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A Special Case of a Conjecture of Hellerstein, Shen, and Williamson

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Dedicated to the memory of I.V. Ostrovskii

The paper proves a special case of a conjecture of Hellerstein, Shen, and Williamson concerning non-real zeros of derivatives of real meromorphic functions.

Key words: meromorphic function, non-real zeros

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1. Introduction

This paper concerns the problem of classifying real meromorphic functions in the plane which, together with some of their derivatives, have only real zeros and poles [10,13,14]. Here a meromorphic function f is called real if $f(\mathbb{R}) \subseteq \mathbb{R} \cup \{\infty\}$, and it is known that if f is real entire and f and f'' have only real zeros then f belongs to the Laguerre–Polya class LP, as conjectured by Wiman and proved in [2,27,32]. For the meromorphic case, the following conjecture was advanced in [10].

Conjecture 1.1 ([10])**.** *Let f be a real transcendental meromorphic function in the plane with at least one pole, and assume that all zeros and poles of f, f', and f'' are real, and that all poles of f are simple. Then f satisfies*

$$f(z) = C \tan(az + b) + Dz + E, \quad a, b, C, D, E \in \mathbb{R}. \tag{1.1}$$

In the absence of the assumption that f has only simple poles, further examples arise for which f, f', and f'' have only real zeros and poles [12]. Conjecture 1.1 is known to be true if any of the following additional hypotheses holds:

(a) f' omits some finite value [10,15,19,21,23,29,31];
(b) f has infinitely many poles and f''/f' has finitely many zeros [24, Theorem 1.5];
(c) f has infinitely many zeros and poles, all real, simple and *interlaced* — that is, between any two consecutive poles of f there is a zero, and between consecutive zeros of f lies a pole [25].

The following theorem implies a further special case of Conjecture 1.1.

Theorem 1.2. *Let f be a real meromorphic function in the plane, such that f and f' have no zeros or poles in $\mathbb{C} \setminus \mathbb{R}$, while f''/f has no zeros in $\mathbb{C}$. Then there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, with $\alpha_1 \alpha_2 \neq 0$, such that $g(z) = \alpha_1 f(\alpha_2 z + \alpha_3)$ satisfies one of the following:*

(i) $g(z) = f_1(z) = \sin z$;

(ii) $g(z) = f_2(z) = e^z$;

(iii) $g(z) = f_3(z) = \tan z$;

(iv) $g(z) = f_4(z)$, *where*

$$f_4(z) = \sum_{k=0}^{\infty} \frac{z^{k+1}}{k!(k+1)!} = z + \frac{z^2}{2} + \frac{z^3}{12} + \cdots \tag{1.2}$$

solves

$$zy''(z) = y(z); \tag{1.3}$$

(v) $g(z) = F_1(z) = z^Q$ *for some* $Q \in \mathbb{Z} \setminus \{0, 1\}$;

(vi) $g(z) = F_2(z)$, *where*

$$F_2(z) = \frac{d^{n-2}}{dz^{n-2}} \left(z^{n-1}(z-1)^{n-1} \right) \tag{1.4}$$

for some integer $n \geq 2$, *and* F_2 *solves*

$$z(z-1)y''(z) = n(n-1)y(z); \tag{1.5}$$

(vii) $g(z) = F_3(z)$, *where* F_3 *is given by*

$$F_3(z) = (z - K) H_n \left(\frac{K+1}{K-1} - \frac{2K}{(K-1)z} \right) \tag{1.6}$$

for some integer $n \geq 1$ *and* $K \in \mathbb{R} \setminus \{0, 1\}$, *in which*

$$H_n(w) = \frac{d^n}{dw^n} \left((w-1)^{n-1}(w+1)^{n+1} \right), \tag{1.7}$$

while F_3 *solves*

$$z^2(z-1)(z-K)y''(z) = Kn(n+1)y(z); \tag{1.8}$$

(viii) $g(z) = F_4(z)$, *where*

$$F_4(z) = H_n\left(1 - \frac{2}{z}\right), \tag{1.9}$$

in which $1 \leq n \in \mathbb{N}$ *and* H_n *is given by* (1.7), *while* F_4 *solves*

$$z^2(z-1)y''(z) = -n(n+1)y(z). \tag{1.10}$$

Conversely, the equations (1.3), (1.5), (1.8) (for $K > 1$), and (1.10) all supply examples satisfying the hypotheses of the theorem. The function f_4 in (1.2) and its connection to Bessel functions will be discussed in Section 2.1, while the rational functions F_2, F_3, F_4 in (vi), (vii), and (viii), which are linked to hypergeometric functions, will be treated in detail in Sections 2.2, 2.3, and 2.4.

Of course, the condition that f''/f has no zeros means that Theorem 1.2 treats only a very special case of Conjecture 1.1, albeit without the assumption that f is transcendental and all poles of f are simple, but the fact that the proofs of all the resolved special cases are lengthy tends to suggest that the full conjecture is difficult. The result may also be viewed as a special case of the problem of determining all meromorphic functions f such that f''/f has no zeros: in this direction, it was proved in [17] that if f is entire of order less than 1, or meromorphic of order less than 1/2, and f''/f is transcendental, then f''/f has infinitely many zeros.

Note that the corresponding problem for the case where f is strictly non-real, that is, f is not a constant multiple of a real meromorphic function, was completely settled in [9], the main result of which classified all strictly non-real meromorphic functions f in the plane for which f, f', and f'' have only real zeros and poles.

In common with much of the work on non-real zeros of derivatives, this paper relies heavily on key results and methods developed by B.Ja. Levin and I.V. Ostrovskii, as set out in the paper [27] and the textbook [5] — in particular, the factorisation of the logarithmic derivative (Section 3.1) and an integral inequality linking the Tsuji and Nevanlinna proximity functions (Lemma 3.4).

2. Preliminaries and examples

First, let D be a real entire function, whose zeros x_k are all real and simple. Then a standard application of the Mittag–Leffler theorem gives a real entire function C such that

$$\frac{e^{C(z)}}{D(z)} = \frac{-2}{z - x_k} + O(|z - x_k|)$$

as $z \to x_k$, for each k. The formula $g'/g = e^C/D$ then defines a real meromorphic function g, such that g and g' have no zeros at all, while for each k there exists $c_k \in \mathbb{R} \setminus \{0\}$ with $g(z) = c_k(z - x_k)^{-2} + O(1)$ as $z \to x_k$. Hence there exists a real meromorphic function f with $f' = g$ and $f'f''$ zero-free. However, this construction of course gives no control over the location of the zeros of f itself.

The remainder of this section will make use of the following standard lemma.

Lemma 2.1. *Let P be a polynomial with a simple zero at $a \in \mathbb{C}$. If the equation*

$$P(z)y''(z) = y(z) \tag{2.1}$$

has a solution f which is meromorphic in the plane and has $f(a) \in \mathbb{C}$, then every solution which is meromorphic in the plane is a constant multiple of f.

Proof. It may be assumed that $a = 0$. The assumptions force $f(0) = 0$, and the zero of f at 0 must be simple, because otherwise $P = f/f''$ has a double zero at a. Hence $c_1 = f'(0) \neq 0$, and it follows from (2.1) that $2c_2 = f''(0) \neq 0$. A second solution g may then be obtained by writing

$$\left(\frac{g}{f}\right)'(z) = \frac{1}{f(z)^2} = \frac{1}{(c_1 z + c_2 z^2 + \cdots)^2} = \frac{1}{c_1^2 z^2}(1 - 2(c_2/c_1)z + \cdots),$$

and integration clearly gives rise to a logarithm. □

2.1. The equation (1.3). Let f_4 be as in (1.2). Then differentiating f_4 twice leads to

$$z f_4''(z) = \sum_{k=1}^{\infty} \frac{z^k}{(k-1)!k!} = \sum_{k=0}^{\infty} \frac{z^{k+1}}{k!(k+1)!} = f_4(z),$$

after replacing k by $k+1$, and so f_4 is a solution of (1.3). Lemma 2.1, with $a = 0$, shows that any solution of (1.3) which is meromorphic in $\mathbb{C}$ is a constant multiple of f_4.

It turns out that f_4 has a representation in terms of Bessel functions: write $z = w^2$ and

$$f_4(z) = \sum_{k=0}^{\infty} \frac{w^{2k+2}}{k!(k+1)!} = \frac{w}{i} \sum_{k=0}^{\infty} \frac{(-1)^k (2iw)^{2k+1}}{2^{2k+1} k!(k+1)!} = \frac{w}{i} J_1(2iw),$$

where J_1 is the Bessel function of the first kind of order 1 [11]. This relation can be used to prove that all zeros of f_4 are real and non-positive, but the following approach applies Green's transform [11, pp. 286–288] directly to f_4 and (1.3).

Suppose then that $R > 0$ and $s \in \mathbb{R}$ and Re^{is} is a zero of f_4. Set

$$F(r) = f_4(re^{is}), \quad H(r) = \overline{F(r)} F'(r).$$

This yields, for $r > 0$, by (1.3),

$$H'(r) = |F'(r)|^2 + \overline{F(r)} F''(r) = |F'(r)|^2 + e^{2is} \overline{F(r)} f_4''(re^{is}) = |F'(r)|^2 + \frac{e^{is}|F(r)|^2}{r}.$$

Since $H(R) = H(0) = 0$, integration from 0 to R results in

$$\int_0^R |F'(r)|^2 \, dr = -e^{is} \int_0^R \frac{|F(r)|^2}{r} \, dr,$$

which forces $e^{is} = -1$, so that Re^{is} lies on the negative real axis.

Next, a straightforward application of the Wiman–Valiron theory [8] in (1.3) shows that the order of f_4 is $1/2$, and so a standard generalisation of the Gauss–Lucas theorem [34] implies that all zeros of f_4' are also real and non-positive. This completes the proof of the following.

