Journal of Mathematical Physics, Analysis, Geometry
2024, Vol. 20, No. 3, pp. 271–278
doi:

Inscribed and Circumscribed Radius of κ-Convex Hypersurfaces in Hadamard Manifolds

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Let P be a convex polygon in a Hadamard surface M with curvature K satisfying $-k_2^2 \geq K \geq -k_1^2$. We give an upper bound of the circumradius of P in terms of a lower bound of the curvature of P at its vertices.

Key words: convex hypersurface, Hadamard manifolds, Blashke theorem, inscribed and circumscribed radius

Mathematical Subject Classification 2020: 52A40

1. Introduction and main result

Let M be a complete m-dimensional Riemannian manifold. Given a domain $\Omega \subset M$, an inscribed ball or inball is a ball in M contained in Ω with maximum radius, which is called the inradius of Ω, and we denote it by r. A circumscribed ball of Ω is a ball in M containing Ω with minimum radius, which is called the circumradius, and we denote it by R.

In the book of Blaschke [2] it is proved that *if Γ is a closed convex regular curve in the Euclidean plane that bounds a compact convex region Ω, and the curvature κ of Γ is bounded from below by some constant $\kappa_0 > 0$, then the circumradius R of Ω is bounded from above by $\dfrac{1}{\kappa_0}$.*

This result was extended by H. Karcher [8] for other space forms. Further developments of the related Blaschke theorems were done by obtaining conditions under which a convex set in $\mathbb{R}^n$ can be included in other [6, 7, 10].

In all these theorems, the hypothesis of strong convexity ($\kappa \geq \kappa_0 > 0$) is necessary, the theorem is not true for $\kappa \geq 0$. Then it cannot be applied to closed convex polygons. In [4], we used two definitions of curvature at the vertex of a polygon which allowed us to obtain a version of the Blaschke theorem for polygons in space forms. Here we use the same definitions to obtain a less precise upper bound of the circumradius R for polygons in Hadamard surfaces with curvature $-k_2^2 \geq K \geq -k_1^2$.

In order to do that, we use comparison theorems to bound the inradius r of a domain and apply Theorem 3.1 from [5] to get an upper bound of R. Then we use this to obtain the upper bound of R for polygons. This is done by a combination of the comparison theorems mentioned above with the comparison of the angles given by the Toponogov theorem, which has to be done carefully because these inequalities go in opposite sense.

Let us recall some known definitions and properties before giving precise statements.

We shall work on an n-dimensional Hadamard manifold M^n, that is, a simply connected complete n-dimensional Riemannian manifold with sectional curvature bounded from above by a constant $-k_2^2 \leq 0$). In such a manifold, we define

Definition 1.1. Let $\lambda > 0$. An orientable smooth (C^2 or more) hypersurface L of a Hadamard manifold M^n is called λ-convex if there is a suitable selection of the unit normal vector of L such that the normal curvatures k_N of L satisfy $k_N \geq \lambda$.

Definition 1.2. A domain $\Omega \subset M$ is called convex if each shortest path with endpoints in Ω lies in Ω.

If Ω is convex in M, then $\partial\Omega$ is a topological embedded hypersurface which is smooth except for a set of zero measure.

Definition 1.3. A convex domain Ω of M is λ-convex if for every point $p \in \partial\Omega$ there is a smooth λ-convex hypersurface L through p such that there is a neighborhood of p in Ω contained in he convex side of L (that is, the side in M where the unit normal vector to L points).

It is known that if $\partial\Omega$ is smooth, then Ω is λ-convex if and only if $\partial\Omega$ is λ-convex.

Definition 1.4. A topological immersion $f : P \to M$ is called locally convex at a point $x \in P$ if x has a neighborhood U in P such that $f(U)$ is part of the boundary of a convex domain of M.

