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I.V. Ostrovskii's Work on Arithmetic of Probability Laws

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The first part of the paper gives a brief overview of the development of the arithmetic of probability laws from H. Cramér's paper on the components of Gaussian law to the investigations of Yu.V. Linnik in the 1950s. The second part describes I.V. Ostrovskii's contribution to the arithmetic of probability laws and the theory of analytic characteristic functions.

Key words: Probability law, characteristic function, infinitely divisible law, class I_0, arithmetic of probability laws

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1. Cramér, Lévy, Khinchin, Raikov

The arithmetic of probability laws is a branch of probability theory that appeared in the 1930s. Already the first results and facts demonstrated its close connection to the theory of entire functions. The present paper aims to describing the contribution of I.V. Ostrovskii in the development of this field during 1960s–1990s.

Let us consider the semigroup of probability laws on the real line with respect to the operation of ordinary convolution

$$(P_1 * P_2)(E) := \int_{\mathbb{R}} P_1(E-x)P_2(dx).$$

The units of this semigroup are the laws δ_a, $a \in \mathbb{R}$, that assign the unit mass to the point a. A *component* of a law P is any law P_1 for which there exists a law P_2 such that $P_1 * P_2 = P$. The laws δ_a (they are called *degenerate*) are components of each law.

Recall that the characteristic function of the law P is a function of a real variable t defined by

$$\varphi(t,P) = \int_{\mathbb{R}} e^{ixt}\, P(dx). \tag{1.1}$$

In 1936, G. Cramér [6] proved that every non-degenerate component of a Gaussian law is again a Gaussian law. This result led to the birth of a new theory, the arithmetic of probability laws.

The proof given by G. Cramér, for the first time in probability theory, was complex-analytic and used a fairly deep and relatively recently established Hadamard's factorization theorem on representation of entire functions of finite order. J. Hadamard used this theorem to prove the asymptotic law for distribution of prime numbers.

Cramér's proof of this theorem is as follows. Let P_1 be a non-degenerate component of a Gaussian law P. The tail of the law P,

$$T_P(x) := P(\mathbb{R} \setminus (-x, x)),$$

decreases rapidly as $x \to \infty$, approximately like $\exp(-cx^2)$, $c > 0$. Using this, it is easy to deduce that the tail $T_{P_1}(x)$ of P_1 also decreases at infinity as $\exp(-c_1 x^2)$, $c_1 > 0$. This means that the characteristic function $\varphi(t, P_1)$ of the law P_1 is an entire function of order 2. In terms of characteristic functions, the equality $P_1 * P_2 = P$ means that $\varphi(t, P_1)\varphi(t, P_2) = \varphi(t, P)$ for all complex numbers t. Since $\varphi(t, P)$ does not vanish in the complex plane, the same is true for $\varphi(t, P_1)$. So, $\varphi(t, P_1)$ is an entire function of order 2 without zeros. By Hadamard's theorem, $\varphi(t, P_1) = \exp(Q(t))$, where $Q(t)$ is a polynomial of degree 2. From the simplest properties of characteristic functions, it follows that this polynomial is equal to $-\gamma t^2 + i\beta t$, where $\gamma > 0$ and β is a real number.

Soon after the publication of Cramér's article, D.A. Raikov [80], who was at that time a graduate student of A.Ya. Khinchin, proved that the Poisson laws possess a similar property: All non-degenerate components of a Poisson law are also Poisson laws. Moreover, the proof (following Cramér's one) was also complex-analytic and used Hadamard's theorem. Thus, already the first theorems of the new field indicated its close connection to the theory of entire functions.

In the same year A.Ya. Khinchin [40] proved two fundamental theorems of the arithmetic of probability laws. To formulate them, let us present necessary definitions. A non-degenerate law P is called *indecomposable* if given a representation in the form of convolution $P = P_1 * P_2$, it follows that one of the laws P_1, P_2 is degenerate. The theorems of Cramér and Raikov show that Gaussian and Poisson laws do not have indecomposable components. The class of all such laws is denoted by I_0. A law P is said to be *infinitely divisible* if for every natural number n it can be represented as the convolution of n identical laws, $P = Q_n * Q_n * \cdots * Q_n$. The class of all infinitely divisible laws is denoted by I. Obviously, it contains the Gaussian and Poisson laws.

Theorem 1.1 (The first Khinchin theorem)**.** *Each non-degenerate law P can be represented as a finite or infinite convolution*

$$P = P_0 * P_1 * P_2 * \cdots,$$

where P_0 is a law that does not have indecomposable components (*i.e.* $P_0 \in I_0$) *and $P_1, P_2, \dots$ are indecomposable laws.*

Note that in the above representation, P_0 or all laws $P_1, P_2, \dots$ may be absent. Infinite convolution should be understood in the sense of convergence of the distribution of finite convolutions.

This theorem can be considered as an analogue of the fundamental theorem of arithmetic. However, unlike the latter, the representation of the law in Khinchin's theorem is, generally speaking, not unique.

Theorem 1.2 (The second Khinchin theorem)**.** *Every law that has no indecomposable components is infinitely divisible, that is,* $I_0 \subset I$.

As Khinchin and Raikov showed, the inclusion in this theorem is strict.