Lemma 2.2. *The real entire function f_4 given by (1.2) is a solution of (1.3), and all zeros of f_4 and f_4' are real and non-positive. Moreover, any solution of (1.3) which is meromorphic in $\mathbb{C}$ is a constant multiple of f_4.*

2.2. The equation (1.5). Let $n \geq 2$ be an integer, and consider the equation (1.5). MAPLE gives solutions in terms of hypergeometric functions, but an explicit solution will be derived as follows. Let $F = F_2$ be given by (1.4) and write

$$\begin{aligned}
F(z) &= \frac{d^{n-2}}{dz^{n-2}}\left(z^{n-1}(z-1)^{n-1}\right) \\
&= \frac{d^{n-2}}{dz^{n-2}}\left(\sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} z^{k+n-1}(-1)^{n-1-k}\right) \\
&= \sum_{k=0}^{n-1} \frac{(n-1)!(k+n-1)!}{k!(n-1-k)!(k+1)!} z^{k+1}(-1)^{n-1-k} \\
&= \sum_{k=0}^{n-1} a_k z^{k+1}, \quad a_k \in \mathbb{R}, \quad a_0 \neq 0.
\end{aligned}$$

It is then clear that F is a polynomial of degree n, with a simple zero at 0, and that all zeros of F and F' lie in $[0,1]$, by repeated application of the Gauss–Lucas theorem. Moreover, a_k satisfies

$$\frac{a_{k+1}}{a_k} = -\frac{(k+n)(n-1-k)}{(k+1)(k+2)} \quad \text{for } k = 0, \ldots, n-2.$$

Hence differentiating F twice yields

$$\begin{aligned}
z(z-1)F''(z) &= (z^2 - z)\sum_{k=0}^{n-1}(k+1)k a_k z^{k-1} \\
&= \sum_{k=0}^{n-1}(k+1)k a_k z^{k+1} - \sum_{k=1}^{n-1}(k+1)k a_k z^{k} \\
&= \sum_{k=0}^{n-1}(k+1)k a_k z^{k+1} - \sum_{k=0}^{n-2}(k+2)(k+1) a_{k+1} z^{k+1} \\
&= \sum_{k=0}^{n-1}(k+1)k a_k z^{k+1} + \sum_{k=0}^{n-2}(k+n)(n-1-k) a_k z^{k+1} \\
&= \sum_{k=0}^{n-1}(k+1)k a_k z^{k+1} + \sum_{k=0}^{n-1}(k+n)(n-1-k) a_k z^{k+1} \\
&= \sum_{k=0}^{n-1}(n^2 - n) a_k z^{k+1} = n(n-1)F(z).
\end{aligned}$$

Thus F solves (1.5). Applying Lemma 2.1, with $a = 0$, completes the proof of the following.

Lemma 2.3. *For $2 \leq n \in \mathbb{Z}$, the real polynomial F_2 given by (1.4) has degree n and solves (1.5). Moreover, all zeros of F_2 and F_2' are real, and any solution of (1.5) which is meromorphic in the plane must be a constant multiple of F_2.*

2.3. The equation (1.8). Let $n \geq 1$ be an integer and let $K \in \mathbb{R} \setminus \{0,1\}$. MAPLE gives solutions of (1.8) in terms of hypergeometric functions, and the following direct determination of a rational solution was found via properties of the related Jacobi polynomials [30, p. 254]. Using the change of variables

$$w = \frac{K+1}{K-1} - \frac{2K}{(K-1)z}, \quad z = \phi(w) = \frac{2K}{K+1-(K-1)w}, \tag{2.2}$$

write $y(z) = (z-K)h(w)$, so that

$$\begin{aligned} y'(z) &= h(w) + \frac{2K(z-K)}{(K-1)z^2} h'(w), \\ y''(z) &= \frac{2K}{(K-1)z^2} h'(w) + \frac{2K}{(K-1)z^2} h'(w) \\ &\quad - \frac{4K(z-K)}{(K-1)z^3} h'(w) + \frac{4K^2(z-K)}{(K-1)^2 z^4} h''(w) \\ &= \frac{4K^2}{(K-1)z^3} h'(w) + \frac{4K^2(z-K)}{(K-1)^2 z^4} h''(w). \end{aligned}$$

Observe next that, by (2.2),

$$\frac{z-1}{z} = \frac{(K-1)(w+1)}{2K}, \qquad \frac{z-K}{z} = \frac{(K-1)(w-1)}{2}.$$

Thus substituting for y and y'' delivers

$$\begin{aligned} R(z) &= Kn(n+1)y(z) - z^2(z-1)(z-K)y''(z) \\ &= Kn(n+1)(z-K)h(w) \\ &\quad - z^2(z-1)(z-K)\left(\frac{4K^2}{(K-1)z^3} h'(w) + \frac{4K^2(z-K)}{(K-1)^2 z^4} h''(w)\right), \\ &= K(z-K)\left[n(n+1)h(w) - \left(\frac{4K}{K-1}\right)\left(\frac{z-1}{z}\right) h'(w)\right] \\ &\quad - K(z-K)\left[\frac{4K}{(K-1)^2}\left(\frac{(z-1)(z-K)}{z^2}\right) h''(w)\right] \\ &= K(z-K)\left[n(n+1)h(w) - 2(w+1)h'(w) + (1-w^2)h''(w)\right]. \end{aligned}$$

Thus y solves (1.8) if and only if h solves

$$(1-w^2)h''(w) - 2(w+1)h'(w) + n(n+1)h(w) = 0. \tag{2.3}$$

Lemma 2.4. *Let $H(w) = H_n(w)$, with H_n as in (1.7). Then H is a polynomial of degree n and solves (2.3). Moreover, $H(-1) = 0$, all n zeros of H are simple, and they all lie in $[-1,1)$.*

Proof. Write (1.7) in the form

$$H(w) = \frac{d^n}{dw^n}\left((w+1-2)^{n-1}(w+1)^{n+1}\right)$$

$$= \frac{d^n}{dw^n}\left(\sum_{k=0}^{n-1}\binom{n-1}{k}(-2)^{n-1-k}(w+1)^{n+1+k}\right)$$
$$= \sum_{k=0}^{n-1}\binom{n-1}{k}(-2)^{n-1-k}\frac{(n+1+k)!}{(k+1)!}(w+1)^{k+1},$$

which leads to

$$H(w) = \sum_{k=0}^{n-1} b_k(w+1)^{k+1}, \quad b_k = \frac{(n-1)!(n+1+k)!}{k!(n-k-1)!(k+1)!}(-2)^{n-1-k}, \tag{2.4}$$

in which $b_0 \neq 0$ and

$$\frac{b_{k+1}}{b_k} = \frac{(n-k-1)(n+2+k)}{(k+1)(k+2)(-2)} = \frac{(k+1-n)(k+2+n)}{2(k+1)(k+2)} \tag{2.5}$$

for $k = 0, \dots, n-2$. Substitution of (2.4) into the right-hand side of (2.3), followed by application of (2.5), delivers

$$\begin{aligned}
Q(w) &= (1-w^2)H''(w) - 2(w+1)H'(w) + n(n+1)H(w) \\
&= (2-(w+1))(w+1)\sum_{k=0}^{n-1}(k+1)kb_k(w+1)^{k-1} \\
&\quad - 2(w+1)\sum_{k=0}^{n-1}(k+1)b_k(w+1)^k + n(n+1)\sum_{k=0}^{n-1}b_k(w+1)^{k+1} \\
&= 2\sum_{k=1}^{n-1}(k+1)kb_k(w+1)^k \\
&\quad + \sum_{k=0}^{n-1}(n(n+1)-(k+1)k-2(k+1))b_k(w+1)^{k+1} \\
&= 2\sum_{k=0}^{n-2}(k+2)(k+1)b_{k+1}(w+1)^{k+1} \\
&\quad + \sum_{k=0}^{n-1}(n(n+1)-(k+2)(k+1))b_k(w+1)^{k+1} \\
&= \sum_{k=0}^{n-2}(k+1-n)(k+2+n)b_k(w+1)^{k+1} \\
&\quad + \sum_{k=0}^{n-1}(n(n+1)-(k+2)(k+1))b_k(w+1)^{k+1} \\
&= \sum_{k=0}^{n-1}(k+1-n)(k+2+n)b_k(w+1)^{k+1} \\
&\quad + \sum_{k=0}^{n-1}(n(n+1)-(k+2)(k+1))b_k(w+1)^{k+1} = 0.
\end{aligned}$$

Thus $H(w)$ is a polynomial solution of (2.3), of degree n, with a simple zero at -1, since $b_0 \neq 0$ in (2.4). Repeated application of the Gauss–Lucas theorem to $G(w) = (w-1)^{n-1}(w+1)^{n+1}$ shows that all zeros of $H(w)$ lie in $[-1,1]$. Moreover, since G has a zero of multiplicity $n-1$ at 1, all zeros of $E = G^{(n-1)}$ lie in $[-1,1)$ and therefore so do all zeros of $H = E'$. Finally, all zeros of H in $(-1,1)$ are simple, by the existence-uniqueness theorem and (2.3). □

Lemma 2.5. *With H_n as in (1.7) and $n \geq 1$, and with w defined by (2.2), the function $F_3(z)$ in (1.6) is a rational solution of (1.8) with $F_3(1) = F_3(K) = 0$ and all its zeros real and simple, and every solution of (1.8) which is meromorphic in the plane is a constant multiple of F_3. Moreover, $F_3(z) = P(z)z^{-n}$, where P is a real polynomial with $P(0) \neq 0$, and P has degree n or $n+1$.*

Furthermore, if P has degree $n+1$ and all zeros of P lie in $(0,+\infty)$, then F_3, F_3', and F_3'' have no zeros or poles in $\mathbb{C} \setminus \mathbb{R}$. In particular, this holds if $K > 1$.

Proof. First, F_3 solves (1.8) and has a pole at $z = 0$ of order n, since $z = 0$ corresponds to $w = \infty$ and H has degree n. Clearly, F_3 has no other poles in $\mathbb{C}$.

Next, $H(-1) = 0$ and all n zeros of H are simple and lie in $[-1,1)$. Since one of them may be mapped to ∞ by $z = \phi(w)$, it follows from (2.2) that $F_3(\infty) \neq 0$ and F_3 has n or $n+1$ zeros in $\mathbb{C}$. In particular, F_3 has zeros at $z = 1$ and $z = K$, which correspond to $w = -1$ and $w = 1$ respectively, and any solution of (1.8) which is meromorphic in the plane is a constant multiple of F_3, by Lemma 2.1 with $a = K$ or $a = 1$.

