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On Eigenvalue Multiplicities of Self-Adjoint Regular Sturm–Liouville Operators

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Dedicated, with great admiration, to the memory of Iosif Vladimirovich Ostrovskii (1934–2020)

We provide a complete discussion of multiplicities of eigenvalues of all self-adjoint regular Sturm–Liouville problems on compact intervals $[a,b] \subset \mathbb{R}$.

Key words: Sturm–Liouville operators, eigenvalue multiplicities

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1. Introduction

The principal aim of this note on three-coefficient regular Sturm–Liouville operators is a clarification of the following seemingly elementary-sounding question: *"Under which circumstances are eigenvalues simple or twice degenerate?"*

To set the stage we recall that three-coefficient regular Sturm–Liouville differential expressions are of the form

$$\tau = \frac{1}{r(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right] \quad \text{for a.e. } x \in [a,b] \subset \mathbb{R}, \tag{1.1}$$

where the coefficients p, q, r satisfy the integrability conditions listed in Hypothesis 2.1. All self-adjoint $L^2((a,b);r\,dx)$-realizations associated with τ then require separated or coupled boundary conditions involving the two interval endpoints, a and b.

Explicitly (see Theorems 2.4 and 2.5), the separated boundary conditions for elements g in the domain of the underlying self-adjoint operator $T_{\alpha,\beta}$ in $L^2((a,b);r\,dx)$ are of the form

$$\begin{aligned} &\sin(\alpha)g^{[1]}(a) + \cos(\alpha)g(a) = 0, \\ &\sin(\beta)g^{[1]}(b) + \cos(\beta)g(b) = 0, \quad \alpha,\beta \in [0,\pi), \end{aligned} \tag{1.2}$$

(see (2.2) for the definition of $g^{[1]}$), and all corresponding self-adjoint operators $T_{\varphi,R}$ in $L^2((a,b);r\,dx)$ with coupled boundary conditions are of the type

$$\begin{pmatrix} g(b) \\ g^{[1]}(b) \end{pmatrix} = e^{i\varphi} R \begin{pmatrix} g(a) \\ g^{[1]}(a) \end{pmatrix}, \tag{1.3}$$

where $\varphi \in [0, \pi]$, and R is a 2×2 matrix with real-valued entries and $\det_{\mathbb{C}^2}(R) = 1$, that is, $R \in SL(2, \mathbb{R})$.

Floquet theory in this second-order context, that is, the theory of second-order periodic differential equations with periodic coefficients p, q, r on $\mathbb{R}$ with period $\omega > 0$ is then naturally associated with the special case $\varphi \in [0, \pi]$, $R = I_2$, $a = 0$, and $b = \omega$ in (1.3) (see, e.g., [6, Sect. 7.5]).

Next, introducing

$$Y_0(z,x,a) = \begin{pmatrix} \theta_0(z,x,a) & \phi_0(z,x,a) \\ \theta_0^{[1]}(z,x,a) & \phi_0^{[1]}(z,x,a) \end{pmatrix}, \quad z \in \mathbb{C}, \ x \in [a,b], \tag{1.4}$$

with $\theta_0(z, \cdot\,, a)$, $\phi_0(z, \cdot\,, a)$ a fundamental system of solutions of $\tau y(z, \cdot\,) = zy(z, \cdot\,)$, normalized by

$$Y_0(z,a,a) = I_2, \ \text{ and satisfying } \det_{\mathbb{C}^2}(Y_0(z,x,a)) = 1, \ z \in \mathbb{C}, \ x \in [a,b], \tag{1.5}$$

the analog of the well-known Floquet discriminant can be defined by

$$\begin{aligned} \Delta_R(z) &= \operatorname{tr}_{\mathbb{C}^2}\big(R^{-1} Y_0(\lambda,b,a)\big)/2 \\ &= \big[R_{1,1}\phi_0^{[1]}(z,b,a) + R_{2,2}\theta_0(z,b,a) - R_{2,1}\phi_0(z,b,a) - R_{1,2}\theta_0^{[1]}(z,b,a)\big]/2, \\ &\qquad R = (R_{j,k})_{1\leqslant j,k\leqslant 2} \in SL(2,\mathbb{R}), \ z \in \mathbb{C}. \end{aligned} \tag{1.6}$$

The principal results of this note then read as follows (see Theorems 3.1 and 3.2):

(i) For any $\alpha, \beta \in [0, \pi)$ and any $\varphi \in (0, \pi)$, $R \in SL(2, \mathbb{R})$, the eigenvalues of $T_{\alpha,\beta}$ and $T_{\varphi,R}$ are simple. (In particular, all eigenvalues in the case of separated boundary conditions are simple.)

(ii) For $\lambda \in \mathbb{R}$ to be a twice degenerate eigenvalue of $T_{\varphi,R}$, the latter must be of the form $T_{0,R}$ or $T_{\pi,R}$ for some $R \in SL(2, \mathbb{R})$.

(iii) The following items (a)–(c) are equivalent:
   (a) $\lambda \in \mathbb{R}$ is a twice degenerate eigenvalue of $T_{0,R}$.
   (b) $\Delta_R(\lambda) = 1$ and $\dot{\Delta}_R(\lambda) = 0$; in this case, $\ddot{\Delta}_R(\lambda) < 0$.
   (c) $Y_0(\lambda, b, a) = R$.

(iv) The following items (a)–(c) are equivalent:
   (a) $\lambda \in \mathbb{R}$ is a twice degenerate eigenvalue of $T_{\pi,R}$.
   (b) $\Delta_R(\lambda) = -1$ and $\dot{\Delta}_R(\lambda) = 0$; in this case, $\ddot{\Delta}_R(\lambda) > 0$.
   (c) $Y_0(\lambda, b, a) = -R$.

In Section 2 we recall the necessary background on regular three-coefficient Sturm–Liouville operators. The above results (i)–(iv) on multiplicities of eigenvalues are then contained in our principal Section 3.

Notation. The inner product in a separable (complex) Hilbert space $\mathcal{H}$ is denoted by $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$, and it is assumed to be linear with respect to the second argument. If T is a linear operator mapping (a subspace of) a Hilbert space into another, then $\operatorname{dom}(T)$ denotes the domain of T. The resolvent set and the spectrum of a closed linear operator in $\mathcal{H}$ will be denoted by $\rho(\,\cdot\,)$ and $\sigma(\,\cdot\,)$, respectively. The Banach space of bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. For $p \in [1,\infty)$, the corresponding ℓ^p-based trace ideals will be denoted by $\mathcal{B}_p(\mathcal{H})$ with norms abbreviated by $\|\cdot\|_{\mathcal{B}_p(\mathcal{H})}$. Finally, we abbreviate $\mathbb{C}_{\pm} = \{z \in \mathbb{C} \,|\, \operatorname{Im}(z) \gtrless 0\}$.

2. Background material on regular Sturm–Liouville operators

The following summary of background information on regular three-coefficient Sturm–Liouville operators is taken from [6, Ch. 4].

