Journal of Mathematical Physics, Analysis, Geometry
2024, Vol. 20, No. 3, pp. 298–312
doi: https://doi.org/10.15407/mag20.03.03

Inclination of Subspaces and Decomposition of Electromagnetic Fields into Potential and Vortex Components

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Using the notion of inclination of two subspaces L and M of a Hilbert space $\mathcal{H}$, we prove the theorem on the extension of linear continuous functionals defined on the subspace L to $\mathcal{H}$ so that the extended functionals vanish on the subspace M. We apply this theorem to study the question of decomposition of the electromagnetic field in resonator with ideally conducting boundary into potential and vortex components and derive the Korn-type inequality for vortex fields.

Key words: Hilbert space, inclination of subspaces, extension of functionals, decomposition of electromagnetic field

Mathematical Subject Classification 2020: 46B20, 46C15

1. Introduction

The present study is motivated by the following question. Let L and M be two linear subspaces of a Hilbert space $\mathcal{H}$. What conditions must these subspaces satisfy so that any vector $x \in \mathcal{H}$ can be represented as a sum

$$x = x^L + x^M, \tag{1.1}$$

where $x^L \in L$, $x^M \in M$, and the inequality is fulfilled

$$\|x^L\| \leq A_1 \|x\|, \quad \|x^M\| \leq A_2 \|x\|, \tag{1.2}$$

with constants A_1, A_2 independent of x?

This is an abstract statement of a problem appearing in various natural sciences. In particular, in electrodynamics it is related to decomposition of an electromagnetic field in the domain with perfectly conducting boundary into the vortex and the potential components [1]. Notice that for $L \cap M = \{0\}$ the decomposition (1.1) is unambiguous.

If the sum of subspaces L and M coincides with $\mathcal{H}$, equality (1.1) is evidently true. On the other hand, for the inequalities (1.2) to be valid, we need additional information about L and M and estimates for constants A_1 and A_2.

That is why the question posed above involves, in the first place, research of the closure of the sum $L + M$ of subspaces L and M, and verification that its orthogonal complement is trivial: $(L + M)^{\perp} = 0$.

The question of the closure of the sum of subspaces was studied earlier in a number of papers [3–5, 8–11]. In the papers [3, 4] and the monograph [5], this question was studied for Banach spaces in very general settings. The notion of inclination $\gamma_B(L, M)$ of two spaces L and M of a Banach space B was introduced, and necessary and sufficient conditions for the closure of the sum of these subspaces were formulated in its terms. In the case of Hilbert spaces $\mathcal{H} = B$, the corresponding inclination $\gamma_{\mathcal{H}}(L, M)$ of the subspaces $L \subset \mathcal{H}$ and $M \subset \mathcal{H}$ is expressed in terms of the cosine of the angle $\varphi(L, M)$ between these subspaces by the formula $\gamma_{\mathcal{H}}(L, M) = \sqrt{1 - \cos^2 \varphi(L, M)}$. The definition of the angle $\varphi(L, M)$ between two subspaces of the Hilbert space was first given by K. Fridrichs for studying the problem of the characteristic values of functions [6]. Subsequently, the notions of inclination and angle between subspaces were further developed and successfully applied in a number of branches of mathematics: the theory of bases, the theory of approximation and splines, operator theory [3–13].

In this paper, we use the definition of the inclination of subspaces $L, M \subset \mathcal{H}$, which takes into account the Hilbert structure of the space $\mathcal{H}$ that is equivalent to the definition given, for example, in [8]. In the first section, the notion of inclination $c(L, M)$ of subspaces L and $M \subset \mathcal{H}$ is used to describe conditions on these subspaces under which representation (1.1) with estimates (1.2) holds. The main result is formulated in Theorem 2.2.

In Section 2, the notion of inclination of subspaces is applied to the study of the possibility of extending linear continuous functionals $f \in L^*$ given on a subspace L and vanishing on $L \cap M$ to functionals $\tilde{f} \in \mathcal{H}^*$ that vanish on a subspace $M \in \mathcal{H}$. It is proved that it is necessary and sufficient that the inclination of subspaces L and M be less than 1 (Theorem 3.1). As a corollary of this theorem, it is shown that conditions 1 and 2, formulated in Theorem 2.2, as sufficient for the validity of decomposition (1.1)–(1.2), are also necessary.

We note that the results obtained in Sections 2 and 3 are proved in this paper by quite elementary methods, although they may also be obtained as corollaries of profound results of previous papers (for example, [8–10]). In Section 3, the results obtained in Sections 2 and 3 are applied to the study of the decomposition of the electric component of the electromagnetic field in a domain with a perfectly conducting boundary into potential and vortex components. The inequality of Korn's type is derived for the vortex component of the field.

2. The inclination of two subspaces of Hilbert space and decomposition of vectors into components from these subspaces

Denote by $\mathcal{H}$ the Hilbert space (complex) with the scalar product (u, v) and the norm $\|u\| = (u, u)^{1/2}$, $u, v \in \mathcal{H}$. Let L and M be two linear subspaces in it, Q be their intersection (the case $Q = \{0\}$ is not excluded), and $L \ominus Q$ and $M \ominus Q$ be orthogonal complements to Q in L and M, respectively. We will assume that subspaces $L \ominus Q$ and $M \ominus Q$ are non-trivial. Taking into account the Hilbert structure of the space $\mathcal{H}$, we introduce the inclination of subspaces L and M by the formula

$$c(L,M) = \operatorname*{supp}_{\substack{0 \neq u \in L \ominus Q \\ 0 \neq v \in M \ominus Q}} \frac{|(u,v)|}{\|u\|\|v\|}. \tag{2.1}$$

Obviously, $0 \leq c(L,M) \leq 1$ and if $M \perp L$ or $(M \ominus Q) \perp (L \ominus Q)$, then $c(L,M) = 0$. This is the reason for the name "inclination" since for $(M \ominus Q) \perp (L \ominus Q)$ we can say that the subspaces L and M are not inclined to each other. We exclude the cases $M \subseteq L$ and $L \subseteq M$ from consideration, since in these cases $c(L,M)$ is not defined by formula (2.1), although it is natural to assume $c(L,M) = 1$ in these cases. The definition of the inclination (2.1) actually coincides with the definition given in [8] and expressed in terms of the angle $\varphi(L,M)$ between the subspaces L and M by the formula $c(L,M) = \cos\varphi(L,M)$. We will denote by $L + M = \{x \in \mathcal{H} : x = x^L + x^M, x^L \in L, x^M \in M\}$ the sum of the subspaces L and M in $\mathcal{H}$.