Now suppose that P has degree $n+1$ and all zeros of P lie in $(0,+\infty)$. This will certainly hold if $K > 1$, because in this case the function $z = \phi(w)$ is finite and increasing for $-1 \leq w < 1$, and maps $[-1,1)$ to $[1,K)$, so that $F_3(z) = (z-K)H(w)$ inherits all n zeros of $H(w)$, as well as having a zero at $z = K$. Under these assumptions, a consideration of leading terms shows that $zP'(z) - nP(z)$ has degree $n+1$, and so

$$F_3'(z) = \frac{zP'(z) - nP(z)}{z^{n+1}}$$

has $n+1$ zeros in $\mathbb{C}$. Of these, n arise from Rolle's theorem and lie in $(0,+\infty)$, while one lies in $(-\infty,0)$ because, with $x \in \mathbb{R}$,

$$\frac{F_3'(x)}{F_3(x)} \sim \frac{1}{x} < 0 \quad \text{as } x \to -\infty, \qquad \frac{F_3'(x)}{F_3(x)} \sim -\frac{n}{x} > 0 \quad \text{as } x \to 0-.$$

Thus F_3 and F_3' have no zeros in $\mathbb{C} \setminus \mathbb{R}$, and nor has F_3'', because of (1.8). □

Taking $K = 2$ delivers $F_3(z) = (z-K)H_n(w) = (z-2)H_n(3-4/z)$ and (with help from MAPLE) the following:

$$n = 1, \quad g_1(z) = \frac{8(z-1)(z-2)}{z};$$
$$n = 2, \quad g_2(z) = \frac{144(z-1)(z-4/3)(z-2)}{z^2};$$

$$n = 3, \quad g_3(z) = \frac{384(z-1)(z-2)(11z^2 - 30z + 20)}{z^3}.$$

In these examples, g_j, g_j', and g_j'' have no zeros in $\mathbb{C} \setminus \mathbb{R}$, by Lemma 2.5. On the other hand, choosing $K = -1$ leads to $F_3(z) = (z-K)H_n(w) = (z+1)H_n(-1/z)$, as well as:

$$n = 1, \quad h_1(z) = \frac{2(z^2-1)}{z};$$
$$n = 2, \quad h_2(z) = \frac{-12(z^2-1)}{z^2};$$
$$n = 3, \quad h_3(z) = \frac{-24(z^2-1)(z^2-5)}{z^3};$$
$$n = 4, \quad h_4(z) = \frac{720(z^2-1)(z^2-7/3)}{z^4}.$$

Here h_j, h_j', and h_j'' have no zeros in $\mathbb{C} \setminus \mathbb{R}$ for $j = 2, 4$, but h_j' has non-real zeros for $j = 1, 3$.

2.4. The equation (1.10). A solution of (1.10) is obtained by the following limiting process with K real: let $n \geq 1$ and let F_3, H_n be as in (1.6) and (1.7), and set

$$\begin{aligned} F_4(z) &= \lim_{K \to +\infty} \frac{F_3(z)}{-K} \\ &= \lim_{K \to +\infty} \left(\frac{z-K}{-K} \right) H_n \left(\frac{K+1}{K-1} - \frac{2K}{(K-1)z} \right) = H_n \left(1 - \frac{2}{z} \right). \end{aligned}$$

Since all zeros of H_n and H_n' lie in $[-1, 1)$, by Lemma 2.4, F_4 and F_4' have no zeros in $\mathbb{C} \setminus [1, +\infty)$, and F_4 has a pole of order n at 0. Applying Weierstrass' theorem yields, since F_3 solves (1.8),

$$\frac{F_4''(z)}{F_4(z)} = \lim_{K \to +\infty} \frac{F_3''(z)}{F_3(z)} = \frac{-n(n+1)}{z^2(z-1)}$$

as required. Furthermore, F_4 has a simple zero at $z = 1$, inherited from the simple zero of H_n at $w = -1$, which completes the proof of the following.

Lemma 2.6. *With H_n as in (1.7) and $n \geq 1$, the function $F_4(z) = H_n(1 - 2/z)$ is a rational solution of (1.10) with a simple zero at 1, and every solution of (1.10) which is meromorphic in the plane is a constant multiple of F_4. Furthermore, F_4, F_4', and F_4'' have no zeros or poles in $\mathbb{C} \setminus \mathbb{R}$.*

Calculating $F_4(z) = H_n(1 - 2/z)$ using MAPLE delivers:

$$n = 1, \quad p_1(z) = \frac{4(z-1)}{z};$$
$$n = 2, \quad p_2(z) = \frac{24(z-1)(z-2)}{z^2};$$
$$n = 3, \quad p_3(z) = \frac{192(z-1)(z^2-5z+5)}{z^3}.$$

3. Lemmas needed for the proof of Theorem 1.2

Lemma 3.1. *Let g be a real meromorphic function in the plane, of order at most 1, and with infinitely many zeros, all but finitely many of them real, and assume that g has finitely many poles. Then*

$$\lim_{\substack{y\to+\infty \\ y\in\mathbb{R}}} \frac{\log|g(iy)|}{\log y} = +\infty.$$

Proof. It is enough to prove this when g is real entire, with only real zeros, and with $g(0)\neq 0$. The hypotheses then imply that

$$g(z) = e^{\alpha z+\beta}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{z/a_n},$$

with α,β,a_n real. As $y\to+\infty$ with $y\in\mathbb{R}$, this gives

$$\begin{aligned} 2\log|g(iy)| &\geq 2\sum_{n=1}^{\infty}\log\left|1-\frac{iy}{a_n}\right| - O(1) = \sum_{n=1}^{\infty}\log\left(1+\frac{y^2}{a_n^2}\right) - O(1) \\ &\geq \sum_{|a_n|\leq\sqrt{y}}\log(1+y) - O(1) \geq n(\sqrt{y},1/g)\log y - O(1). \end{aligned}$$ □

Lemma 3.2 ([4])**.** *Let $D\subseteq\mathbb{C}$ be a domain and let $\mathcal{F}$ be a family of meromorphic functions f on D such that f and f'' have no zeros in D. Then the family $\{f'/f : f\in\mathcal{F}\}$ is normal on D.*

Next, suppose that G is a transcendental meromorphic function in the plane, and that $G(z)\to a\in\mathbb{C}\cup\{\infty\}$ as $z\to\infty$ along a path γ; then the inverse G^{-1} is said to have a transcendental singularity over the asymptotic value a [1, 28]. If $a\in\mathbb{C}$ then for each $\varepsilon>0$ there exists a component $\Omega=\Omega(a,\varepsilon,G)$ of the set $\{z\in\mathbb{C} : |G(z)-a|<\varepsilon\}$ such that $\gamma\setminus\Omega$ is bounded: these components are referred to as neighbourhoods of the singularity [1]. Two such paths γ,γ' on which $G(z)\to a$ determine distinct singularities if the corresponding components $\Omega(a,\varepsilon,G)$, $\Omega'(a,\varepsilon,G)$ are disjoint for some $\varepsilon>0$. The singularity is called direct [1] if $\Omega(a,\varepsilon,G)$, for some $\varepsilon>0$, contains finitely many zeros of $G-a$, and indirect otherwise. A direct singularity is called logarithmic if there exists $\varepsilon>0$ such that $w=\log 1/(G(z)-a)$ is a conformal bijection from $\Omega(a,\varepsilon,G)$ to the half-plane $\mathrm{Re}\,w>\log 1/\varepsilon$. Finally, transcendental singularities over ∞ may be classified using $1/G$, and a transcendental singularity will be referred to as lying in an open set D if $\Omega(a,\varepsilon,G)\subseteq D$ for some $\varepsilon>0$.

The next lemma combines [20, Lemma 2.4] and [23, Lemma 2.2], and links asymptotic values approached on paths in the upper half-plane H^+ with the growth of the Tsuji characteristic $\mathfrak{T}(r,g)=\mathfrak{m}(r,g)+\mathfrak{N}(r,g)$ for functions g that are meromorphic on the closed upper half-plane [2, 5, 35].

Lemma 3.3 ([20,23])**.** *Let $L\not\equiv 0$ be a real meromorphic function in the plane such that $\mathfrak{T}(r,L)=O(\log r)$ as $r\to\infty$, and let $F(z)=z-1/L(z)$. Assume that* *at least one of L and $1/L$ has finitely many non-real poles. Then there exist finitely many $\alpha \in \mathbb{C}$ such that $F(z)$ or $L(z)$ tends to α as z tends to infinity along a path in $\mathbb{C} \setminus \mathbb{R}$.*

Moreover, there exists at most one direct transcendental singularity of F^{-1} lying in H^+.

The following result of Levin and Ostrovskii [27] (see also [5, Ch. 6, Lemma 5.2] and [20, Lemma 2.4]) will be required.

Lemma 3.4 ([27]). *Let G be a meromorphic function in the plane: then, for each $R \geq 1$,*

$$\frac{1}{2\pi} \int_R^{+\infty} \frac{1}{r^3} \int_0^\pi \log^+ |G(re^{i\theta})| \, d\theta \, dr \leq \int_R^{+\infty} \frac{\mathfrak{m}(r, G)}{r^2} \, dr.$$

If, in addition, G is real meromorphic with finitely many poles, and satisfies $\mathfrak{T}(r, G) = O(\log r)$ as $r \to \infty$, then $T(R, G) = O(R \log R)$ as $R \to +\infty$.

Lemma 3.5. *There exists a positive constant c_0 such that if the function ψ maps the upper half-plane H^+ analytically into itself then, for $r \geq 1$ and $\theta \in (0, \pi)$,*

$$\frac{|\psi(i)| \sin\theta}{5r} < |\psi(re^{i\theta})| < \frac{5r|\psi(i)|}{\sin\theta} \quad \text{and} \quad \left| \frac{\psi'(re^{i\theta})}{\psi(re^{i\theta})} \right| \leq \frac{c_0}{r \sin\theta}. \tag{3.1}$$

Both of these estimates are standard: the first is essentially just Schwarz' lemma [26, Chap. I.6, Theorem 8′], while the second follows from Bloch's theorem applied to $\log \psi$.

3.1. The Levin–Ostrovskii factorisation.

The following constructions are standard [2, 27]. Suppose that $(u_k), (v_k)$ are sequences satisfying $u_k < v_k < u_{k+1}$ for $-\infty \leq M < k < N \leq +\infty$. Then there exists $k_0 \in \mathbb{N}$ such that u_k and v_k have the same sign for $|k| \geq k_0$, and

$$\psi(z) = \prod_{|k| \geq k_0} \frac{1 - z/v_k}{1 - z/u_k}$$

converges on $\mathbb{C}$ by the alternating series test. Furthermore, ψ satisfies, for z in the upper half-plane H^+,

$$\arg \psi(z) = \sum_{|k| \geq k_0} \arg \frac{1 - z/v_k}{1 - z/u_k} = \sum_{|k| \geq k_0} \arg \frac{v_k - z}{u_k - z} \in (0, \pi).$$

This leads to the Levin–Ostrovskii factorisation [2, 27] of the logarithmic derivative of a real entire function f with real zeros. If f has finitely many zeros, set $\psi(z) = 1$, while if f has infinitely many zeros u_k then zeros of f' given by Rolle's theorem can be labelled v_k so that $u_k < v_k < u_{k+1}$, whereupon ψ may be constructed as above. It follows that $f'/f = P\psi$, where P is real meromorphic with finitely many poles and either $\psi \equiv 1$ or $\psi(H^+) \subseteq H^+$.

