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Multiple Solutions for Problems Involving $p(x)$-Laplacian and $p(x)$-Biharmonic Operators

Abdelhakim Sahbani, Abdeljabbar Ghanmi, and Rym Chammem

In this paper, we consider the following $p(x)$-biharmonic problem with Hardy nonlinearity:

$$\begin{cases} \Delta^2_{p(x)} u - \Delta_{p(x)} u = \lambda \dfrac{|u|^{p(x)-2} u}{\delta(x)^{2p(x)}} + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u|^{p(x)-2} \dfrac{\partial u}{\partial n} = g(x,u) & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$, $\Delta_{p(x)}$ is the $p(x)$-Laplacian and $\Delta^2_{p(x)}$ is the $p(x)$-biharmonic operator. More precisely, under some appropriate conditions on the nonlinearities f and g, we combine the variational methods with the theory of the generalized Lebesgue and Sobolev spaces to prove the existence and the multiplicity of solutions.

Key words: $p(x)$-biharmonic operator, $p(x)$-Laplacian, symmetric mountain pass theorem, generalized Sobolev space

Mathematical Subject Classification 2020: 35J20, 35J60, 35G30, 35J35

1. Introduction

In this paper, we are interested in the existence and the multiplicity of solutions for the following $p(x)$-biharmonic problem:

$$\begin{cases} \Delta^2_{p(x)} u - \Delta_{p(x)} u = \lambda \dfrac{|u|^{p(x)-2} u}{\delta(x)^{2p(x)}} + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u|^{p(x)-2} \dfrac{\partial u}{\partial n} = g(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain, $\frac{\partial}{\partial n}$ is the outer unit normal derivative, $\delta(x)$ denotes the Euclidian distance from x to the boundary $\partial\Omega$,

$\Delta^2_{p(\cdot)}u = \Delta(|\Delta u|^{p(\cdot)-2}\Delta u)$ is the $p(\cdot)$-biharmonic operator, and $(-\Delta)_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(\cdot)$-Laplacian operator.

Hereafter, we assume that p is a continuous function on $\overline{\Omega}$ satisfying the inequality

$$1 < p^- = \inf_{\overline{\Omega}} p(x) \le p^+ = \sup_{\overline{\Omega}} p(x) < \frac{N}{2}. \tag{1.2}$$

Also, we assume that $0 < \lambda < C_H$, where C_H is a positive constant in the $p(\cdot)$-Hardy inequality given by

$$C_H = \frac{p^-}{p^+} \min\left(\left(\frac{N(p^- - 1)(N - 2p^-)}{(p^-)^2}\right)^{p^-}, \left(\frac{N(p^+ - 1)(N - 2p^+)}{(p^+)^2}\right)^{p^+}\right).$$

We recall that the $p(\cdot)$-Hardy inequality is given by

$$\int_\Omega \frac{|\Delta u(x)|^{p(x)}}{p(x)}\, dx \ge C_H \int_\Omega \frac{|u(x)|^{p(x)}}{p(x)\delta(x)^{2p(x)}}\, dx \tag{1.3}$$

for all $u \in W_0^{2,p(x)}(\Omega)$, where $W_0^{2,p(x)}(\Omega)$ is defined in Section 2 below.

We note that the $p(x)$-biharmonic and $p(x)$-Laplacian operators present a more complicated non-linearity. Additionally, problems involving these types of operators have been widely studied. This is due to their important applications in several fields such as non-Newtonian fluids, viscous fluids, and chemical heterogeneous. For more information, we refer readers to the references [7, 16, 22].

Very recently, problems like (1.1) attracted a considerable attention from several researchers. See, for example, the papers [1, 8, 11, 13–15, 17–19, 21]. In particular, in [15], the authors proved the existence result for the following eigenvalue problem:

$$\begin{cases} \Delta^2_{p(x)}u = \lambda \dfrac{|u|^{p(x)-2}u}{\delta(x)^{2p(x)}} + \mu|u|^{p(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Later, by using a variational approach and the symmetric mountain pass theorem, in [12], M. Jennane proved that the problem

$$\begin{cases} \Delta^2_{p(x)}u - \Delta_{p(x)}u = a(x)|u|^{\alpha(x)-2}u + \lambda\left(b_1(x)(x)|u|^{\beta(x)-2}u - b_2(x)|u|^{\gamma(x)-2}u\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega, \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega \end{cases}$$

admits infinitely many solutions.

Motivated by the above-mentioned works, we will use the mountain pass theorem to prove the existence of a nontrivial solution for problem (1.1). Moreover, we will use the $\mathbb{Z}_2$-mountain pass theorem to prove that under additional assumptions, problem (1.1) possesses infinitely many weak solutions.

The rest of this paper is organized as follows. In Section 2, we recall some notions and basic results on the generalized Sobolev and Lebesgue spaces. In Section 3, we state and prove the main results of this paper.