Lemma 2.1. *If $c(L,M) < 1$, then $L+M$ is a closed set in $\mathcal{H}$, and hence $L+M$ is a closed linear subspace of $\mathcal{H}$.*

The statement of the lemma follows from Theorem 13 in [8] (see also [9, p. 1424, Example 3.2]). But for convenience, we present it here by a simpler elementary method, which we will use in what follows.

Proof of Lemma 2.1. Let $\{x_n \in L+M, n = 1, 2, \ldots\}$ be a convergent sequence of vectors x_n and

$$x_n \to x \in L + M \subset \mathcal{H} \quad \text{as } n \to \infty. \tag{2.2}$$

Since $L+M = Q+(L\ominus Q)+(M\ominus Q)$, each vector $x_n \in L+M$ can be represented in a unique way in the form

$$x_n = \hat{y}_n + \hat{x}_n^L + \hat{x}_n^M. \tag{2.3}$$

Here, $\hat{y}_n$ is the orthogonal projection x_n onto the subspace $Q = L \cap M$, $\hat{x}_n^L \in L \ominus Q$, $\hat{x}_n^M \in M \ominus Q$. Therefore,

$$\|x_n\|^2 = \|\hat{y}_n\|^2 + \left\|x_n^L\right\|^2 + \left\|x_n^M\right\|^2 + 2\mathrm{Re}\left(\hat{x}_n^L, \hat{x}_n^M\right). \tag{2.4}$$

Taking into account (2.1), we write the inequality

$$\left|\left(\hat{x}_n^L, \hat{x}_n^M\right)\right| \leq c(L,M)\left\|\hat{x}_n^L\right\|\left\|\hat{x}_n^M\right\| \leq \frac{c(L,M)}{2}\left(\varepsilon\left\|\hat{x}_n^L\right\|^2 + \varepsilon^{-1}\left\|\hat{x}_n^M\right\|^2\right),$$

where ε is any positive number. Then from (2.4), it follows that

$$\begin{aligned}\|x_n\|^2 &\geq \left\|\hat{y}_n\right\|^2 + \left\|\hat{x}_n^L\right\|^2 + \left\|\hat{x}_n^M\right\|^2 - 2\left|\left(\hat{x}_n^L, \hat{x}_n^M\right)\right| \\ &\geq \left\|\hat{y}_n\right\|^2 + \left\|\hat{x}_n^L\right\|^2\left(1 - \varepsilon c(L,M)\right) + \left\|\hat{x}_n^M\right\|^2\left(1 - \varepsilon^{-1} c(L,M)\right).\end{aligned}$$

Hence, assuming that $0 < c(L,M) < 1$ and setting $\varepsilon = c(L,M)$ or $\varepsilon = c^{-1}(L,M)$, we obtain the inequalities

$$\|\hat{y}_n\| \le \|x_n\|, \quad \|\hat{x}_n^L\| \le \frac{\|x_n\|}{\sqrt{1-c^2(L,M)}}, \quad \|\hat{x}_n^M\| \le \frac{\|x_n\|}{\sqrt{1-c^2(L,M)}}. \tag{2.5}$$

For $c(l,M) = 0$, these inequalities are obvious since the vectors $\hat{y}_n$, $\hat{x}_n^L$, $\hat{x}_n^M$ are mutually orthogonal. In a similar way, we estimate the differences

$$\begin{aligned}\|\hat{y}_n - \hat{y}_m\| &\le \|x_n - x_m\|,\\ \|\hat{x}_n^L - \hat{x}_m^L\| &\le \frac{\|x_n - x_m\|}{\sqrt{1-c^2(L,M)}},\\ \|\hat{x}_n^M - \hat{x}_m^M\| &\le \frac{\|x_n - x_m\|}{\sqrt{1-c^2(L,M)}}.\end{aligned}$$

Since $\|x_n - x_m\| \to 0$ as $n, m \to \infty$, from these inequalities it follows that the sequences $\{\hat{y}_n\}_{n=1}^\infty$, $\{\hat{x}_n^L\}_{n=1}^\infty$, $\{\hat{x}_n^M\}_{n=1}^\infty$ are fundamental in the spaces Q, $L \ominus Q$, $M \ominus Q$, respectively. Since these spaces are complete, then

$$\hat{y}_n \to \hat{y} \in Q, \quad \hat{x}_n^L \to \hat{x}^L \in L \ominus Q, \quad \hat{x}_n^M \to \hat{x}^M \in M \ominus Q,$$

as $n \to \infty$, and therefore,

$$x_n \to \hat{y} + \hat{x}^L + \hat{x}^M \in L + M.$$

By recalling (2.2), we get $x = \hat{y} + \hat{x}^L + \hat{y}^M \in L + M$ for any $x \in \overline{L+M}$, and hence $L + M = \overline{L+M}$. The proof is complete. □

Let us now formulate the main result of this section.

Theorem 2.2. *Let L and M be subspaces in $\mathcal{H}$ satisfying the conditions*:

1. $c = c(L,M) < 1$;
2. $(L+M)^\perp = \{0\}$, *where $(L+M)^\perp$ is the orthogonal complement of $L+M$ in $\mathcal{H}$.*

Then every vector $x \in \mathcal{H}$ can be represented as in (1.1) *so that the inequalities* (1.2) *hold with constants*

$$A_1 = a_1 + \frac{1}{\sqrt{1-c^2}}, \quad A_2 = a_2 + \frac{1}{\sqrt{1-c^2}},$$

where $0 \le a_k \le 1$ $(k = 1,2)$ and $a_k = 0$ if $L \cap M = \{0\}$.