4. Proof of Theorem 1.2: first steps

Let f be as in the hypotheses and write

$$L = \frac{f'}{f}, \quad F(z) = z - \frac{f(z)}{f'(z)}, \quad F' = \frac{ff''}{(f')^2}. \tag{4.1}$$

Lemma 4.1. *Let $0 < \delta < \pi/2$ and $\delta < \sigma < \pi - \delta$.*

(I) *If $rL(re^{i\sigma})$ is bounded as $r \to +\infty$ then $zL(z)$ is bounded as $z \to \infty$ with $\delta < \arg z < \pi - \delta$.*

(II) *If $\lim_{r\to+\infty} rL(re^{i\sigma}) = 0$, then $zL(z) \to 0$ uniformly as $z \to \infty$ with $\delta < \arg z < \pi - \delta$.*

Proof. The functions $u_R(z) = RL(Rz)$, $R \geq 1$, form a normal family on the domain $D_1 = \{z \in \mathbb{C} : 1/2 < |z| < 2, \delta/2 < \arg z < \pi - \delta/2\}$: this follows from Lemma 3.2 applied to the functions $f(Rz)$. Take a sequence $R_n \to +\infty$ such that (u_{R_n}) converges locally spherically uniformly on D_1. In case (I), (u_{R_n}) cannot have ∞ as limit, while in case (II), the limit function must vanish identically, by the identity theorem. □

Lemma 4.2. *Poles of F in $\mathbb{C}$ coincide with zeros of $L = f'/f$, all of which are real and simple. All zeros of F' in $\mathbb{C}$ are real zeros of f and super-attracting fixpoints of F; furthermore, simple zeros of F' in $\mathbb{C}$ are zeros of f which are not zeros of f'', while multiple zeros of F' in $\mathbb{C}$ have multiplicity 2 and are common simple zeros of f and f''.*

Proof. This is standard, and all assertions follow from (4.1). First, any multiple zero of $L = f'/f$ would be a zero of f'', and hence of f, and thus a pole of f'/f, an obvious contradiction. Next, zeros of F' are zeros of f or f'', and hence of f. But multiple zeros of f are not zeros of F', and so all zeros of F' must be simple zeros of f, and since f''/f has no zeros they cannot be zeros of f'' of multiplicity greater than 1. □

Define the sets W^+ and W^- using

$$H^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \quad W^\pm = \{z \in H^+ : \pm F(z) \in H^+\}. \tag{4.2}$$

The next lemma is fairly standard and goes back to Sheil-Small [32].

Lemma 4.3. *Let $x_0 \in \mathbb{R}$ be a zero of f'/f. If $(f'/f)'(x_0) < 0$ then $x_0 \in \partial W^-$, while if $(f'/f)'(x_0) > 0$ then $x_0 \in \partial W^+$. Poles of f are repelling fixpoints of F and lie in $\partial W^+ \setminus \partial W^-$.*

Proof. The first two assertions hold since as $z \to x_0$ from within H^+ the sign of $\operatorname{Im}(-f(z)/f'(z))$ is the same as that of $\operatorname{Im}(f'(z)/f(z))$. Furthermore, if x_1 is a pole of f of multiplicity m_1 then $F(x_1) = x_1$ and $F'(x_1) = 1 + 1/m_1 > 1$. □

Lemma 4.4. *The following statements hold.*

(i) *If F is transcendental and has finitely many asymptotic values then all but finitely many zeros of f are simple.*

(ii) *If F is rational and either $F(\infty) = \infty$ or ∞ is not a multiple point of F, then all zeros of f in $\mathbb{C}$ are simple.*

Proof. This uses standard facts involving iteration [33]. To prove (i), observe that a multiple zero of f is an attracting, but not super-attracting, fixpoint of F, and so under iteration of F attracts a critical or asymptotic value of F, while zeros of F' in $\mathbb{C}$ are fixpoints of F. Now (ii) follows since the only singular values of F^{-1} are the values taken by F at multiple points in $\mathbb{C} \cup \{\infty\}$, all of which are fixpoints of F by Lemma 4.2 and the assumptions of (ii). □

Denote by ∂D the boundary of a domain D with respect to $\mathbb{C}$.

Lemma 4.5. *Let C, D be domains with $C \subseteq D \subseteq H^+$ and $\mathbb{R} \subseteq \partial D$, such that F maps C univalently onto D. Then ∂C contains at most one point which is a pole of f.*

Proof. Suppose that $y_1, y_2 \in \partial C$ are distinct poles of f. Each y_j is a real repelling fixpoint of F and the branch of F^{-1} mapping D to C extends by reflection to a small neighbourhood U_j of y_j, with an attracting fixpoint at y_j. The iterates $(F^{-1})^n$ of $F^{-1} : D \to C \subseteq D \subseteq H^+$ form a normal family on D, but as $n \to \infty$ they tend to the constant y_j on $D \cap U_j$, a contradiction. □

Lemma 4.6. *Let A be a component of W^+, and suppose that a closed interval $[a, b]$ lies in $\partial A \cap \mathbb{R}$, with $a < b$ and $f(a), f(b) \in \{0, \infty\}$, and with $f(x) \neq 0, \infty$ on (a, b). Then one of the following holds:*

(A) *$f(a) \neq f(b)$ and $L = f'/f$ has no zeros in (a, b);*

(B) *$f(a) = f(b) = \infty$, the function L has exactly one zero c in (a, b), and c satisfies $L'(c) > 0$, while F does not map A univalently onto H^+.*

Proof. Observe first that all zeros of L in $\mathbb{C}$ are simple, by Lemma 4.2, and that if $f(a) = f(b) = \infty$ then F cannot map A univalently onto H^+, by Lemma 4.5. Moreover, if $f(a) \neq f(b)$ then $L = f'/f$ has an even number of zeros in (a, b). It follows that if neither (A) nor (B) holds then there exists at least one zero d of L in (a, b) with $L'(d) < 0$, contradicting Lemma 4.3. □

Lemma 4.7. *Let A be a component of W^+ which is mapped univalently onto H^+ by F, and assume that $x_1 \in \partial A \cap \mathbb{R}$ is a zero of $L = f'/f$. Then at least one of $(-\infty, x_1]$ and $[x_1, +\infty)$ lies in ∂A.*

Proof. Assume the contrary. Since all multiple points of F in $\mathbb{C}$ are zeros of f, by Lemma 4.2, it is possible to start at x_1 and follow $\mathbb{R}$ in each direction until the first encounter with a zero or pole of f, giving a closed interval $[a, b] \subseteq \partial A \cap \mathbb{R}$, with $a < x_1 < b$, satisfying the hypotheses of Lemma 4.6; this is impossible, since alternative (A) is incompatible with the existence of x_1 and (B) with F mapping A univalently onto H^+. □

Lemma 4.8. *Let A be a component of W^+. Then A is unbounded.*

Proof. Assume the contrary: since F has no critical values in $\mathbb{C} \setminus \mathbb{R}$, the mapping $F : A \to H^+$ is univalent and onto. Thus F must have a pole on $\partial A \cap \mathbb{R}$, which contradicts Lemma 4.7. □

Lemma 4.9. *Let A be a bounded component of W^-. Then $-F$ maps A univalently onto H^+, and ∂A consists of a closed interval $[a, b]$, where $-\infty < a < b < +\infty$ and $f(a) = f(b) = 0$, together with a Jordan curve λ which joins a to b via H^+. Moreover, ∂A contains precisely one zero $x_0 \in (a, b)$ of $L = f'/f$.*

Proof. First, F must have a pole on ∂A, and so on $\partial A \cap \mathbb{R}$. Second, the mapping is univalent since F has no critical values in $\mathbb{C} \setminus \mathbb{R}$. Finally, the nature of the boundary follows from the absence of bounded components of W^+. □

Definition 4.10. A finite chain D of bounded components of W^- will mean the following:

(a) D is the union of $N \in \mathbb{N}$ bounded components $C_1, \ldots, C_N$ of W^-, each as in Lemma 4.9;

(b) the boundary of each C_j consists of a closed interval $[a_j, b_j]$, where $-\infty < a_j < b_j < +\infty$, together with a Jordan curve λ_j which joins a_j to b_j via H^+;

(c) the boundaries of the C_j are disjoint except that $b_{j-1} = a_j$.

Such a finite chain D will be called maximal if $D' = D$ whenever D' is a finite chain of bounded components of W^- with $D \subseteq D'$.

Lemma 4.11. *Let D be a maximal finite chain of bounded components of W^- as in Definition 4.10. Then $a_2 = b_1, \ldots, a_N = b_{N-1}$ are common simple zeros of f and f'', and double zeros of F', and there exists a component A of W^+ such that*

$$\lambda_1 \cup \cdots \cup \lambda_N \subseteq \partial A.$$

Moreover, if $x^* = a_1$ or $x^* = b_N$ then x^* satisfies exactly one of the following: (i) x^* is a simple zero of F' and a simple zero of f, but not a zero of f''; (ii) x^* is a double zero of F', and a common zero of f and f'', lying on the boundary of an unbounded component B of W^-.

Proof. This follows from the maximality of D and Lemmas 4.2 and 4.9. □

5. The case where f is a rational function

Proposition 5.1. *Assume that f is a rational function which satisfies the hypotheses of Theorem 1.2. Then there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ with $\alpha_1\alpha_2 \neq 0$ such that $g(z) = \alpha_1 f(\alpha_2 z + \alpha_3)$ is one of the functions F_j in (v)–(viii) of Theorem 1.2.*

The whole of this section will be occupied with the proof of Proposition 5.1. First, f/f'' is a polynomial and

$$L(z) = \frac{f'(z)}{f(z)} = \frac{m}{z} + O(|z|^{-2}) \quad \text{as } z \to \infty, \tag{5.1}$$

where m is the number of zeros minus the number of poles, counting multiplicities, of f in the finite plane. Further, $F(\infty)$ exists and is real or infinite, and all components of $W^{\pm}$ are mapped univalently onto H^+ by $\pm F$.

5.1. The case where $m \neq 0, 1$. Assume that $m \neq 0, 1$ in (5.1): then

$$\frac{f''(z)}{f(z)} = L'(z) + L(z)^2 = \frac{m(m-1)}{z^2} + O(|z|^{-3}) \quad \text{as } z \to \infty. \tag{5.2}$$

Thus f/f'' has degree 2 and so has either one real double zero, or two simple real zeros.

Suppose that f/f'' has a double zero. Then applying a real translation in the z plane leads to $f(z)/f''(z) = cz^2$ for some real constant c, and comparison with (5.2) forces f to satisfy

$$z^2 y''(z) = m(m-1)y(z),$$

which has linearly independent solutions z^{d_j}, where $d_1 = m \neq 0, 1$ and $d_2 = 1 - m \neq 0, 1$. If f is a constant multiple of z^{d_j}, for some j, then clearly f satisfies conclusion (v) of Theorem 1.2. The only remaining possibility in this subcase is that there exist $a_1, a_2 \in \mathbb{C} \setminus \{0\}$ with

$$f(z) = a_1 z^m + a_2 z^{1-m} = z^m(a_1 + a_2 z^{1-2m}).$$

Since f has only real zeros, the odd integer $1 - 2m$ must be ± 1, and either possibility gives $m = 0$ or $m = 1$, a contradiction.