2. Preliminaries and variational setting

In this section, we recall some necessary properties of variable exponent Lebesgue and Sobolev spaces. For more details, see [10, 23].

Let Ω be a bounded domain in $\mathbb{R}^N, N \geq 3$. We consider the set

$$C_+(\overline{\Omega}) = \left\{p \in C(\overline{\Omega}) \mid \forall x \in \overline{\Omega} \;\; p(x) > 1\right\}.$$

For all $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \;\middle|\; u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty\right\}.$$

Equipped with the Luxemburg norm

$$|u|_{p(x)} = \inf\left\{\mu > 0 \;\middle|\; \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\},$$

$L^{p(x)}(\Omega)$ becomes a separable and reflexive Banach space if and only if

$$1 < p^- \leq p^+ < \infty.$$

Moreover, in the following proposition, we see that the Hölder inequality holds.

Proposition 2.1 ([10, 23])**.** *For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, we have*

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) |u|_{p(x)} |v|_{p'(x)}.$$

A very important role in manipulating generalized Lebesgue spaces with variable exponents is played by the modular of $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$, which is defined by

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

Proposition 2.2 ([10, 23])**.** *For all $u \in L^{p(x)}(\Omega)$, we have:*

1. $|u|_{p(x)} < 1 \; (= 1, > 1) \Leftrightarrow \rho_{p(x)}(u) < 1$(*respectively,* $= 1, > 1$).
2. $|u|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^+}_{p(x)}$.
3. $|u|_{p(x)} < 1 \Rightarrow |u|^{p^+}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)}$.

Another interesting property of the variable exponent Lebesgue space is given in the proposition below.

Proposition 2.3 ([10, 23])**.** *Let p and q be measurable functions such that $q \in L^\infty(\mathbb{R}^N)$ and $1 \leq p(x), q(x) \leq \infty$ for all $x \in \mathbb{R}^N$. Let $u \in L^{p(x)}(\mathbf{R}^N)$ with $u \neq 0$, then we have*:

1. $|u|_{p(x)q(x)} \leq 1 \Rightarrow |u|^{p^+}_{p(x)q(x)} \leq \||u|^{p(x)}|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)}$.

2. $|u|_{p(x)q(x)} \geq 1 \Rightarrow |u|^{p^-}_{p(x)q(x)} \leq \||u|^{p(x)}|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)}$.

For $u \in L^p(x)(\partial\Omega)$, we put

$$\rho_\partial(u) = \int_{\partial\Omega} |u(x)|^{p(x)} d\sigma.$$

Proposition 2.4 ([10, 23]). *For all $u \in L^{p(x)}(\partial\Omega)$, we have:*

1. $|u|_{L^{p(x)}(\partial\Omega)} > 1 \Rightarrow |u|^{p^-}_{L^{p(x)}(\partial\Omega)} \leq \rho_\partial(u) \leq |u|^{p^+}_{L^{p(x)}(\partial\Omega)}$.
2. $|u|_{L^{p(x)}(\partial\Omega)} < 1 \Rightarrow |u|^{p^+}_{L^{p(x)}(\partial\Omega)} \leq \rho_\partial(u) \leq |u|^{p^-}_{L^{p(x)}(\partial\Omega)}$.

Now, the generalized Sobolev space $W^{k,p(x)}(\Omega)$ is defined for any positive integer k by

$$W^{k,p(x)}(\Omega) = \left\{ \varphi \in L^{p(x)}(\Omega) \;\middle|\; \frac{\partial^{\alpha_1+\cdots+\alpha_n}\varphi}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_n}x_n} \in L^{p(x)}(\Omega),\ \sum_{i=1}^{n} \alpha_i \leq k \right\}.$$

In this space, we introduce the norm

$$\|\varphi\|_{k,p(x)} = \sum_{\alpha_1+\cdots+\alpha_n \leq k} \left| \frac{\partial^{\alpha_1+\cdots+\alpha_n}\varphi}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_n}x_n} \right|_{p(x)}.$$

We recall that $(W^{k,p(x)}(\Omega), \|\cdot\|_{k,p(x)})$ is a separable and reflexive Banach space. Moreover, the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$ denoted by $W_0^{k,p(x)}(\Omega)$ is also a separable and reflexive Banach space.

In the rest of this paper, we will deal with the generalized Sobolev space

$$E = W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$$

equipped with the norm

$$\|u\|_{p(x)} = |\Delta u|_{p(x)} + |\nabla u|_{p(x)}.$$

We note that the norms $\|u\|_{p(x)}$ and $|\Delta u|_{p(x)}$ are equivalent. Moreover, the norm

$$\|u\| = \inf \left\{ \mu > 0 \;\middle|\; \int_\Omega \left| \frac{\Delta u(x)}{\mu} \right|^{p(x)} + \left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}$$

is also equivalent to $\|u\|_{p(x)}$ and $|\Delta u|_{p(x)}$.

