z_3).
\end{aligned}$$

Thus,

$$\begin{aligned}
\left(\mathcal{L}^1_{X_1} g_1\right)(Y_1, Z_1) &= 2(\lambda + \rho R - \mu_1 + \frac{n_2}{f}\sigma + \frac{n_3}{h}\psi) g_1(Y_1, Z_1), \\
\left(\mathcal{L}^2_{X_2} g_2\right)(Y_2, Z_2) &= \frac{2}{f^2}(\lambda f^2 + \rho R f^2 - \mu_2 + f^\sharp + \frac{n_3}{h}\psi f^2 - f X_1(f)) g_2(Y_2, Z_2), \\
\left(\mathcal{L}^3_{X_3} g_3\right)(Y_3, Z_3) &= \frac{2}{h^2}(\lambda h^2 + \rho R h^2 - \mu_3 + h^\sharp - h(X_1 + X_2)(h)) g_3(Y_3, Z_3).
\end{aligned}$$

Hence, X_1, X_2 and X_3 are conformal vector fields on M_1, M_2 and M_3, respectively. □

Theorem 3.7. *Let $M = (M_1 \times_f M_2) \times_h M_3$ be a sequential warped product equipped with the metric $g = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$ and let (M, g, X, λ, ρ) be an RBS such that $X = \nabla u$. Then*

(i) *$(M_1, g_1, \nabla\phi_1, \lambda_1, \rho_1)$ is a gradient RBS when $\phi_1 = u_1 - n_2 \ln f - n_3 \ln h_1$, $u_1 = u$ and $\lambda_1 = \lambda + \rho R - \rho_1 R_1$ is a constant.*

(ii) *$(M_3, g_3, \nabla\phi_3, \lambda_3, \rho_3)$ is a gradient RBS when $\phi_3 = u$ and $\lambda_3 = \lambda h^2 + \rho R h^2 + h^\sharp - \rho_3 R_3$ is a constant.*

Proof. Assume that (M, g, X, λ, ρ) is an RBS with the structure of the sequential warped product such that $X = \nabla u$. Then, for $Y, Z \in \mathfrak{X}(M)$,

$$\mathrm{Ric}(Y, Z) + \mathrm{Hess}\, u(Y, Z) = \lambda g(Y, Z) + \rho R g(Y, Z) \tag{3.2}$$

is satisfied. Now, let $Y = Y_1$ and $Z = Z_1$. Then equation (3.2) becomes

$$\begin{aligned}\mathrm{Ric}^1(Y_1, Z_1) - \frac{n_2}{f}\mathrm{Hess}^1 f(Y_1, Z_1) - \frac{n_3}{h}\overline{\mathrm{Hess}}\, h(Y_1, Z_1) + \mathrm{Hess}\, u_1(Y_1, Z_1)\\ = \lambda g_1(Y_1, Z_1) + \rho R g_1(Y_1, Z_1)\end{aligned}$$

or, equivalently,

$$\begin{aligned}\mathrm{Ric}^1(Y_1, Z_1) + \mathrm{Hess}\,\phi_1(Y_1, Z_1) &= \lambda_1 g_1(Y_1, Z_1) + (-\lambda_1 + \lambda + \rho R) g_1(Y_1, Z_1)\\ &= \lambda_1 g_1(Y_1, Z_1) + \rho_1 R_1 g_1(Y_1, Z_1),\end{aligned}$$

where $\phi_1 = u_1 - n_2 \ln f - n_3 \ln h_1$ and $u_1 = u$. In this case, $(M_1, g_1, \nabla\phi_1, \lambda_1, \rho_1)$ is a gradient *RBS* soliton when $\lambda_1 = \lambda + \rho R - \rho_1 R_1$ is a constant. Using the same pattern, (ii) can be verified. □

4. Ricci–Bourguignon solitons on sequential warped product space-times

In this section, we examine Ricci–Bourguignon solitons admitting two space-times, namely standard static space-times and generalized Robertson–Walker space-times.