Assume next that f/f'' has two simple zeros, which implies that f has no poles in $\mathbb{C}$ and is a polynomial of degree $n \geq 2$. A real linear change of variables then leads to

$$z(z-1)f''(z) = df(z), \quad d \in \mathbb{R} \setminus \{0\}.$$

A comparison of leading terms shows that $d = n(n-1)$, giving equation (1.5), so that f satisfies conclusion (vi) of Theorem 1.2, by Lemma 2.3.

5.2. The case $m = 1$. Suppose that $m = 1$ in (5.1): if f has no poles in $\mathbb{C}$ then evidently f is a linear function, contradicting the assumption that f''/f has no zeros. Assume for the remainder of this section that f has at least one pole in $\mathbb{C}$. Then a real linear re-scaling delivers $c \in \mathbb{R} \setminus \{0\}$ and $q \geq 1$ such that, as $z \to \infty$ and $\zeta = 1/z \to 0$,

$$f(z) = z\left(1 + \frac{c}{z^{q+1}} + \cdots\right), \quad J(\zeta) = \frac{1}{f(1/\zeta)} = \zeta(1 - c\zeta^{q+1} + \cdots) = \zeta - c\zeta^{q+2} + \cdots.$$

The flower theorem from complex dynamics [33] (see [3, Lemma 10] for a convenient statement of the theorem as applied here) gives $q+1$ components U_j of the Fatou set of J, each with $0 \in \partial U_j$ and containing a critical value ζ_j of J, such that the iterates J^n tend to 0 on U_j. Moreover the U_j can be labelled so that, as $n \to +\infty$,

$$\arg J^n(\zeta_j) \to \frac{2\pi j - \arg c}{q+1}.$$

Since all critical values of J belong to $\mathbb{R} \cup \{\infty\}$, because those of f do, while $J(\mathbb{R} \cup \{\infty\}) \subseteq \mathbb{R} \cup \{\infty\}$, this gives a contradiction unless $q = 1$.

It follows that, again as $z \to \infty$,

$$\begin{aligned} &f(z) = z + c/z + \cdots, \quad f'(z) = 1 - c/z^2 + \cdots, \quad f''(z) = 2c/z^3 + \cdots, \\ &\frac{f(z)}{f'(z)} = \frac{z(1 + c/z^2 + \cdots)}{1 - c/z^2 + \cdots} = z(1 + 2c/z^2 + \cdots) = z + 2c/z + \cdots, \\ &F(z) = -2c/z + \cdots, \quad \frac{f''(z)}{f(z)} = 2c/z^4 + \cdots. \end{aligned} \tag{5.3}$$

Hence W^+ and W^- have one unbounded component A between them. Moreover, all zeros of f are simple by Lemma 4.4.

Since f has at least one pole in $\mathbb{C}$, Lemmas 4.3, 4.5, and 4.8 imply that A is a component of W^+ and f has exactly one pole x_0 in $\mathbb{C}$, of order n say, and hence $n+1$ zeros in $\mathbb{C}$, all simple. Furthermore, each of these simple zeros u of f is a multiple point of F and so lies in $\partial W^+ \cap \partial W^-$. Because W^- has only bounded components, each u belongs to the boundary of a maximal finite chain D of bounded components of W^- as in Definition 4.10.

Take such a maximal finite chain D: then the unique pole x_0 of f does not lie on ∂D, by Lemma 4.3. Moreover, with the notation of Definition 4.10 and Lemma 4.11, a_1 and b_N are simple poles of f''/f, but the intermediate points $a_j = b_{j-1}$, $j = 2, \ldots, N$, are neither zeros nor poles of f''/f. Hence the closure of each such D contributes exactly 2 to the number of poles of f''/f in $\mathbb{C}$. Since f''/f has a double pole at x_0, and no zeros in $\mathbb{C}$, (5.3) implies that there exists precisely one maximal finite chain D.

Hence among the $n+1$ zeros of f, precisely $n-1$ are also zeros of f'', and f/f'' has degree 4 and two simple zeros at the ends a_1, b_N of D, plus one double zero at x_0. Since $x_0 \in \partial A \setminus \partial D$, a real linear change of independent variable makes it possible to assume that $x_0 = 0$ and $\partial D \cap \mathbb{R} = [1, K]$ for some $K > 1$. Hence f satisfies

$$z^2(z-1)(z-K)f''(z) = df(z), \quad d \in \mathbb{R},$$

and expanding about $z = 0$ shows that $d = Kn(n+1)$, giving equation (1.8). Lemma 2.5 then implies that f satisfies conclusion (vii) of Theorem 1.2. This completes the discussion of the case $m = 1$.

5.3. The case $m = 0$. Suppose that $m = 0$ in (5.1): then f has as many zeros as poles in $\mathbb{C}$, counting multiplicities. Moreover, $f(\infty)$ is finite and real but non-zero, and it may be assumed that $f(\infty) = 1$. Further, there exist $c \in \mathbb{R} \setminus \{0\}$ and $s \geq 1$ such that, as $z \to \infty$,

$$f(z) - 1 \sim cz^{-s}, \quad L(z) = \frac{f'(z)}{f(z)} \sim f'(z) \sim -csz^{-1-s}, \quad F(z) \sim \frac{z^{1+s}}{cs}, \tag{5.4}$$

as well as

$$\frac{f''(z)}{f(z)} = L'(z) + L(z)^2 \sim cs(s+1)z^{-2-s}. \tag{5.5}$$

Thus F has a pole of multiplicity $1 + s \geq 2$ at infinity and so a super-attracting fixpoint there. Moreover, Lemma 4.4 again implies that all zeros of f are simple. Assume that W^+ has p components, all necessarily unbounded by Lemma 4.8, and W^- has q unbounded components, while the polynomial f/f'' has r zeros in $\mathbb{C}$ arising from zeros of f, all of which must be simple zeros of f which are not zeros of f''.

Each component C of W^+ has at most one pole of f on its boundary, by Lemma 4.5, and so precisely one, by Lemma 4.3 and the Denjoy–Wolff theorem [33] applied to the inverse function $F^{-1} : H^+ \to C$, coupled with the fact that ∞ is a super-attracting fixpoint of F (which implies in particular that F is not a Möbius transformation and $C \neq H^+$). Thus f has poles at precisely p points, and each is a double zero of f/f''. It now follows, in light of (5.4) and (5.5), that

$$|p - q| \leq 1, \quad p + q = 1 + s \geq 2, \quad 2p + r = 2 + s = p + q + 1, \\ r = q - p + 1 \in \{0, 1, 2\}. \tag{5.6}$$

Lemma 5.2. *There do not exist $x_1, x_2, x_3 \in \mathbb{R}$ such that $x_1 < x_2 < x_3$ and x_1, x_3 are poles of f while x_2 is a zero of f'/f.*

Proof. Assume that such a triple x_1, x_2, x_3 does exist, and without loss of generality that f has no poles in (x_1, x_3). No zero of f'/f can lie on the boundary of an unbounded component B of $W^\pm$, by the univalence of F on B and the fact that $F(\infty) = \infty$. In particular, by Lemma 4.8, x_2 must lie on the boundary of a bounded component of W^-, and hence on the boundary of a maximal finite chain D of bounded components C_j of W^- joined end to end as in Definition 4.10 and its notation. Then, in view of Lemma 4.11, the C_j all border the same component A of W^+, and A is unbounded, with exactly one pole x_0 of f on ∂A. On the other hand, ∂D contains no poles of f, and so $x_1 < a_1 < x_2 < b_N < x_3$.

Suppose that $x_1 \neq x_0$. Then f has no poles in (x_1, a_1), by the choice of x_1 and x_3, and x_1 lies on the boundary of some component $A' \neq A$ of W^+. Since a_1, which is a zero of f, lies on ∂A, there must exist at least one zero of F', and so of f, in (x_1, a_1): let c_1 be the nearest such zero to a_1. Then there must exist a zero d_1 of $L = f'/f$ with $c_1 < d_1 < a_1$ and $L'(d_1) < 0$, which forces $d_1 \in \partial W^-$, so that d_1 lies on the boundary of a bounded component of W^-. Because f has no zeros or poles in (c_1, a_1), this contradicts the maximality of the finite chain D.

Similar reasoning if $x_3 \neq x_0$ completes the proof of the lemma. □

Lemma 5.3. *The integer s in (5.4) satisfies $s \leq 2$, and if $s = 2$, then $c < 0$.*

Proof. Suppose that $s > 2$, or $s = 2$ and $c > 0$. Then there exist a large positive R and θ_j satisfying $0 \leq \theta_1 < \theta_2 < \theta_3 \leq \pi$, with the property that $(-1)^{j+1}(f(Re^{i\theta_j}) - 1)$ is small, real and positive.

Thus $Re^{i\theta_2}$ lies on a level curve λ_2 in the closed upper half-plane on which $f(z)$ is real and $0 < f(z) < 1$, and following λ_2 in the direction of decreasing f leads to a real zero y_2 of f, possibly via one or more real zeros of f'. Similarly, for $j = 1, 3$, the point $Re^{i\theta_j}$ lies on a level curve γ_j in the closed upper half-plane on which $f(z)$ is real and $1 < f(z) < +\infty$. Follow each γ_j in the direction of increasing f: then γ_j must approach a real pole x_j of f.

Furthermore, γ_1 and γ_3 do not meet λ_2 at all, and do not meet each other in the open half-plane H^+. Hence it must be the case that $x_1 > y_2 > x_3$. Thus y_2 lies in a unbounded component U of the set $\{z \in \mathbb{C} : |f(z)| < 1\}$, which cannot contain a zero of f', by Lemma 5.2. By the Riemann–Hurwitz formula [33], or by analytic continuation of f^{-1}, the function f is univalent on U. But this contradicts the fact that y_2, λ_2 and the reflection of λ_2 across $\mathbb{R}$ must all lie in U. □

If $s = 1$ then $r = 1$ and $p = 1$ by (5.6), and f/f'' has degree 3, and after a real linear re-scaling it may be assumed that f has a pole at 0, of order n say, while the remaining zero of f'/f'' lies at 1. Thus f satisfies (1.10) and is, by Lemma 2.6, a constant multiple of the function F_4 in conclusion (viii) of Theorem 1.2.

Now suppose $s = 2$ and $c < 0$. Then $q = 2$, $p = 1$, and $r = 2$ by (5.4) and (5.6). Hence f/f'' has degree 4, with one double zero at the unique pole of f and two simple zeros. After a linear re-scaling it may be assumed that 0 is a pole of f of order n, and that the simple zeros of f/f'' are 1 and $K \neq 0, 1$. This leads to (1.8) and, in view of Lemma 2.5, to conclusion (vii) of Theorem 1.2, which completes the proof of Proposition 5.1.