Theorem 2.5 ([10, 23]). *We have:*

1. *If $q \in C_+(\overline{\Omega})$ with $q(x) < p^*(x)$, for any $x \in \overline{\Omega}$, then the embedding from E into $L^{q(x)}(\Omega)$ is compact and continuous, where*

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - 2p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

2. *If $q \in C_+(\partial\Omega)$ with $q(x) < p_*(x)$, for any $x \in \partial\Omega$, then the trace embedding from E into $L^{q(x)}(\partial\Omega)$ is compact and continuous, where*

$$p_*(x) = \begin{cases} \dfrac{(N-1)p(x)}{N-2p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

Lemma 2.6 ([12])**.**

1. $\Delta^2_{p(x)} : W_0^{2,p(x)}(\Omega) \to W_0^{-2,p'(x)}(\Omega)$ *is a mapping of type S^+, i.e., if $u_n \rightharpoonup u$ weakly in $W_0^{2,p(\cdot)}(\Omega)$ and $\limsup_{n\to\infty}\langle \Delta^2_{p(x)}(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ strongly in $W_0^{2,p(\cdot)}(\Omega)$.*
2. $-\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) \to W_0^{-1,p'(x)}(\Omega)$ *is a mapping of type S^+.*

For simplicity, we denote

$$\Gamma(u) = \int_\Omega |\Delta u|^{p(x)} + |\nabla u|^{p(x)}\, dx.$$

Proposition 2.7 ([9, 15])**.** *For $u \in E$, we have:*

1. *If $\Gamma(u) \geq 1$, then $\|u\|^{p^-} \leq \Gamma(u) \leq \|u\|^{p^+}$.*
2. *If $\Gamma(u) \leq 1$, then $\|u\|^{p^+} \leq \Gamma(u) \leq \|u\|^{p^-}$.*
3. $\Gamma(u) \geq 1 (= 1, \leq 1) \Leftrightarrow \|u\| \geq 1 (= 1, \leq 1)$.

We recall now the mountain pass theorem and the $\mathbb{Z}_2$-mountain pass theorem that will be used in the proofs of our results.

Definition 2.8. Let E be a Banach space and $\chi \in C^1(E, \mathbb{R}), c \in \mathbb{R}$. We say that χ satisfies the $(PS)_c$ condition if any sequence $u_n \subset E$ such that

$$\chi(u_n) \to c \quad \text{and} \quad \chi'(u_n) \to 0 \quad \text{in} \quad X' \quad \text{as} \quad n \to \infty,$$

contains a subsequence converging to a critical point of χ.

In what follows, we write the $(PS)_c$ condition simply as the (PS) condition if it holds for every level $c \in \mathbb{R}$ for the Palais–Smale condition at level c.

Finally, from [2], we recall the following theorems that we will be used in the proofs of our main results.

Theorem 2.9 (mountain pass theorem)**.** *Let E be a Banach space. Let $\chi \in C^1(E, \mathbb{R})$ satisfy the following conditions*:

1. $\chi(0) = 0$;
2. *χ satisfies the Palais–Smale condition*;
3. *there exist positive constants η and ϱ such that if $||u\| = \eta$, then $\chi(u) \geq \varrho$;*
4. *there exist $e \in E$ with $\|e\| > \eta$ such that $\chi(e) \leq 0$.*

Then χ possesses a critical value $c \geq \varrho$ which can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \chi(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], E) \mid \gamma(0) = 0, \gamma(1) = e\}.$$

Theorem 2.10 ($\mathbb{Z}_2$-mountain pass theorem)**.** *Let E be an infinite dimensional real Banach space. Let $\chi \in C^1(E, \mathbb{R})$ satisfy the following conditions:*

1. *χ is an even functional such that $\chi(0) = 0$;*
2. *χ satisfies the (PS) condition;*
3. *there exist positive constants η and ϱ such that if $\|u\| = \eta$, then $\chi(u) \geq \varrho$;*
4. *for each finite-dimensional subspace $X_1 \subset E$, the set $\{u \in X_1, \chi(u) \geq 0\}$ is bounded in E.*

Then χ has an unbounded sequence of critical values.

3. Main results

In this section, we will state our main results. For this aim, we put

$$f(x,u) = \phi_1(x)\psi_1(u) \quad \text{and} \quad g(x,u) = \phi_2(x)\psi_2(u),$$

where ϕ_1, ψ_1, ϕ_2 and ψ_2 are measurable functions satisfying some integrability conditions. Precisely, we assume the following hypotheses:

(A_1) There exist $c > 0$, α, $S \in C_+(\overline{\Omega})$ such that for all $(x,u) \in \Omega \times \mathbb{R}$, we have

$$\phi_1 \in L^{\frac{S(x)}{S(x)-\alpha(x)}}(\Omega), \quad \psi_1(u) \leq c|u|^{\alpha(x)-1}$$

and

$$p^+ < \alpha(x) < S(x) < p^*(x). \tag{3.1}$$

(A_2) There exist $M_1 > 0, \theta_1 > p^+$ such that for all $x \in \Omega$, we have

$$0 < \theta_1 \phi_1(x) \Psi_1(u) \leq \phi_1(x)\psi_1(u)u, \ \ |u| \geq M_1,$$

where $\Psi_1(t) = \int_0^t \psi_1(s)\, ds$.