A description of class I was obtained in the 1930s in terms of characteristic functions. A law P is infinitely divisible if and only if its characteristic function admits the representation (Lévy's representation)

$$\varphi(t,P) = \exp\left(i\beta t - \gamma t^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\lambda(dx)\right), \tag{1.2}$$

where $\beta \in \mathbb{R}, \gamma \geq 0, \lambda$ is a Borel measure on $\mathbb{R}\setminus\{0\}$ such that

$$\int_{\mathbb{R}\setminus\{0\}} \frac{x^2}{1+x^2}\,\lambda(dx) < \infty.$$

This is equivalent to the following (Khinchin's representation):

$$\varphi(t,P) = \exp\left(i\beta t + \int_{\mathbb{R}} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}\,\chi(dx)\right), \tag{1.3}$$

where $\beta \in \mathbb{R}$, χ is a Borel measure on $\mathbb{R}$ such that $\chi(\mathbb{R}) < \infty$. The measures λ and χ are called the Lévy and Lévy-Khinchin spectral measures, respectively. The Gaussian law is obtained when in formula (1.2) we have $\gamma > 0$ and λ is the zero measure; and in formula (1.3), when $\chi = \gamma\delta_0$. The Poisson law is obtained when in formula (1.2) one has $\gamma = 0$, and $\lambda = b\delta_c$ for some $b > 0$, $c \neq 0$, and in formula (1.3) when $\chi = b\delta_c$ with the same conditions on b and c.

It is a natural problem to describe the class I_0. One can check that an equivalent to given above definition of I_0 is as follows: A law $P \in I_0$ if and only if all (including P itself) components of the law P are infinitely divisible. Thus, the description of class I_0 is reduced to the description of those measures χ in representation (1.3), for which the following holds: The characteristic function of each components P_1 of the law P can be represented in the form (1.3) with a measure χ_1, which is majorized by the measure χ.

After the works of P. Lévy and D.A. Raikov in 1937–1938 on components of finite convolutions of Poisson laws, until the mid-1950s, there was no progress in the study of class I_0. In particular, it was not known whether the convolution of the Gaussian and Poisson laws belongs to class I_0. (That the class I_0 is not a semigroup was known from the results of Lévy and Raikov on decompositions of convolutions of Poisson laws.) The characteristic function of the convolution of Gaussian and Poisson laws

$$\exp(-\gamma t^2 + \lambda(e^{it} - 1))$$

is an entire function of infinite order. If there were no Gaussian component ($\gamma = 0$), then using the substitution $z = e^{it}$ (as in the proof of Raikov's theorem), the problem would be reduced to the case of order 1. But if $\gamma \neq 0$, then this method does not work. New approaches were required. They were found by Yu.V. Linnik in the second half of the 1950s.

2. Linnik

In 1957, an article was published [47] by Yu.V. Linnik, containing a proof that the convolution of Gaussian and Poisson laws belongs to class I_0. The proof was very technical (25 journal pages), but the methods used made it possible to discover new general facts about I_0. The following was proved in [48]:

Theorem 2.1. *If an infinitely divisible law P has a Gaussian component* (*i.e.* $\gamma \neq 0$ *in representation* (1.2)) *and belongs to class I_0, then the spectral measure λ is purely discrete:*

$$\mathit{supp}\,\lambda = \{\mu_n^-\}_{n=-\infty}^{\infty} \cup \{\mu_n^+\}_{n=-\infty}^{\infty},$$

where $\mu_n^- < 0$, $\mu_n^+ > 0$ satisfy

$$-\infty < \cdots < \mu_{n+1}^- < \mu_n^- < \cdots < 0, \quad 0 < \cdots < \mu_n^+ < \mu_{n+1}^+ < \cdots < \infty,$$

and are such that the ratios μ_{n+1}^+/μ_n^+ and μ_{n+1}^-/μ_n^- are integers.

Therefore, if an infinitely divisible law belongs to class I_0 and has a Gaussian component, then its spectral Lévy measure has a very special form. A class of infinitely divisible laws (with or without Gaussian component), whose spectral Lévy measure satisfies the conditions specified in Theorem 2.1 is called the *Linnik class* and denoted by $\mathcal{L}$.

The question arises if it is true that if $P \in \mathcal{L}$ then $P \in I_0$? As Yu.V. Linnik showed [48], the answer is positive for those laws from $\mathcal{L}$ whose spectral Lévy measure decreases rapidly at infinity:

$$\lambda(\{\mu_n^\pm\}) < \exp\bigl(-\exp(|\mu_n^\pm|^{1+\alpha})\bigr), \tag{2.1}$$

for some $\alpha > 0$ and all sufficiently large n. In the book [49], he conjected that condition (2.1) can be replaced by a weaker one:

$$\lambda(\{\mu_n^\pm\}) = O\bigl(\exp(-K(\mu_n^\pm)^2)\bigr), \quad n \to \infty, \quad \forall K > 0. \tag{2.2}$$

His other hypothesis is that there are no laws of class I_0 with a continuous Poisson spectrum. (The *Poisson spectrum* of an infinitely divisible low P is the set of all points $x \in \mathbb{R}\setminus\{0\}$ such that the measure λ of each arbitrarily small neighborhood of x is positive.)