Throughout this paper, we suppose that τ is regular, that is, we assume the following hypotheses:

Hypothesis 2.1. *Let $[a,b] \subset \mathbb{R}$ be a compact interval and suppose that p, q, r are (Lebesgue) measurable functions on (a,b) such that the following items (i)–(iii) hold:*

(i) *$r > 0$ a.e. on (a,b), $r \in L^1((a,b);dx)$.*
(ii) *$p > 0$ a.e. on (a,b), $1/p \in L^1((a,b);dx)$.*
(iii) *q is real-valued a.e. on (a,b), $q \in L^1((a,b);dx)$.*

To describe minimal and maximal $L^2((a,b);r\,dx)$-realizations associated with the regular three-coefficient differential expression τ on the compact interval $[a,b] \subset \mathbb{R}$, where

$$\tau = \frac{1}{r(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right] \quad \text{for a.e. } x \in [a,b] \subset \mathbb{R}, \tag{2.1}$$

we recall our notation for the first quasi-derivative,

$$g^{[1]}(x) = p(x)g'(x) \ \text{ for a.e. } x \in (a,b), \quad g \in AC_{loc}((a,b)). \tag{2.2}$$

Definition 2.2. Assume Hypothesis 2.1.

(i) The differential expression τ of the form (2.1) is called *regular on* $[a,b]$.
(ii) The *maximal operator* $T_{\max}$ in $L^2((a,b);r\,dx)$ associated with τ is defined by

$$\begin{aligned} &T_{\max} f = \tau f, \\ &f \in \operatorname{dom}(T_{\max}) = \big\{g \in L^2((a,b);r\,dx) \,\big|\, \\ &\qquad\qquad g, g^{[1]} \in AC([a,b]);\ \tau g \in L^2((a,b);r\,dx)\big\}. \end{aligned} \tag{2.3}$$

The *minimal operator* $T_{\min}$ in $L^2((a,b);r\,dx)$ associated with τ is defined by

$$T_{\min} f = \tau f,$$

$$
\begin{aligned}
f \in \operatorname{dom}(T_{\min}) = \big\{ g \in L^2((a,b); r\,dx) \,\big|\, g, g^{[1]} \in AC([a,b]);\\
g(a) = g^{[1]}(a) = g(b) = g^{[1]}(b) = 0;\ \tau g \in L^2((a,b); r\,dx) \big\}.
\end{aligned} \tag{2.4}
$$

One recalls the relations,

$$
T_{\min}^* = T_{\max}, \quad T_{\min} = T_{\max}^*. \tag{2.5}
$$

In addition, $T_{\min}$ is symmetric, but $T_{\max}$ is not.

Lemma 2.3. *Assume Hypothesis 2.1 so that τ is regular on $[a,b]$. Fix $z_\pm \in \mathbb{C}_\pm$, then the deficiency indices of $T_{\min}$ are given by*

$$
\begin{aligned}
n_\pm(T_{\min}) &= \dim(\ker(T_{\min}^* - z_\pm I)) = \dim(\ker(T_{\max} - z_\pm I)) \\
&= \dim\big([\operatorname{ran}(T_{\min} + z_\pm I)]^\perp\big) = 2.
\end{aligned} \tag{2.6}
$$

Thus, $T_{\min}$ has a real four parameter family of self-adjoint extensions. In addition, $T_{\min}$ is bounded from below, that is, there exists $C \in \mathbb{R}$ such that

$$
T_{\min} \geqslant CI, \tag{2.7}
$$

that is,

$$
(f, T_{\min} f)_{L^2((a,b);r\,dx)} \geqslant C\|f\|^2_{L^2((a,b);r\,dx)}, \quad f \in \operatorname{dom}(T_{\min}). \tag{2.8}
$$

Consequently, all self-adjoint extensions of $T_{\min}$ in $L^2((a,b); r\,dx)$ are bounded from below.

All self-adjoint extensions of $T_{\min}$ in $L^2((a,b); r\,dx)$ can be characterized as follows:

Theorem 2.4 (See, e.g., [6, Theorem 4.3.6])**.** *Assume Hypothesis 2.1 so that τ is regular on $[a,b]$. Given $A, B \in \mathbb{C}^{2\times 2}$, one introduces the operator $T_{A,B}$ in $L^2((a,b); r\,dx)$ via*

$$
\begin{aligned}
&T_{A,B} f = \tau f,\\
&f \in \operatorname{dom}(T_{A,B}) = \left\{ g \in \operatorname{dom}(T_{\max}) \,\middle|\, A \begin{pmatrix} g(a) \\ g^{[1]}(a) \end{pmatrix} = B \begin{pmatrix} g(b) \\ g^{[1]}(b) \end{pmatrix} \right\}.
\end{aligned} \tag{2.9}
$$

Then the following items (i)–(iii) hold:

(i) *$T_{A,B}$ is a self-adjoint extension of $T_{\min}$ if and only if A and B satisfy*

$$
\operatorname{rank}(A \ \ B) = 2, \quad AJA^* = BJB^*, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.10}
$$

(ii) *For A, B satisfying (2.10) and $z \in \rho(T_{A,B})$, the resolvent of $T_{A,B}$ is of the form*

$$
\big((T_{A,B} - zI)^{-1} f\big)(x) = \int_a^b r(x')dx'\, G_{A,B}(z,x,x') f(x'), \quad f \in L^2((a,b); r\,dx),
$$

where the Green's function $G_{A,B}(z,\,\cdot\,,\,\cdot\,)$ *is given by*

$$G_{A,B}(z,x,x') = \begin{cases} \sum_{j,k=1}^{2} m_{j,k}^{-}(z)u_j(z,x)u_k(z,x'), & a \leqslant x \leqslant x' \leqslant b, \\ \sum_{j,k=1}^{2} m_{j,k}^{+}(z)u_j(z,x)u_k(z,x'), & a \leqslant x' \leqslant x \leqslant b, \end{cases} \tag{2.11}$$

with $\{u_1(z,\,\cdot\,), u_2(z,\,\cdot\,)\}$ *a fundamental system of solutions of* $\tau u = zu$ *and* $m_{j,k}^{\pm}(z)$, $1 \leqslant j,k \leqslant 2$, *appropriate constants.*

(iii) *For* A, B *satisfying* (2.10) *and* $z \in \rho(T_{A,B})$,

$$(T_{A,B} - zI)^{-1} \in \mathcal{B}_1\big(L^2((a,b); r\,dx)\big), \tag{2.12}$$

and hence $T_{A,B}$ *has purely discrete spectrum* (*i.e.*, $\sigma_{ess}(T_{A,B}) = \varnothing$) *with eigenvalues of multiplicity at most two. Moreover, if* $\sigma(T_{A,B}) = \{\lambda_{A,B,j}\}_{j\in\mathbb{N}}$, *then*

$$\sum_{j\in\mathbb{N}} [|\lambda_{A,B,j}| + 1]^{-1} < \infty. \tag{2.13}$$

Turning to the important special cases of separated and coupled boundary conditions which together describe all self-adjoint extensions of $T_{\min}$, one obtains the following facts:

Theorem 2.5 (See, e.g., [6, Theorem 4.3.9])**.** *Assume Hypothesis 2.1 so that* τ *is regular on* $[a,b]$. *Then the following items (i)–(iii) hold*:

(i) *All self-adjoint extensions* $T_{\alpha,\beta}$ *of* $T_{\min}$ *in* $L^2((a,b); r\,dx)$ *with separated boundary conditions are of the form*

$$\begin{aligned} &T_{\alpha,\beta} f = \tau f, \quad \alpha,\beta \in [0,\pi), \\ &f \in \operatorname{dom}(T_{\alpha,\beta}) = \big\{g \in \operatorname{dom}(T_{\max}) \,\big|\, \sin(\alpha) g^{[1]}(a) + \cos(\alpha) g(a) = 0; \\ &\qquad\qquad\qquad\qquad\qquad\qquad \sin(\beta) g^{[1]}(b) + \cos(\beta) g(b) = 0\big\}. \end{aligned} \tag{2.14}$$

In this case one can choose

$$A = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ \cos(\beta) & \sin(\beta) \end{pmatrix} \tag{2.15}$$

in connection with (2.9), (2.10).