In [1, 3], it was proved that

Theorem 1.5 ([1, 3])**.** *If P is a compact orientable locally convex and immersed hypersurface of dimension $n \geq 2$ in a Hadamard manifold M, then P is embedded, homeomorphic to the sphere and it is the boundary of a convex set Ω.*

For $n = 1$, the above theorem is not true even when the immersion is C^2. In this paper, even for dimension 1, we consider only compact embedded convex curves which are the boundary of a convex domain $\Omega \subset M$, that is, for any dimension, we adopt the following

Definition 1.6. A compact orientable locally convex hypersurface P in M is λ-convex if it is the boundary of a λ-convex domain Ω.

By the inradius and the circumradius of P we understand the inradius and circumradius of Ω.

We remark that, with this general definition, P can be λ-convex and, at the same time, contain conical or rudge points, or points where P is only C^1, where it is allowed to say that normal curvature is infinite in some directions.

In the next sections we prove the following:

Theorem 1.7. *Let M be a Hadamard manifold with sectional curvature K satisfying $-k_2^2 \geq K \geq -k_1^2$, $k_2, k_1 > 0$. Let P be a compact hypersurface of M such that*

$$P \text{ is } k_2 \coth(k_2\rho)\text{-convex} \quad \text{and} \quad k_2 \coth(k_2\rho) \geq k_1, \tag{1.1}$$

then

$$r \leq \frac{1}{k_1} \operatorname{arccoth}\left(\frac{k_2}{k_1}\coth(k_2\rho)\right) \text{ and } R \leq \frac{1}{k_1} \operatorname{arccoth}\left(\frac{k_2}{k_1}\coth(k_2\rho)\right) + k_1 \ln 2. \tag{1.2}$$

Definition 1.8. In a surface M, let P be a polygon. If A is a vertex of the polygon, α the interior angle at A and ℓ_1, ℓ_2 the lengths of the sides of P that meet at vertex A, then the curvature of P at A is defined by

$$\kappa_A = \frac{2\,(\pi - \alpha)}{\ell_1 + \ell_2}. \tag{1.3}$$

Theorem 1.9. *Let M be a Hadamard surface with Gauss curvature K satisfying $-k_2^2 > K \geq -k_1^2$, $k_2, k_1 > 0$. Let P be a polygon with sides of lengths ℓ_i and vertices A_i. If $\kappa_{A_i} \geq \frac{\pi}{2} k_1 \coth(k_1\rho)$ and $\coth(k_2\rho) \geq \frac{k_1}{k_2}$, then the inradius r of P satisfy*

$$k_1 \coth(k_1 r) \geq k_2 \coth(k_2\rho), \tag{1.4}$$

that is,

$$r \leq \frac{1}{k_1} \operatorname{arccoth}\left(\frac{k_2}{k_1}\coth(k_2\rho)\right) \text{ and } R \leq \frac{1}{k_1} \operatorname{arccoth}\left(\frac{k_2}{k_1}\coth(k_2\rho)\right) + k_1 \ln 2.$$

2. Proof of Theorem 1.7

We use the following result:

Lemma 2.1 ([5, Theorem 3.1])**.** *Let M be a Hadamard manifold with sectional curvature K satisfying $0 \geq K \geq -k_1^2$. If Ω is a compact k_1-convex domain, then*

$$R - r \leq k_1 \ln \frac{(1+\sqrt{\tau})^2}{1+\tau} < k_1 \ln 2, \tag{2.1}$$

where $\tau = \tanh(k_1 r/2)$. Moreover, this bound is sharp.

Proof. Let S be the geodesic sphere of M which is the boundary of an inball of P. From standard comparison theory (see [9]), we have that the normal curvature of S at any point satisfies

$$k_2 \coth(k_2 r) \leq k_N^S \leq k_1 \coth(k_1 r). \tag{2.2}$$

Let $Q_0 \in S \cap P$. Since $P = \partial\Omega$, which is $k_2 \coth(k_2\rho)$-convex, there is a smooth $k_2 \coth(k_2\rho)$-convex hypersurface L through Q leaving a neighbourhood of Q in Ω (and then in S) in the convex side of L, then S and L are tangent at Q and

$$k_N^S(Q_0) \geq k_N^L(Q_0) \geq k_2 \coth(k_2\rho), \tag{2.3}$$

where $k_N^L(Q_0)$ is the normal curvature of L at Q_0. From (2.2) and (2.3), we obtain that

$$k_2 \coth(k_2\rho) \leq k_N^L(Q_0) \leq k_1 \coth(k_1 r), \tag{2.4}$$

from which we obtain the first inequality of (1.2). Now, by using the hypothesis (1.1), we have that P is k_1-convex. Then we can apply Lemma 2.1 to obtain the second inequality of (1.2). □