The book [49] also poses a question related to Marcinkiewicz's theorem. To formulate Marcinkiewicz's theorem, some preparations are needed. In general, given a probability law P, then speaking about smoothness properties of its characteristic function $\varphi(t, P)$, one can only say that it is uniformly continuous on the real line. However, if the tail $T_P(x) = P(\mathbb{R} \setminus (-x, x))$ of P decreases sufficiently fast as $x \to \infty$, then the function $\varphi(t, P)$ can be extended by formula (1.1) to entire function in the complex t-plane. How quickly can its maximum modulus increase? Excluding the case $\varphi(t, P) \equiv 1$ (i.e. P is concentrated at origin), we can only say that the growth of the entire characteristic function is not lower than the order one, normal type.

In 1938, Marcinkiewicz [54] discovered that there are restrictions on the growth of an entire characteristic function from above, if it is known that its order is finite and if it has few zeros in the sense that the point 0 is a Borel exceptional value. (Recall that this means that the index of convergence of the zeros of a function is strictly less than its order; the index of convergence ρ_1 of zeros $\{a_k\}$ is the infimum of the numbers ρ' for which the series $\sum_k |a_k|^{-\rho'}$ converges (each zeros is counted taking into account its multiplicity)).

Theorem 2.2 ([54]). *If the convergence index ρ_1 of the zeros of an entire characteristic function of finite order ρ is strictly less than ρ, then $\rho \le 2$.*

This estimate is sharp, as the example of Gaussian law shows.

It follows from Marcinkiewicz's theorem that if $\exp(f(t))$ is an entire characteristic function of finite order, then $f(t)$ is a polynomial of degree no greater than 2 (and thus the corresponding law is Gaussian or degenerate). There exist characteristic functions of the form $\exp(f(t))$ where $f(t)$ is an entire function of order 1 of normal type (an obvious example is the characteristic function of Poisson's law $\exp(\lambda(e^{it} - 1))$). Yu.V. Linnik posed the question whether there are entire characteristic functions of the form $\exp(f(t))$, in which $f(t)$ has a minimal type at order 1 [49, p. 255]. Answers to these and other questions by Yu.V. Linnik were the content of many of I.V. Ostrovskii's works in 1960s–1970s on the arithmetic of probability laws.

3. Research related to the Linnik hypothesis in connection with Marcinkiewicz's theorem

In 1962, I.V. Ostrovskii [55, 57] answered Linnik's question related to Marcinkiewicz's theorem, obtaining results on the growth of entire functions of a special form, more general than entire characteristic functions of the form $\exp(f(t))$. Let us denote by $\widetilde{\mathcal{R}}$ the class of entire functions of complex variable $t = \tau + i\eta$ ($\tau, \eta \in \mathbb{R}$) (not necessarily characteristic functions of probability laws), satisfying the condition

$$|\varphi(\tau + i\eta)| \le M(|\eta|, \varphi), \quad \forall\, \tau, \eta \in \mathbb{R}, \tag{3.1}$$

where $M(r, \varphi) = \max\{|\varphi(t)| : |t| = r\}$. The class $\widetilde{\mathcal{R}}$ is wider than the class $\mathcal{R}$ of ridge functions introduced by D. Dugué, defined by the condition

$$|\varphi(\tau + i\eta)| \le |\varphi(i\eta)|, \quad \forall\, \tau, \eta \in \mathbb{R}. \tag{3.2}$$

Theorem 3.1 ([55, 57]). *Let $F(w)$ and $f(t)$ be entire functions where $F(w)$ is not identically constant. Let $\varphi(t) = F(f(t))$ and $\varphi(t) \in \widetilde{\mathcal{R}}$. Then*

1) *either $f(t)$ is a polynomial of degree not greater than 2;*
2) *or the function $f(t)$ has a growth of at least order 1 normal type, i.e.*

$$\limsup_{r\to\infty} r^{-1} \log M(r, f) > 0. \tag{3.3}$$

Thus, regardless of the form of function F, inequality (3.1) imposes restrictions on the growth of the function $f(t)$. For the allowed growth, a gap appears between the polynomial degree 2 and the order 1 of normal type.

For $F(w) = \exp(w)$, Theorem 3.1 gives an answer to Yu.V. Linnik's question: Entire functions of order one minimal type without zeros cannot be characteristic functions. In article [63], I.V. Ostrovskii posed a conjecture that one may improve the estimate for the growth of f in the second part of Theorem 3.1 by replacing in formula (3.3) the upper limit with lower limit. The validity of this hypothesis was proven by his student V.V. Zimoglyad in [89]. Moreover, it turned out that if the function $\varphi(t) = F(f(t))$ satisfies the conditions of Theorem 3.1 and $f(t)$ is not a polynomial, then the smallest possible growth of $f(t)$ can only be realized on functions that grow regularly in the following sense: either

$$\liminf_{r\to\infty} r^{-1} \log M(r, f) = \infty,$$

or

$$0 < \liminf_{r\to\infty} r^{-1} \log M(r, f) = \limsup_{r\to\infty} r^{-1} \log M(r, f) < \infty.$$

Theorem 3.1 contains only a special case of Marcinkiewicz's theorem, when the function $\varphi(t)$ does not vanish. In this regard, the following Theorem 3.2 is interesting, in which $\varphi(t)$ is allowed to have a faster growth than that of finite order.