Let (M_i, g_i) be semi-Riemannian manifolds, $1 \leq i \leq 2$, and let $f : M_1 \to \mathbb{R}^+$, $h : M_1 \times M_2 \to \mathbb{R}^+$ be two smooth functions. The $(n_1 + n_2 + 1)$-dimensional *sequential standard static space-time* [15] $\overline{M}$ is the triple product manifold $\overline{M} = (M_1 \times_f M_2) \times_h I$ endowed with the metric tensor $\overline{g} = (g_1 \oplus f^2 g_2) \oplus h^2(-dt^2)$. Here I is an open, connected subinterval of $\mathbb{R}$ and dt^2 is the usual Euclidean metric tensor on I.

Lemma 4.1 ([15])**.** *Let $(\overline{M} = (M_1 \times_f M_2) \times_h I, \overline{g})$ be a sequential standard static space-time and $X_i, Y_i \in \mathfrak{X}(M_i)$ for $1 \leq i \leq 2$. Then*

1. $\overline{\nabla}_{X_1} Y_1 = \nabla^1_{X_1} Y_1$;
2. $\overline{\nabla}_{X_1} X_2 = \overline{\nabla}_{X_2} X_1 = X_1(\ln f) X_2$;
3. $\overline{\nabla}_{X_2} Y_2 = \nabla^2_{X_2} Y_2 - f g_2(X_2, Y_2) \nabla^1 f$;

4. $\overline{\nabla}_{X_i}\partial_t = \overline{\nabla}_{\partial_t}X_i = X_i(\ln h)\partial_t,\ i = 1, 2;$
5. $\overline{\nabla}_{\partial_t}\partial_t = h\ \nabla h.$

Lemma 4.2 ([15])**.** *Let $(\overline{M} = (M_1 \times_f M_2) \times_h I, \overline{g})$ be a sequential standard static space-time and $X_i, Y_i \in \mathfrak{X}(M_i)$ for $1 \leq i \leq 2$. Then*

1. $\overline{\mathrm{Ric}}(X_1, Y_1) = \mathrm{Ric}^1(X_1, Y_1) - \frac{n_2}{f}\,\mathrm{Hess}^1 f(X_1, Y_1) - \frac{1}{h}\,\overline{\mathrm{Hess}}\, h(X_1, Y_1);$
2. $\overline{\mathrm{Ric}}(X_2, Y_2) = \mathrm{Ric}^2(X_2, Y_2) - f^\sharp g_2\,(X_2, Y_2) - \frac{1}{h}\,\overline{\mathrm{Hess}}\, h(X_2, Y_2);$
3. $\overline{\mathrm{Ric}}(\partial_t, \partial_t) = h\Delta h;$
4. $\overline{\mathrm{Ric}}(X_i, Y_j) = 0$ *when $i \neq j$, where* $f^\sharp = \left(f\Delta^1 f + (n_2 - 1)\left\|\nabla^1 f\right\|^2\right).$

By using Lemma 2.3, it is easy to state the following corollary:

Corollary 4.3. *Let $(\overline{M} = (M_1 \times_f M_2) \times_h I, \overline{g})$ be a sequential standard static space-time. Then*

$$\left(\mathcal{L}_{\overline{X}}\overline{g}\right)(\overline{Y}, \overline{Z}) = \left(\mathcal{L}^1_{X_1}g_1\right)(Y_1, Z_1) + f^2\left(\mathcal{L}^2_{X_2}g_2\right)(Y_2, Z_2) - 2h^2uv\frac{\partial w}{\partial t} + 2fX_1(f)g_2(Y_2, Z_2) - 2uvh(X_1 + X_2)(h),$$

where $\overline{X} = X_1 + X_2 + w\partial_t$, $\overline{Y} = Y_1 + Y_2 + u\partial_t$, $\overline{Z} = Z_1 + Z_2 + v\partial_t \in \mathfrak{X}(M)$.