Special cases: $\alpha = 0$, $g(a) = 0$ *is called the Dirichlet boundary condition at* a; $\alpha = \frac{\pi}{2}$, $g^{[1]}(a) = 0$ *is called the Neumann boundary condition at* a (*analogous facts apply to the endpoint* b).

(ii) *All self-adjoint extensions* $T_{\varphi,R}$ *of* $T_{\min}$ *in* $L^2((a,b); r\,dx)$ *with coupled boundary conditions are of the type*

$$\begin{aligned} &T_{\varphi,R} f = \tau f, \\ &f \in \operatorname{dom}(T_{\varphi,R}) = \left\{ g \in \operatorname{dom}(T_{\max}) \,\middle|\, \begin{pmatrix} g(b) \\ g^{[1]}(b) \end{pmatrix} = e^{i\varphi} R \begin{pmatrix} g(a) \\ g^{[1]}(a) \end{pmatrix} \right\}, \end{aligned} \tag{2.16}$$

where $\varphi \in [0, \pi]$, *and* R *is a* 2×2 *matrix with real-valued entries and* $\det_{\mathbb{C}^2}(R) = 1$, *that is,* $R \in SL(2, \mathbb{R})$. *In this case one can choose*

$$A = e^{i\varphi} R, \quad B = I_2 \tag{2.17}$$

in connection with (2.9), (2.10).

Special cases: $\varphi = 0$, $R = I_2$, $g(b) = g(a)$, $g^{[1]}(b) = g^{[1]}(a)$ *are called* periodic *boundary conditions; similarly,* $\varphi = 0$, $R = -I_2$ (*equivalently,* $\varphi = \pi$, $R = I_2$), $g(b) = -g(a)$, $g^{[1]}(b) = -g^{[1]}(a)$ *are called* antiperiodic *boundary conditions.*

(iii) *Every self-adjoint extension of* $T_{\min}$ *is either of type* (i) (*i.e., separated*) *or of type* (ii) (*i.e., coupled*).

Remark 2.6. In connection with coupled boundary conditions in Theorem 2.5(ii) one notes that the (φ, R)-pairs, $(0, -R)$ and (π, R), $R \in SL(2, \mathbb{R})$, describe the same self-adjoint extension, hence one could use the restriction $\varphi \in [0, \pi)$. However, in certain circumstances, such as in connection with Floquet theory, it is advantageous to fix R and then vary $\varphi \in [0, \pi]$.

In the following we describe the characteristic equations determining the eigenvalues of self-adjoint regular Sturm–Liouville problems. For this purpose we assume that $\phi_0(z, \cdot\,, a)$ and $\theta_0(z, \cdot\,, a)$ constitute a fundamental system of solutions of $\tau u = zu$, which, for fixed $x \in [a, b]$, are entire with respect to $z \in \mathbb{C}$, and satisfy the following initial conditions at $x = a$,

$$\begin{aligned} &\phi_0(z, a, a) = \theta_0^{[1]}(z, a, a) = 0, \\ &\theta_0(z, a, a) = \phi_0^{[1]}(z, a, a) = 1, \quad z \in \mathbb{C}. \end{aligned} \tag{2.18}$$

Next, we introduce

$$\begin{aligned} F_{\alpha,\beta}(z) = \cos(\alpha)\big[\sin(\beta)\phi_0^{[1]}(z, b, a) + \cos(\beta)\phi_0(z, b, a)\big] \\ - \sin(\alpha)\big[\sin(\beta)\theta_0^{[1]}(z, b, a) + \cos(\beta)\theta_0(z, b, a)\big], \quad \alpha, \beta \in [0, \pi), \; z \in \mathbb{C}, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} F_{\varphi,R}(z) = -e^{i\varphi}\big[R_{1,1}\phi_0^{[1]}(z, b, a) + R_{2,2}\theta_0(z, b, a) - R_{2,1}\phi_0(z, b, a) \\ - R_{1,2}\theta_0^{[1]}(z, b, a) - 2\cos(\varphi)\big], \; \varphi \in [0, \pi], \; R \in SL(2, \mathbb{R}), \; z \in \mathbb{C}, \end{aligned} \tag{2.20}$$

and recall the following connection between $T_{\alpha,\beta}$, $T_{\varphi,R}$ and appropriate Fredholm determinants and traces of resolvents.

Theorem 2.7 (See, e.g., [6, Theorem 12.3.2])**.** *Assume Hypothesis* 2.1 *and denote by* $T_{\alpha,\beta}$ *and* $T_{\varphi,R}$ *the self-adjoint extensions of* $T_{\min}$ *as described in cases* (i) *and* (ii) *of Theorem* 2.5, *respectively.*

(i) *Suppose* $z_0 \in \rho(T_{\alpha,\beta})$, $\alpha, \beta \in [0, \pi)$, *then*

$$\begin{aligned} \det_{L^2((a,b);rdx)}\big(I - (z - z_0)(T_{\alpha,\beta} - z_0 I)^{-1}\big) \\ = F_{\alpha,\beta}(z)/F_{\alpha,\beta}(z_0), \quad z \in \mathbb{C}. \end{aligned} \tag{2.21}$$

In addition,

$$\operatorname{tr}_{L^2((a,b);rdx)}\big((T_{\alpha,\beta} - zI)^{-1}\big) = -(d/dz)\ln(F_{\alpha,\beta}(z)), \quad z \in \rho(T_{\alpha,\beta}). \tag{2.22}$$

(ii) *Suppose* $z_0 \in \rho(T_{\varphi,R})$, $\varphi \in [0, \pi]$, $R \in SL(2, \mathbb{R})$, *then*

$$\begin{aligned} \det_{L^2((a,b);rdx)}\big(I - (z - z_0)(T_{\varphi,R} - z_0 I)^{-1}\big) \\ = F_{\varphi,R}(z)/F_{\varphi,R}(z_0), \quad z \in \mathbb{C}. \end{aligned} \tag{2.23}$$

In addition,

$$\operatorname{tr}_{L^2((a,b);rdx)}\big((T_{\varphi,R} - zI)^{-1}\big) = -(d/dz)\ln(F_{\varphi,R}(z)), \quad z \in \rho(T_{\varphi,R}). \tag{2.24}$$

Thus, one confirms that for all $\alpha, \beta \in [0, \pi)$,

$$\lambda \in \sigma(T_{\alpha,\beta}) \text{ if and only if } F_{\alpha,\beta}(\lambda) = 0. \tag{2.25}$$

Moreover,

$$\begin{aligned} &\text{if (2.25) holds, } \lambda \text{ is a simple (necessarily discrete) eigenvalue of } T_{\alpha,\beta} \\ &\quad \text{and } \dot{F}_{\alpha,\beta}(\lambda) \neq 0. \end{aligned} \tag{2.26}$$

Indeed, by Theorem 2.7(i), $F_{\alpha,\beta}(z)/F_{\alpha,\beta}(z_0)$ is the Fredholm determinant associated with the trace class operator $(z - z_0)(T_{\alpha,\beta} - z_0 I)^{-1}$, $z \in \mathbb{C}$, $z_0 \in \rho(T_{\alpha,\beta})$, and hence the multiplicity of every zero of $F_{\alpha,\beta}(\,\cdot\,)$ coincides with the multiplicity of the underlying eigenvalue of $T_{\alpha,\beta}$ and thus is necessarily simple. The latter claim can be shown as follows: The assumption of two linearly independent eigenfunctions y_k, $k = 1, 2$, of $T_{\alpha,\beta} y = \lambda y$ for some $\lambda \in \sigma(T_{\alpha,\beta})$, and the constancy of the Wronskian $W(y_1, y_2)(x)$, $x \in [a, b]$, permits one to compute the Wronskian at $x = a$, say, and hence yields the contradiction

$$\begin{aligned} W(y_1, y_2)(a) &= f_1(a) f_2^{[1]}(a) - f_1^{[1]}(a) f_2(a) \\ &= \begin{cases} f_1(a)[-\cot(\alpha)] f_2(a) + \cot(\alpha) f_1(a) f_2(a), & \alpha \in (0, \pi), \\ 0 \cdot f_2^{[1]}(a) - f_1^{[1]}(a) \cdot 0, & \alpha = 0 \end{cases} \\ &= 0. \end{aligned}$$

Thus, the eigenvalue $\lambda \in \sigma(T_{\alpha,\beta})$, $\alpha, \beta \in [0, \pi)$, is necessarily simple. For an alternative and a bit shorter argument for this fact, see the proof of Theorem 3.1(iv).