(A_3) There exist $c' > 0$, $\beta, T \in C_+(\partial\Omega)$ such that for all $(x,u) \in \partial\Omega \times \mathbb{R}$, we have

$$\phi_2 \in L^{\frac{T(x)}{T(x)-\beta(x)}}(\partial\Omega), \quad \psi_2(u) \leq c'|u|^{\beta(x)-1},$$

and

$$p^+ < \beta(x) < T(x) < p_*(x). \tag{3.2}$$

(A_4) There exist $M_2 > 0$, $\theta_2 > p^+$ such that for all $x \in \partial\Omega$, we have

$$0 < \theta_2 \phi_2(x) \Psi_2(u) \leq \phi_2(x)\psi_2(u)u, \quad |u| \geq M_2,$$

where $\Psi_2(t) = \int_0^t \psi_2(s)\, ds$.

(A_5) For all $x \in \overline{\Omega}$ and for all $y \in \partial\Omega$, we have

$$\psi_1(-x) = -\psi_1(x) \quad \text{and} \quad \psi_2(-y) = -\psi_2(y).$$

Definition 3.1. A function $u \in E$ is a weak solution for problem (1.1) if for any $v \in E$ we have

$$\int_\Omega |\Delta u|^{p(x)-2}\nabla u \nabla v\, dx + \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\, dx - \lambda \int_\Omega \frac{|u|^{p(x)-2}uv}{\delta(x)^{2p(x)}}\, dx \\ - \int_\Omega \phi_1(x)\psi_1(u)v(x)\, dx - \int_{\partial\Omega} \phi_2(x)\psi_2(u)v(x)\, dx = 0.$$

Now we are ready to state our main results.

Theorem 3.2. *Under hypotheses (A_1)–(A_4), there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, problem (1.1) has nontrivial weak solutions.*

Theorem 3.3. *Under hypotheses (A_1)–(A_5), there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$ problem (1.1) has infinitely many solutions.*

The energy functional associated with problem (1.1) is defined as follows:

$$\chi(u) = \int_\Omega \frac{|\Delta u(x)|^{p(x)}}{p(x)}\, dx + \int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)}\, dx - \lambda \int_\Omega \frac{|u(x)|^{p(x)}}{p(x)\delta(x)^{2p(x)}}\, dx \\ - \int_\Omega \phi_1(x)\Psi_1(u)\, dx - \int_{\partial\Omega} \phi_2(x)\Psi_2(u)\, dx.$$

Remark 3.4. From hypotheses (A_1), (A_4), the Hölder inequality, Proposition 2.2 and Proposition 2.7, we can see that $\chi \in C^1(E, \mathbb{R})$. Moreover,

$$\langle \chi'(u), v\rangle = \int_\Omega |\Delta u|^{p(x)-2}\nabla u\nabla v\, dx + \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx \\ - \lambda \int_\Omega \frac{|u|^{p(x)-2}uv}{\delta(x)^{2p(x)}} dx - \int_\Omega \phi_1(x)\psi_1(u)v(x)\, dx - \int_{\partial\Omega} \phi_2(x)\psi_2(u)v(x)\, dx.$$

Hence, if $u \in E$ is a critical point of the functional χ, then u is a weak solution of problem (1.1).

Lemma 3.5. *Under (A_1), (A_3) and for all $0 < \lambda < C_H$, there exist $\eta, \varrho > 0$ such that for $u \in E$,*

$$\|u\| = \eta \Rightarrow \chi(u) \geq \varrho.$$

Proof. By (A_1) and (A_3), for all $x \in \Omega$, we have

$$\phi_1(x)\Psi_1(u) \leq c \int_0^u |\phi_1(x)||s|^{\alpha(x)-1}\, ds \leq \frac{c}{\alpha(x)}|\phi_1(x)||u|^{\alpha(x)}, \tag{3.3}$$

$$\phi_2(x)\Psi_2(u) \leq c \int_0^u |\phi_2(x)||s|^{\beta(x)-1}\, ds \leq \frac{c'}{\beta(x)}|\phi_2(x)||u|^{\beta(x)}. \tag{3.4}$$

Also, by (1.3), we have

$$\frac{\lambda}{C_H}\int_\Omega \frac{|\Delta u(x)|^{p(x)}}{p(x)}\,dx \geq \lambda\int_\Omega \frac{|u(x)|^{p(x)}}{p(x)\delta(x)^{2p(x)}}\,dx.$$