Proof. By virtue of condition 1, according to Lemma 2.1, $L + M = \overline{L+M}$, that is, $L+M$ is a closed linear subspace in $\mathcal{H}$, and according to condition 2, any vector from $\mathcal{H}$ orthogonal to $L+M$ is zero. Therefore, $L + M = \mathcal{H}$ and every vector $x \in \mathcal{H}$ can be represented in a form similar to (2.3),

$$x = \hat{y} + \hat{x}^L + \hat{x}^M, \tag{2.6}$$

where $\hat{y}$ is the orthogonal projection of x onto the subspace $Q = L \cap M$, $\hat{x}^L \in L \ominus Q$, $\hat{x}^M \in M \ominus Q$. Hence, similarly to (2.5), we obtain the inequalities

$$\|\hat{y}\| \leq \|x\|, \quad \left\|\hat{x}^L\right\| \leq \frac{\|x\|}{\sqrt{1-c^2}}, \quad \left\|\hat{x}^M\right\| \leq \frac{\|x\|}{\sqrt{1-c^2}}. \tag{2.7}$$

Now we represent the equality (2.6) as (1.1),

$$x = x^L + x^M,$$

where $x^L = \hat{x}^L + \hat{a}_1\hat{y} \in L$, $x^M = \hat{x}^M + \hat{a}_2\hat{y} \in M$, $\hat{a}_1$, $\hat{a}_2$ are arbitrary non-negative numbers such that $\hat{a}_1 + \hat{a}_2 = 1$.

Taking into account (2.7), we get

$$\left\|x^L\right\| \leq \hat{a}_1\|\hat{y}\| + \frac{1}{\sqrt{1-c^2}}\|x\| \leq A_1\|x\|,$$
$$\left\|x^M\right\| \leq \hat{a}_2\|\hat{y}\| + \frac{1}{\sqrt{1-c^2}}\|x\| \leq A_2\|x\|.$$

Therefore, every $x \in \mathcal{H}$ can be represented in the form (1.1) with estimates (1.2), where

$$A_k = a_k + \frac{1}{\sqrt{1-c^2}} \quad (k = 1, 2),$$

$0 \leq a_k \leq 1$, and $a_k = 0$ if $\hat{y} = 0$, and, in particular, if $L \cap M = \{0\}$. □

Remark 2.3. Theorem 2.2 establishes the sufficiency of conditions 1 and 2, which must be satisfied by subspaces $L, M \subset \mathcal{H}$ in order to realize for all $x \in \mathcal{H}$ decomposition (1.1) with estimates (1.2). It turns out that these conditions are also necessary. The necessity of 2 is evident, and the necessity of condition 1 follows from Theorem 3.1 of the next section. Then Theorem 2.2 implies the necessity of condition 1 for the closedness of the sum $L + M$ (see Lemma 2.1).

3. On the extension of linear continuous functionals

Consider the set $F_Q \subset L^*$ of linear continuous functionals given on the subspace $L \subset \mathcal{H}$ and vanishing on the subspace $Q = L \cap M$ in L, i.e.,

$$F_Q = \{f \in L^* : f(x) = 0 \text{ as } x \in Q\}.$$

The following question is posed:
Is it possible to extend $f \in F_Q$ to a linear continuous functional $\tilde{f} \subset F_M \subset \mathcal{H}^*$, where

$$F_M = \Big\{\tilde{f} \in \mathcal{H}^* : \tilde{f}(x) = f(x) \text{ as } x \in L,\ f(x) = 0 \text{ as } x \in M,$$
$$\|\tilde{f}\|_{\mathcal{H}^*} \leq A\|f\|_L,\ A = A(L, M) < \infty \text{ does not depend on } f \in F_Q\Big\}?$$

The answer is given by the following theorem.

Theorem 3.1. *In order to have an extension* $\tilde{f} \in F_M \subset \mathcal{H}^*$ *for the functional* $f \in L^*$ *(i.e., a mapping* F_Q *to* F_M*) it is necessary and sufficient that the inclination* $c(L,M)$ *of the subspaces* L *and* M *satisfy the inequality* $c(L,M) < 1$. *Moreover, if* $\tilde{f} \in F_M$ *is an extension of* $f \in F_Q$*, then*

$$\|\tilde{f}\|_{\mathcal{H}^*} \le \frac{1}{\sqrt{1-c^2(L,M)}}\|f\|_{L^*}. \tag{3.1}$$

Proof. Let $c(L,M) < 1$. Then from Lemma 2.1 it follows that the sum of the subspaces L and M is a closed linear subspace of $\mathcal{H}$. For a given functional $f \in F_Q \subset L^*$, we define an extension $\hat{f} \in F_M \subset (L+M)^*$ by

$$\hat{f}(x) = \begin{cases} f(x), & x \in L, \\ 0, & x \in M, \end{cases}$$

i.e., for $x \in L+M$, $x = x^L + x^M = \hat{y} + \hat{x}^L + \hat{x}^M$ $\big(\hat{y} \in Q = L \cap M$, $\hat{x}^L \in L \ominus Q$, $\hat{x}^M \in M \ominus Q\big)$ we assume that $\hat{f}(x) = f\big(\hat{x}^L\big)$. This definition is correct because $f(x) = 0$ as $x \in Q$.

Let us estimate the norm of this functional in $L+M$. Using the inequality $\|x\|^2 \ge (1-c^2(L,M))\|\hat{x}^L\|^2$ (see (2.7)), we get

$$\begin{aligned}\|\hat{f}\|_{(L+M)^*} &= \sup_{0\ne x\in L+M}\frac{|\hat{f}(x)|}{\|x\|} = \sup_{0\ne x^L\in L+M}\frac{|f(x^L)|}{\|x^L\|} \le \frac{1}{\sqrt{1-c^2}}\sup_{x^L\in L\ominus Q}\frac{|f(x^L)|}{\|x^L\|} \\ &\le \frac{1}{\sqrt{1-c^2}}\sup_{0\ne x\in L}\frac{|f(x)|}{\|x\|} \le \frac{1}{\sqrt{1-c^2}}\|f\|_{L^*}. \end{aligned} \tag{3.2}$$

Now we extend the functional $\hat{f} \in (L+M)^*$ to the whole space $\mathcal{H}$ so that $\tilde{f}(x) = \hat{f}(x)$ as $x \in L+M$ and

$$\|\tilde{f}\|_{\mathcal{H}^*} = \|\hat{f}\|_{(L+M)^*}. \tag{3.3}$$

For the Hilbert spaces $\mathcal{H}$ and $L+M \subseteq \mathcal{H}$, the possibility of such an extension follows from the Riesz theorem on the general form of the linear functional [14].