The eigenvalue equation $F_{\alpha,\beta}(\,\cdot\,) = 0$ is also called the *characteristic equation* associated with separated self-adjoint boundary conditions.

Next, we turn to the analogue of (2.25), (2.26) for the case of coupled boundary conditions, that is, with $T_{\alpha,\beta}$, $\alpha,\beta\in[0,\pi)$, replaced by $T_{\varphi,R}$, $\varphi\in[0,\pi]$, $R\in SL(2,\mathbb{R})$. One then confirms that for all $\varphi\in[0,\pi]$, $R\in SL(2,\mathbb{R})$,

$$\lambda\in\sigma(T_{\varphi,R}) \text{ if and only if } F_{\varphi,R}(\lambda)=0, \tag{2.27}$$

and all (necessarily discrete) eigenvalues of $T_{\varphi,R}$ have multiplicity equal to one or two (once again, since $T_{\varphi,R}$ is self-adjoint and τ has order two). Moreover,

$$\text{if (2.27) holds and } \varphi\in(0,\pi), \text{ then } \lambda \text{ is a simple eigenvalue of } T_{\varphi,R} \\ \text{and } \dot{F}_{\varphi,R}(\lambda)\neq 0. \tag{2.28}$$

Indeed, by Theorem 2.7(ii), $F_{\varphi,R}(z)/F_{\varphi,R}(z_0)$ is the Fredholm determinant associated with the trace class operator $(z-z_0)(T_{\varphi,R}-z_0I)^{-1}$, $z\in\mathbb{C}$, $z_0\in\rho(T_{\varphi,R})$, and hence the multiplicity of every zero of $F_{\varphi,R}(\,\cdot\,)$ coincides with the multiplicity of the underlying eigenvalue of $T_{\varphi,R}$ and thus is necessarily simple for $\varphi\in(0,\pi)$ as the following elementary Wronskian argument, assuming the existence of two linearly independent eigenfunctions y_k, $k=1,2$, of $T_{\varphi,R}y=\lambda y$ for some $\lambda\in\sigma(T_{\varphi,R})$, shows:

$$\begin{aligned}
W(y_1,y_2)(b) &= \det_{\mathbb{C}^2}\left(\left(e^{i\varphi}R\begin{pmatrix} y_1(a)\\ y_1^{[1]}(a)\end{pmatrix}\ e^{i\varphi}R\begin{pmatrix} y_2(a)\\ y_2^{[1]}(a)\end{pmatrix}\right)\right)\\
&= e^{2i\varphi}\det_{\mathbb{C}^2}\left(R\begin{pmatrix} y_1(a) & y_2(a)\\ y_1^{[1]}(a) & y_2^{[1]}(a)\end{pmatrix}\right)\\
&= e^{2i\varphi}\det_{\mathbb{C}^2}(R)\det_{\mathbb{C}^2}\left(\begin{pmatrix} y_1(a) & y_2(a)\\ y_1^{[1]}(a) & y_2^{[1]}(a)\end{pmatrix}\right)\\
&= e^{2i\varphi}W(y_1,y_2)(a) = e^{2i\varphi}W(y_1,y_2)(b),
\end{aligned}\tag{2.29}$$

employing $\det_{\mathbb{C}^2}(R)=1$ and the constancy of the Wronskian of y_1 and y_2 on $[a,b]$. Thus,

$$\big[1-e^{2i\varphi}\big]\,W(y_1,y_2)(b)=0, \tag{2.30}$$

implying the contradiction $W(y_1,y_2)(\,\cdot\,)=0$ on $[a,b]$ for $\varphi\in(0,\pi)$. Thus, the eigenvalue $\lambda\in\sigma(T_{\varphi,R})$, $\varphi\in(0,\pi)$, $R\in SL(2,\mathbb{R})$, is necessarily simple. Once again, for an alternative and a bit shorter argument for this fact, see the proof of Theorem 3.1(v).

Once more, the eigenvalue equation $F_{\varphi,R}(\,\cdot\,)=0$ represents the characteristic equation associated with coupled self-adjoint boundary conditions.

Remark 2.8. As an interesting example of coupled boundary conditions we briefly mention the case of the Krein–von Neumann extension. In this case one has

$$\varphi=0, \quad R_K=\begin{pmatrix}\theta_0(0,b,a) & \phi_0(0,b,a)\\ \theta_0^{[1]}(0,b,a) & \phi_0^{[1]}(0,b,a)\end{pmatrix}, \tag{2.31}$$

and one confirms that

$$F_{0,R_K}(z) = -2[D_K(z) - 1], \quad z \in \mathbb{C}, \tag{2.32}$$

where

$$\begin{aligned} D_K(z) = \big[&\phi_0^{[1]}(0,b,a)\theta_0(z,b,a) + \theta_0(0,b,a)\phi_0^{[1]}(z,b,a) - \phi_0(0,b,a)\theta_0^{[1]}(z,b,a) \\ &- \theta_0^{[1]}(0,b,a)\phi_0(z,b,a)\big]/2, \quad z \in \mathbb{C}. \end{aligned} \tag{2.33}$$

3. On eigenvalue multiplicities

In this, our principal section, we now take a close look at multiplicities of eigenvalues of all self-adjoint regular Sturm–Liouville operators.

We will break the discussion into two parts, Theorems 3.1 and 3.2.

To set the stage, we recall the operator $T_{A,B}$ in (2.9) and introduce the quantity

$$\delta_{A,B}(z) = \det_{\mathbb{C}^2}(A - BY_0(z,b,a)), \quad z \in \mathbb{C}. \tag{3.1}$$

Here Y_0 abbreviates a particularly normalized fundamental system matrix of solutions of

$$\begin{aligned} &\underline{y}'(z,\cdot\,) = C(z,\cdot\,)\underline{y}(z,\cdot\,), \quad \underline{y}(z,\cdot\,) = \begin{pmatrix} y(z,\cdot\,) \\ y^{[1]}(z,\cdot\,) \end{pmatrix}, \\ &C(z,\cdot\,) = \begin{pmatrix} 0 & 1/p \\ q - zr & 0 \end{pmatrix} \text{ a.e. on } (a,b),\ z \in \mathbb{C}, \end{aligned} \tag{3.2}$$

given by

$$Y_0(z,x,a) = \begin{pmatrix} \theta_0(z,x,a) & \phi_0(z,x,a) \\ \theta_0^{[1]}(z,x,a) & \phi_0^{[1]}(z,x,a) \end{pmatrix}, \quad z \in \mathbb{C},\ x \in [a,b], \tag{3.3}$$

that satisfies

$$Y_0(z,a,a) = I_2, \quad \det_{\mathbb{C}^2}(Y_0(z,x,a)) = 1, \quad z \in \mathbb{C},\ x \in [a,b]. \tag{3.4}$$