This implies that

$$\int_\Omega \frac{|\Delta u(x)|^{p(x)}}{p(x)}\,dx - \lambda\int_\Omega \frac{|u(x)|^{p(x)}}{p(x)\delta(x)^{2p(x)}}\,dx \geq \left(1-\frac{\lambda}{C_H}\right)\int_\Omega \frac{|\Delta u(x)|^{p(x)}}{p(x)}\,dx. \quad (3.5)$$

Let $u \in E$ with $\|u\| < 1$. By (3.5) and by using the fact that $0 < \lambda < C_H$, we have

$$\begin{aligned}\chi(u) &\geq \left(1-\frac{\lambda}{C_H}\right)\int_\Omega \frac{|\Delta u(x)|^{p(x)}}{p(x)}\,dx + \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx \\ &\quad - \int_\Omega \phi_1(x)\Psi_1(u)\,dx - \int_{\partial\Omega}\phi_2(x)\Psi_2(u)\,dx \\ &\geq \frac{\left(1-\frac{\lambda}{C_H}\right)}{p^+}\Gamma(u) - \int_\Omega \phi_1(x)\Psi_1(u)\,dx - \int_{\partial\Omega}\phi_2(x)\Psi_2(u)\,dx. \qquad (3.6)\end{aligned}$$

By (3.6), (3.3), (3.4), Hölder inequality and Proposition 2.4, there exist constants $c, c' > 0$ such that

$$\begin{aligned}\chi(u) &\geq \frac{\left(1-\frac{\lambda}{C_H}\right)}{p^+}\Gamma(u) - \frac{c}{\alpha^-}\int_\Omega |\phi_1(x)||u|^{\alpha(x)}\,dx - \frac{c'}{\beta^-}\int_{\partial\Omega}|\phi_2(x)||u|^{\beta(x)}\,dx \\ &\geq \frac{\left(1-\frac{\lambda}{C_H}\right)}{p^+}\Gamma(u) - \frac{c_1}{\alpha^-}|\phi_1|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}(\Omega)}\left||u|^{\alpha(x)}\right|_{L^{\frac{S(x)}{\alpha(x)}}(\Omega)} \\ &\quad - \frac{c_2}{\beta^-}|\phi_2|_{L^{\frac{T(x)}{T(x)-\beta(x)}}(\partial\Omega)}\left||u|^{\beta(x)}\right|_{L^{\frac{T(x)}{\beta(x)}}(\partial\Omega)} \\ &\geq \frac{\left(1-\frac{\lambda}{C_H}\right)}{p^+}\Gamma(u) - \frac{c_1}{\alpha^-}|\phi_1|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}(\Omega)}\max\left(|u|^{\alpha^-}_{L^{S(x)}(\Omega)}, |u|^{\alpha^+}_{L^{S(x)}(\Omega)}\right) \\ &\quad - \frac{c_2}{\beta^-}|\phi_2|_{L^{\frac{T(x)}{T(x)-\beta(x)}}(\partial\Omega)}\max\left(|u|^{\beta^-}_{L^{T(x)}(\partial\Omega)}, |u|^{\beta^+}_{L^{T(x)}(\partial\Omega)}\right).\end{aligned}$$

Using $1 < S(x) < p^*(x)$, $1 < T(x) < p_*(x)$, by Proposition 2.2, there exist $c_1, c_2 > 0$ such that

$$|u|_{L^{S(x)}(\Omega)} \leq c_1\|u\| \quad \text{and} \quad |u|_{L^{T(x)}(\partial\Omega)} \leq c_2\|u\|. \qquad (3.7)$$

By (3.7), we obtain

$$\begin{aligned}\chi(u) &\geq \frac{\left(1-\frac{\lambda}{C_H}\right)}{p^+}\|u\|^{p^+} - \frac{c_1}{\alpha^-}|\phi_1|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}(\Omega)}\|u\|^{\alpha^-} \\ &\quad - \frac{c_2}{\beta^-}\mid\phi_2|_{L^{\frac{T(x)}{T(x)-\beta(x)}}(\partial\Omega)}\|u\|^{\beta^-}\end{aligned}$$

$$\geq \|u\|^{p^+}\left(\frac{\left(1-\frac{\lambda}{C_H}\right)}{p^+}-\frac{c_1}{\alpha^-}|\phi_1|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}(\Omega)}\|u\|^{\alpha^- - p^+}\right.$$
$$\left.-\frac{c_2}{\beta^-}\mid \phi_2|_{L^{\frac{T(x)}{T(x)-\beta(x)}}(\partial\Omega)}\|u\|^{\beta^- - p^+}\right)$$
$$\geq \|u\|^{p^+}\left(\frac{\left(1-\frac{\lambda}{C_H}\right)}{p^+}-\kappa\|u\|^{\min(\alpha^- - p^+,\beta^- - p^+)}\right),$$

where

$$\kappa=\frac{c_1}{\alpha^-}|\phi_1|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}(\Omega)}+\frac{c_2}{\beta^-}|\phi_2|_{L^{\frac{T(x)}{T(x)-\beta(x)}}(\partial\Omega)}.$$