Now we consider an RBS with the structure of the sequential standard static space-times. By using Theorem 3.1, the following result can be given:

Theorem 4.4. *Let $\overline{M} = (M_1 \times_f M_2) \times_h I$ be a sequential standard static space-time equipped with the metric $\overline{g} = (g_1 \oplus f^2 g_2) \oplus h^2(-dt^2)$. If $(\overline{M}, \overline{g}, \overline{X}, \overline{\lambda}, \overline{\rho})$ is an RBS with $\overline{X} = X_1 + X_2 + w\partial_t$, where $X_i \in \mathfrak{X}(M_i)$ for $1 \leq i \leq 2$ and $w\partial_t \in \mathfrak{X}(I)$, then*

(i) *$(M_1, g_1, X_1, \lambda_1, \rho_1)$ is an RBS when $\mathrm{Hess}\, f = \sigma\overline{g}$ and $\overline{\mathrm{Hess}}\, h = \psi\overline{g}$ and $\lambda_1 = \overline{\lambda} + \overline{\rho}R + \frac{n_2}{f}\sigma + \frac{1}{h}\psi - \rho_1 R_1$ is a constant;*

(ii) *M_2 is an Einstein manifold when X_2 a Killing vector field and $\overline{\mathrm{Hess}}\, h = \psi\overline{g}$;*

(iii) $-\frac{\Delta h}{h} + \frac{\partial w}{\partial t} + \frac{1}{h}(X_1 + X_2)(h) = \overline{\lambda} + \overline{\rho}R.$

Proof. Let $(\overline{M}, \overline{g}, \overline{X}, \overline{\lambda}, \overline{\rho})$ be an RBS with the structure of the sequential warped product. Then, for $\overline{Y}, \overline{Z} \in \mathfrak{X}(\overline{M})$, the equation

$$\overline{\mathrm{Ric}}(\overline{Y}, \overline{Z}) + \frac{1}{2}(\mathcal{L}_{\overline{X}}\overline{g})(\overline{Y}, \overline{Z}) = (\overline{\lambda} + \overline{\rho}R)\overline{g}(Y, Z)$$

is satisfied. Using Lemma 4.2 and Corollary 4.3 for vector fields $\overline{Y} = Y_1 + Y_2 + u\partial_t$ and $\overline{Z} = Z_1 + Z_2 + v\partial_t$, we get

$$\mathrm{Ric}^1(Y_1, Z_1) - \frac{n_2}{f}\,\mathrm{Hess}^1 f(Y_1, Z_1) - \frac{1}{h}\,\overline{\mathrm{Hess}}\, h(Y_1, Z_1) + \mathrm{Ric}^2(Y_2, Z_2) - f^\sharp g_2(Y_2, Z_2) - \frac{1}{h}\,\overline{\mathrm{Hess}}\, h(Y_2, Z_2) + h\Delta huv + \frac{1}{2}\left(\mathcal{L}^1_{X_1}g_1\right)(Y_1, Z_1) + \frac{1}{2}f^2\left(\mathcal{L}^2_{X_2}g_2\right)(Y_2, Z_2) - h^2\frac{\partial w}{\partial t}uv$$

$$+ fX_1(f)g_2(Y_2,Z_2) - uvh(X_1+X_2)(h)$$
$$= (\overline{\lambda}+\overline{\rho}\overline{R})g_1(Y_1,Z_1) + (\overline{\lambda}+\overline{\rho}\overline{R})f^2g_2(Y_2,Z_2) - (\overline{\lambda}+\overline{\rho}\overline{R})h^2uv. \tag{4.1}$$

When the arguments are restricted to the factor manifolds, we obtain

$$\mathrm{Ric}^1(Y_1,Z_1) - \frac{n_2}{f}\sigma g_1(Y_1,Z_1) - \frac{1}{h}\psi g_1(Y_1,Z_1) + \frac{1}{2}\left(\mathcal{L}^1_{X_1}g_1\right)(Y_1,Z_1)$$
$$= (\overline{\lambda}+\overline{\rho}\overline{R})g_1(Y_1,Z_1), \tag{4.2}$$
$$\mathrm{Ric}^2(Y_2,Z_2) - f^\sharp g_2(Y_2,Z_2) - \frac{1}{h}\overline{\mathrm{Hess}}\,h(Y_2,Z_2)$$
$$+ \frac{1}{2}f^2\left(\mathcal{L}^2_{X_2}g_2\right)(Y_2,Z_2) + fX_1(f)g_2(Y_2,Z_2)$$
$$= (\overline{\lambda}+\overline{\rho}\overline{R})f^2g_2(Y_2,Z_2), \tag{4.3}$$

and
$$h\Delta huv - h^2\frac{\partial w}{\partial t}uv - h(X_1+X_2)(h)uv = -(\overline{\lambda}+\overline{\rho}\overline{R})h^2uv,$$
which imply (iii).