One notes that (3.2) is equivalent to

$$-(py'(z,\cdot\,))' + [q - zr]y(z,\cdot\,) = 0 \text{ a.e. on } (a,b),\ z \in \mathbb{C}. \tag{3.5}$$

More generally, for $y_k \in \mathbb{C}$, $k = 0,1$,

$$\begin{aligned} &\underline{y}'(z,\cdot\,) = C(z,\cdot\,)\underline{y}(z,\cdot\,) \text{ a.e. on } (a,b), \quad \underline{y}(z,\cdot\,) = \begin{pmatrix} y(z,\cdot\,) \\ y^{[1]}(z,\cdot\,) \end{pmatrix}, \\ &\qquad \begin{pmatrix} y(z,a) \\ y^{[1]}(z,a) \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}; \quad z \in \mathbb{C}, \end{aligned} \tag{3.6}$$

is equivalent to

$$\begin{aligned} &-(py'(z,\cdot\,))' + [q - zr]y(z,\cdot\,) = 0 \text{ a.e. on } (a,b), \\ &y^{[k]}(z,a) = y_k,\ k = 0,1;\ z \in \mathbb{C}. \end{aligned}$$

Moreover, any $\lambda \in \sigma(T_{A,B})$, which is necessarily a real and discrete eigenvalue of $T_{A,B}$ (see the first paragraph in the proof of Theorem 3.1 below), has a corresponding eigenfunction $y(\lambda, \cdot)$, that is,

$$-(py'(\lambda, \cdot))' + [q - \lambda r]y(\lambda, \cdot) = 0, \quad y(\lambda, \cdot) \in \mathrm{dom}(T_{A,B}), \tag{3.7}$$

if and only if

$$\begin{aligned} &\underline{y}'(\lambda, \cdot) = C(\lambda, \cdot)\underline{y}(\lambda, \cdot) \text{ a.e. on } (a,b), \quad \underline{y}(\lambda, \cdot) = \begin{pmatrix} y(\lambda, \cdot) \\ y^{[1]}(\lambda, \cdot) \end{pmatrix}, \\ &A\underline{y}(\lambda, a) - B\underline{y}(\lambda, b) = [A - BY_0(\lambda, b, a)]\underline{y}(\lambda, a) = 0. \end{aligned} \tag{3.8}$$

With these preliminaries out of the way, we can proceed to the following result:

Theorem 3.1. *Assume Hypothesis 2.1, and recall the definition of $T_{A,B}$ in (2.10) and (2.9). Then the following items (i)–(v) hold:*

(i) *$\lambda \in \mathbb{C}$ is an eigenvalue of $T_{A,B}$ if and only if $\delta_{A,B}(\lambda) = 0$; in this case, necessarily $\lambda \in \mathbb{R}$.*

(ii) *Geometric and algebraic multiplicities of all eigenvalues of $T_{A,B}$ coincide. The multiplicity of an eigenvalue $\lambda \in \sigma(T_{A,B})$ of $T_{A,B}$ equals the number of linearly independent solutions $\underline{y}_j$, $j = 1, 2$, of the linear algebraic system*

$$[A - BY_0(\lambda, b, a)]\underline{y} = 0, \quad \underline{y} \in \mathbb{C}^2. \tag{3.9}$$

In particular, each eigenvalue of $T_{A,B}$ has multiplicity at most two.

(iii) *$\lambda \in \sigma(T_{A,B})$ is an eigenvalue of multiplicity two if and only if $A = BY_0(\lambda, b, a)$.*

(iv) *For any $\alpha, \beta \in [0, \pi)$, all eigenvalues of $T_{\alpha,\beta}$ in (2.14), that is, all eigenvalues in the case of separated self-adjoint boundary conditions, are simple.*

(v) *For any $\varphi \in (0, \pi)$, $R \in SL(2, \mathbb{R})$, the eigenvalues of $T_{\varphi,R}$ in (2.16) are simple.*

Proof. Since $T_{A,B}$ has a trace class, and hence compact, resolvent, see relation (2.12), its spectrum is purely discrete. In addition, since $T_{A,B}$ is self-adjoint, $\sigma(T_{A,B}) \subset \mathbb{R}$, and the geometric and algebraic multiplicity of its eigenvalues coincide. Thus, it suffices to focus on the notion of "multiplicity" alone. The fact that $T_{A,B}$ is self-adjoint and of second order implies that the multiplicity of its eigenvalues is at most two, but this fact is also immediate from (3.9).

To prove item (i), suppose $\lambda \in \mathbb{R}$ is such that $\delta_{A,B}(\lambda) = 0$. Then $[A - BY_0(\lambda, b, a)]\underline{y}_0(\lambda) = 0$ for some $0 \neq \underline{y}_0(\lambda) \in \mathbb{C}^2$, and hence

$$\underline{y}'(\lambda, \cdot) = C(\lambda, \cdot)\underline{y}(\lambda, \cdot) \text{ a.e. on } (a,b), \quad \underline{y}(\lambda, a) = \underline{y}_0(\lambda), \tag{3.10}$$

has a unique solution $\underline{y}(\lambda, \cdot)$ satisfying

$$A\underline{y}(\lambda, a) - B\underline{y}(\lambda, b) = [A - BY_0(\lambda, b, a)]\underline{y}(\lambda, a) = [A - BY_0(\lambda, b, a)]\underline{y}_0(\lambda) = 0, \tag{3.11}$$

implying $\lambda \in \sigma(T_{A,B})$. Conversely, suppose that $\lambda \in \sigma(T_{A,B})$ with associated eigenvector $y(\lambda, \cdot\,)$. Then $\underline{y}(\lambda, \cdot\,) = \begin{pmatrix} y(\lambda, \cdot\,) \\ y^{[1]}(\lambda, \cdot\,) \end{pmatrix}$ satisfies

$$\underline{y}(\lambda, b) = Y_0(\lambda, b, a)\underline{y}(\lambda, a) \text{ a.e. on } (a, b), \quad [A - BY_0(\lambda, b, a)]\underline{y}(\lambda, a) = 0, \tag{3.12}$$

with $\underline{y}(\lambda, a) \neq 0$ (since $y(\lambda, \cdot\,)$ is a nontrivial eigenfunction of $T_{A,B}$). Thus, $\delta_{A,B}(\lambda) = 0$, completing the proof of item (i).