Theorem 3.2 ([55,57]). *Let $f(t)$ be an entire transcendental function of finite order ρ, and $g(t)$ be an entire function satisfying the condition*

$$\limsup_{r\to\infty} (\log r)^{-1} \log\log\log M(r, g) < \rho \qquad \text{if } \rho > 0;$$
$$\limsup_{r\to\infty} (\log r)^{-1} \log\log M(r, g) < \infty \qquad \text{if } \rho = 0.$$

Let $\varphi(t) = g(t)\exp(f(t)) \in \widetilde{\mathcal{R}}$. Then the growth of $f(t)$ is not lower than order 1 normal type.

Articles [55, 57] gave rise to a number of studies on generalizing and finding other variants of Marcinkiewicz's theorem. At the same time, in some papers the condition of being a characteristic function was replaced by the weaker condition of being a ridge function, since Marcinkiewicz's theorem holds in the class of ridge functions, that is wider than the class of characteristic functions. I.V. Ostrovskii together with his student I.P. Kamynin in [36] studied the question of whether the condition $\rho_1 < \rho$ in Marcinkiewicz's theorem can be replaced by the condition $\delta(0,\varphi) > 0$, where $\delta(0,\varphi)$ is the Nevanlinna defect at the origin of entire characteristic function φ. (For the concept of Nevanlinna defect, see [23].) The answer turned out to be negative, because, as shown in [36],

For every $2 < \rho < \infty$ there is an entire ridge function $\varphi(t)$ of order ρ such that for every $\varepsilon > 0$ the inequality holds

$$\delta(0,\varphi) \geq c_\varepsilon \rho^{-(2+\varepsilon)\rho}, \tag{3.4}$$

where the constant c_ε does not depend on ρ.

However, if we assume that $\delta(0,\varphi) = 1$, then the statement of Marcinkiewicz's theorem remains valid.

Theorem 3.3 ([36]). *If φ is an entire ridge function of finite lower order such that condition $\delta(0,\varphi) = 1$ is satisfied, then the order of φ does not exceed 2.*

In this regard, in [36] the problem is set to find for each $\rho > 2$ the quantity

$$C(\rho) = \sup \delta(0,\varphi), \tag{3.5}$$

where the supremum is taken over the set of all entire ridge functions of order ρ.

A.E. Eremenko [10] proved that Teorem 3.3 has no analogue for exceptional values in the sense of Valiron.

The condition $\rho_1 < \rho$ in Marcinkiewicz's theorem imposes a restriction on the absolute values of the zeros of φ. As A.A. Goldberg and I.V. Ostrovskii showed in [24], the statement of Marcinkiewicz's theorem remains valid if this condition is replaced by a condition that imposes a restriction only on the *arguments* of the zeros. Instead of the condition $\rho_1 < \rho$, their work assumes that all zeros of φ are real.

Theorem 3.4 ([24]). *Let $\varphi(t)$ be an entire ridge function of finite order having only real zeros. Then its order does not exceed 2. Moreover, it admits the representation*

$$\varphi(t) = C \exp\big(-\gamma t^2 + i\beta t\big) \prod_k \left(1 - \frac{t^2}{a_k^2}\right),$$

where $C \in \mathbb{C}$, $\gamma \geq 0$, $\beta \in \mathbb{R}$, $a_k > 0$, $\sum_k a_k^{-2} < \infty$.

In particular, this theorem shows that the condition $\sum_k a_k^{-2} < \infty$ characterizes the zero sets lying on the real axis of entire characteristic functions of finite order.

The requirement that the order of the function φ be finite in Theorem 3.4 cannot be discarded, since there are entire ridge functions of infinite order without zeros. But it can be weakened somewhat by imposing an additional restriction on the absolute values of zeros:

If φ is an entire ridge function such that all its zeros are real, $\rho_1 < \infty$ and

$$\lim_{r\to\infty} r^{-1} \log\log M(r,\varphi) = 0,$$

then Theorem 3.4 remains valid.

In the same paper a hypothesis is put forward:

If the entire ridge function has only real zeros a_k, then the condition $\sum_k a_k^{-2} < \infty$ is satisfied (zeros are counted with multiplicities).

The work [24] gave rise to a number of articles by I.V. Ostrovskii and his students. For example, in [34] a result is obtained that contains both the Marcinkiewicz theorem and the Goldberg–Ostrovskii theorem.

Theorem 3.5 ([34])**.** *Let φ be an entire ridge function of order ρ and let, for some $B > 0$, the convergence index of its zeros lying outside the strip $|\mathrm{Im}\, t| < B$, be strictly less than ρ. Then $\rho \le 2$.*

In the same article, analogues of Marcinkiewicz's theorem were obtained for the ridge functions that are analytic in a half-plane.

In [76], in order to generalize Marcinkiewicz's theorem, I.V. Ostrovskii used a different, simpler method than the one used in [57]. The method is based on integral representation formulas for analytic functions in the half-plane. Moreover, the theorems obtained by this method contain all previously obtained generalizations. One of the consequences of the results of article [76] can be formulated as follows.