Inequality (3.1) follows from (3.2) and (3.3). Thus we have proved the sufficiency of the condition $c(L,M) < 1$ for the mapping $F_Q \to F_M$.

To verify its necessity, we use the method of proving by contradiction. Let us suppose that $c(L,M) = 1$ and there exists a mapping F_Q onto F_M such that $f \to \tilde{f}$, $\|\tilde{f}\|_{\mathcal{H}^*} \le A\|f\|_{L^*}$, where A does not depend on $f \in F_Q$. Since $c(L,M) = 1$, from (2.1) it follows that there exist subsequences $\{u'_n\}_{n=1}^\infty$ and $\{v'_n\}_{n=1}^\infty$ such that $u'_n \in L \ominus Q$, $v'_n \in M \ominus Q$, $\|u'_n\| = \|v'_n\| = 1$, and $|(u'_n, v'_n)| \to 1$ as $n \to \infty$. We denote $\varphi'_n = \arg(u'_n, v'_n)$, so that $(u'_n, v'_n) = |(u'_n, v'_n)|e^{i\varphi'_n}$, and set $u_n = u'_n e^{-i\frac{\varphi'_n}{2}}$, $v_n = v'_n e^{i\frac{\varphi'_n}{2}}$. Then $(u_n, v_n) = e^{-i\varphi'_n}(u'_n, v'_n) = |(u'_n, v'_n)|$. Consequently, there are sequences $u_n \in L \ominus Q$, $v_n \in M \ominus Q$ such that $\|u_n\| = \|v_n\| = 1$, $\operatorname{Im}(u_n, v_n) = 0$, $(u_n, v_n) \to 1$ as $n \to \infty$. Therefore,

$$\|u_n - v_n\|^2 = \|u_n\|^2 + \|v_n\|^2 - 2(u_n, v_n) \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

Let us introduce a sequence of functionals $\{f_n \in L^*\}$ assuming $f_n(x) = (x, u_n)$. Then $f_n(u_n) = 1$ for $x \in L$, $\|f_n\|_{L^*} = 1$ and $f_n(y) = 0$ for $y \in Q = L \cap M$, i.e., $f_n \in F_Q$. By assumption, there are extensions $\tilde{f}_n \in F_M \subset \mathcal{H}^*$ of functionals f_n to the whole space $\mathcal{H}$ such that $\tilde{f}_n(x) = f_n(x)$ for $x \in L$, $\tilde{f}_n(x) = 0$ for $x \in M$ and

$$\|\tilde{f}_n\|_{\mathcal{H}^*} \leq A\|f_n\|_{L^*} = A < \infty, \tag{3.5}$$

where the constant A does not depend on n.

Let us calculate the value of the functional $\tilde{f}_n \in F_M$ on the vector $w_n = u_n - v_n \in \mathcal{H}$. Since $u_n \in L \ominus Q$ and $v_n \in M \ominus Q$, $\tilde{f}_n(w_n) = \tilde{f}_n(u_n) - \tilde{f}_n(v_n) = 1$. Therefore, taking into account (3.4), we get

$$\|\tilde{f}_n\|_{\mathcal{H}^*} = \sup_{0 \neq x \in \mathcal{H}} \frac{|\tilde{f}_n(x)|}{\|x\|} \geq \frac{|\tilde{f}_n(w_n)|}{\|w_n\|} \to \infty, \quad n \to \infty.$$

This contradicts (3.5), and thus $c(L, M) < 1$. □

Remark 3.2. From Theorem 3.1, it follows that condition 1 in Theorem 2.2 is necessary. Indeed, if the decomposition (1.1) is valid for all $x \in \mathcal{H}$ with estimates (1.2), i.e., $x = x^L + x^M$, $x^L \in L$, $x^M \in M$ and $\|x^L\| \leq A_1\|x\|$, then any functional $f \in F_Q$ can be extended to a functional $\tilde{f} \in F_M$ by setting $\tilde{f}(x) = f(x^L)$. Then $\tilde{f}(x) = f(x)$ for $x \in L$, $f(x) = 0$ for $x \in M$ and $|\tilde{f}(x)| = |f(x^L)| \leq \|f\|_{L^*}\|x^L\| \leq A_1\|f\|_{L^*}\|x\|$ for all $x \in \mathcal{H}$, i.e., $\|\tilde{f}\|_{\mathcal{H}^*} \leq A_1\|f\|_{L^*}$, and therefore, $\tilde{f} \in F_M$. Thus, Theorem 3.1 implies that $c(L, M) < 1$.

Let us give now the simplest example illustrating the notion of inclination and the results of Theorems 2.2 and 3.1. Let $\mathcal{H} = l^2(\mathbb{N})$, where the vectors $u = \{u_i\}_{i=1}^\infty$ from $l^2(\mathbb{N})$ are real-valued (i.e., $u_i \in \mathbb{R}$) and the scalar product is defined by the formula

$$(u, v) = \sum_{i=1}^\infty u_i v_i.$$

Consider two subspaces in $\mathcal{H} = l^2(\mathbb{N})$,

$$L = \{v = (v_1, v_2, \ldots) \in l^2(\mathbb{N}) : v_{2n-1} \in \mathbb{R},\ v_{2n} = 0\}, \tag{3.6}$$

$$M = \{w = (w_1, w_2, \ldots) \in l^2(\mathbb{N}) : w_{2n-1} \in \mathbb{R},\ w_{2n-1} = w_{2n}\theta_{2n},\ \theta_{2n} \in \mathbb{R},\ 0 < |\theta_{2n}| < \infty\}. \tag{3.7}$$