Regarding item (ii), suppose that $[A - BY_0(\lambda, b, a)]\underline{y}(\lambda) = 0$ has either precisely one solution $0 \neq \underline{y}_1(\lambda) \in \mathbb{C}^2$ (up to constant multiples), or two linearly independent solutions $0 \neq \underline{y}_j(\lambda) \in \mathbb{C}^2$, $j = 1, 2$. In either case one then solves the first-order 2×2 system

$$\begin{aligned} &\underline{y}'(\lambda, \cdot\,) = A(\lambda, \cdot\,)\underline{y}(\lambda, \cdot\,) \text{ a.e. on } (a, b), \quad \underline{y}(\lambda, \cdot\,) = \begin{pmatrix} y(\lambda, \cdot\,) \\ y^{[1]}(\lambda, \cdot\,) \end{pmatrix} \\ &A\underline{y}(\lambda, a) - B\underline{y}(\lambda, b) = [A - BY_0(\lambda, b, a)]\underline{y}(\lambda, a) = 0, \end{aligned} \tag{3.13}$$

with either

$$\underline{y}(\lambda, a) = \underline{y}_1(\lambda), \tag{3.14}$$

or

$$\underline{y}(\lambda, a) = \underline{y}_j(\lambda), \quad j = 1, 2, \tag{3.15}$$

giving rise to a solution $\underline{y}_1(\lambda, \cdot\,) = \big(y_1(\lambda, \cdot\,) \;\; y_1^{[1]}(\lambda, \cdot\,)\big)^\top$ of (3.13), or to two linearly independent solutions (due to constancy of the Wronskian) $\underline{y}_j(\lambda, \cdot\,) = \big(y_j(\lambda, \cdot\,) \;\; y_j^{[1]}(\lambda, \cdot\,)\big)^\top$, $j = 1, 2$, of (3.13). This, in turn, gives rise to a solution $y_1(\lambda, \cdot\,)$, or to two linearly independent solutions $y_j(\lambda, \cdot\,)$, $j = 1, 2$, of

$$\begin{aligned} &-(py'(\lambda, \cdot\,))' + [q - \lambda r]y(\lambda, \cdot\,) = 0 \text{ a.e. on } (a, b), \\ &A\underline{y}(\lambda, a) - B\underline{y}(\lambda, b) = [A - BY_0(\lambda, b, a)]\underline{y}(\lambda, a) = 0. \end{aligned} \tag{3.16}$$

Hence, $T_{A,B}$ has either a simple or a twice degenerate eigenvalue $\lambda \in \sigma(T_{A,B})$. Conversely, if $T_{A,B}$ has either a simple or a twice degenerate eigenvalue λ, then

$$-(py'(\lambda, \cdot\,))' + [q - \lambda r]y(\lambda, \cdot\,) = 0 \text{ a.e. on } (a, b) \tag{3.17}$$

has either one solution $y_1(\lambda, \cdot\,)$ or two linearly independent solutions $y_j(\lambda, \cdot\,)$, $j = 1, 2$, which necessarily satisfy the boundary conditions in $\operatorname{dom}(T_{A,B})$. The latter being of the form

$$A\underline{y}(\lambda, a) - B\underline{y}(\lambda, b) = [A - BY_0(\lambda, b, a)]\underline{y}(\lambda, a) = 0, \tag{3.18}$$

giving rise to either one solution $\underline{y}_1(\lambda, a)$, or necessarily two linearly independent solutions $\underline{y}_j(\lambda, a)$, $j = 1, 2$, of (3.18). This proves item (ii).

Next, assume that there are two linearly independent solutions $y_j(\lambda, \cdot\,)$, $j = 1, 2$, of (3.16). Then, as in the proof of item (ii), they give rise to two linearly independent solutions $\underline{y}_j(\lambda, \cdot\,)$, $j = 1, 2$, of (3.13). Thus, $Y(\lambda, \cdot\,) =$

$(\underline{y}_1(\lambda,\,\cdot\,)\;\; \underline{y}_2(\lambda,\,\cdot\,))$ constitutes a 2×2 fundamental system of solutions of (3.13) such that

$$[A - BY_0(\lambda, b, a)]Y(\lambda, a) = 0. \tag{3.19}$$

Since $Y(\lambda, a)$ is invertible, this yields $A - BY_0(\lambda, b, a) = 0$. Conversely, if $A = BY_0(\lambda, b, a)$, choosing any basis $\underline{y}_j$, $j = 1, 2$, in $\mathbb{C}^2$, gives rise to two linearly independent solutions $\underline{y}_j(\lambda,\,\cdot\,)$ of (3.13), satisfying $\underline{y}_j(\lambda, a) = \underline{y}_j$, $j = 1, 2$, and hence again gives rise to two solutions $y_j(\lambda,\,\cdot\,)$, $j = 1, 2$, of (3.16) (employing once more the equivalence of (3.13) and (3.16)). Thus, $T_{A,B}$ has the eigenvalue λ of multiplicity two, completing the proof of item (iii).

Arguing by contradiction, we now fix $\alpha, \beta \in [0, \pi)$ and suppose that $\lambda \in \sigma(T_{\alpha,\beta})$ has multiplicity two. In this case both $\theta_0(\lambda,\,\cdot\,, a)$ and $\phi_0(\lambda,\,\cdot\,, a)$ are eigenfunctions of $T_{\alpha,\beta}$ corresponding to the eigenvalue λ. Thus, considering $\theta_0(\lambda,\,\cdot\,, a)$, the boundary condition for $\theta_0(\lambda,\,\cdot\,, a) \in \operatorname{dom}(T_{\alpha,\beta})$ at $x = a$ yields

$$\sin(\alpha)\cdot 0 + \cos(\alpha)\cdot 1 = 0, \ \text{ implying } \ \alpha = \pi/2. \tag{3.20}$$

Similarly, regarding $\phi_0(\lambda,\,\cdot\,, a)$, the boundary condition for $\phi_0(\lambda,\,\cdot\,, a) \in \operatorname{dom}(T_{\alpha,\beta})$ at $x = a$ then yields the contradiction

$$1 = \sin(\alpha)\cdot 1 + \cos(\alpha)\cdot 0 = 0, \ \text{ since } \ \alpha = \pi/2, \tag{3.21}$$

proving item (iv).

Regarding item (v) we may choose $B = I_2$ and $A = e^{i\varphi_0}R_0$ for some $\varphi_0 \in (0, \pi)$, and $R_0 \in SL(2, \mathbb{R})$. Arguing once again by contradiction, we assume that $\lambda \in \sigma(T_{\varphi_0, R_0})$ has multiplicity two. Then an application of item (iii) implies that

$$e^{i\varphi} I_2 = R_0^{-1} Y_0(\lambda, b, a). \tag{3.22}$$

However, R_0^{-1} and $Y_0(\lambda, b, a)$ have only real entries contradicting the left-hand side of (3.22). □

As a consequence of Theorem 3.1, $T_{A,B}$ can only have twice degenerate eigenvalues in the case of coupled boundary conditions and then only if $T_{A,B}$ is of the form $T_{0,R}$ or $T_{\pi,R}$ for some $R \in SL(2, \mathbb{R})$.

In the following we will determine the precise circumstances under which a twice degenerate eigenvalue becomes possible.

To set the stage, we now introduce

$$\begin{aligned}
\Delta_R(z) &= \operatorname{tr}_{\mathbb{C}^2}\big(R^{-1}Y_0(z, b, a)\big)/2 \\
&= \big[R_{1,1}\phi_0^{[1]}(z, b, a) + R_{2,2}\theta_0(z, b, a) - R_{2,1}\phi_0(z, b, a) - R_{1,2}\theta_0^{[1]}(z, b, a)\big]/2, \\
R &= (R_{j,k})_{1\leqslant j,k\leqslant 2} \in SL(2, \mathbb{R}), \quad z \in \mathbb{C},
\end{aligned} \tag{3.23}$$

which, for $R = I_2$ reduces to the well-known Floquet discriminant.

For the following result we recall that we use the abbreviation $\dot{} = d/dz$.