Theorem 3.6. *Let φ be an entire characteristic function without zeroes that satisfies the condition*

$$\liminf_{r\to\infty} r^{-1} \log\log M(r,\varphi) = 0. \tag{3.6}$$

Then φ is the characteristic function of a Gaussian law, possible degenerate.

Recently A.E. Eremenko and A.E. Fryntov [11] obtained the following stability theorem for Theorem 3.6:

Let φ be an entire characteristic function of a random variable X with mean 0 and variance 1. Assume that $\varphi(t) \neq 0$ in a strip $|\mathrm{Re}\, t| < A$ and satisfies condition (3.6). Then the distance in the uniform metric between the distribution functions of X and the standard Gaussian random variable can be estimated above by the value C/A, where C is an absolute constant.

Another stability theorem for the Marcinkiewicz-Ostrovskii theorem is proved in [9].

I.V. Ostrovskii with his students A.M. Vishnyakova and A.M. Ulanovskii [88] gave a short and elementary solution to Linnik's problem. The result obtained is stronger than the one on non-existence of characteristic functions of the form $\exp(f(t))$, where $f(t)$ is entire function of order one minimal type. It is related to analogues of Marcinkiewicz's theorem for entire ridge functions with restrictions on the arguments of the zeros.

Theorem 3.7 ([88])**.** *Assume an entire ridge function $\varphi(t)$ satisfies condition (3.6) and does not vanish in the angles $|\arg t \pm \pi/2| < \alpha$, for some $0 < \alpha \le \pi/2$. Then the order ρ of function φ is finite.*

An exact upper estimate for the value of order ρ in terms of α was found in [20, 86, 87].

4. Relationship between $\mathcal{L}$ and I_0

Article [60] of I.V. Ostrovskii contains a new short proof of the theorem Yu.V. Linnik on the components of the composition of Gaussian and Poisson laws. Its beginning, as in the proof of Yu.V. Linnik, is based on the concept of ridgeness. It is used to obtain preliminary estimates on the real part of the logarithm of the characteristic function of the component of the original law (convolution of the Gaussian and Poisson laws). Starting from a certain place, Yu.V. Linnik used real analysis, while I.V. Ostrovskii noticed that the real part of logarithm of the characteristic function of a component, which is a harmonic function on the real plane, for one variable admits analytic continuation to the whole complex plane, and is an entire function to which it is convenient to apply the Phragmen-Lindelöf theorem for an angle. The problem is quickly reduced to solving simple finite-difference equations, and the proof turned out to be transparent and relatively short. Although it, of course, cannot be called elementary, it made it possible to make a significant progress in the problem of describing class I_0. In [56,61], I.V. Ostrovskii proved the validity of Linnik's conjecture (see (2.2)) with an improvement in the estimate in (2.2) by replacing the quantifier $\forall$ with the quantifier $\exists$. All further studies of the question, under what conditions $P \in \mathcal{L}$ implies $P \in I_0$, were based on the approach proposed in [56, 61]. In work [58] I.V. Ostrovskii showed that condition (2.2) with the quantifier $\exists$ can be further weakened by imposing an additional condition on the law $P \in \mathcal{L}$, requiring that it's support is a lattice, i.e. it lies on some arithmetic progression. (In particular, this means that $\chi((-a, a)) = 0$ for some $a > 0$, where χ is the Lévy-Khinchin spectral measure.) It suffices to require that

$$T_\chi(x) = o\left(\exp(-2(x/d)\log(x/d))\right), \quad x \to +\infty, \tag{4.1}$$

where d is the smallest difference of arithmetic progressions containing the support of P.

The answer to Yu.V. Linnik's question, whether the inclusion $\mathcal{L} \subset I_0$ is true, was given in [25] by A.A. Goldberg and I.V. Ostrovskii. It turned out to be negative. It was shown in [25] that the infinite convolution of Poisson laws with the characteristic function

$$\exp\left(\sum_{k=1}^{\infty} \exp\left(-2^k\right)\Big(\exp\left(2^k it\right) - 1\Big)\right),$$

obviously belonging to the class $\mathcal{L}$, possesses non-infinitely divisible components, and therefore does not belong to the class I_0.

Later, students of I.V. Ostrovskii, A.E. Fryntov and G.P. Chistyakov [21] proved, following mainly the reasoning from [50, p. 191–211], that the assumption on the support in the theorem of I.V. Ostrovskii [58] can be removed, and condition (4.1) can be replaced with the following one

$$T_\chi(x) = O\big(\exp(-Kx\log x)\big), \quad x \to +\infty, \tag{4.2}$$

for some $K > 0$.

In 1987, G.P. Chistyakov [2, 3] showed that condition (4.2) can be replaced by the condition

$$T_\chi(x) = O\big(\exp(-Kx)\big), \quad x \to +\infty,$$

for every $K > 0$, which means that the characteristic function of the law P is entire. This condition can no longer be improved, as the discussed above example of A.A. Goldberg and I.V. Ostrovskii shows.

5. On infinitely divisible laws without Gaussian component

It follows from results of Raikov and Lévy that if an infinitely divisible law P does not have Gaussian component, then the condition $P \in \mathcal{L}$ is not necessary for it to belong to class I_0. But by 1965 it was not known whether the Poisson spectrum needed to be finite or countable. In [59] I.V. Ostrovskii described two broad classes of laws from I_0 with continuous Poisson spectrum.