The vectors from L and M have the following structures: $v = (v_1, 0, v_2, 0, \ldots)$, $w = (w_1, w_1\theta_2, w_3, w_3\theta_4, \ldots)$. Thus, $Q = L \cap M = \{0\}$ and the inclination of these subspaces is defined by the equality

$$c(L, M) = \sup_{\substack{0 \neq v \in L \\ 0 \neq w \in M}} \frac{\left|\sum_{i=1}^\infty v_{2n-1}w_{2n-1}\right|}{\left(\sum_{n=1}^\infty v_{2n-1}^2\right)^{1/2}\left(\sum_{n=1}^\infty w_{2n-1}^2(1+\theta_{2n}^2)\right)^{1/2}} = \frac{1}{(1+\theta)^{1/2}}, \tag{3.8}$$

where $\theta = \inf \theta_{2n}^2$. Therefore, $c(L, M) < 1$ if $\theta > 0$ and $c(L, M) = 1$ if $\theta = 0$.

For $\theta > 0$, we have $L + M = \mathcal{H}$. Indeed, any vector $u = (u_1, u_2, \ldots) \in l^2(\mathbb{N}$ can be represented in the form $u = v + w$, where $v_{2n-1} = u_{2n-1} - \frac{u_{2n}}{\theta_{2n}}$, $v_{2n} = 0$, i.e., $v \in L$, and $w_{2n-1} = \frac{u_{2n}}{\theta_{2n}}$, $w_{2n} = u_{2n}$, i.e., $w \in M$. Hence, we obtain the following estimates for the vectors v and w:

$$\|v\| \leq \sqrt{\frac{\theta+1}{\theta}}\|u\|, \quad \|w\| \leq \sqrt{\frac{\theta+1}{\theta}}\|u\|.$$

Since, by virtue of (3.8),

$$\frac{\theta+1}{\theta} = \frac{1}{1-c^2}, \tag{3.9}$$

these estimates coincide with the estimates of Theorem 2.2.

Consider now the functional $f \in L^*$. By the Riesz theorem, there exists a vector $a \in L$, $a = (a_1, 0, a_2, 0, \ldots) \in l^2(\mathbb{N})$ such that

$$f(v) = (v, a) = \sum_{n=1}^{\infty} v_{2n-1} a_{2n-1} \tag{3.10}$$

for $v \in L$.

Extension $\tilde{f} \in F_M$ of the functional f to the whole space $\mathcal{H} = l^2$ can be represented in the form

$$\tilde{f}(u) = (u, \tilde{a}) \quad \text{for all } u \in \mathcal{H}, \tag{3.11}$$

where $\tilde{a} = (\tilde{a}_1, \tilde{a}_2, \ldots) \in l^2(\mathbb{N})$, $\tilde{a}_{2n-1} = a_{2n-1}$, $\tilde{a}_{2n} = -a_{2n-1}\theta_{2n}^{-1}$, $n = 1, 2, \ldots$.

Indeed, according to (3.6), (3.7), (3.10), (3.11), $\tilde{f}(v) = f(v)$ for $v \in L$; $\tilde{f}(w) = 0$ for $w \in M$, and by virtue of (3.9)–(3.11), the estimate

$$\|\tilde{f}\|_{\mathcal{H}^*} = \|\tilde{a}\| \leq \sqrt{\frac{\theta+1}{\theta}}\|a\| = \frac{1}{\sqrt{1-c^2}}\|a\| = \frac{1}{\sqrt{1-c^2}}\|f\|_{L^*}$$

holds and it coincides with the estimate of Theorem 3.1.

4. On decomposition of electric component of electromagnetic field in domain with perfectly conducting boundary into potential and vortex components

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega = \Gamma$. The electric component of the electromagnetic field in the domain Ω with a perfectly conducting boundary Γ is a vector field $u(x) \in L_2(\Omega, \mathbb{C}^3)$ with a finite norm

$$\|u\|_\Omega = \left\{\int_\Omega \left(|\mathrm{rot} u|^2 + |\mathrm{div} u|^2 + |u|^2\right) dx\right\}^{1/2} < \infty \tag{4.1}$$

which satisfies the condition

$$u_\tau(x) = 0, \quad x \in \Gamma. \tag{4.2}$$

Here and below, we denote by $|\cdot|$ the norms of vectors from $\mathbb{C}^3$ (or $\mathbb{C}$), by $\langle\,\cdot\,\rangle$ the standard scalar product in $\mathbb{C}^3$ ($\mathbb{C}$), and by $u_\tau(x)$ and $u_\nu(x)$ the tangential and normal components of the field $u(x)$ on the surface Γ at a point $x \in \Gamma$. For non-smooth vector functions satisfying condition (4.1), these components are defined as elements of the space $H^{-1/2}(\Gamma)$ [15], and therefore, the boundary condition (4.2) in the general case is understood in the generalized sense:

$$u_\tau = 0 \Leftrightarrow \int_\Omega \langle u, \operatorname{rot} v\rangle\, dx = \int_\Omega \langle \operatorname{rot} u, v\rangle\, dx \quad \text{for all } v \in H^1\big(\Omega, \mathbb{C}^3\big),$$

or, equivalently,

$$f_u(v) = \int_\Gamma \langle v \wedge \nu, u\rangle\, d\Gamma = 0 \quad \text{for all } v \in H^{1/2}\big(\Gamma, \mathbb{C}^3\big),$$

where $\nu = \nu(x)$ is the unit vector of the outer normal to the surface Γ at the point $x \in \Gamma$, $\wedge$ is the vector product in $\mathbb{C}^3$, and since $u \in H^{-1/2}(\Gamma)$, the integral over Γ is understood as a functional $f_u \in \big(H^{1/2}\big(\Gamma, \mathbb{C}^3\big)\big)^*$.