In equation (4.2), by following the same pattern as in Theorem 3.1, we arrive that $(M_1,g_1,X_1,\lambda_1,\rho_1)$ is an RBS when

$$\lambda_1 = \overline{\lambda}+\overline{\rho}\overline{R} + \frac{n_2}{f}\sigma + \frac{1}{h}\psi - \rho_1R_1$$

is a constant. Moreover, in equation (4.3), if X_2 is a Killing vector field and $\overline{\mathrm{Hess}}\,h = \psi\overline{g}$, we obtain that M_2 is an Einstein manifold, which completes the proof. □

Now, as an application of Theorems 3.4–3.6, we can give the following results:

Theorem 4.5. *Let $\overline{M} = (M_1\times_f M_2)\times_h I$ be a sequential standard static space-time and let $(\overline{M},\overline{g},\overline{X},\overline{\lambda},\overline{\rho})$ be an RBS with $\overline{X} = X_1+X_2+w\partial_t$, where $X_i\in\mathfrak{X}(M_i)$ for $1\le i\le 2$ and $w\partial_t\in\mathfrak{X}(I)$. Assume that $\overline{X}$ is a conformal vector field on $\overline{M}$. If $\mathrm{Hess}\,f = \sigma\overline{g}$ and $\overline{\mathrm{Hess}}\,h = \psi\overline{g}$, then M_1 and M_2 are Einstein manifolds with factors $\mu_1 = -\frac{\Delta h}{h} + \frac{n_2}{f}\sigma + \frac{1}{h}\psi$ and $\mu_2 = -\frac{\Delta h}{h}f^2 + f^\sharp + \frac{1}{h}\psi f^2$.*

Proof. Assume that $(\overline{M},\overline{g},\overline{X},\overline{\lambda},\overline{\rho})$ is an RBS and $\overline{X}$ is a conformal vector field on $\overline{M}$ with factor 2α. Then

$$\overline{\mathrm{Ric}}(\overline{Y},\overline{Z}) = (\overline{\lambda}+\overline{\rho}\overline{R}-\alpha)\overline{g}(Y,Z).$$

If $\mathrm{Hess}\,f = \sigma\overline{g}$ and $\overline{\mathrm{Hess}}\,h = \psi\overline{g}$, the above equation turns into

$$\mathrm{Ric}^1(Y_1,Z_1) - \frac{n_2}{f}\sigma g_1(Y_1,Z_1) - \frac{1}{h}\psi g_1(Y_1,Z_1) + \mathrm{Ric}^2(Y_2,Z_2)$$
$$- f^\sharp g_2(Y_2,Z_2) - \frac{1}{h}\psi f^2g_2(Y_2,Z_2) + h\Delta huv$$
$$= (\overline{\lambda}+\overline{\rho}\overline{R}-\alpha)g_1(Y_1,Z_1) + (\overline{\lambda}+\overline{\rho}\overline{R}-\alpha)f^2g_2(Y_2,Z_2) - (\overline{\lambda}+\overline{\rho}\overline{R}-\alpha)h^2uv.$$

Hence we find

$$\mathrm{Ric}^1(Y_1, Z_1) = (\overline{\lambda} + \overline{\rho}\overline{R} - \alpha + \frac{n_2}{f}\sigma + \frac{1}{h}\psi)g_1(Y_1, Z_1),$$
$$\mathrm{Ric}^2(Y_2, Z_2) = (\overline{\lambda}f^2 + \overline{\rho}\overline{R}f^2 - \alpha f^2 + \frac{1}{h}\psi f^2 + f^\sharp)g_2(Y_2, Z_2)$$

and $h\Delta huv = -(\overline{\lambda} + \overline{\rho}\overline{R} - \alpha)h^2 uv$. So, M_1 and M_2 are Einstein manifolds with factors $\mu_1 = -\frac{\Delta h}{h} + \frac{n_2}{f}\sigma + \frac{1}{h}\psi$ and $\mu_2 = -\frac{\Delta h}{h}f^2 + f^\sharp + \frac{1}{h}\psi f^2$. □