Theorem 5.1 ([59]). *Every infinitely divisible law without Gaussian component whose Poisson spectrum lies on an interval* $[a, b]$ *satisfying* $0 < a < b \leq 2a < \infty$*, belongs to class* I_0*.*

Theorem 5.2 ([59]). *Every infinitely divisible law without Gaussian component whose Poisson spectrum is a closed bounded set with linearly independent over* $\mathbb{Q}$ *points lying on* $(0, \infty)$ *belongs to class* I_0*.*

From each of these theorems it follows that there are infinitely divisible laws of class I_0 with a continuous Poisson spectrum. It also follows from Theorem 5.1 that every infinitely divisible law is finite or infinite convolution of lows from class I_0.

Work [59] initiated numerous studies aimed at generalization of its results. In particular, I.V. Ostrovskii in [68] obtained a generalization of Theorem 5.2, from which the density of class I_0 in class I followed.

Theorem 5.3 ([59, 68]). *Every infinitely divisible law is a finite or infinite convolution of laws from class* I_0*. Class* I_0 *is dense in class* I *of all infinitely divisible laws.*

For generalizations of Theorem 5.2, see also [4, 5, 7, 13, 14, 19, 69]. For other special classes of characteristic functions studied by I.V. Ostrovskii, see [67,70,74].

6. On decompositions of multidimensional probability laws

Yu.V. Linnik mentioned in the book [49] as an important problem to extend results on arithmetic of one-dimensional probability laws to the multidimensional case. A number of works by I.V. Ostrovskii is dedicated to this problem. In [62] he obtained a multidimensional analogue of Linnik's theorem on the components of the convolution of Gaussian and Poisson laws.

In paper [75] I.V. Ostrovskii studied the question under what conditions an n-dimensional infinitely divisible law P, which is a Cartesian product of one-dimensional laws, has the property: all its components are also Cartesian products of one-dimensional ones. It turned out that it is sufficient to require that the Lévy spectral measure λ_P of the law P satisfies condition

$$\lambda_P\big(\{x \in \mathbb{R}^n : \|x\| > r\}\big) = O\big(\exp(-Kr^2)\big), \quad r \to +\infty, \tag{6.1}$$

for some $K > 0$, and that the law P does not have a Gaussian component. In the same paper, the following result was obtained about the class I_0:

If a n-dimensional infinitely divisible law P is a Cartesian product of one-dimensional laws of the class I_0, and its spectral Lévy measure λ_P satisfies condition (6.1), then $P \in I_0$.

In [64, 65], I.V. Ostrovskii generalized and extended to the multidimensional case the theorem of Raikov on characteristic functions analytical in a neighborhood of the zero of the complex plane:

1) If the characteristic function of the one-dimensional law P is holomorphic in the disk $|t| < R$, then it is holomorphic in the strip $|\mathrm{Im}\, t| < R$;
2) If the characteristic function of the law P is holomorphic in the strip $|\mathrm{Im}\, t| < R$, then the same is true for the characteristic function of each component of this law.

Here is also presented the following generalization to the multidimensional case of another Raikov's theorem:

Theorem 6.1 ([64, 65]). *Let P be an infinitely divisible law in $\mathbb{R}^n$ without a Gaussian component whose Lévy spectral measure is concentrated in some bounded convex open set $A \subset \mathbb{R}^n$ such that $A \cap (A + A) = \varnothing$. Then $P \in I_0$.*

The results of articles [51,52,66] relate to the same range of problems. I.V. Ostrovskii and his student L.Z. Livshits in [53, 71] obtained the following, in some sense unimprovable, result on infinitely divisible laws without Gaussian component whose spectral Lévy measure is the restriction of the Lebesgue measure to a certain set.

Theorem 6.2 ([53, 71]). *Let P be an infinitely divisible n-dimensional law without Gaussian component whose spectral Lévy measure λ_P is equal to*

$$\lambda_P(E) = k l_n(E \cap A) \quad (E \textit{ is any Borel set}),$$

where $k > 0, l_n$ is the Lebesgue measure in $\mathbb{R}^n$, A is a bounded open set. Then the following is true:

$$P \in I_0 \Longleftrightarrow A^* \cap (2)M^+(A) = \varnothing,$$

where A^* is the convex hull of the set A, $(m)A = \{a_1 + \cdots + a_m : a_1, \ldots, a_m \in A\}$, $M^+(A) = \bigcup_{m=1}^{\infty}(m)A$.

7. On class I_0^α

In connection with some questions of mathematical statistics, Yu.V. Linnik introduced a concept of α-component of the probability law, which generalizes the concept of ordinary component. The probability law P_1 is called an α-*component* of the law P if there are laws $P_2, \ldots, P_k$, positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ and a sequence of real numbers t_n tending to zero, such that for all $t = t_n$ the equality is true

$$\varphi(t,P) = \varphi^{\alpha_1}(t,P_1)\varphi^{\alpha_2}(t,P_2)\cdots\varphi^{\alpha_k}(t,P_k).$$

Yu.V. Linnik posed the problem of describing class I_0^α of those infinitely divisible laws that have only infinitely divisible α-components.