The properties of such vector functions were studied in papers [17–20] in connection with the study of the Maxwell operator in domains with a non-smooth ideally conducting boundary. For domains with a smooth boundary, they were studied in [21, 22]. In particular, it was proved that in the case of a smooth boundary of the domain, the set of vector functions satisfying conditions (4.1), (4.2) is a closed subspace $H_0^1\big(\Omega, \mathbb{C}^3; \tau\big)$ of the Sobolev space $H^1\big(\Omega, \mathbb{C}^3\big)$ of vector functions $v(x)$ satisfying the condition $v_\tau(x) = 0$ on Γ. Hence, by virtue of the embedding theorem, $u_\tau \in H^{1/2}(\Gamma)$, $u_\nu \in H^{1/2}(\Gamma)$ and condition (4.2) can be understood in the usual sense.

Introducing in the subspace $H_0^1(\Omega, \mathbb{C}^3; \tau) \subset H^1(\Omega, \mathbb{C}^3)$ the scalar product $(\cdot,\cdot)_\Omega$ compatible in the standard way $(u,u)_\Omega^{1/2} = \|u\|_\Omega$ with the norm (4.1), we obtain a complete Hilbert space, which we denote by $\mathscr{H}$. Consider two linear subspaces in it:

$$\begin{aligned}\mathscr{L} &= \{u \in \mathscr{H} : u(x) = 0,\ x \in \Gamma\},\\ \mathscr{M} &= \{u \in \mathscr{H} : u = \nabla\varphi,\ \varphi \in H_0^1(\Omega, \mathbb{C}),\ \Delta\varphi \in L_2(\Omega)\}.\end{aligned}$$

It can be shown that these subspaces are closed in $\mathscr{H}$, and

$$\begin{aligned}\mathscr{L} &= \big\{u \in H_0^1\big(\Omega, \mathbb{C}^3\big)\big\},\\ \mathscr{M} &= \big\{u = \nabla\varphi,\ \varphi \in H_0^1(\Omega, \mathbb{C}) \cap H^2(\Omega, \mathbb{C})\big\},\end{aligned}$$

where H_0^1, H^2 are the standard notation for Sobolev spaces $\overset{\circ}{W}{}_2^1 = H_0^1$, $W_2^2 = H^2$ (see, for example, [16]). For this purpose, we use the well-known equality

$$\|\nabla u\|_{L_2}^2 = \|\operatorname{rot} u\|_{L_2}^2 + \|\operatorname{div} u\|_{L_2}^2 \quad \text{for all } u \in C_0^1\big(\Omega, \mathbb{C}^3\big),$$

and inequality

$$\|\varphi\|_{H^2} \le C_2(\varkappa)\|\Delta\varphi\|_{L_2} \quad \text{for all } \varphi \in C_0^2(\Omega),$$

where $C_2(\varkappa)$ is a constant depending on the curvature $\varkappa$ of the surface Γ [20].

Let us show now that for $\mathcal{H} = \mathscr{H}$, $L = \mathscr{L}$, $M = \mathscr{M}$, conditions 1 and 2 of Theorem 2.2 are satisfied, i.e., the inclination $c(\mathscr{L}, \mathscr{M})$ of subspaces $\mathscr{L}$ and $\mathscr{M}$ is less than 1, and any vector $u \in \mathscr{H}$ orthogonal to the sum $\mathscr{L} + \mathscr{M}$ is zero. To verify that condition 1 of Theorem 2.2 is satisfied, we use Theorem 3.1.

Let $f \in \mathscr{L}^*$ be a linear continuous functional, defined on the space $\mathscr{L}$, that vanishes on $Q = \mathscr{L} \cap \mathscr{M}$, i.e., $f \in F_Q$. By virtue of the Riesz theorem, there exists a vector function $w \in \mathscr{L} = H_0^1(\Omega, \mathbb{C}^3)$ such that

$$f(u) = (u, w)_\Omega = \int_\Omega \{\langle \operatorname{rot} u, \operatorname{rot} w\rangle + \langle \operatorname{div} u, \operatorname{div} w\rangle + \langle u, w\rangle\}\, dx. \tag{4.3}$$

With the help of this equality, taking into account that $f(u) = 0$ for all $u \in Q = \mathscr{L} \cap \mathscr{M} = \{u = \nabla\varphi : \varphi \in H_0^1(\Omega),\ \nabla\varphi \in H_0^1(\Omega, \mathbb{C}^3)\}$, we conclude that $w(x)$ is a generalized solution of the following boundary value problem:

$$\begin{cases} \operatorname{rot}\operatorname{rot} w(x) - \nabla \operatorname{div} w(x) + w(x) = j(x), & x \in \Omega, \\ w(x) = 0, & x \in \Gamma, \end{cases} \tag{4.4}$$

where $j(x)$ is a vector function from $H^{-1}(\Omega, \mathbb{C}^3)$ that satisfies the equation $\operatorname{div} j(x) = 0$ in Ω in the sense of distributions

$$j(x) \in J(\Omega) = \{j \in H^{-1}(\Omega, \mathbb{C}^3),\ \operatorname{div} j(x) = 0\}.$$

From this, it follows that $w(x)$ satisfies the equation

$$-\Delta \operatorname{div} w(x) + \operatorname{div} w(x) = 0, \quad x \in \Omega, \tag{4.5}$$

and thus, $\|\Delta \operatorname{div} w\|_{L_2(\Omega)} = \|\operatorname{div} w\|_{L_2(\Omega)} < \infty$.

Denote by $W(\Omega)$ the set of solutions of the boundary value problem (4.4) for all $j \in J(\Omega)$ and assume that

$$S = \sup_{W(\Omega)} \frac{\|\nabla \operatorname{div} w\|_{L_2(\Omega)}}{\|w\|_{H^1(\Omega)}} < \infty. \tag{4.6}$$

Apparently, this inequality indeed holds since all $w(x)$ from $W(\Omega)$ belong to the space $H_0^1(\Omega, \mathbb{C}^3)$ and satisfy equation (4.5) in the domain Ω.

Under this assumption, let us show that any functional $f \in F_Q \subset \mathscr{L}^*$ defined on the subspace $\mathscr{L}$ by formula (4.3) can be extended on the whole space $\mathscr{H}$ to a functional $\tilde{f} \in \mathscr{H}^*$ such that $\tilde{f}(u) = f(u)$ for $u \in \mathscr{L}$, $\tilde{f}(v) = 0$ for $v \in \mathscr{M}$, and $\|\tilde{f}\|_{\mathscr{H}^*} \le \hat{C}(S)\|f\|_{\mathscr{L}^*}$, where $\hat{C}(S)$ does not depend on $f \in F_Q$.