Since $\alpha^-,\beta^- > p^+$ and $0<\lambda<C_H$, we can choose $\|u\|=\eta$ small enough such that

$$\frac{(1-\frac{\lambda}{C_H})}{p^+}-\kappa\eta^{\min(\alpha^- - p^+,\beta^- - p^+)}>0.$$

Then

$$\chi(u)\geq \eta^{p^+}(\frac{(1-\frac{\lambda}{C_H})}{p^+}-\kappa\eta^{\min(\alpha^- - p^+,\beta^- - p^+)})=\varrho>0. \qquad \square$$

Next, denote $\varphi' : E \to E^*$ defined by

$$\langle \varphi'(u), v\rangle=\int_\Omega \frac{|u(x)|^{p(x)-2}}{\delta(x)^{2p(x)}}u(x)v(x)\,dx.$$

Lemma 3.6 ([15])**.** *The functional φ' is sequentially weakly-strongly continuous, namely,*

$$\big(u_n \rightharpoonup u \quad in\ E\big) \Rightarrow \big(\varphi'(u_n)\to\varphi'(u) \quad in\ E^*\big).$$

Lemma 3.7. *Assume that (A_1)–(A_4) are satisfied. Then there exists $0<\lambda^*<C_H$ such that for any $\lambda\in(0,\lambda^*)$, χ satisfies the (PS) condition.*

Proof. Suppose that $\{u_n\}\subset E$ such that

$$\chi(u_n)\to c, \quad \chi'(u_n)\to 0 \quad \text{in } E^* \text{ as } n\to\infty,$$

where c is a positive constant. Then, for n large enough, there exists $M_1>0$ such that

$$|\chi(u_n)|\leq M_1. \tag{3.8}$$

Also, since $\chi'(u_n)\to 0$ in E^*, we have $\langle\chi'(u_n),u_n\rangle\to 0$. In particular, $\langle\chi'(u_n),u_n\rangle$ is bounded. Thus, there exists $M_2>0$ such that

$$|\langle\chi'(u_n),u_n\rangle|\leq M_2. \tag{3.9}$$

We claim that the sequence $\{u_n\}$ is bounded. If it is not true, by passing a subsequence, if necessary, we may assume that $\|u_n\| \to \infty$. Without loss of generality, we can also assume that $\|u_n\| \geq 1$. So, from (3.8) and (3.6), we get

$$M_1 \geq \chi(u_n) \geq \frac{\left(1-\frac{\lambda}{C_H}\right)}{p^+}\Gamma(u_n) - \int_\Omega \phi_1(x)\Psi_1(u_n)\,dx - \int_{\partial\Omega} \phi_2(x)\Psi_2(u_n)\,dx.$$

From (3.9), we obtain

$$M_2 \geq -\langle \chi'(u_n), u_n\rangle = -\Gamma(u_n) + \int_\Omega \phi_1(x)\psi_1(u_n)u_n\,dx + \int_{\partial\Omega} \phi_2(x)\psi_2(u_n)u_n\,dx.$$

By virtue of assumptions (A_2) and (A_4), we have

$$\begin{aligned}
\theta M_1 + M_2 &\geq \left(\left(1-\frac{\lambda}{C_H}\right)\frac{\theta}{p^+} - 1\right)\Gamma(u_n) \\
&\quad + \int_\Omega (\phi_1(x)\psi_1(u_n)u_n - \theta\phi_1(x)\Psi_1(u_n))\,dx \\
&\quad + \int_{\partial\Omega} (\phi_2(x)\psi_2(u_n)u_n - \theta\phi_2(x)\Psi_2(u_n))dx \\
&\geq \left(\left(1-\frac{\lambda}{C_H}\right)\frac{\theta}{p^+} - 1\right)\Gamma(u_n) \geq \left(\left(1-\frac{\lambda}{C_H}\right)\frac{\theta}{p^+} - 1\right)\|u_n\|^{p^-}, \qquad (3.10)
\end{aligned}$$

where $\theta = \min(\theta_1, \theta_2)$.

Now, let $\lambda^* = \left(1 - \dfrac{p^+}{\theta}\right) C_H$. Then, by the fact that $\theta > p^+$ for all $\lambda \in (0, \lambda^*)$, one has

$$\left(1-\frac{\lambda}{C_H}\right)\frac{\theta}{p^+} - 1 > 0.$$

So, by letting n tend to infinity in equation (3.10), we get a contradiction. We conclude that $\{u_n\}$ is bounded in E. So, there exists a subsequence $\{u_n\}$ and u in E such that $\{u_n\}$ converges weakly to u in E. Using Proposition 2.2 and the fact that $S(x) < p^*(x)$, we conclude that the sequence $\{u_n\}$ converges strongly to u in $L^{S(x)}(\Omega)$.