By 1970 it was known (Yu.V. Linnik, I.V. Ostrovskii) that if $P \in \mathcal{L}$ and for some $C > 0$ the condition is satisfied

$$T_\chi(x) = O\Big(\exp\big(-Cx^2\big)\Big), \quad x \to +\infty, \tag{7.1}$$

where χ is the Lévi-Khinchin spectral measure of the law P, then $P \in I_0^\alpha$.

I.V. Ostrovskii in [68,69] gave sufficient conditions for a law to belong to class I_0^α that are different from the one above, and that generalize Raikov's conditions for belonging to the class I_0 of finite convolutions of Poisson laws.

Theorem 7.1. *Assume that the characteristic function of the law P have the form*

$$\varphi(t,P) = \exp\left(i\beta t + \int_{\mathbb{R}} \left(e^{itx} - 1\right)\right) \chi(dx),$$

where $\beta \in \mathbb{R}, \chi$ is a finite discrete measure on the line. Assume also that the discrete spectrum of χ consists of points linearly independent over the field $\mathbb{Q}$ and that condition (7.1) is satisfied for some $C > 0$. Then $P \in I_0^\alpha$.

Theorem 7.1 implies that the class I_0^α is dense in the class I in the topology of weak convergence. This work also solved the question of what the spectrum of Poisson laws from the class I_0^α can be.

8. Properties of entire characteristic functions

I.V. Ostrovskii paid much attention to studying various properties of entire characteristic functions of probability laws. In 1982, A.A. Goldberg and I.V. Ostrovskii pubilshed the article [26] containing a deep analysis of the class of entire characteristic functions of finite order. The following result from this work gives a description of indicators of such functions (for brevity, the formulation given for the case of functions of order strictly greater than 1).

Theorem 8.1 ([26]). *Let $\rho(r)$ be the refined order $\rho(r) \to \rho > 1$ $(r \to \infty)$. In order for there to be an entire characteristic function of refined order $\rho(r)$ with indicator $h(\theta)$ relative to $r^{\rho(r)}$, it is necessary and sufficient that the function $h(\theta)$ satisfies the following conditions*:

1) $h(\theta)$ *is a 2π-periodic ρ-trigonometrically convex function*;

2) *$h(\theta + \pi/2)$ is an even function*;

3) $\max\{h(\theta) : 0 \le \theta \le 2\pi\} = \max\{h(\pi/2), h(-\pi/2)\} > 0$;

4) *the inequalities are satisfied*

$$h(\theta) \le \begin{cases} h(\pi/2)(\sin\theta)^\rho & \text{for } 0 \le \theta \le \pi, \\ h(-\pi/2)(|\sin\theta|)^\rho & \text{for } \pi \le \theta \le 2\pi. \end{cases}$$

The article presents numerous applications of Theorem 8.1. Sufficient conditions are presented for a set of points in $\mathbb{C}$ to be the zero set of an entire characteristic function of a given refined order; it is shown that the entire characteristic functions of finite order are not uniquely determined (modulo Gaussian multiplier) by their zero sets; an answer is given to the question on the possible number of negativity intervals for the indicator of entire characteristic functions of finite order ρ in the case $\rho > 1$; relations between the Nevanlinna growth characteristic $T(r, \varphi)$ and $\log M(r, \varphi)$ are considered for entire characteristic functions φ of finite order (we note here work [27], which considers the Paley effect for entire characteristic functions of finite order); the estimate for the value of $C(\rho)$ (see definition of $C(\rho)$ in (3.5)) has been improved (see (3.4)) and the hypothesis is stated:

$$C(\rho) = 1 - (1 + o(1))\sqrt{2\pi/\rho}, \quad \rho \to \infty.$$

A complete description of the zero sets of entire characteristic functions of one-dimensional probability laws without restrictions on the growth of the function was obtained in [37, 38].

Theorem 8.2 ([37, 38])**.** *Let A be at most a countable set in $\mathbb{C}$ (points of finite multiplicity are allowed). In order for the set A to be the zero set of some entire characteristic function, it is necessary and sufficient that*

1) *A does not intersect the imaginary axis, it is symmetric about it (and the multiplicities of points a and $-\bar{a}$ must be the same)*;

2) *for every $0 < H < \infty$ the condition is satisfied*

$$\log n(A; r, H) = o(r), \quad r \to \infty,$$

where $n(A; r, H)$ is a number (taking into account multiplicities) of points of A in the rectangle $\{|\mathrm{Re}\, t| < r, |\mathrm{Im}\, t| < H\}$.

In particular, the zero sets of entire functions having sufficiently slow growth and positive on the imaginary axis, are also the zero sets of entire characteristic functions:

If $f(t)$ is an entire function, $f(0) = 1$, $f(t) > 0$ for $\mathrm{Re}\, t = 0$ and $\log\log M(r, f) = o(r)$, $r \to \infty$, then there is an entire characteristic function $\psi(t)$ without zeros such that $\psi(t)f(t)$ is a characteristic function.

In the n-dimensional case ($n > 1$), this work provides a description of algebraic zero surfaces (in the case of entire characteristic functions of finite order this was done in [22]), and the following general theorem was proved.