Theorem 4.6. *Let $\overline{M} = (M_1 \times_f M_2) \times_h I$ be a sequential standard static space-time. Assume that $(\overline{M}, \overline{g}, \overline{X}, \overline{\lambda}, \overline{\rho})$ is an RBS with $\overline{X} = X_1 + X_2 + w\partial_t$, where $X_i \in \mathfrak{X}(M_i)$ for $1 \le i \le 2$ and $w\partial_t \in \mathfrak{X}(I)$. Then $(\overline{M}, \overline{g})$ is an Einstein manifold if one of the following conditions holds*:

(i) *$\overline{X} = w\partial_t$ and it is a Killing vector field on I*;

(ii) *X_1 is a Killing vector field on M_1, X_2 and $w\partial_t$ are conformal vector fields on M_2 and I with factors $-2X_1(\ln f)$ and $-2(X_1 + X_2)(\ln h)$*;

(iii) *$X = X_2 + w\partial_t$ and $X_2, w\partial_t$ are Killing vector fields on M_2 and I, and $X_2(h) = 0$.*

Theorem 4.7. *Let $\overline{M} = (M_1 \times_f M_2) \times_h I$ be a sequential standard static space-time and let $(\overline{M}, \overline{g}, \overline{X}, \overline{\lambda}, \overline{\rho})$ be an RBS with $\overline{X} = X_1 + X_2 + w\partial_t$, where $X_i \in \mathfrak{X}(M_i)$ for $1 \le i \le 2$ and $w\partial_t \in \mathfrak{X}(I)$. Assume that* $\mathrm{Hess}\, f = \sigma\overline{g}$ *and* $\overline{\mathrm{Hess}}\, h = \psi\overline{g}$. *If M_1 and M_2 are Einstein manifolds, then X_1 and X_2 are conformal vector fields on M_1 and M_2.*

Proof. Let $(\overline{M}, \overline{g}, \overline{X}, \overline{\lambda}, \overline{\rho})$ be an RBS and let M_1 and M_2 be Einstein manifolds with factors μ_1 and μ_2. If $\mathrm{Hess}\, f = \sigma\overline{g}$ and $\overline{\mathrm{Hess}}\, h = \psi\overline{g}$, then from equation (4.1), we can write

$$\begin{aligned}
&\mu_1 g_1(Y_1, Z_1) - \frac{n_2}{f}\sigma g_1(Y_1, Z_1) - \frac{1}{h}\psi g_1(Y_1, Z_1) + \mu_2 g_2(Y_2, Z_2) - f^\sharp g_2(Y_2, Z_2)\\
&\quad - \frac{1}{h}\psi f^2 g_2(Y_2, Z_2) + h\Delta huv + \frac{1}{2}\mathcal{L}^1_{X_1} g_1(Y_1, Z_1)\\
&\quad + \frac{1}{2}f^2\left(\mathcal{L}^2_{X_2} g_2\right)(Y_2, Z_2) - h^2\frac{\partial w}{\partial t}uv + fX_1(f)g_2(Y_2, Z_2) - uvh(X_1 + X_2)(h)\\
&\quad = (\overline{\lambda} + \overline{\rho}\overline{R})g_1(Y_1, Z_1) + (\overline{\lambda} + \overline{\rho}\overline{R})f^2 g_2(Y_2, Z_2) - (\overline{\lambda} + \overline{\rho}\overline{R})h^2 uv.
\end{aligned}$$