6. Continuation of the proof in the transcendental case

Assume henceforth that f is transcendental and satisfies the hypotheses of Theorem 1.2. Since all zeros and poles of f and f'' are real, the Tsuji characteristic of $L = f'/f$ satisfies [2]

$$\mathfrak{T}(r, L) = O(\log r) \quad \text{as } r \to +\infty. \tag{6.1}$$

Lemma 6.1. *The function f/f'' is real entire and has order of growth at most* 1*. If f/f'' is a polynomial then f satisfies conclusion* (i)*,* (ii)*, or* (iv) *of Theorem* 1.2*.*

Proof. The growth estimate follows from (6.1) and Lemma 3.4. If f/f'' is a polynomial then f has finitely many poles and a standard application of the Wiman–Valiron theory [8] implies that f/f'' has degree at most 1. If f/f'' is constant then evidently f satisfies conclusion (i) or (ii), whereas if f/f'' has degree 1 a real linear change of variables leads to f satisfying equation (1.3), in which case Lemma 2.2 delivers conclusion (iv). □

Assume henceforth that f and f/f'' are both transcendental. Then Lemmas 3.1 and 6.1 together imply that

$$\lim_{\substack{y\to+\infty \\ y\in\mathbb{R}}} \frac{\log|f(iy)/f''(iy)|}{\log y} = +\infty. \tag{6.2}$$

Lemma 6.2. *f has infinitely many zeros.*

Proof. Suppose that f has finitely many zeros. Then so has f'', and hence f'/f is a rational function, by the main result of [16], and so is f''/f, contrary to the assumption just made. □

Lemma 6.3. *f has infinitely many poles.*

Proof. Suppose that f has finitely many poles. Then Section 3.1 shows that $L = f'/f$ has a representation $L = P\psi$, where P and ψ are real meromorphic functions such that $\psi(H^+) \subseteq H^+$ and P has finitely many poles. Combining (3.1) with (6.1) delivers $\mathfrak{T}(r, P) = O(\log r)$ as $r \to +\infty$, and so P has order at most 1, by Lemma 3.4.

Suppose first that P is transcendental with infinitely many zeros. Then Lemma 3.1 implies that $P(z)$ tends to infinity as $z \to \infty$ on $i\mathbb{R}^+$, faster than any power of $|z|$, and so does $L(z)$, by (3.1). The fact that P has real zeros implies that, again as $z \to \infty$ on $i\mathbb{R}^+$,

$$\begin{aligned}
\frac{L'(z)}{L(z)} &= \frac{P'(z)}{P(z)} + \frac{\psi'(z)}{\psi(z)} = O(|z|),\\
\frac{f''(z)}{f(z)} &= L(z)^2 + L'(z) = L(z)^2 + O(|z|)L(z) \to \infty,
\end{aligned}$$

which contradicts (6.2).

Next, if P is transcendental with finitely many zeros then $zL(z)$ tends to 0 on one of the rays $\arg z = \pi/4, 3\pi/4$ and to ∞ on the other, by (3.1), contradicting Lemma 4.1.

Hence P must be a rational function, and f has finite order [2]. Moreover, it follows from (3.1) and (6.2) that, as $z \to \infty$ on $i\mathbb{R}^+$,

$$L(z)^2 + L'(z) = L(z)^2 + O\left(\frac{1}{|z|}\right) L(z) = \frac{f''(z)}{f(z)} = O\left(\frac{1}{|z|^2}\right), \quad zL(z) = O(1).$$

Let δ be small and positive. Then Lemma 4.1 implies that, as $z \to \infty$ with $\delta < \arg z < \pi - \delta$, $zL(z)$ is bounded and $\log|f(z)| = O(\log|z|)$. An application of the Phragmén–Lindelöf principle now shows that f is a rational function, contrary to assumption. □

Lemma 6.4. *The following statements hold for asymptotic values $\beta \in \mathbb{C} \cup \{\infty\}$ of F, that is, values β such that $F(z) \to \beta$ as $z \to \infty$ on a path Γ_β.*

(i) *There exist at most two $\beta \in \mathbb{C} \cup \{\infty\}$ such that $\Gamma_\beta \cap \mathbb{R}$ is unbounded.*

(ii) *There exist finitely many $\beta \in \mathbb{C}$ for which $\Gamma_\beta \cap \mathbb{R}$ is bounded, and F has finitely many asymptotic values.*
(iii) *All transcendental singularities of F^{-1} over finite values are logarithmic.*
(iv) *F has at most one asymptotic value $\beta \in \mathbb{C} \setminus \mathbb{R}$ with $\Gamma_\beta \setminus H^+$ bounded.*

Proof. To prove (i) just note that if $\Gamma_\beta \cap \mathbb{R}$ is unbounded then $\beta \in \mathbb{R} \cup \{\infty\}$ and it may be assumed that Γ_β lies in the closed upper half plane; hence there is at most one β such that $\Gamma_\beta \cap \mathbb{R}^+$ is unbounded, and at most one for which $\Gamma_\beta \cap \mathbb{R}^-$ is unbounded. Next, the first assertion of (ii) follows from Lemma 3.3, and on combination with (i) shows that F has finitely many asymptotic values. Since all critical points of F are fixpoints of F, all finite singular values of F^{-1} are isolated, so that (iii) is a consequence of the argument from [28, p.287]. The fact that F^{-1} has at most one direct singularity lying in H^+, by Lemma 3.3, then delivers (iv). □

Lemma 6.5. *Let D be a neighbourhood of a logarithmic singularity of F^{-1} over $\beta \in \mathbb{R}$, such that $D \cap \mathbb{R}^+$ is unbounded. Then there exists $a \in \mathbb{R}$ with $[a, +\infty) \subseteq D$, and f has finitely many zeros and poles on $\mathbb{R}^+$. Moreover, there cannot exist a neighbourhood $E \subseteq \mathbb{C} \setminus \mathbb{R}$ of a transcendental singularity of F^{-1} over a finite value $\gamma \neq \beta$.*

Proof. The first two assertions hold since D is simply connected and symmetric with respect to $\mathbb{R}$, while all zeros and poles of f are fixpoints of F.

Next, assume that E and γ do exist, without loss of generality with $E \subseteq H^+$. There must exist a path tending to infinity in $D \cap H^+$ on which $F(z) \to \beta$, and so F^{-1} has a direct singularity over γ, lying in H^+, by Lemma 6.4, plus one over ∞, which contradicts Lemma 3.3. □

Lemma 6.6. *The finite asymptotic values of F comprise either a pair $\beta, \overline{\beta}$, where $\beta \in \mathbb{C} \setminus \mathbb{R}$, or one value $\beta \in \mathbb{R}$. Furthermore, all but finitely many zeros of f are simple.*

Proof. Suppose that $\beta, \gamma \in \mathbb{C}$ are distinct asymptotic values of F: then there exist simply connected neighbourhoods D, E of logarithmic singularities of F^{-1} over β, γ respectively, by Lemma 6.4. If $D \cap \mathbb{R}^+$ and $E \cap R^-$ are both unbounded then β, γ are real and Lemma 6.5, applied to $f(z)$ and $f(-z)$, implies that f has finitely many poles, contrary to assumption.

It may therefore be assumed that either D or E lies in $\mathbb{C} \setminus \mathbb{R}$, and hence that both do, by Lemma 6.5 again. But then it must be the case that one of D, E lies in H^+ and the other in the lower half-plane H^-, by Lemma 3.3, and moreover that $\gamma = \overline{\beta}$.

The last assertion then follows from Lemma 4.4. □

Lemma 6.7. *Suppose that f'/f has finitely many zeros. Then f satisfies conclusion (iii) of Theorem 1.2.*

Proof. This can be deduced from [19] but the following proof is included in order to keep the account self-contained. The function f/f' has finitely many poles, and so has order at most 1 by (6.1) and Lemma 3.4. On the other hand, f/f' is transcendental, by Lemma 6.2.

Since all but finitely many zeros of f are simple, by Lemma 6.6, the function f can be written in the form $f = f_1/f_2$, in which f_1, f_2 are real entire functions with real zeros and no common zeros, and f_1 has order at most 1. Here each f_j has infinitely many zeros, by Lemmas 6.2 and 6.3. Use the Levin–Ostrovskii factorisation of f_j'/f_j to write

$$\frac{f'}{f} = \frac{f_1'}{f_1} - \frac{f_2'}{f_2} = \phi_1\psi_1 - \phi_2\psi_2,$$

in which ϕ_j and ψ_j are real meromorphic, while $\psi_j(H^+) \subseteq H^+$ and ϕ_j has finitely many poles. Since f_1 has finite order, (3.1) leads to $m(r, \phi_1) = O(\log r)$ as $r \to \infty$, and so ϕ_1 must be a rational function. It then follows from (3.1), (6.1) and standard properties of the Tsuji characteristic that $\mathfrak{T}(r, \phi_2) = O(\log r)$ as $r \to +\infty$ and so ϕ_2 has order at most 1 by Lemma 3.4.

Let δ be small and positive and apply Lemma 3.1 to f/f'. On combination with Lemma 4.1 and a standard estimate for f_1'/f_1 [6], this yields, as $z \to \infty$ with $\delta < \arg z < \pi - \delta$,

$$zL(z) \to 0, \quad \frac{f_1'(z)}{f_1(z)} = O\left(|z|\right) \quad \text{and} \quad \frac{f_2'(z)}{f_2(z)} = O\left(|z|\right). \tag{6.3}$$

It then follows in view of (3.1) that $\log^+ |\phi_2(z)| \leq 3 \log |z|$ as $z \to \infty$ in the same sector. Since δ may be chosen arbitrarily small, an application of the Phragmén–Lindelöf principle now shows that ϕ_2 is a rational function, so that f_2 has finite order [2] and so has f.

The next step is to show that f and f'' have, with finitely many exceptions, the same zeros. Since f''/f has no zeros, and all but finitely many zeros of f are simple, it suffices to show that all but finitely many zeros of f are zeros of f''. Suppose then that $x_1, x_2, x_3 \in \mathbb{R}$ are zeros of f but not of f'', such that $x_1 < x_2 < x_3$, while $|x_1|$ and $|x_3|$ are large and $x_1x_3 > 0$. Thus x_2 is a simple zero of F' and lies on the boundary of a component of A of W^-. Hence it is possible to move along the real axis, away from x_2, while remaining on ∂A. Since f cannot have a pole on ∂A, by Lemma 4.3, it follows that continuing along $\mathbb{R}$ in the same direction until the first encounter with a pole or zero of f gives rise to a closed interval $I \subseteq \partial A$, its endpoints being zeros of f. This interval I must then contain a zero of f'/f, a contradiction.