We define the functional $\tilde{f}$ by the formula

$$\tilde{f}(u) = \int_\Omega \{\langle \operatorname{rot} u, \operatorname{rot} w\rangle + \langle \operatorname{div} u, \operatorname{div} w\rangle + \langle u, w\rangle\}\, dx - \int_\Gamma \langle u, \nu \operatorname{div} w\rangle\, d\Gamma \quad \text{for all } u \in \mathscr{H}, \tag{4.7}$$

where $w = w(x)$ is the same vector function as in the functional (4.3), ν is the unit vector of the outward normal to the surface Γ. The surface integral in (4.7) is well defined since $u \in H^1(\Omega, \mathbb{C}^3, \tau)$ and $\operatorname{div} w \in H^1\big(\Omega, \mathbb{C}^3\big)$ in view of assumption (4.6). Taking this into account and using the embedding theorem for $H^1(\Omega)$ in $L_2(\Gamma)$, we obtain the inequality

$$\left| \int_\Gamma \langle u, \nu \operatorname{div} w\rangle \, d\Gamma \right| \leq \| \operatorname{div} w\|_{L_2(\Gamma)} \|u_\nu\|_{L_2(\Gamma)} \leq C(S+1)\|w\|_{H^1(\Omega)}\|u\|_{H^1(\Omega)}.$$

Due to this inequality, from (4.7), it follows that

$$|\tilde{f}(u)| \leq (1 + C(S+1))\|w\|_{H^1(\Omega)}\|u\|_{H^1(\Omega)}. \tag{4.8}$$

As it follows from [21,22] (see also [17,18]), in the case of smooth boundary Γ there exists a continuous linear mapping $\mathscr{H} \to H_0^1\big(\Omega, \mathbb{C}^3; \tau\big)$ and $\mathscr{L} \to H_0^1\big(\Omega, \mathbb{C}^3\big)$ such that

$$\|u\|_{H_0^1(\Omega)} \leq C_1\|u\|_{\mathscr{H}} \ (1 \leq C_1 < \infty) \quad \text{for all } u \in \mathscr{H},$$

where the constant C_1 does not depend on $u \in \mathscr{H}$, and

$$\|w\|_{H_0^1(\Omega)} = \|w\|_{\mathscr{L}} \quad \text{for all } w \in \mathscr{L}.$$

Moreover, according to (4.3),

$$\|w\|_{\mathscr{L}} = \|f\|_{\mathscr{L}^*}.$$

Considering all this, with the help of (4.8) we obtain

$$\|\tilde{f}\|_{\mathscr{H}^*} = \sup_{u \in \mathscr{H}} \frac{|\tilde{f}(u)|}{\|u\|_{\mathscr{H}}} \leq C_1(1 + C(S+1))\|f\|_{\mathscr{L}^*} = \hat{C}(S)\|f\|_{\mathscr{L}^*}, \tag{4.9}$$

and thus the required inequality for $\tilde{f}$ is established.

Further, according to (4.7) and (4.3), it is obvious that

$$\tilde{f}(u) = f(u) \quad \text{for } u \in \mathscr{L} = H_0^1\big(\Omega, \mathbb{C}^3\big), \tag{4.10}$$

and for $u \in \mathscr{M} = \{u = \nabla\varphi : \varphi \in H_0^1(\Omega, \mathbb{C}) \cap H^2(\Omega, \mathbb{C})\}$,

$$\begin{aligned} \tilde{f}(u) &= \int_\Omega \{\langle \Delta\varphi, \operatorname{div} w\rangle + \langle \nabla\varphi, w\rangle\} \, dx - \int_\Gamma \left\langle \frac{\partial\varphi}{\partial\nu}, \operatorname{div} w \right\rangle d\Gamma \\ &= \int_\Omega \langle \varphi, \Delta \operatorname{div} w - \operatorname{div} w\rangle \, dx \end{aligned}$$

and, according to (4.5),

$$\tilde{f}(u) = 0 \quad \text{for } u \in \mathscr{M}. \tag{4.11}$$

Combining (4.9)–(4.10), we conclude that any functional $f \in F_Q \subset \mathscr{L}^*$ ($Q = \mathscr{L} \cap \mathscr{M}$) can be extended to the functional $\tilde{f} \in F_{\mathscr{M}} \subset \mathscr{H}^*$. Therefore, according to Theorem 3.1, the inclination of subspaces $\mathscr{L}$ and $\mathscr{M}$ is less than 1, i.e., condition 1 of Theorem 2.2 is satisfied.

Remark 4.1. In a quite simple proof of this fact presented above, it was assumed that condition (4.1) is satisfied. The proof that does not use this assumption is rather cumbersome and is not given here.

Let us now show that condition 2 of Theorem 2.2 is also satisfied. Let $u \in (\mathscr{L} + \mathscr{M})^{\perp} \subset \mathscr{H}$. Then $(u, v)_{\Omega} = 0$ for all $v \in \mathscr{L} = H_0^1(\Omega, \mathbb{C}^3)$ and $(u, v)_{\Omega} = 0$ for all $v \in \mathscr{M} = \{v = \nabla\varphi, \varphi \in H_0^1(\Omega) \cap H^2(\Omega)\}$. Using these equalities and assuming that $u \in H_0^1(\Omega, \mathbb{C}^3; \tau) \cap H^2(\Omega, \mathbb{C}^3)$, we conclude that $u(x)$ is a solution of the following boundary value problem:

$$\begin{aligned} \operatorname{rot}\operatorname{rot} u(x) - \nabla \operatorname{div} u(x) + u(x) &= 0, \qquad x \in \Omega, \\ \operatorname{div} u(x) &= 0, \qquad x \in \Gamma, \\ u_{\tau}(x) &= 0, \qquad x \in \Gamma. \end{aligned}$$

Hence, it follows that

$$\int_{\Omega} \left\{ |\operatorname{rot} u|^2 + |\operatorname{div} u|^2 + |u|^2 \right\} dx = 0,$$

and therefore, $u \equiv 0$, and condition 2 of Theorem 2.2 is satisfied.