3. Proof of Theorem 1.9

Let ℓ_i, ℓ_{i+1} be the lengths of the sides having A_i as a common vertex. As in the case of constant curvature (see [4]), we consider the segments of circles C_i from A_{i-1} to A_i of radius ρ_i and center O_i and C_{i+1} from A_i to A_{i+1} of radius ρ_{i+1} and center O_{i+1}. Now, on the Hyperbolic space $\overline{M}^2_\lambda$ of constant sectional curvature $\lambda = -k_1^2$, we consider the geodesic triangles $\overline{O}_i\overline{A}_{i-1}\overline{A}_i$, $\overline{O}_{i+1}\overline{A}_i\overline{A}_{i+1}$ with sides with the same lengths as those of the corresponding triangles $O_iA_{i-1}A_i$, $O_{i+1}A_iA_{i+1}$ in M. From the Toponogov comparison theorem on the angles of a triangle, we have that interior angles of the triangles in M are bigger than the corresponding ones in $\overline{M}^2_\lambda$.

We want the curve, obtained by the union of the segments of circle C_i, to be convex. This may happen if and only if the angle $\widehat{O_{i+1}A_iO_i} \in [0, \pi]$, but this occurs if and only if

$$\widehat{A_{i-1}A_iO_i} + \widehat{O_{i+1}A_iA_{i+1}} \geq \widehat{A_{i-1}A_iA_{i+1}}. \tag{3.1}$$

If $\delta_i = \frac{\pi}{2} - \widehat{A_{i-1}A_iO_i}$ and $\delta_{i+1} = \frac{\pi}{2} - \widehat{O_{i+1}A_iA_{i+1}}$, we can use the definition (1.3) to write inequality (3.1) in the form

$$\delta_i + \delta_{i+1} \leq \pi - \widehat{A_{i-1}A_iA_{i+1}} = \kappa_{A_i}(\ell_i + \ell_{i+1})/2. \tag{3.2}$$

On the other hand, we have $\delta_i = \pi/2 - \widehat{A_{i-1}A_iO_i} \leq \pi/2 - \widehat{\overline{A}_{i-1}\overline{A}_i\overline{O}_i} =: \overline{\delta}_i$, $\delta_{i+1} = \pi/2 - \widehat{O_{i+1}A_iA_{i+1}}) \leq \pi/2 - \widehat{\overline{O}_{i+1}\overline{A}_i\overline{A}_{i+1}} =: \overline{\delta}_{i+1}$. Then inequality (3.2) is satisfied if

$$\overline{\delta}_i + \overline{\delta}_{i+1} \leq \kappa_{A_i}(\ell_i + \ell_{i+1})/2 \tag{3.3}$$

is satisfied. Now we are going to check that the hypothesis on the lower bound of κ_{A_i} implies (3.3).

From hyperbolic trigonometry applied to the triangles $\overline{O}_i\overline{A}_{i-1}\overline{A}_i$, $\overline{O}_{i+1}\overline{A}_i\overline{A}_{i+1}$ one has that $\tanh(k_1\ell_i/2) = \tanh(k_1\rho_i)\sin(\overline{\delta}_i)$ for i and for $i+1$.