Hence we have

$$\left(\mathcal{L}^1_{X_1} g_1\right)(Y_1, Z_1) = 2(\overline{\lambda} + \overline{\rho}\overline{R} - \mu_1 + \frac{n_2}{f}\sigma + \frac{1}{h}\psi)g_1(Y_1, Z_1),$$
$$\left(\mathcal{L}^2_{X_2} g_2\right)(Y_2, Z_2) = \frac{2}{f^2}((\overline{\lambda} + \overline{\rho}\overline{R})f^2 - \mu_2 + f^\sharp + \frac{1}{h}\psi f^2 - fX_1(f))g_2(Y_2, Z_2)$$

and

$$h\Delta h - h^2\frac{\partial w}{\partial t} - h(X_1 + X_2)(h) = -(\overline{\lambda} + \overline{\rho}\overline{R})h^2,$$

which imply that X_1 and X_2 are conformal vector fields on M_1 and M_2. □

Now we consider an RBS with the structure of the sequential generalized Robertson–Walker space-times. Firstly, we define the notion of the sequential generalized Robertson-Walker space-time.

Let (M_i, g_i) be semi-Riemannian manifolds, $2 \leq i \leq 3$, and let $f : I \to \mathbb{R}^+$, $h : I \times M_2 \to \mathbb{R}^+$ be two smooth functions. The (n_2+n_3+1)-dimensional *sequential generalized Robertson–Walker space-time* $\overline{M}$ is the triple product manifold $\overline{M} = (I \times_f M_2) \times_h M_3$ endowed with the metric tensor $\overline{g} = (-dt^2 \oplus f^2 g_2) \oplus h^2 g_3$ [15]. Here I is an open, connected subinterval of $\mathbb{R}$ and dt^2 is the usual Euclidean metric tensor on I.

Lemma 4.8 ([15])**.** *Let $(\overline{M} = (I \times_f M_2) \times_h M_3, \overline{g})$ be a sequential generalized Robertson–Walker space-time and $X_i, Y_i \in \mathfrak{X}(M_i)$ for $2 \leq i \leq 3$. Then*

1. $\overline{\nabla}_{\partial_t}\partial_t = 0$;
2. $\overline{\nabla}_{\partial_t}X_i = \nabla_{X_i}\partial_t = \frac{\dot{f}}{f}X_i$, $i = 2, 3$;
3. $\overline{\nabla}_{X_2}Y_2 = \nabla^2_{X_2}Y_2 - f\dot{f}g_2(X_2, Y_2)\partial_t$;
4. $\overline{\nabla}_{X_2}X_3 = \overline{\nabla}_{X_3}X_2 = X_2(\ln h)X_3$;
5. $\overline{\nabla}_{X_3}Y_3 = \overline{\nabla}^3_{X_3}Y_3 - hg_3(X_3, Y_3)\nabla h$.

Lemma 4.9 ([15])**.** *Let $(\overline{M} = (I \times_f M_2) \times_h M_3, \overline{g})$ be a sequential generalized Robertson–Walker space-time and $X_i, Y_i \in \mathfrak{X}(M_i)$ for $2 \leq i \leq 3$. Then*

1. $\overline{\mathrm{Ric}}(\partial_t, \partial_t) = \frac{n_2}{f}\ddot{f} + \frac{n_3}{h}\frac{\partial^2 h}{\partial t^2}$;
2. $\overline{\mathrm{Ric}}(X_2, Y_2) = \mathrm{Ric}^2(X_2, Y_2) - f^{\diamond}g_2(X_2, Y_2) - \frac{n_3}{h}\overline{\mathrm{Hess}}\, h(X_2, Y_2)$;
3. $\overline{\mathrm{Ric}}(X_3, Y_3) = \mathrm{Ric}^3(X_3, Y_3) - h^{\sharp}g_3(X_3, Y_3)$;
4. $\overline{\mathrm{Ric}}(X_i, Y_j) = 0$ *when* $i \neq j$, *where* $f^{\diamond} = -f\ddot{f} + (n_2 - 1)\dot{f}^2$ *and* $h^{\sharp} = h\Delta h + (n_3 - 1)\|\nabla h\|^2$.