Since f'/f has finitely many zeros and all but finitely many zeros of f are simple, it now follows that the function

$$R = \frac{f''}{ff'}$$

has finite order and finitely many poles. As $z \to \infty$ in $\delta < \arg z < \pi - \delta$, integration of f'/f using (6.3), coupled with a standard estimate for f''/f' from

[6], yields

$$\log^+ |R(z)| \le \log^+ \frac{1}{|f(z)|} + \log^+ \left| \frac{f''(z)}{f'(z)} \right| = O(\log |z|).$$

The Phragmén–Lindelöf principle forces R to be a real rational function, and so all but finitely many poles of f are simple. Now write $R = 2/S$ and

$$2ff' = Sf'', \quad (f^2 - Sf' + S'f)' = 2ff' - Sf'' - S'f' + S'f' + S''f = S''f.$$

Hence $S''f$ is the derivative of a meromorphic function and, since f has infinitely many simple poles, by Lemma 6.3, the rational function S'' must vanish identically and $f^2 - Sf' + S'f$ must be a constant c. This yields a Riccati equation

$$Sf' = f^2 + S'f - c = P_2(f) = (f - A_1)(f - A_2), \quad A_j \in \mathbb{C}, \quad -c = A_1 A_2. \tag{6.4}$$

If $A_1 = A_2$, then $1/S$ is the derivative of the transcendental meromorphic function $-(f - A_1)^{-1}$, which is obviously impossible. Assume that $A_1 \neq A_2$: then A_1, A_2 are distinct Picard values, and hence asymptotic values, of f and so $A_2 = \overline{A}_1$ by Lemma 6.6. Moreover, partial fractions yields

$$\frac{f'}{f - A_1} - \frac{f'}{f - A_2} = \frac{A_1 - A_2}{S}.$$

Thus S must be constant, since otherwise f is rational, a contradiction. It now follows from (6.4) that $Sf' = f^2 - c$ and so $A_1 + A_2 = 0$. Hence A_1 and $A_2 = \overline{A}_1$ are purely imaginary, while $-c = A_1 A_2 > 0$, and conclusion (iii) of Theorem 1.2 follows easily. □

It may be assumed henceforth that

$$f \text{ and } f'/f \text{ each have infinitely many zeros and infinitely many poles.} \tag{6.5}$$

7. Non-real asymptotic values

Proposition 7.1. *F has no finite non-real asymptotic values.*

To prove Proposition 7.1, assume for the remainder of this section that F has an asymptotic value $\beta \in \mathbb{C} \setminus \mathbb{R}$. Then by Lemma 6.6 and the argument of [28, p. 287], it may be assumed that the only finite asymptotic values of F are β and $\overline{\beta}$ and that there exists an unbounded component A of $W^\pm$ which contains no β-points of F, and which is mapped "infinite to one" by F onto $H^\pm \setminus \{\beta\}$, where H^- denotes the open lower half-plane. Moreover, the corresponding transcendental singularity of F^{-1} over β is logarithmic, and A is simply connected. The first main step in the proof of Proposition 7.1 will be accomplished via the following lemma.

Lemma 7.2. *There does not exist a component B of $W^\pm$ such that $\pm F$ maps B univalently onto H^+ and $F(z) \to \infty$ as $z \to \infty$ on a path in B.*

Proof. Assume the contrary, let δ be small and positive, and set

$$u(z) = \log^+ \frac{\delta}{|F(z) - \beta|} \quad (z \in A), \quad u(z) = 0 \quad (z \notin A),$$

as well as $B(t, u) = \max\{u(z) : |z| = t\}$. Then u is subharmonic and non-constant in the plane and Lemma 3.4 yields, for $R \geq 1$,

$$\begin{aligned}
\int_R^{+\infty} \frac{B(r/2, u)}{r^3}\, dr &\leq \frac{3}{2\pi} \int_R^{+\infty} \left(\int_0^\pi u(re^{i\theta})\, d\theta \right) \frac{dr}{r^3} \\
&\leq \frac{3}{2\pi} \int_R^{+\infty} \left(\int_0^\pi \log^+ 1/|F(re^{i\theta}) - \beta|\, d\theta \right) \frac{dr}{r^3} \\
&\leq 3 \int_R^{+\infty} \frac{\mathfrak{m}(r, 1/(F-\beta))}{r^2}\, dr \\
&\leq 3 \int_R^{+\infty} \frac{\mathfrak{T}(r, F) + O(\log r)}{r^2}\, dr \leq O\left(\frac{\log R}{R} \right).
\end{aligned}$$

Since $B(r/2, u)$ is non-decreasing this yields $B(R, u) = O(R \log R)$ as $R \to +\infty$.

Let δ and $1/r_0$ be small and positive and denote by $\theta_A(r)$, $\theta_B(r)$ the angular measure of the intersection with the circle $|z| = r \geq r_0$ of A, B respectively. Suppose first that $\theta_A(r) < \pi(1 - \delta)$ on a set F_1 of upper logarithmic density at least δ. Then, since $A \subseteq H^+$, all sufficiently large $r \in F_1$ satisfy [2, Lemma 2.1]

$$\begin{aligned}
(1 + o(1)) \log r \geq \log B(2r, u) &\geq \int_{r_0}^r \frac{\pi dt}{t\theta_A(t)} - O(1) \\
&\geq \int_{[r_0,r]\cap F_1} \frac{dt}{(1-\delta)t} + \int_{[r_0,r]\setminus F_1} \frac{dt}{t} - O(1) \\
&\geq \int_{[r_0,r]\cap F_1} \frac{\delta\, dt}{(1-\delta)t} + \log r - O(1) \geq \left(\frac{\delta^2}{2(1-\delta)} + 1 \right) \log r,
\end{aligned}$$

an evident contradiction.

Hence there exists a set E_1 of lower logarithmic density at least $1 - \delta$ on which $\theta_A(r) \geq \pi(1 - \delta)$ and so $\theta_B(r) \leq \pi\delta$, since A, B are evidently not the same component of $W^\pm$. The function $w = \pm iF(z)$ maps B conformally onto the right half-plane: let $z = G(w)$ be the inverse mapping, and let γ_0 be the image in B under G of the real interval $[1, +\infty)$, starting from $z_0 = G(1)$. Then γ_0 tends either to infinity or to a pole of F, and so to infinity since F is univalent on B. Let $r^* = |z_0|$ and let $z = G(X) \in \gamma_0$, with $X \geq 1$ and $r = |z|$ large. Then applying Koebe's quarter theorem to G on the disc of centre $w \in [1, X]$ and radius w leads to

$$\begin{aligned}
\log |F(z)| = \log X = \int_{[1,X]} \frac{|dw|}{|w|} &= \int_{z_0}^z \frac{|dz|}{|w||G'(w)|} \\
&\geq \int_{z_0}^z \frac{|dz|}{4|z|\theta_B(|z|)} \geq \int_{r^*}^r \frac{dt}{4t\theta_B(t)} \\
&\geq \int_{[r^*,r]\cap E_1} \frac{dt}{4\pi\delta t} \geq \left(\frac{1 - 2\delta}{4\pi\delta} \right) \log r \geq 2 \log r,
\end{aligned}$$

in which the second integral is from z_0 to z along γ_0. This delivers, as $z \to \infty$ on γ_0,

$$|F(z)| = |z(1 - 1/zL(z))| \geq |z|^2, \quad zL(z) \to 0.$$

On the other hand, there evidently exists a path γ_1, tending to infinity in A, on which $F(z) \to \beta$ and hence $zL(z) \to 1$. Since L has only real poles, the inverse of $zL(z)$ must have a direct singularity over ∞, lying in H^+ and separating γ_0 from γ_1. But L has only real zeros, and so the inverse of $zL(z)$ must have a direct singularity over 0, lying in H^+ and separating the singularity over ∞ from γ_1. This contradicts Lemma 3.3. □

Because the component A of $W^{\pm}$ is unbounded and simply connected and F has no finite real asymptotic values, the boundary of A consists of countably many pairwise disjoint piecewise analytic simple curves γ_j, each going to infinity in both directions and mapped by F onto $\mathbb{R}$ or $\mathbb{R} \cup \{\infty\}$, and if $F(\gamma_j) = \mathbb{R} \cup \{\infty\}$ then γ_j must meet $\mathbb{R}$, since F has only real poles. Suppose that one of these curves, γ say, lies wholly in H^+; then γ is mapped by F onto $\mathbb{R}$ and forms part of the boundary of a component $A' \neq A$ of $W^{\mp}$. Let $z^* \in A'$: since F^{-1} cannot have two logarithmic singularities lying in H^+, by Lemma 3.3, analytic continuation of a local branch of the inverse of $\mp F$ shows that z^* lies in a component of $W^{\mp}$ which is mapped univalently onto H^+, and which must be A', so that $\gamma = \partial A'$ and $F(z)$ tends to ∞ along a path in A', contradicting Lemma 6.6. Hence each γ_j meets $\mathbb{R}$, and there is only one, because if γ_j meets $\mathbb{R}$ then it must separate any other $\gamma_{j'}$ from $\mathbb{R}$.

Thus ∂A consists of a single curve, which is mapped "infinite to one" onto $\mathbb{R} \cup \{\infty\}$, and passes in each direction through infinitely many poles of F, all of which are real. In particular, $F^{-1}(\{\infty\})$ and $\partial A \cap \mathbb{R}$ are neither bounded above nor bounded below, and neither W^+ nor W^- has any unbounded component other than A. Since W^+ has no bounded components, by Lemma 4.7, while f has by (6.5) infinitely many poles, all of which lie in ∂W^+ by Lemma 4.3, it must be the case that $A = W^+$ and $\beta \in H^+$, and all components of W^- must be bounded.

Let $K_0 = \{\beta + it : 0 \leq t < +\infty\}$. Then each pole of F on ∂A is the starting point of a simple curve Λ which tends to infinity in A and is mapped injectively onto $\{\beta + it : 0 < t \leq +\infty\}$ by F. There are infinitely many of these Λ and they are pairwise disjoint. Moreover, at most finitely many such Λ meet the vertical line segment $I_0 = [\mathrm{Re}\,\beta, \beta]$, because otherwise $\mathrm{Re}\,F$ would be constant on I_0 and on the curves Λ, contradicting the absence of non-real critical points of F. Choose a component I_1 of $\mathbb{R} \setminus \{\mathrm{Re}\,\beta\}$ which contains infinitely many zeros of f: this is possible by Lemma 6.2. Because ∂A passes in each direction through infinitely many real poles of F, one of the curves Λ can be chosen to start at a pole $y_1 \in \partial A \cap I_1$ of F and not meet I_0. If this curve is labelled K_1 then the set $H^+ \setminus K_1$ is the union of two disjoint domains U_1, U_2, with $\beta \in U_1$ and with infinitely many zeros of f lying on the boundary of U_2.

Lemma 7.3. *Choose a simple path $\Gamma = K_2$ which starts at β, tends to infinity and lies in U_1, such that K_2 does not meet K_0 except at β itself. If $y_3 \in \partial A \cap$*

∂U_2 is a pole of F with $y_3 \neq y_1$, then there exists a path $K_3 \subseteq A \cup \{y_3\}$, starting at y_3 and tending to infinity, which is mapped injectively by F onto $K_2 \cup \{\infty\} \setminus \{\beta\}$. Moreover, K_3 lies in $U_2 \cup \{y_3\}$.