Now, we will show that $\{u_n\}$ converges strongly to u in E. We know that

$$\begin{aligned}
\langle \chi'(u_n), u_n - u\rangle &= \langle \Delta^2_{p(x)}(u_n), u_n - u\rangle - \langle \Delta_{p(x)}, (u_n)u_n - u\rangle - \lambda\langle \varphi'(u_n), u_n - u\rangle \\
&\quad - \int_\Omega \phi_1(x)\psi_1(u_n)(u_n - u)\,dx - \int_{\partial\Omega} \phi_2(x)\psi_2(u_n)(u_n - u)\,d\sigma. \qquad (3.11)
\end{aligned}$$

Using hypothesis (A_1), the Hölder inequality, Proposition 2.2 and Proposition 2.7, we have

$$\begin{aligned}
\int_\Omega \phi_1(x)\psi_1(u_n)(u_n - u)dx &\leq \int_\Omega C|\phi_1(x)||u_n|^{\alpha(x)-1}|u_n - u|\,dx \\
&\leq C|u_n - u|_{L^{S(x)}}|\phi_1(x)|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}}\left||u_n|^{\alpha(x)-1}\right|_{L^{\frac{S(x)}{\alpha(x)-1}}}
\end{aligned}$$

$$\leq C|u_n - u|_{L^{S(x)}} |\phi_1(x)|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}} \max\left(|u_n|^{\alpha^+-1}|_{L^{S(x)}}, |u_n|^{\alpha^+-1}|_{L^{S(x)}}\right)$$
$$\leq C'|u_n - u|_{L^{S(x)}} |\phi_1(x)|_{L^{\frac{S(x)}{S(x)-\alpha(x)}}} \max\left(\|u_n\|^{\alpha^+-1}, \|u_n\|^{\alpha^+-1}\right)$$

for some constants $C > 0$ and $C' > 0$.

From the last inequalities, we deduce that

$$\lim_{n\to\infty} \int_\Omega \phi_1(x)\psi_1(u_n)(u_n - u)\, dx = 0. \tag{3.12}$$

Similarly, we can obtain

$$\lim_{n\to\infty} \int_{\partial\Omega} \phi_2(x)\psi_2(u_n)(u_n - u)\, d\sigma = 0. \tag{3.13}$$

Next, by Lemma 3.6, we will show that

$$\lim_{n\to\infty} \langle \varphi'(u_n), u_n - u\rangle = 0. \tag{3.14}$$

Indeed, since, $\langle \chi'(u_n), u_n - u\rangle \to 0$, then by equations (3.12), (3.13) and (3.14), we conclude that

$$\langle \Delta^2_{p(x)}(u_n), u_n - u\rangle - \langle \Delta_{p(x)}, (u_n)u_n - u\rangle \to 0.$$

Finally, since $\Delta^2_{p(x)} - \Delta_{p(x)}$ is of type (S^+), (see Lemma 2.6), we conclude that $u_n \to u$ strongly in E. Therefore, χ satisfies the (PS) condition. □

Lemma 3.8. *Assuming (A_1)–(A_4), there exists $u_* \in E$ such that $\|u_*\| > \eta$ and $\chi(u_*) < 0$, where η is given by Lemma 3.5.*

Proof. By (A_2) and (A_4), there exist two positive constants m_1 and m_2 such that

$$\phi_1(x)\Psi_1(t) \geq m_1\, |t|^{\theta_1} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}. \tag{3.15}$$
$$\phi_2(x)\Psi_2(t) \geq m_2\, |t|^{\theta_2} \quad \text{for all } (x,t) \in \partial\Omega \times \mathbb{R}. \tag{3.16}$$

Let $e \in E$ with $\int_\Omega |e|^{\theta_i} dx > 0$, $i = 1, 2$ and let $t > 1$. Then we have

$$\chi(te) = \int_\Omega \frac{|\Delta(te)|^{p(x)}}{p(x)}\, dx - \lambda \int_\Omega \frac{|te|^{p(x)}}{p(x)\delta(x)^{2p(x)}}\, dx + \int_\Omega \frac{|\nabla(te)|^{p(x)}}{p(x)}\, dx$$
$$- \int_\Omega \phi_1(x)\Psi_1(te)\, dx - \int_{\partial\Omega} \phi_2(x)\Psi_2(te)\, dx.$$

We deduce from (3.15) and (3.16) that

$$\chi(te) \leq \frac{t^{p^+}}{p^-} \int_\Omega \left(|\Delta(e)|^{p(x)} + |\nabla(e)|^{p(x)}\right) dx - m_1 t^{\theta_1} \int_\Omega |e|^{\theta_1} dx - m_2 t^{\theta_2} \int_{\partial\Omega} |e|^{\theta_2}\, dx.$$

Since $\min(\theta_1, \theta_2) > p^+$, then we get

$$\chi(te) \to -\infty \quad \text{as } t \to \infty.$$

We can choose $t_0 > 0$ such that the function $u_* = t_0 e$ satisfies $\|u_*\| > \eta$, and $\chi(u_*) < 0$. □

Now we are ready to prove the first main result of this paper.