Let B^n denote the class of entire functions $f(t)$, $t \in \mathbb{C}^n$, such that:

1) $f(0) = 1, f(t) > 0$ as $\operatorname{Re} t = 0$;
2) $\sup\{|f(t)| : |\operatorname{Im} t| < H\} < \infty$ for every $H > 0$.

Theorem 8.3 ([37,38]). *For any function $f(t) \in B^n$ there is an entire characteristic function $\psi(t)$ of a n-dimensional law that does not vanish and is such that $\psi(t)f(t)$ is a characteristic function.*

Thus, the zero sets of entire characteristic functions of n-dimensional probability laws are the same as those of entire functions of class B^n. The works [35,77,78] study the zero sets of other special classes of entire characteristic functions.

9. Special semigroups with Khinchin's theorems

In the late 1960s, after the works of D. Kendall [39], R. Davidson [8] and K. Urbanik [85], interest arose in semigroups in which analogues of two Khinchin's fundamental theorems from the arithmetic of probabilistic laws on the line hold true. I.V. Ostrovskii wrote in [73] that Yu.V. Linnik drew his attention to one of such semigroups, introduced by J. Kingman [41] and associated with symmetric random walks.

Let S be the set of all probability measures on the half-axis $[0, +\infty)$ equipped with the binary operation $\circ$ (depending on the real parameter $n > 1$) defined by the formula (f is an arbitrary continuous bounded function on $[0, +\infty)$)

$$\int_0^\infty f(x)(\sigma_1 \circ \sigma_2)(dx) = \int_0^\infty \int_0^\infty \int_{-1}^1 f((u^2+v^2+2uv\lambda)^{1/2}) p_n(\lambda)\, d\lambda\, \sigma_1(du)\, \sigma_2(dv),$$

where

$$p_n(\lambda) = \frac{\Gamma(n/2)}{\Gamma((n-1)/2)\sqrt{\pi}}(1-\lambda^2)^{(n-3)/2}, \quad -1 < \lambda < 1.$$

The validity of analogues of Khinchin's fundamental theorems from the arithmetic of probability laws on the line in the semigroup $S_n = (S, \circ)$ was proved by N. Bingham, class $I(S_n)$ of infinitely divisible elements of the semigroup S_n was described by J. Kingman. Class $I_0(S_n)$ was described in 1973 by I.V. Ostrovskii [72,73].

Theorem 9.1 ([72,73]). *The class $I_0(S_n)$ consists of Rayleigh distributions, i.e. distributions with the density*

$$r_a(x) = \frac{2a^n}{\Gamma(n/2)} x^{n-1} \exp(-a^2x^2), \quad x \geq 0,$$

where $0 < a < \infty$, and the distribution concentrated at zero (which is the weak limit of distributions with density $r_a(x)$ as $a \to +\infty$).

When n is an integer, $n \geq 2$, the semigroup S_n is isomorphic to the semigroup of spherically symmetric (that is, invariant under all rotations of the space $\mathbb{R}^n$ around the origin) distribution laws in $\mathbb{R}^n$ with the operation of ordinary convolution. Rayleigh distributions from the semigroup S_n correspond to spherically symmetric Gaussian distributions in $\mathbb{R}^n$. Therefore, in the semigroup of spherically symmetric distribution laws in $\mathbb{R}^n (n \geq 2)$ with the operation of ordinary convolution, class I_0 consists of spherically symmetric Gaussian laws and a law concentrated at the origin.

The methods developed by I.V. Ostrovskii in this work were used by L.S. Kudina in her research devoted to finding the conditions under which any component of a spherically symmetric distribution law in $\mathbb{R}^n (n \geq 2)$ or a law that is a Cartesian product of distribution laws of smaller dimensions, has the same form as the original one, up to a shift [42–45]. The ideas of this work were applied by I.V. Ostrovskii and his students in the study of the arithmetic of other special semigroups [28–33, 79, 81–84].

I.V. Ostrovskii did not publish articles on the stability of decompositions of probability laws and the arithmetic of probability laws on general locally compact Abelian groups (see necessary definitions in [46]). However, he contributed in every possible way to the development of these areas in Kharkiv. He actively supported G.P. Chistyakov who was working on problems of stability of decompositions and G.M. Fel'dman who developed the theory of arithmetic of probability laws on groups. Mainly due to their works, these fields have become vast areas of research with highly nontrivial theorems and deep connections to other mathematical theories. We refer the reader to review [1] for contribution of G.P. Chistyakov to the theory of stability of expansions of probability laws. The results of the works of G.M. Feldman on the arithmetic of probability distributions on abstract Abelian groups are presented in detail in the monograph [17]. See also [12, 15, 16, 18].

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Про роботи Й.В. Островського з арифметики ймовірнісних законів

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Перша частина статті є коротким оглядом розвитку арифметики ймовірнісних законів від роботи Г. Крамера про компоненти закону Гаусса до досліджень Ю.В. Лінника 1950-х років. У другій частині дано опис внеску Й.В. Островського в арифметику ймовірнісних законів та теорію аналітичних характеристичних функцій.

Ключові слова: ймовірнісний закон, характеристична функція, безмежно подільний закон, клас I_0, арифметика ймовірнісних законів