Proof. Here K_2 can be constructed using the fact that K_1 does not meet the line segment I_0: just follow I_0 vertically downwards from β and then go to infinity within U_1, keeping sufficiently close to the real axis to avoid K_1.

The existence of K_3 follows from analytic continuation along K_2 of the branch of F^{-1} which maps ∞ to y_3. The path K_3 lies in $A \cup \{y_3\}$ and meets U_2, but cannot meet K_1 because K_3 and K_1 start at different poles of F and

$$F(K_3 \cap K_1) \subseteq F(K_3) \cap F(K_1) = (K_2 \cap K_0) \cup \{\infty\} \setminus \{\beta\} = \{\infty\}.$$

It follows that $K_3 \subseteq U_2 \cup \{y_3\}$. □

Take a zero $x_1 \in \partial U_2$ of f with $|x_1|$ large, which is possible by the choice of U_2. Then x_1 is a simple zero of f and a multiple point of F, and x^* lies on $\partial W^+ = \partial A$. Moreover, $F(z)$ describes $\mathbb{R} \cup \{\infty\}$ monotonely and "infinite to one" as the curve ∂A is followed in each direction: let y_3, y_4 be the first poles of F which are thereby reached. Since $|x_1|$ is large it may be assumed that $|y_3|$ and $|y_4|$ are large and $y_3 < x_1 < y_4$, and that $y_3, y_4 \in \partial U_2$.

Let $\Omega = H^+ \setminus K_2$. Then, since $F(x_1) = x_1 \in \mathbb{R}$, the point x_1 lies on the boundary of a component $C \subseteq A$ of $F^{-1}(\Omega)$, and F maps C univalently onto Ω, by analytic continuation of F^{-1} and the fact that $\beta \in K_2$. Furthermore, the parts of ∂A described in reaching y_3, y_4 from x_1 belong also to ∂C. In particular, y_3, y_4 both lie in $\partial U_2 \cap \partial C \cap \partial A$.

Lemma 7.3 gives paths K_3, K_4 with $K_j \subseteq (A \cap U_2) \cup \{y_j\}$, each starting at y_j and tending to infinity, mapped by F onto $K_2 \cup \{\infty\} \setminus \{\beta\}$. Since K_2 lies in U_1, and no path in C can cross K_3 or K_4, it follows that C lies in U_2 and thus in Ω.

Now start at y_3, which is a simple pole of F in $\partial A \cap \mathbb{R}$, and let z follow ∂A in each direction until the first encounter with a pole of F or a zero or a pole of f: then z does not leave $\mathbb{R}$ as this is done, since all critical points of F are zeros of f. Neither of the points so reached can be a pole of F, by Lemmas 4.2 and 4.3, since if zeros of L are not separated by by a zero or pole of f then the values of L' at these two zeros must differ in sign. Thus Lemma 4.6 implies that that both these points must be poles of f, and one of them, y_3' say, lies on the part of the curve ∂A between y_3 and x_1, which also lies in ∂C. Doing the same for y_4 shows that ∂C contains at least two distinct poles y_3', y_4' of f. But this conclusion is incompatible with the choice $D = \Omega$ in Lemma 4.5, giving a contradiction and hence completing the proof of Proposition 7.1.

8. Completion of the proof in the transcendental case

Lemma 8.1. *All components of $W^\pm$ are mapped univalently onto H^+ by $\pm F$, and if x_1 is a zero of f'/f with $|x_1|$ large, then $L'(x_1) < 0$ and x_1 does not lie on the boundary of a component of W^+.*

Proof. This follows from Lemmas 4.3 and 4.7, in conjunction with Proposition 7.1. □

Proposition 8.2. *There do not exist sequences x_j of zeros of f'/f and y_j of poles of f both tending to $+\infty$.*

Proof. Assume the contrary: then it is possible to choose a large positive X_0 and enumerate all the zeros x_j of f'/f and distinct poles y_j of f in $(X_0, +\infty)$ as

$$X_0 < x_0 < x_1 < x_2 < \cdots, \quad X_0 < y_0 < y_1 < y_2 < \cdots.$$

Let A_j be the component of W^+ with $y_j \in \partial A_j$. By Lemmas 4.5 and 8.1, it may be assumed that the A_j are distinct and their boundaries contain no zeros of f'/f. It then follows that each A_j contains a path tending to infinity on which $F(z) \to \infty$. Hence at most finitely many of these A_j also contain a path tending to infinity on which $F(z)$ tends to a finite real asymptotic value, because otherwise F^{-1} would have at least two direct singularities over ∞ lying in H^+, contradicting Lemma 3.3. Thus it may be assumed further, for each j, that ∞ is the one and only asymptotic value approached by F along a path tending to infinity in A_j.

Similar reasoning shows that it may now also be assumed that each x_j lies on the boundary of a component B_j of W^-, these B_j being distinct and mapped univalently onto H^+ by $-F$. Hence no B_j contains a path tending to infinity on which $F(z) \to \infty$, and again Lemma 3.3 implies that at most finitely many B_j contain a path on which $F(z)$ tends to a finite real asymptotic value. Thus each of these B_j may be assumed to be bounded.

After re-labelling if necessary, poles y_1, y_2 of f and a zero x_m of f'/f may be chosen with y_1 large and positive and $y_1 < x_m < y_2$. Then x_m lies on the boundary of a bounded component B_m of W^-, and hence on the boundary of a maximal finite chain D of bounded components of W^- as in Definition 4.10 and its notation, and Lemma 4.11 applies to D, with $[a_1, b_N] \subseteq \partial D \cap (y_1, y_2)$. Let A be the component of W^+ given by Lemma 4.11: then F maps A univalently onto H^+.

Suppose first that a_1, b_N are both simple zeros of F'. Then there exist $c_1 < a_1$ and $d_N > b_N$ with $[c_1, a_1] \cup [b_N, d_N] \subseteq \partial A$. Continue along $\mathbb{R}$ leftwards from a_1 and rightwards from b_N until the first encounter with a zero or pole of f: this is possible since $y_1 < a_1 < b_N < y_2$. But then conclusion (A) of Lemma 4.6 must hold, which gives at least two poles of f on ∂A, contradicting Lemma 4.5.

Hence Lemma 4.11 forces some $x^* \in \{a_1, b_N\}$ to be a zero of F' of multiplicity 2, lying on the boundary of an unbounded component B of W^-. Starting from x^*, follow the real axis, in the direction away from $[a_1, b_N]$, until the first encounter with a zero or pole of f, again possible since $y_1 < a_1 < b_N < y_2$. Then the point so reached lies on ∂B and must be a zero of f, by Lemma 4.3. But this gives a zero of $L = f'/f$ and hence a pole of F lying on ∂B, so that B is one of the B_j and hence bounded, a contradiction. □

Lemma 8.3. *It may be assumed that:*

(I) *f has finitely many positive poles but infinitely many negative zeros;*

(II) *f'/f has infinitely many positive zeros but finitely many negative zeros;*

(III) *there exists a large $X_1 \in (0, +\infty)$ such that the zeros of f and f'/f in $(X_1, +\infty)$ are simple and interlaced in the sense that if $X_1 < a < b$ and a, b are zeros of f, then f'/f has a zero in (a, b), while if if $X_1 < a < b$ and a, b are zeros of f'/f then f has a zero in (a, b).*

Proof. It can certainly be assumed, by (6.5) and an application of Proposition 8.2 to $f(z)$ and $f(-z)$, that f has finitely many positive poles but infinitely many negative poles, while f'/f has finitely many negative zeros and infinitely many positive zeros. It then follows that f must also have infinitely many negative zeros, which proves (I) and (II). Together (I) and (II) imply (III), on combination with Lemmas 4.2, 4.3, 6.6, and 8.1 and the fact that if $X_1 > 0$ is large and two zeros of f'/f in $(X_1, +\infty)$ are not separated by a pole or zero of f or f'/f then one of them has $(f'/f)' > 0$. ☐

It is now possible to write

$$\frac{f'}{f} = P\psi, \tag{8.1}$$

in which: ψ is formed as Section 3.1 using zeros $0 < u_1 < u_2 < \cdots$ of f and zeros $v_j \in (u_j, u_{j+1})$ of f'/f, and ψ satisfies $\psi(H^+) \subseteq H^+$; the function P is real meromorphic, with finitely many zeros in $\mathbb{C}$, and finitely many positive poles, but infinitely many negative poles.

Lemma 8.4. *The function P has order of growth at most 1 and satisfies*

$$\lim_{\substack{x \to +\infty \\ x \in \mathbb{R}}} \frac{\log |P(x)|}{\log x} = -\infty.$$

Proof. The first assertion follows from (6.1), (8.1) and Lemma 3.4, applied to $1/P$. Next, since P is transcendental, applying Lemma 3.1 to $1/P$ leads to

$$\lim_{\substack{y \to +\infty \\ y \in \mathbb{R}}} \frac{\log |P(iy)|}{\log y} = -\infty, \qquad \lim_{\substack{y \to +\infty \\ y \in \mathbb{R}}} yL(iy) = 0. \tag{8.2}$$

Now let δ be small and positive. Then (3.1), (8.1), and Lemma 4.1 imply that $zL(z) \to 0$ and $P(z) \to 0$ as $z \to \infty$ with $\delta < |\arg z| < \pi - \delta$. Because P has finite order and finitely many poles on $\mathbb{R}^+$, it follows from the Phragmén–Lindelöf principle that $P(z) \to 0$ as $z \to \infty$ with $|\arg z| \leq \delta$. If $N_1 \in \mathbb{N}$ is large then (8.2) and the Phragmén-Lindelöf principle now show that $z^{N_1} P(z)$ tends to 0 on the sector $|\arg z| \leq \pi/2$, which completes the proof. ☐

Since ψ maps H^+ into itself there exists a series representation [26]

$$\psi(z) = az + b + \sum_{k=1}^{\infty} A_k \left(\frac{1}{u_k - z} - \frac{1}{u_k} \right),$$

in which the u_k are the poles of ψ, all of which are positive and zeros of f, while a, b, A_k are real and $A_k > 0$, $\sum_{k=1}^{\infty} A_k u_k^{-2} < \infty$. On combination with Lemma 8.4 this implies that if k is large then the residue of $f'/f = P\psi$ at u_k is $-P(u_k)A_k = o(u_k^{-2})o(u_k^2) = o(1)$, an obvious contradiction. This completes the proof of Theorem 1.2.

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Спеціальний випадок гіпотези Геллерштайна, Шена і Вільямсона

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У роботі доведено спеціальний випадок гіпотези Геллерштайна, Шена і Вільямсона щодо недійсних нулів похідних дійсних мероморфних функцій.

Ключові слова: мероморфна функція, недійсні нулі