Since $\overline{\delta}_i \in [0, \pi/2]$, $\overline{\delta}_i \leq \pi/2 \sin(\overline{\delta}_i)$. Moreover, we take $\rho_i = \rho$. Then we have

$$\overline{\delta}_i + \overline{\delta}_{i+1} \leq \frac{\pi}{2}(\sin\overline{\delta}_i + \sin\overline{\delta}_{i+1}) = \frac{\pi/2}{\tanh(k_1\rho)}(\tanh(k_1\ell_i/2) + \tanh(k_1\ell_{i+1}/2))$$

$$
\begin{aligned}
&= \frac{\pi/2}{\tanh(k_1\rho)} \frac{\tanh(k_1\ell_i/2)+\tanh(k_1\ell_{i+1}/2)}{(\ell_i+\ell_{i+1})/2}(\ell_i+\ell_{i+1})/2 \\
&= \frac{\tanh(k_1\ell_i/2)+\tanh(k_1\ell_{i+1}/2)}{(\ell_i+\ell_{i+1})/2} \frac{\pi/2}{\tanh(k_1\rho)\kappa_{A_i}} \kappa_{A_i}(\ell_i+\ell_{i+1})/2 \\
&\le \frac{\tanh(k_1\ell_i/2)+\tanh(k_1\ell_{i+1}/2)}{k_1(\ell_i+\ell_{i+1})/2} \kappa_{A_i}(\ell_i+\ell_{i+1})/2 \\
&\le \kappa_{A_i}(\ell_i+\ell_{i+1})/2, \qquad (3.4)
\end{aligned}
$$

which is the desired condition.

Now, let us take C_ε, which is parallel to C at distance ε. C_ε is the union of segments of circles C_i' of radius $\rho+\varepsilon$ and center O_i and circles of radius ε centered at A_i, then it is $C^{1,1}$ and its normal curvature k_N satisfies $k_2\coth(k_2\varepsilon) \le k_N \le k_1\coth(k_1\varepsilon)$ at points of C_ε at distance ε from the vertices A_i and $k_2\coth(k_2\rho+\varepsilon) \le k_N \le k_1\coth(k_1\rho+\varepsilon)$ for others. Then, for every ε, $k_2\coth(k_2\rho_i+\varepsilon) \le k_N$, and we can apply Theorem 1.7 to conclude $k_1\coth(k_1 r) \ge k_2\coth(k_2(\rho+\varepsilon))$ for C_ε, then for P, because the domain bounded by P is included in the domain bounded by C_ε. Taking $\varepsilon \to 0$, we obtain $k_1\coth(k_1 r) \ge k_2\coth(k_2\rho)$, which is (1.4). □

4. Remarks

4.1. Bounds for Theorem 1.7 in terms of only k_1. We stated Theorem 1.7 under a form that has a direct application for the proof of Theorem 1.9. This form implies some restrictions for the number ρ $\left(\rho \le \frac{1}{k_2}\operatorname{arccoth}\left(\frac{k_1}{k_2}\right)\right)$, which appears in the lower bound of the normal curvatures of the hypersurface P. The same arguments as for the proof of Theorem 1.7 allow us to obtain another upper bound for r with hypotheses that impose no restriction on ρ. The statement of this other result is:

Theorem 4.1. *Let M be a Hadamard manifold with sectional curvature K satisfying $0 \ge K \ge -k_1^2$, $k_1 > 0$. Let P be a compact $k_1\coth(k_1\rho)$-convex hypersurface of M, then*

$$r \le \rho \quad \text{and} \quad R \le \rho + k_1 \ln 2. \qquad (4.1)$$

4.2. The theorems when $k_2 = 0$. In this case, in the hypotheses, the lower bounds for k_N or k_{a_i} should be $1/\rho$ instead of $k_2\coth(k_2\rho)$, and ρ has to satisfy $1/\rho \ge k_1$. Then the statement of the theorems, under similar proof that the ones given before, is