Proof of Theorem 3.2. By Lemma 3.5, we have

$$\inf_{\|u\|=\eta} \chi(u) \geq \varrho > 0 = \chi(0).$$

On the other hand, Lemma 3.8 implies the existence of $u_* \in E$ such that $\|u_*\| > \eta$, for some $\eta > 0$, and

$$\chi(u_*) < 0 = \chi(0). \tag{3.17}$$

By Lemma 3.7, we know that χ is a C^1 function satisfying the Palais–Smale conditions. Hence, from Theorem 2.9, χ has a critical point, which is a weak solution for problem (1.1). Finally, due to (3.17), we see that u_* is nontrivial. □

Next, we prove the second main result of this paper. Actually, we need to prove the following lemma.

Lemma 3.9. *Under hypotheses (A_1)–(A_4), if X is a finite dimensional subspace of E, then the set*

$$H = \{u \in X \mid \chi(u) \geq 0\}$$

is bounded in E.

Proof. Let $u \in H$. Firstly, we have

$$\chi(u) \leq \frac{1}{p^-} \int_\Omega |\Delta u|^{p(x)} dx + \frac{1}{p^-} \int_\Omega |\nabla u|^{p(x} dx - \int_\Omega \phi_1(x) \Psi_1(u) dx.$$

Then, by (3.15), (3.16) and Proposition 2.3, we have

$$\begin{aligned} \chi(u) &\leq \frac{1}{p^-}\Gamma(u) - m_1 \int_\Omega |u|^{\theta_1}\, dx - m_2 \int_{\partial\Omega} |u|^{\theta_2}\, dx \\ &\leq \frac{1}{p^-}\Gamma(u) - m_1 |u|^{\theta_1}_{L^{\theta_1}} \leq \frac{1}{p^-}\big(\|u\|^{p^+} + \|u\|^{p^-}\big) - m_1 |u|^{\theta_1}_{L^{\theta}}. \end{aligned}$$

Since X is a finite dimensional subspace, then $|\cdot|_{L^{\theta_1}}$ and $\|\cdot\|$ are equivalent. Hence, there exists a positive constant C such that

$$\|u\|^{\theta_1} \leq C |u|^{\theta_1}_{L^{\theta_1}}.$$

Then

$$\chi(u) \leq \frac{1}{p^-}\big(\|u\|^{p^+} + \|u\|^{p^-}\big) - \frac{m_1}{C}\|u\|^{\theta_1}.$$

Using the fact that $p^- < p^+ < \theta_1$, we deduce that the set H is bounded in E. □

Proof of Theorem 3.3. It is clear that $\chi(0) = 0$. On the other hand, from hypothesis (A_5), we see that χ is an even functional. So, from Lemma 3.5, Lemma 3.7 and Lemma 3.9, we conclude that all conditions of Theorem 2.10 are satisfied. Hence χ has an unbounded sequence of critical values which are an unbounded sequence of nontrivial solutions for problem (1.1). □

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Abdelhakim Sahbani,

University Tunis El Manar, Faculty of Sciences, Mathematics department, Tunis 2092, Tunisia,
E-mail: abdelhakim.sahbani@gmail.com

Abdeljabbar Ghanmi,

University Tunis El Manar, Faculty of Sciences, Mathematics department, Tunis 2092, Tunisia,
E-mail: Abdeljabbar Ghanmi

Rym Chammem,

University Tunis El Manar, Faculty of Sciences, Mathematics department, Tunis 2092, Tunisia,
E-mail: Rym.chemmam@gmail.com

Множинні розв'язки для задач, що містять $p(x)$-лапласіан і $p(x)$-бігармонічний оператор

Abdelhakim Sahbani, Abdeljabbar Ghanmi, and Rym Chammem

У роботі розглянуто таку $p(x)$-бігармонічну задачу з нелінійністю Гарді:

$$\begin{cases} \Delta^2_{p(x)} u - \Delta_{p(x)} u = \lambda \dfrac{|u|^{p(x)-2} u}{\delta(x)^{2p(x)}} + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u|^{p(x)-2} \dfrac{\partial u}{\partial n} = g(x,u) & \text{on } \partial\Omega, \end{cases}$$

де $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$, $\Delta_{p(x)}$ є $p(x)$-лапласіаном і $\Delta^2_{p(x)}$ є $p(x)$-бігармонічним оператором. Точніше, для доведення існування і множинності розв'язків варіаційні методи скомбіновано з теорією узагальнених просторі Лебега і Соболєва за відповідних умов на нелінійності f і g.

Ключові слова: $p(x)$-бігармонічний оператор, $p(x)$-лапласіан, теорема про симетричний гірський перевал, узагальнений простір Соболєва