By using Lemma 2.3, it is easy to state the following corollary:

Corollary 4.10. *Let $(\overline{M} = (I \times_f M_2) \times_h M_3, \overline{g})$ be a sequential generalized Robertson–Walker space-time. Then*

$$(\mathcal{L}_{\overline{X}}\overline{g})(\overline{Y}, \overline{Z}) = -2\frac{\partial w}{\partial t}uv + f^2\left(\mathcal{L}^2_{X_2}g_2\right)(Y_2, Z_2) + h^2\left(\mathcal{L}^3_{X_3}g_3\right)(Y_3, Z_3)$$
$$+ 2wf\frac{\partial f}{\partial t}g_2(Y_2, Z_2) + 2wh(\frac{\partial h}{\partial t} + X_2(h))g_3(Y_3, Z_3),$$

where $\overline{X} = w\partial_t + X_2 + X_3$, $\overline{Y} = u\partial_t + Y_2 + Y_3$ and $\overline{Z} = v\partial_t + Z_2 + Z_3 \in \mathfrak{X}(\overline{M})$.

First, we give the following theorem as an application of Theorem 3.1.

Theorem 4.11. *Let $\overline{M} = (I \times_f M_2) \times_h M_3$ be a sequential generalized Robertson–Walker space-time. Assume that $(\overline{M}, \overline{g}, \overline{X}, \overline{\lambda}, \overline{\rho})$ is an RBS with $\overline{X} = w\partial_t + X_2 + X_3$ on $\overline{M}$, where $X_i \in \mathfrak{X}(M_i)$ for $2 \leq i \leq 3$ and $w\partial_t \in \mathfrak{X}(I)$. Then*

(i) $-\frac{n_2}{f}\ddot{f} - \frac{n_3}{h}\frac{\partial^2 h}{\partial t^2} + \frac{\partial w}{\partial t} = \overline{\lambda} + \overline{\rho}\overline{R}$;

(ii) *when* $\overline{\mathrm{Hess}}\, h = \psi\overline{g}$ *and* $\lambda_2 = \overline{\lambda}f^2 + \overline{\rho}\overline{R}f^2 + f^{\diamond} - wf\dot{f} + \frac{n_3}{h}\psi f^2 - \rho_2 R_2$ *is a constant,* $(M_2, g_2, f^2X_2, \lambda_2, \rho_2)$ *is an RBS;*

(iii) $(M_3, g_3, h^2X_3, \lambda_3, \rho_3)$ *is an RBS when* $\lambda_3 = \overline{\lambda}h^2 + \overline{\rho}\overline{R}h^2 + h^\sharp - wh\frac{\partial h}{\partial t}$ $-whX_2(h) - \rho_3 R_3$ *is a constant.*

Proof. Assume that $(\overline{M}, \overline{g}, \overline{X}, \overline{\lambda}, \overline{\rho})$ is an RBS soliton with the structure of the generalized Robertson-Walker space-time $\overline{M} = (I \times_f M_2) \times_h M_3$. By Lemma 4.9 and Corollary 4.10, the proof is clear. □

The next result can be considered as a consequence of Theorem 3.4.

Theorem 4.12. *Let* $\overline{M} = (I \times_f M_2) \times_h M_3$ *be a sequential generalized Robertson–Walker space-time and let* $(\overline{M}, \overline{g}, \overline{X}, \overline{\lambda}, \overline{\rho})$ *be an RBS soliton with* $\overline{X} = w\partial_t + X_2 + X_3$. *Assume that* $\overline{X}$ *is a conformal vector field on* $\overline{M}$. *If* $\overline{\text{Hess}}\, h = \psi\overline{g}$, *then* M_2 *and* M_3 *are Einstein manifolds with factors*

$$\mu_1 = \left(-\frac{n_2}{f}\ddot{f} - \frac{n_3}{h}\frac{\partial^2 h}{\partial t^2}\right) f^2 + f^\diamond + \frac{n_3}{h}\psi f^2$$

and

$$\mu_2 = \left(-\frac{n_2}{f}\ddot{f} - \frac{n_3}{h}\frac{\partial^2 h}{\partial t^2}\right) h^2 + h^\sharp.$$

Proof. The proof is similar to those of Theorem 3.4 and Theorem 4.5. □

Now we give the following result for the gradient RBS with the structure of the generalized Robertson–Walker space-time.