Theorem 4.2. *Let M be a Hadamard manifold with sectional curvature K satisfying $0 \ge K \ge -k_1^2$, $k_1 > 0$. Let P be a compact hypersurface of M such that*

$$P \text{ is } \frac{1}{\rho}\text{-convex} \quad \text{and} \quad \frac{1}{\rho} \ge k_1, \qquad (4.2)$$

then

$$r \leq \frac{1}{k_1} \operatorname{arccoth}\left(\frac{1}{k_1\ \rho}\right) \quad \textit{and} \quad R \leq \frac{1}{k_1} \operatorname{arccoth}\left(\frac{1}{k_1\ \rho}\right) + k_1 \ln 2. \tag{4.3}$$

Theorem 4.3. *Let M be a Hadamard surface with Gauss curvature K satisfying $0 > K \geq -k_1^2$, $k_1 > 0$. Let P be a polygon with n sides of lengths ℓ_i and vertices A_i. If $\kappa_{A_i} \geq \frac{\pi}{2} k_1 \coth(k_1\rho)$ and $\frac{1}{\rho} \geq k_1$, then the inradius r of P satisfy*

$$k_1 \coth(k_1 r) \geq \frac{1}{\rho},$$

that is,

$$r \leq \frac{1}{k_1} \operatorname{arccoth}\left(\frac{1}{k_1\rho}\right) \quad \textit{and} \quad R \leq \frac{1}{k_1} \operatorname{arccoth}\left(\frac{1}{k_1\rho}\right) + k_1 \ln 2.$$

4.3. Theorem 1.9 with the other definition of k_A. In [4], another definition of the curvature of a polygon in a surface of constant sectional curvature is given. It coincides with (1.3), but in spaces of constant sectional curvature $-k_1^2$ it takes the form

$$\kappa_A = \frac{(\pi - \alpha)}{\frac{1}{k_1}\tanh(k_1\ell_1/2) + \frac{1}{k_1}\tanh(k_1\ell_2/2)}. \tag{4.4}$$

Taking this definition for surfaces with $0 \geq K \geq -k_1^2$ seems as natural as taking definition (1.3). With the computations that we have done, the result is again Theorem 1.9. The reason is that in the last inequality of (3.4), we have bounded

$$\frac{\tanh(k_1\ell_i/2) + \tanh(k_1\ell_{i+1}/2)}{k_1(\ell_i + \ell_{i+1})/2}$$

by $1/k_1$, but with the new definition of κ_{A_i}, inequality (3.3) changes to $\overline{\delta}_i + \overline{\delta}_{i+1} \leq \kappa_{A_i}\left(\frac{1}{k_1}(\tanh(\ell_i/2) + \tanh(\ell_{i+1})/2\right)$, and the above quotient is $1/k_1$, i.e., the same value of the bound that we took.

4.4. The theorems when $k_2 = k_1$. For Theorem 1.7, the hypothesis is only that P is $k_2 1 \coth(k_1\rho)$-convex and, as a result, $r \leq \rho$ and $R \leq \rho + k_1 \ln 2$. If we compare this with the corresponding theorem by Karcher when P is C^2, then, with the same hypothesis, we obtain $R \leq \rho$. We can see that our bound is not the best one: we bounded r for which there should be a bound of R. Thus our bound is far from being the best one.

Similar remarks on Theorem 1.9: The hypothesis is “the same ”, and the conclusion is $r \leq \rho$. In [4], with the same hypothesis, we obtained $R \leq \rho$, again the same difference than with Theorem 1.7.

Acknowledgments. The research is partially supported by Grant PID2019-105019GB-C21 funded by MCIN/AEI/ 10.13039/501100011033 and by ERDF "A way of making Europe", and by the Grant AICO 2021 21/378.01/1 funded by the Generalitat Valenciana.

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Received August 25, 2023.

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Вписаний і описаний радіус κ-опуклих гіперповерхонь у многовиді Адамара

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Нехай P є опуклим багатокутником на поверхні Адамара M з кривиною K, яка задовольняє нерівності $-k_2^2 \geq K \geq -k_1^2$. Доведено оцінку зверху радіусу описаного кола P через оцінку знизу кривини P в його вершинах.

Ключові слова: опукла гіперповерхня, многовиди Адамара, теорема Бляшке, вписаний і описаний радіус