Theorem 3.2. *Assume Hypothesis 2.1, recall the definition of $T_{\varphi,R}$ in (2.16), and let $\varphi \in [0, \pi]$, $R \in SL(2, \mathbb{R})$ (cf. Remark 2.6). Then the following items (i)–(iv) hold:*

(i) *$\lambda \in \mathbb{C}$ is an eigenvalue of $T_{\varphi,R}$ if and only if $\Delta_R(\lambda) = \cos(\varphi)$; in this case, necessarily $\lambda \in \mathbb{R}$.*

(ii) *For $\lambda \in \mathbb{R}$ to be a twice degenerate eigenvalue of $T_{A,B}$, the latter must be of the form $T_{0,R}$ or $T_{\pi,R}$ for some $R \in SL(2,\mathbb{R})$.*

(iii) *The following items (a)–(c) are equivalent:*

  (a) *$\lambda \in \mathbb{R}$ is a twice degenerate eigenvalue of $T_{0,R}$.*

  (b) *$\Delta_R(\lambda) = 1$ and $\dot{\Delta}_R(\lambda) = 0$; in this case, $\ddot{\Delta}_R(\lambda) < 0$.*

  (c) *$Y_0(\lambda, b, a) = R$.*

(iv) *The following items (a)–(c) are equivalent:*

  (a) *$\lambda \in \mathbb{R}$ is a twice degenerate eigenvalue of $T_{\pi,R}$.*

  (b) *$\Delta_R(\lambda) = -1$ and $\dot{\Delta}_R(\lambda) = 0$; in this case, $\ddot{\Delta}_R(\lambda) > 0$.*

  (c) *$Y_0(\lambda, b, a) = -R$.*

Proof. Item (i) is an immediate consequence of Theorem 3.1(i), in particular,

$$\delta_{e^{i\varphi}R,I_2}(z) = 2e^{i\varphi}[\cos(\varphi) - \Delta_R(z)], \quad z \in \mathbb{C}, \tag{3.24}$$

and hence,

$$\delta_{e^{i\varphi}R,I_2}(\lambda) = 0 \text{ is equivalent to } \Delta_R(\lambda) = \cos(\varphi). \tag{3.25}$$

Similarly, item (ii) is clear from Theorem 3.1 and hence we focus on the proof of items (iii) and (iv). By Theorem 3.1(iii), items (iii)(a) and (iii)(c) are equivalent, and so are items (iv)(a) and (iv)(c). Hence, it suffices to prove the equivalence of items (iii)(b) and (iii)(c) and that of items (iv)(b) and (iv)(c), to which we turn next.

We start by recalling the fundamental solution $Y_0(z, \cdot\,, a)$ introduced in (3.2)–(3.4) and note that the function $\dot{Y}_0(z, \cdot\,, a)$ satisfies the inhomogeneous equation

$$\begin{aligned} &\dot{Y}_0'(z,x,a) = C(z,x)\dot{Y}_0(z,x,a) + \dot{C}(z,x)Y_0(z,x,a) \text{ for a.e. } x \in [a,b], \\ &\dot{Y}_0(z,a,a) = 0, \end{aligned}$$

where $C(z, \cdot\,)$ is as in (3.2) and hence

$$\dot{C}(z,x) = \begin{pmatrix} 0 & 0 \\ -r(x) & 0 \end{pmatrix} \text{ for a.e. } x \in (a,b),\, z \in \mathbb{C}. \tag{3.26}$$

It follows that for $z \in \mathbb{C}$, $x \in [a,b]$,

$$\dot{Y}_0(z,x,a) = Y_0(z,x,a) \int_a^x dx'\, Y_0(z,x',a)^{-1}\dot{C}(z,x')Y_0(z,x',a). \tag{3.27}$$

Then by (3.23) one has

$$\begin{aligned} 2\dot{\Delta}_R(z) &= \operatorname{tr}_{\mathbb{C}^2}\left(R^{-1}\dot{Y}_0(z,b,a)\right) \\ &= \operatorname{tr}_{\mathbb{C}^2}\left(R^{-1}Y_0(z,b,a) \int_a^b dx\, Y_0(z,x,a)^{-1}\dot{C}(z,x)Y_0(z,x,a)\right). \end{aligned} \tag{3.28}$$

If $Y_0(\lambda,b,a)=\pm R$, then $\Delta_R(\lambda)=\pm 1$ by (3.23) and using the cyclic property of the trace one derives from (3.28) and (3.26) that

$$\begin{aligned} 2\dot{\Delta}_R(\lambda) &= \pm\int_a^b dx\, \operatorname{tr}_{\mathbb{C}^2}\big(Y_0(\lambda,x,a)^{-1}\dot{C}(\lambda,x)Y_0(\lambda,x,a)\big) \\ &= \pm\int_a^b dx\, \operatorname{tr}_{\mathbb{C}^2}\big(\dot{C}(\lambda,x)Y_0(\lambda,x,a)Y_0(\lambda,x,a)^{-1}\big) \\ &= \pm\int_a^b dx\, \operatorname{tr}_{\mathbb{C}^2}\big(\dot{C}(\lambda,x)\big) = 0. \end{aligned} \tag{3.29}$$

Conversely, suppose $\Delta_R(\lambda)=\pm 1$ and $\dot{\Delta}_R(\lambda)=0$. It suffice to show that

$$Q = R^{-1}Y_0(\lambda,b,a)\mp I_2 \tag{3.30}$$

is the zero matrix. By assumption, $\operatorname{tr}_{\mathbb{C}^2}(R^{-1}Y_0(\lambda,b,a))=2\Delta_R(\lambda)=\pm 2$ and, by (3.4), $\det_{\mathbb{C}^2}(R^{-1}Y_0(\lambda,b,a))=1$, hence both eigenvalues of $R^{-1}Y_0(\lambda,b,a)$ equal ± 1. It follows that $\det_{\mathbb{C}^2}(Q)=0$ and $\operatorname{tr}_{\mathbb{C}^2}(Q)=0$. Next, arguing by contradiction, assume that $Q\neq 0$. Then Q is rank one and hence there are nonzero vectors $u,v\in\mathbb{R}^2$ such that $Q=uv^\top$. Using the cyclic property of the trace one notes that $0=\operatorname{tr}_{\mathbb{C}^2}(Q)=v^\top u$ so u and v are orthogonal vectors in $\mathbb{R}^2$ and hence $v=\alpha(u_2,-u_1)^\top$ for some $\alpha\neq 0$. Thus,

$$Q = \alpha(u_1,u_2)^\top(u_2,-u_1). \tag{3.31}$$

It follows from (3.3) and (3.26) that

$$\begin{aligned} &Y_0(z,x,a)^{-1}\dot{C}(z,x)Y_0(z,x,a) \\ &\quad = r(x)\begin{pmatrix} \phi_0(z,x,a)\theta_0(z,x,a) & \phi_0(z,x,a)^2 \\ -\theta_0(z,x,a)^2 & -\phi_0(z,x,a)\theta_0(z,x,a) \end{pmatrix} \\ &\quad = r(x)\big(\phi_0(z,x,a),-\theta_0(z,x,a)\big)^\top\big(\theta_0(z,x,a),\phi_0(z,x,a)\big). \end{aligned} \tag{3.32}$$

Let

$$\begin{aligned} f(x) &= (u_2,-u_1)\big(\phi_0(z,x,a),-\theta_0(z,x,a)\big)^\top \\ &= \big(\theta_0(z,x,a),\phi_0(z,x,a)\big)(u_1,u_2)^\top \end{aligned} \tag{3.33}$$

and note that $f(x)$ is real-valued and $f\neq 0$ in $L^2((a,b);r dx)$ since $\phi_0(z,\,\cdot\,,a)$ and $\theta_0(z,\,\cdot\,,a)$ are linearly independent and $u\neq 0$. Then substituting (3.30)–(3.32) into (3.28) and utilizing the cyclic property of the trace one obtains

$$\begin{aligned} 0 = 2\dot{\Delta}_R(\lambda) &= \int_a^b dx\, \operatorname{tr}_{\mathbb{C}^2}\big((Q\pm I_2)Y_0(z,x,a)^{-1}\dot{C}(z,x)Y_0(z,x,a)\big) \\ &= \int_a^b dx\, \operatorname{tr}_{\mathbb{C}^2}\big(QY_0(z,x,a)^{-1}\dot{C}(z,x)Y_0(z,x,a)\big) = \alpha\int_a^b r(x)dx\, f(x)^2, \end{aligned} \tag{3.34}$$

a contradiction. Thus, $Q=0$ and hence $Y_0(\lambda,b,a)=\pm R$.