Theorem 4.13. *Let* $(\overline{M} = (I \times_f M_2) \times_h M_3, \overline{g}, \nabla u, \overline{\lambda}, \overline{\rho})$ *be a sequential generalized Robertson–Walker space-time and let* $(\overline{M}, \overline{g}, \nabla u, \overline{\lambda}, \overline{\rho})$ *be an RBS, where*

$$u = \int_a^t f(r)\, dr$$

for some constant $a \in I$. *Then* $\overline{M}$ *is an Einstein manifold with factor* $(\overline{\lambda} + \overline{\rho}\overline{R} - \dot{f})$.

Proof. Suppose that $X = \nabla u$. Then $X = f\partial_t$.

Let $\{\partial_t, \partial_1, \partial_2, \ldots, \partial_{n_2}, \partial_{n_2+1}, \ldots, \partial_{n_2+n_3}\}$ be an orthonormal basis for $\mathfrak{X}(\overline{M})$. The Hessian of u is given by $\text{Hess}\, u(Y, Z) = \overline{g}(\overline{\nabla}_Y \nabla u, Z)$. Here we have the following six cases:

i) If $Y = Z = \partial_t$, we get

$$\text{Hess}(\partial_t, \partial_t) = \overline{g}(\overline{\nabla}_{\partial_t} \nabla u, \partial_t) = \dot{f}\overline{g}(\partial_t, \partial_t).$$

ii) If $Y = \partial_t$ and $Z = \partial_i$, $1 \le i \le n_2$, we have

$$\text{Hess}\, u(\partial_t, \partial_i) = \overline{g}(\overline{\nabla}_{\partial_t} \nabla u, \partial_i) = \dot{f}\overline{g}(\partial_t, \partial_i).$$

iii) If $Y = \partial_t$ and $Z = \partial_k$, $n_2 + 1 \le k \le n_2 + n_3$, we get $\text{Hess}\, u = \dot{f}\overline{g}$.

iv) If $Y = \partial_i$ and $Z = \partial_j$, $1 \leq i, j \leq n_2$, we have

$$\operatorname{Hess} u(\partial_i, \partial_j) = \overline{g}(\overline{\nabla}_{\partial_i} \nabla u, \partial_j) = f\overline{g}(\frac{\dot{f}}{f}\partial_i, \partial_j) = \dot{f}\overline{g}(\partial_i, \partial_j).$$

v) If $Y = \partial_i$, $1 \leq i \leq n_2$, and $Z = \partial_k$, $n_2 + 1 \leq k \leq n_2 + n_3$, we obtain $\operatorname{Hess} u = \dot{f}\overline{g}$.

vi) Finally, if $Y = \partial_k$ and $Z = \partial_l$, $n_2 + 1 \leq k$, $l \leq n_2 + n_3$, we have

$$\operatorname{Hess} u(\partial_k, \partial_l) = \overline{g}(\overline{\nabla}_{\partial_k} \nabla u, \partial_l) = f\overline{g}(\frac{\dot{f}}{f}\partial_k, \partial_l) = \dot{f}\overline{g}(\partial_k, \partial_l).$$

Hence, $\operatorname{Hess} u(Y, Z) = \dot{f}\overline{g}(Y, Z)$ and $(\mathcal{L}_X \overline{g})(Y, Z) = 2 \operatorname{Hess} u(Y, Z) = 2\dot{f}\overline{g}(Y, Z)$. Therefore, $\overline{\text{Ric}} = (\overline{\lambda} + \overline{\rho}\overline{R} - \dot{f})\overline{g}$ is satisfied, which completes the proof. □

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Received June 7, 2023, revised February 25, 2024.

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Солітони Річчі–Бургіньона на многовидах із секвенціально викривленим добутком

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Ми вивчаємо солітони Річчі–Бургіньона на многовидах із секвенціально викривленим добутком. Одержано необхідні умови того, що солітон Річчі–Бургіньона із структурою секвенціально викривленого добутку є многовидом Ейнштейна, коли потенційне поле розглядається як поле Кіллінга або конформне векторне поле.

Ключові слова: солітон Річчі–Бургіньона, многовид з викривленим добутком, многовид із секвенціально викривленим добутком, векторне поле Кіллінга, конформне векторне поле