According to Theorem 2.2, any vector function from $\mathscr{H}$ can be represented as a sum of two vector functions from subspaces $\mathscr{M}$ and $\mathscr{L}$ with the corresponding estimates, see (1.1) and (1.2). This representation is obviously not unique if $\mathscr{M} \cap \mathscr{L} \neq \varnothing$.

Let us call vector functions from the subspace $\mathscr{M}$ potential fields, and those from the subspace

$$\hat{\mathscr{L}} = \mathscr{L} \ominus (\mathscr{M} \cap \mathscr{L}),$$

vortex fields. It is clear from above that we can represent any vector function from $\mathscr{H}$ as a sum of two terms from subspaces $\mathscr{M}$ and $\mathscr{L}$; this decomposition is unique and the estimates of the form (1.2) are valid.

Note that in the physical literature the terms “potential field” and “vortex field” are quite common, but their definitions are of a local nature that does not take into account the boundary conditions for these vector fields (see, for example, [23]). Our definitions of potential and vortex fields as subspaces in the Hilbert space $\mathscr{H}$ ensure that these fields have zero tangent components at the boundary of domain Ω.

In conclusion, we show that the Korn-type inequality [24] holds for vortex fields:

$$\|u\|^2_{H_0^1(\Omega)} \leq \frac{1}{1 - c^2(\hat{\mathscr{L}}, \mathscr{M})} \left(\|\operatorname{rot} u\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Omega)} \right). \tag{4.12}$$

First, we note that according to definition (2.1) and Theorem 3.1, the inclinations $c(\hat{\mathscr{L}}, \mathscr{M})$ and $c(\mathscr{L}, \mathscr{M})$ of subspaces $(\hat{\mathscr{L}}, \mathscr{M})$ and $(\mathscr{L}, \mathscr{M})$ in $\mathscr{H}$ are equal and less than 1,

$$c = c(\hat{\mathscr{L}}, \mathscr{M}) = c(\mathscr{L}, \mathscr{M}) < 1$$

and

$$|(u, v)_{\Omega}| \leq c\|u\|_{\mathscr{H}}\|v\|_{\mathscr{H}}, \tag{4.13}$$

where $c = c(\hat{\mathscr{L}}, \mathscr{M})$ for all $u \in \hat{\mathscr{L}}$ and $v \in \mathscr{M}$

Let us denote by $P_{\mathscr{M}}$ the orthogonal projection operator in $\mathscr{H}$ onto the subspace $\mathscr{M}$. Then for $u \in \hat{\mathscr{L}}$,

$$\sup_{v\in\mathscr{M}} \frac{|(u,v)_\Omega|}{\|v\|_{\mathscr{H}}} = \|P_{\mathscr{M}} u\|_{\mathscr{H}},$$

and thus, according to (4.13),

$$\|P_{\mathscr{M}} u\|^2_{\mathscr{H}} = \int_\Omega \left\{ |\operatorname{div} P_{\mathscr{M}} u|^2 + |P_{\mathscr{M}} u|^2 \right\} dx \le c^2 \|u\|^2_{\mathscr{H}}. \tag{4.14}$$

Let us represent $u \in \hat{\mathscr{L}} = \mathscr{L} \ominus (\mathscr{L} \cap \mathscr{M})$ in the form

$$u = P_{\mathscr{M}} u + u', \quad u' \in \mathscr{H} \ominus (\mathscr{L} \cap \mathscr{M}). \tag{4.15}$$

Evidently, $P_{\mathscr{M}} u' = 0$, and hence, since $\nabla\varphi \in \mathscr{M}$, the equality $(u', \nabla\varphi)_\Omega = 0$ is true for all $\varphi \in H^1_0(\Omega) \cap H^2(\Omega)$. Using this equality, we may conclude that

$$\operatorname{div} u'(x) = 0, \ \ x \in \Omega. \tag{4.16}$$

Taking into account (4.15), (4.16), we can rewrite inequality (4.14) in the form

$$\int_\Omega |\operatorname{div} u|^2 \, dx + \int_\Omega |u - u'|^2 \, dx \le \int_\Omega \left\{ |\operatorname{rot} u|^2 + |\operatorname{div} u|^2 + |u|^2 \right\} dx.$$

Whence it follows that

$$\left(1 - c^2\right) \int_\Omega |\operatorname{div} u|^2 \, dx \le c^2 \int_\Omega |\operatorname{rot} u|^2 dx + c^2 \int_\Omega |u|^2 dx.$$

Since $c < 1$, we rewrite this inequality in the form

$$\int_\Omega \left\{ |\operatorname{rot} u|^2 + |\operatorname{div} u|^2 + |u|^2 \right\} dx \le \frac{1}{1-c^2} \left(\|\operatorname{rot} u\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Omega)} \right)$$

and, recalling the well-known equality

$$\int_\Omega |\nabla u|^2 \, dx = \int_\Omega \left\{ |\operatorname{rot} u|^2 + |\operatorname{div} u|^2 \right\} dx \quad \text{for all } u \in H^1_0\left(\Omega, \mathbb{C}^3\right),$$

we obtain the required inequality (4.12).

Acknowledgments. The authors are grateful to the referee for many useful comments which improved the paper.

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Received March 28, 2023, revised July 21, 2024.

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Нахил підпросторів і розкладання електромагнітних полів на потенційну та вихрову складові

Maria Goncharenko and Evgen Khruslov

Використовуючи поняття нахилу двох підпросторів L і M гільбертового простору $\mathcal{H}$, доведено теорему про продовження лінійних неперервних функціоналів, визначених на підпросторі L, до $\mathcal{H}$ так, що розширені функціонали дорівнюють нулю на підпросторі M. Ми застосували цю теорему для дослідження питання розкладання електромагнітного поля в резонаторі з ідеально провідною межею на потенційну та вихрову складові та вивели нерівність типу Корна для вихрових полів.

Ключові слова: гільбертів простір, нахил підпросторів, розширення функціоналів, розкладання електромагнітного поля