Finally, we prove the assertions regarding $\ddot{\Delta}_R(\lambda)$. If $\Delta_R(\lambda) = \pm 1$ and $\dot{\Delta}_R(\lambda) = 0$, then, as shown above, $Y_0(\lambda, b, a) = \pm R$. Differentiating (3.28) with respect to z and employing (3.27), $Y_0(\lambda, b, a) = \pm R$, and $\ddot{C}(\lambda, x) = 0$, which is a consequence of (3.26), one obtains

$$\begin{aligned} 2\ddot{\Delta}_R(\lambda) = \pm \operatorname{tr}_{\mathbb{C}^2} \bigg(& \bigg[\int_a^b dx\, Y_0(\lambda, x, a)^{-1} \dot{C}(\lambda, x) Y_0(\lambda, x, a) \bigg]^2 \\ & - \int_a^b dx\, Y_0(\lambda, x, a)^{-1} \dot{Y}_0(\lambda, x, a) Y_0(\lambda, x, a)^{-1} \dot{C}(\lambda, x) Y_0(\lambda, x, a) \\ & + \int_a^b dx\, Y_0(\lambda, x, a)^{-1} \dot{C}(\lambda, x) \dot{Y}_0(\lambda, x, a) \bigg). \end{aligned} \tag{3.35}$$

Bringing the trace under the integral in the last two terms and using the cyclic property of the trace to rearrange the order of matrices, one notes that the last two terms in (3.35) cancel. In addition, noting that $\operatorname{tr}_{\mathbb{C}^2}(M^2) = [\operatorname{tr}_{\mathbb{C}^2}(M)]^2 - 2\det_{\mathbb{C}^2}(M)$ for any 2×2 matrix M, one then obtains

$$\begin{aligned} \pm 2\ddot{\Delta}_R(\lambda) = & \bigg[\operatorname{tr}_{\mathbb{C}^2} \bigg(\int_a^b dx\, Y_0(\lambda, x, a)^{-1} \dot{C}(\lambda, x) Y_0(\lambda, x, a) \bigg) \bigg]^2 \\ & - 2\det_{\mathbb{C}^2} \bigg(\int_a^b dx\, Y_0(\lambda, x, a)^{-1} \dot{C}(\lambda, x) Y_0(\lambda, x, a) \bigg). \end{aligned} \tag{3.36}$$

A computation as in (3.29) shows that the first term is zero. Thus, using (3.32) and the Cauchy–Schwarz inequality, one obtains

$$\begin{aligned} \pm \ddot{\Delta}_R(\lambda) = & \bigg(\int_a^b r(x) dx\, \phi_0(\lambda, x, a) \theta_0(\lambda, x, a) \bigg)^2 \\ & - \bigg(\int_a^b r(x) dx\, \phi_0(\lambda, x, a)^2 \bigg) \bigg(\int_a^b r(x) dx\, \theta_0(\lambda, x, a)^2 \bigg) \leqslant 0. \end{aligned} \tag{3.37}$$

In fact, the inequality must be strict since $\phi_0(\lambda, \cdot, a)$ and $\theta_0(\lambda, \cdot, a)$ are linearly independent. Thus, $\ddot{\Delta}_R(\lambda) \lessgtr 0$, completing the proof. □

Remark 3.3. Regarding Theorem 3.1 we refer to [17, Lemmas 3.2.2 and 3.2.3], see also [10, Sects. 3.3, 3.4], [12, Ch. I, § 2]. Somehow, parts (b) and (b) in Theorem 3.2 appear to have escaped notice in the pertinent literature we are aware of. For relevant literature in connection with Theorem 3.2 we refer to [1,5,8,9], [4, Sect. 2.3], and [17, Sects. 4.7 and 4.8]. In connection with Remark 3.4 see also [7]. For additional literature in this connection see, for instance, [15,18].

Remark 3.4. We led the reader through the scenic, but rather long, route in connection with eigenvalues and their multiplicities in this section. A much quicker tour, but at the expense of a number of details, relies on the Fredholm determinant approach. For instance, in the case of coupled boundary conditions, one infers that

$$F_{\varphi,R}(z)/F_{\varphi,R}(z_0) = [\Delta_R(z) - \cos(\varphi)]/[\Delta_R(z_0) - \cos(\varphi)]$$

$$
\begin{aligned}
&= \det_{L^2((a,b);r dx)}\big(I - (z-z_0)(T_{\varphi,R} - z_0 I)^{-1}\big) \\
&= \det_{L^2((a,b);r dx)}\big((T_{\varphi,R} - zI)(T_{\varphi,R} - z_0 I)^{-1}\big) \\
&= \bigg(\frac{z}{z_0}\bigg)^{m(0;T_{\varphi,R})} \prod_{\substack{j\in\mathbb{J} \\ \lambda_j \in \sigma(T_{\varphi,R})\backslash\{0\}}} \bigg(\frac{1-(z/\lambda_j)}{1-(z_0/\lambda_j)}\bigg)^{m(\lambda_j;T_{\varphi,R})} \\
&= \prod_{j\in\mathbb{J}} \bigg(\frac{\lambda_j - z}{\lambda_j - z_0}\bigg)^{m(\lambda_j;T_{\varphi,R})},
\end{aligned} \tag{3.38}
$$

where $\mathbb{J} \subseteq \mathbb{N}$ is an appropriate index set and $m(\lambda_j;T_{\varphi,R})$ represents the multiplicity of the eigenvalue λ_j of $T_{\varphi,R}$. Thus, λ_j is a (twice) degenerate eigenvalue of $T_{0,R}$ if and only if $F_{0,R}(\lambda_j) = 0$ and $\dot{F}_{0,R}(\lambda_j) = 0$. Since $T_{\varphi,R}$ is a second-order operator, $1 \leqslant m(\lambda_j;T_{\varphi,R}) \leqslant 2$, and thus one also concludes that $\ddot{F}_{0,R}(\lambda_j) \neq 0$. Thus, some parts of multiplicity theory for eigenvalues of regular Sturm–Liouville operators follow a bit quicker if one is willing to systematically invoke Fredholm determinant theory.

Remark 3.5. Theorem 3.2 is patterned after the well-known periodic case, or, Floquet theory, and the standard Floquet discriminant D now corresponds to the special case $R = I_2$ in (3.23), that is,

$$
D(z) = \Delta_{I_2}(z), \quad z \in \mathbb{C}, \tag{3.39}
$$

see, for instance, [2, Ch. 1], [3, Sect. 3.5], [4, Chs. 1, 2], and [11, Chs. I, II].

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Про кратність власних значень самоспряжених регулярних операторів Штурма–Ліувілля

Fritz Gesztesy, Roger Nichols, and Maxim Zinchenko

Ми надаємо вичерпне дослідження кратності власних значень усіх самоспряжених регулярних задач Штурма–Ліувілля на компактних інтервалах $[a, b] \subset \mathbb{R}$.

Ключові слова: оператори Штурма–Ліувілля, кратність власних значень